Resolution-based Theorem Proving for SHn-Logics

EXTENDED ABSTRACT
Viorica Sofronie-Stokkermans
Max-Planck-Institut für Informatik, D-66123 Saarbrücken, Germany
e-mail: sofronie@mpi-sb.mpg.de

Abstract

In this paper we illustrate by means of an example, namely SHn-logics, a method for translation to clause form and automated theorem proving for first-order many-valued logics based on distributive lattices with operators.

1 Introduction

The main goal of this paper is to present a method for translation to clause form and automated theorem proving in finitely-valued logics having as algebras of truth values distributive lattices with certain types of operators. Many non-classical logics that occur in practical applications fall in this class. One of the advantages of distributive lattices (with well-behaved operators) is the existence, in such cases, of good representation theorems, such as the Priestley representation theorem. The method for translation to clause form we present uses the Priestley dual of the algebra of truth values. The ideas behind this method are very natural, even if the algebraic notions used may at first sight seem involved. This is why in this paper we illustrate the ideas by one example, namely SHn-logics. The particular properties of SHn-logics allow us to further improve the efficiency of the automated theorem proving procedure for certain types of formulae in SHn-logics, by exploiting the structure of the Priestley dual of the algebra of truth values. This effect is difficult to explain in a general setting; the case of SHn-logics can be considered a case-study, in which we take first steps in this direction.

The main sources of inspiration for our work are the many-valued resolution method of Baaz and Fermüller [2], and the results due to Hähnle [5]. In this paper we use the particular structure of the algebra of truth values: our method leads to a reduction of the number of clauses compared to methods using the algebra of truth values, when the difference between the number of elements of the algebra of truth values and the number of elements of its Priestley dual is large. Moreover, resolution procedures are discussed (in particular, a negative hyperresolution procedure that extends the results established for regular clauses in [5]).

The paper is structured as follows. In Section 2 we briefly present the main notions needed in the paper. In Section 3 SHn-logics are defined. In Section 4 our method for translation to clause form for SHn-logics is presented. In Section 5 resolution-based methods for automated theorem proving are briefly presented.

2 Preliminaries

Partially-ordered sets and lattices. In what follows we assume known standard notions, such as partially-ordered set and lattice, order-filter and order-ideal in partially-ordered sets, meet- and join-irreducible elements, and (prime) filter and (prime) ideal in lattices. For the definitions and further informations we refer to [3]. Given a partially-ordered set $(P, \leq)$, by $\mathcal{O}(P)$ we will denote the set of order-filters of P; for every $i \in P$, we will use the following notations: $\uparrow i = \{j \in P \mid j \geq i\}$, and $\downarrow i = \{j \in P \mid i \geq j\}$. Note also that every ideal in a finite distributive lattice is of the form $\downarrow i$ and every filter is of the form $\uparrow i$ for some element i in the lattice); such an ideal (resp. filter) is prime iff i is meet-irreducible (resp. join-irreducible).

Priestley duality. The Priestley representation theorem [3] states that every distributive lattice A is isomorphic to the lattice of clopen (i.e. closed and open) order filters of the ordered topological space having as points the prime filters of A, ordered by inclusion, and the topology generated by the sets of the form $X_a = \{F \mid F \text{ prime filter}, a \in F\}$ and their complements as a subbasis. The partially ordered set of all prime filters of A, ordered by inclusion, and endowed with the topology mentioned above will be denoted $D(A)$ (we will refer to it as the dual of A). Given an ordered topological space $X = (X, \leq, \tau)$, its lattice of clopen order filters will be denoted by $\mathsf{ClopenOF(X)}$ or $E(X)$. In particular, if A is finite, the topology on $D(A)$ is discrete, and A is isomorphic to $\mathcal{O}(D(A))$.

Many-valued logics. We briefly define the syntax and semantics of many-valued logics. Let A be a (finite) set of truth values. The semantics of a many-valued logic $\mathcal{L}$ with language $(X, O, P, \Sigma, \{Q_1, \dots, Q_k\})$ (where X is an infinite (countable) set of variables; O a set of function symbols; P a set of predicate symbols; $\Sigma = \{\sigma_1, \dots, \sigma_r\}$ a set of logical operators; and $Q_1, \dots, Q_k$ are (one-place) quantifiers) is given as follows: (i) to every $\sigma \in \Sigma$ with arity n we associate a truth function $\sigma_A : A^n \to A$; (ii) to every quantifier Q we associate a truth function $\overline{Q} : \mathcal{P}(A) \backslash \emptyset \to A$. An interpretation for a language $(X, O, P, \Sigma, \{Q_1, \dots, Q_k\})$ and a set of truth values A is a tuple (D, I, d) where D is a non-empty set, the *domain*, I is a signature interpretation, i.e. a function assigning a function $I(f) : D^n \to D$ to every n-ary function symbol $f \in O$, and a function $I(R) : D^n \to A$ to every n-ary predicate symbol $R \in P$, and $d : X \to D$ a *variable assignment*. Every interpretation $\mathcal{I} = (D, I, d)$ induces a valuation $v_{\mathcal{I}} : \mathsf{Fma}(\mathcal{L}) \to A$ on the set $\mathsf{Fma}(\mathcal{L})$ of formulae of $\mathcal{L}$. For details we refer to [2]. A formula ϕ will be called *valid* iff for all interpretations $\mathcal{I}$, $v_{\mathcal{I}}(\phi) = 1$; ϕ will be called *satisfiable* iff there exists an interpretation $\mathcal{I}$ with $v_{\mathcal{I}}(\phi) = 1$.

3 SHn-logics

The propositional SHn-logics were introduced by Iturrioz in [7]. The language of SHn-logics is a propositional language, whose formulas are built from propositional variables taken from a set Var, with operations $\vee$ (disjunction), $\wedge$ (conjunction), $\Rightarrow$ (intuitionistic implication), $\sim, \neg$ (a De Morgan resp. an intuitionistic negation), and a family $\{S_i \mid i = 1, \dots, n-1\}$ of unary operations (expressing the degree of truth of a formula). Recently, Iturrioz noticed that SHn-logics provide a relevant source of examples for L_{t_f}-logics, which have been introduced by Rasiowa to formalize the reasoning of a poset of intelligent agents. A Hilbert style axiomatization of SHn-logics is given in [8] and [9]. We present it below:

Axioms: (A1) $a \Rightarrow (b \Rightarrow a)$
(A2) $(a \Rightarrow (b \Rightarrow c)) \Rightarrow ((a \Rightarrow b) \Rightarrow (a \Rightarrow c))$
(A3) $(a \wedge b) \Rightarrow a$
(A4) $(a \wedge b) \Rightarrow b$
(A5) $(a \Rightarrow b) \Rightarrow ((a \Rightarrow c) \Rightarrow (a \Rightarrow (b \wedge c)))$
(A6) $a \Rightarrow (a \vee b)$
(A7) $b \Rightarrow (a \vee b)$
(A8) $(a \Rightarrow c) \Rightarrow ((b \Rightarrow c) \Rightarrow ((a \vee b) \Rightarrow c))$
(A9) $\sim\sim a \leftrightarrow a$
(A10) $S_i(a \wedge b) \leftrightarrow S_i(a) \wedge S_i(b)$
(A11) $S_i(a \Rightarrow b) \leftrightarrow (\bigwedge_{k=i}^{n} S_k(a) \Rightarrow S_k(b))$
(A12) $S_i(S_j(a)) \leftrightarrow S_j(a)$, for every $i, j = 1, \ldots, n-1$
(A13) $S_1(a) \Rightarrow a$
(A14) $S_i(\sim a) \leftrightarrow \sim S_{n-i}a$, for $i = 1, \ldots, n-1$
(A15) $S_1(a) \vee \neg S_1(a)$

where $\neg(a) = (a \Rightarrow \sim (a \Rightarrow a))$ and $a \leftrightarrow b$ is an abbreviation for $(a \Rightarrow b) \wedge (b \Rightarrow a)$.

Inference rules:

$$\text{(R1)}\ \frac{a, \quad a \Rightarrow b}{b} \qquad \text{(R2)}\ \frac{a \Rightarrow b}{\sim b \Rightarrow \sim a} \qquad \text{(R3)}\ \frac{a \Rightarrow b}{S_1(a) \Rightarrow S_1(b)}$$

3.1 Algebraic semantics for propositional SHn-logics

In [7, 8], Iturrioz gave a lattice-based semantics for SHn-logics, by means of symmetrical Heyting algebras of order n, or for short SHn-algebras.

Definition 1 *An abstract algebra* $A = (A, 0, 1, \wedge, \vee, \Rightarrow, \neg, \sim, S_1, \ldots, S_{n-1})$ *is said to be a* symmetric Heyting algebra of order n *(SHn-algebra for short) if:*

(1) $(A, 0, 1, \wedge, \vee, \Rightarrow, \neg)$ *is a Heyting algebra,*

(2) $\sim$ *is a De Morgan negation on* A*,*

(3) For every $a, b \in A$ *and for all* $i, j \in \{1, \ldots, n-1\}$*, the following equations hold:*

(S1) $S_i(a \wedge b) = S_i(a) \wedge S_i(b)$,
(S2) $S_i(a \Rightarrow b) = (\bigwedge_{k=i}^{n} S_k(a) \Rightarrow S_k(b))$,
(S3) $S_i(S_j(a)) = S_j(a)$, *for every* $i, j = 1, \ldots, n-1$,
(S4) $S_1(a) \vee a = a$,
(S5) $S_i(\sim a) = \sim S_{n-i}(a)$, *for* $i = 1, \ldots, n-1$,
(S6) $S_1(a) \vee \neg S_1(a) = 1$, *with* $\neg a = a \Rightarrow 0$.

The class of SHn-algebras is a variety, which will be denoted SHn in what follows. Iturrioz [8] proved that this variety is generated by one finite SHn-algebra S_{n^2} (represented in the left-hand side of Figure 1) defined as follows.

Definition 2 *Let* $n \geq 2$ *and let* S_{n^2} *be the cartesian product* $L_n \times L_n$ *where* $L_n = \{0, \frac{1}{n-1}, \ldots, \frac{n-1}{n-1}\}$*. Consider the following operations on* S_{n^2}*:*

(1) $(x_1, y_1) \wedge (x_2, y_2) = (\mathsf{min}(x_1, x_2), \mathsf{min}(y_1, y_2))$,

(2) $(x_1, y_1) \vee (x_2, y_2) = (\mathsf{max}(x_1, x_2), \mathsf{max}(y_1, y_2))$,

(3) $\sim (x, y) = (1 - y, 1 - x)$ *for every* $(x, y) \in S_{n^2}$,

(4) $S_i(x, y) = (S_i(x), S_i(y))$, *where* $S_i(\frac{j}{n-1}) = \begin{cases} 1 & \text{if } i + j \geq n, \\ 0 & \text{if } i + j < n, \end{cases}$

(5) $(x_1, y_1) \Rightarrow (x_2, y_2) = (x_1 \Rightarrow x_2, y_1 \Rightarrow y_2)$, *where* $\Rightarrow$ *is the Heyting relative pseudocomplementation*[1] *on* L_n,

(6) $\neg(x, y) = (x \Rightarrow 0, y \Rightarrow 0)$.

In [7] it is proved that SHn-logics are sound and complete with respect to the variety of SHn-algebras. Since the variety SHn is generated by S_{n^2} it follows that the SHn-logics can be regarded as many-valued logics having S_{n^2} as an algebra of truth values.

A Priestley representation theorem for SHn-algebras is established in [8], see also [12]. In particular we give a description of the dual of the generator S_{n^2} of the variety of SHn-algebras. The Priestley dual of S_{n^2}, $D(S_{n^2})$ is the set of prime filters of S_{n^2}, with the

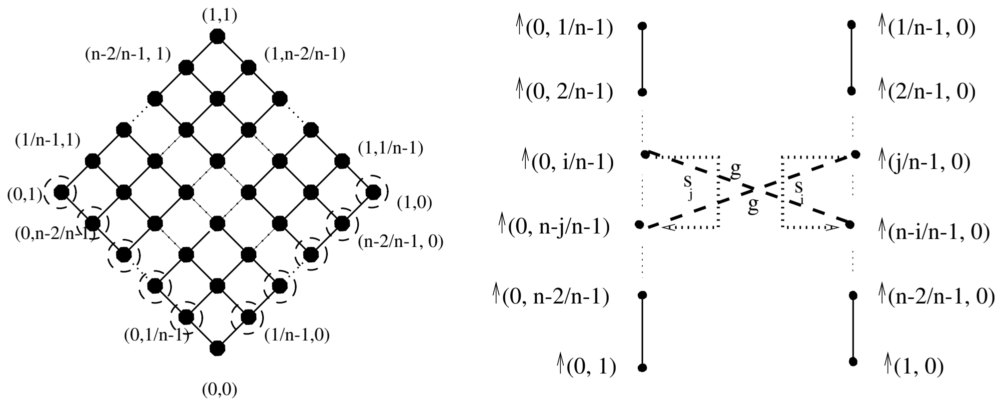

Figure 1: S_{n^2} and its Priestley dual $D(S_{n^2})$

discrete topology, and ordered by inclusion. Every prime filter of S_{n^2} is of the form $\uparrow a$ where a is a join-irreducible element of S_{n^2}. The set of join-irreducible elements of S_{n^2} (encircled in Figure 1) is $\{(0, \frac{i}{n-1}) \mid i = 1, \ldots, n-1\} \cup \{(\frac{i}{n-1}, 0) \mid i = 1, \ldots, n-1\}$ with the order defined pointwise. Thus, $D(S_{n^2})$ (the right-hand side of Figure 1) is the set $\{\uparrow (0, \frac{i}{n-1}) \mid i = 1, \ldots, n-1\} \cup \{\uparrow (\frac{i}{n-1}, 0) \mid i = 1, \ldots, n-1\}$. Additional operations $g, s_1, \ldots, s_{n-1}$ are defined for every prime filter F of S_{n^2} by $g(F) = S_{n^2} \setminus \sim^{-1} (F)$, and $s_i(F) = S_i^{-1}(F)$. Therefore, $g(\uparrow (0, \frac{i}{n-1})) = \ \uparrow (\frac{n-i}{n-1}, 0)$; $g(\uparrow (\frac{i}{n-1}, 0)) = \ \uparrow (0, \frac{n-i}{n-1})$; and, for every $j = 1, \ldots, n-1$, $s_j(\uparrow (0, \frac{i}{n-1})) = \ \uparrow (0, \frac{n-j}{n-1})$; $s_j(\uparrow (\frac{i}{n-1}, 0)) = \ \uparrow (\frac{n-j}{n-1}, 0)$. By

[1]The Heyting relative pseudocomplementation on L_n is defined by: $a \Rightarrow b$ is the largest element c of L_n such that $a \wedge c \leq b$. Hence, for every $a, b \in L_n$, $a \Rightarrow b = \begin{cases} 1 & \text{if } a \leq b \\ b & \text{if } a > b \end{cases}$. The pseudocomplementation induced by $\Rightarrow$ is defined by $\neg a := a \Rightarrow 0$. Hence, $\neg a = \begin{cases} 1 & \text{if } a = 0 \\ 0 & \text{if } a > 0 \end{cases}$.

the Priestley representation theorem for SHn-algebras, since the topology of $D(S_{n^2})$ is discrete, it follows that there exists an isomorphism $\eta : S_{n^2} \simeq \mathcal{O}(D(S_{n^2}))$.

3.2 A finite Kripke-style frame for SHn-logics

In [9], Iturrioz and Orłowska give a completeness theorem for SHn-logics with respect to a Kripke-style semantics. We will show that one can regard $D(S_{n^2})$ as a particular Kripke-style frame for SHn-logics, and that a formula is an SHn-theorem if and only if it is valid in this frame. As in [9] where Kripke-style models for SHn-logics are investigated, we will assume that all meaning functions for $D(S_{n^2})$ have as values order-filters.

Definition 3 *Let $m : \mathsf{Var} \to \mathcal{O}(D(S_{n^2}))$ be a meaning function, and let $\overline{m} : \mathsf{Fma}(\mathsf{Var}) \to \mathcal{O}(D(S_{n^2}))$ be the unique extension of m to formulae. We define: (i) $D(S_{n^2}) \overset{r}{\models}_{m,x} \phi$ iff $x \in \overline{m}(\phi)$; (ii) $D(S_{n^2}) \overset{r}{\models}_m \phi$ iff $\overline{m}(\phi) = D(S_{n^2})$; (iii) $D(S_{n^2}) \overset{r}{\models} \phi$ iff $D(S_{n^2}) \overset{r}{\models}_m \phi$ for every m: $\mathsf{Var} \to \mathcal{O}(D(S_{n^2}))$.*

Proposition 1 ([12]) *Every formula ϕ of SHn-logic is a SHn-theorem iff $D(S_{n^2}) \overset{r}{\models} \phi$.*

$(D(S_{n^2}), \leq, g, s_1, \ldots, s_{n-1})$ is in particular a Kripke-style frame as defined in [9], and the relation $\overset{r}{\models}$ defined above agrees with the satisfiability relation for SHn-models defined in [9], i.e. for every meaning function $m : \mathsf{Var} \to \mathcal{O}(D(A))$, the following hold:

$$
\begin{array}{lll}
D(S_{n^2}) \overset{r}{\models}_{m,x} p & \text{iff} & x \in m(p), \qquad \text{for } p \in \mathsf{Var}, \\
D(S_{n^2}) \overset{r}{\models}_{m,x} \phi \vee \psi & \text{iff} & D(S_{n^2}) \overset{r}{\models}_{m,x} \phi \text{ or } D(S_{n^2}) \overset{r}{\models}_{m,x} \psi, \\
D(S_{n^2}) \overset{r}{\models}_{m,x} \phi \wedge \psi & \text{iff} & D(S_{n^2}) \overset{r}{\models}_{m,x} \phi \text{ and } D(S_{n^2}) \overset{r}{\models}_{m,x} \psi, \\
D(S_{n^2}) \overset{r}{\models}_{m,x} \phi \Rightarrow \psi & \text{iff} & \text{for all } y, \text{ if } x \leq y \text{ and } D(S_{n^2}) \overset{r}{\models}_{m,y} \phi \text{ then } D(S_{n^2}) \overset{r}{\models}_{m,y} \psi, \\
D(S_{n^2}) \overset{r}{\models}_{m,x} \neg \phi & \text{iff} & \text{for all } y, \text{ if } x \leq y \text{ then } D(S_{n^2}) \overset{r}{\not\models}_{m,y} \phi, \\
D(S_{n^2}) \overset{r}{\models}_{m,x} S_i(\phi) & \text{iff} & D(S_{n^2}) \overset{r}{\models}_{m,s_i(x)} \phi, \\
D(S_{n^2}) \overset{r}{\models}_{m,x} \sim \phi & \text{iff} & D(S_{n^2}) \overset{r}{\not\models}_{m,g(x)} \phi.
\end{array}
$$

3.3 First-order SHn-logics

Following [2] we define first-order SHn-logics as many-valued logics having S_{n^2} as set of truth values.

A first-order SHn-logic is defined by specifying a set X of variables, a set O of operation symbols and a set P of predicate symbols. The set of logical connectives is $\{\vee, \wedge, \Rightarrow, \neg, \sim\}$, and they are interpreted in S_{n^2} as explained in Section 3.1. We allow only two quantifiers, namely $\exists$ and $\forall$ (interpreted as a generalized join, resp. meet).

Frames and interpretations are defined in the usual way; note that due to the fact that S_{n^2} is isomorphic to $\mathcal{O}(S_{n^2})$, an alternative (and equivalent) notion of frame (and resp. interpretation) can be defined, where the signature interpretation I assigns with every n-ary predicate symbol $R \in P$ a function $I(R) : D^n \to \mathcal{O}(D(A))$. For every interpretation in $\mathcal{O}(D(A))$, $\mathcal{I} = (D, I, d)$, the induced valuation function $v_{\mathcal{I}} : \mathsf{Fma}(\mathcal{L}) \to \mathcal{O}(D(A))$ is defined in the usual way; so are the notions of validity and satisfiability. Taking into account the fact that the largest element of the lattice $\mathcal{O}(D(A))$ is $D(A)$, it follows that a formula ϕ is valid iff for all interpretations $\mathcal{I}$ in $\mathcal{O}(D(A))$, $v_{\mathcal{I}}(\phi) = D(A)$; and satisfiable iff $v_{\mathcal{I}}(\phi) = D(A)$ for some interpretation $\mathcal{I}$ in $\mathcal{O}(D(A))$.

4 Translation into clause form

The main idea of our approach is to use signed literals, where the signs are "possible worlds", i.e. elements of $D(S_{n^2})$ (corresponding to prime filters of truth values) instead of truth values (as done in [2]) or arbitrary sets of truth values (as done in [4, 5]). The idea of using "valuations in $\{0,1\}$" instead of values appears already for instance in [11] for the case of Łukasiewicz logics. In what follows, given an interpretation $\mathcal{I} = (D, I, d)$ in $\mathcal{O}(D(S_{n^2}))$ we use the following notation:

$\boxed{\alpha}\ \phi^t$ in $\mathcal{I}$ means "ϕ is true at α" in the interpretation $\mathcal{I}$ (i.e. $\alpha \in v_{\mathcal{I}}(\phi)$)
$\boxed{\alpha}\ \phi^f$ in $\mathcal{I}$ means "ϕ is false at α" in the interpretation $\mathcal{I}$ (i.e. $\alpha \notin v_{\mathcal{I}}(\phi)$)

where α is an element of $D(S_{n^2})$. Note that if $\boxed{\beta}\ \phi^t$ in $\mathcal{I}$ and $\beta \leq \alpha$ then $\boxed{\alpha}\ \phi^t$ in $\mathcal{I}$.

Definition 4 (Literal, Clause, Signed CNF, Satisfiability) *Let L be an atomic formula and $\alpha \in D(S_{n^2})$. Then $\boxed{\alpha}\ L^t$ is a* positive literal (with sign $\boxed{\alpha}$) *and $\boxed{\alpha}\ L^f$ is a* negative literal (with sign $\boxed{\alpha}$). *A set of (positive or negative) signed literals is called a* (signed) clause. *A* formula in signed conjunctive normal form (CNF) *is a finite set of signed clauses. We require that the clauses in a formula have disjoint variables.*

A positive literal $\boxed{\alpha}\ L^t$ (resp. a negative literal $\boxed{\alpha}\ L^f$) is satisfiable if for some interpretation $\mathcal{I}$ of $\mathcal{L}$ in $\mathcal{O}(D(A))$, L is true (resp. false) in $\mathcal{I}$ at α. A signed clause is satisfiable if at least one of its literals is satisfiable. A formula Φ in signed CNF is satisfiable if all clauses in Φ are simultaneously satisfiable by the same interpretation.

We give a *structure-preserving* transformation method to clause form in first-order logic. The main idea of such methods is to introduce, for every non-atomic subformula ψ of ϕ, a new atomic formula of the form $P_\psi(\overline{x})$, where P_ψ is a new predicate symbol and $\overline{x}$ are the free variables in ψ.

The next result is a consequence of the fact that a formula ϕ is valid iff there exists no interpretation $\mathcal{I}$ in $\mathcal{O}(D(S_{n^2}))$ such that ϕ is false in $\mathcal{I}$ at some world $\alpha \in D(S_{n^2})$ (and, thus, at some minimal world), taking into account the renamings mentioned before.

Proposition 2 *The formula ϕ is valid if and only if there exists no interpretation $\mathcal{I} = (D, I, d)$ of $\mathcal{L}$ in $\mathcal{O}(D(S_{n^2}))$ such that*

$$\left\{\begin{array}{c}
\boxed{\uparrow(0,1)}\ P_\phi^f \vee \boxed{\uparrow(1,0)}\ P_\phi^f \ \textit{in } \mathcal{I}, \\[2ex]
(\boxed{\alpha}\ P_\psi(x_1,\ldots,x_n)^t \vee \boxed{\alpha}\ \sigma(P_{\psi_1},\ldots,P_{\psi_m})^f) \\
(\boxed{\alpha}\ P_\psi(x_1,\ldots,x_n)^f \vee \boxed{\alpha}\ \sigma(P_{\psi_1},\ldots,P_{\psi_m})^t) \\
\textit{in } \mathcal{I};\ \textit{for every } \alpha \in D(A) \textit{ for all subformulae } \psi = \sigma(\psi_1,\ldots,\psi_m) \\
\textit{of } \phi \textit{ and all instantiations in } D \textit{ of the free variables } x_1,\ldots,x_n \textit{ of } \psi, \\[2ex]
(\boxed{\alpha}\ P_\psi(x_1,\ldots,x_n)^t \vee \boxed{\alpha}\ (Qx)P_{\psi_1}(x,x_1,\ldots,x_n)^f) \\
(\boxed{\alpha}\ P_\psi(x_1,\ldots,x_n)^f \vee \boxed{\alpha}\ (Qx)P_{\psi_1}(x,x_1,\ldots,x_n)^t) \\
\textit{in } \mathcal{I};\ \textit{for every } \alpha \in D(A) \textit{ for all subformulae } \psi = (Qx)\psi_1(x,x_1,\ldots,x_n) \\
\textit{of } \phi \textit{ and all instantiations in } D \textit{ of the free variables } x_1,\ldots,x_n \textit{ of } \psi.
\end{array}\right.$$

The elimination rules for the operators and quantifiers use the correspondence between the operations on S_{n^2} and the corresponding operations on $D(S_{n^2})$, as showed below.

Lemma 3 *For any atomic formulae $L, L_1, L_2, \ldots, L_n$, in any given interpretation $\mathcal{I}$:*

$$
\begin{array}{llllllll}
(\vee, \mathrm{t}) & \boxed{\alpha}\,(L_1 \vee L_2)^t & \text{iff} & \boxed{\alpha}\,L_1^t \vee \boxed{\alpha}\,L_2^t; & (\vee, \mathrm{f}) & \boxed{\alpha}\,(L_1 \vee L_2)^f & \text{iff} & \boxed{\alpha}\,L_1^f \wedge \boxed{\alpha}\,L_2^f. \\
(\wedge, \mathrm{t}) & \boxed{\alpha}\,(L_1 \wedge L_2)^t & \text{iff} & \boxed{\alpha}\,L_1^t \wedge \boxed{\alpha}\,L_2^t; & (\wedge, \mathrm{f}) & \boxed{\alpha}\,(L_1 \wedge L_2)^f & \text{iff} & \boxed{\alpha}\,L_1^f \vee \boxed{\alpha}\,L_2^f. \\
(S_j, \mathrm{t}) & \boxed{\alpha}\,(S_j(L_1))^t & \text{iff} & \boxed{s_j(\alpha)}\,L_1^t; & (S_j, \mathrm{f}) & \boxed{\alpha}\,(S_j(L_1))^f & \text{iff} & \boxed{s_j(\alpha)}\,L_1^f. \\
(\sim, \mathrm{t}) & \boxed{\alpha}\,(\sim (L_1))^t & \text{iff} & \boxed{g(\alpha)}\,L_1^f; & (\sim, \mathrm{f}) & \boxed{\alpha}\,(\sim (L_1))^f & \text{iff} & \boxed{g(\alpha)}\,L_1^t. \\
(\neg, \mathrm{t}) & \boxed{\alpha}\,(\neg L)^t & \text{iff} & \forall \beta \geq \alpha, \boxed{\beta}\,L^f; & (\neg, \mathrm{f}) & \boxed{\alpha}\,(\neg L)^f & \text{iff} & \exists \beta \geq \alpha \text{ with } \boxed{\beta}\,L^t. \\
(\Rightarrow, \mathrm{t}) & \boxed{\alpha}\,(L_1 \Rightarrow L_2)^t & \text{iff} & \forall \beta \geq \alpha, \boxed{\beta}\,L_1^f \vee \boxed{\beta}\,L_2^t. & & & & \\
(\Rightarrow, \mathrm{f}) & \boxed{\alpha}\,(L_1 \Rightarrow L_2)^f & \text{iff} & \multicolumn{5}{l}{\boxed{\mathsf{max}\{\beta \mid \beta \geq \alpha\}}\,L_1^t, \boxed{\alpha}\,L_2^f, \text{ and } \forall \alpha_1 \neq \alpha_2 \geq \alpha, \boxed{\alpha_1}\,L_1^t \vee \boxed{\alpha_2}\,L_2^f.}
\end{array}
$$

Proof: The cases $(\vee, t)$, $(\vee, f)$, $(\wedge, t)$, $(\wedge, f)$, (S_j, t), (S_j, f), $(\sim, t)$, $(\sim, f)$, $(\neg, t)$, $(\neg, f)$, and $(\Rightarrow, t)$ follow immediately. We present the case $(\Rightarrow, f)$ in detail. We know that $\boxed{\alpha}\,(p_1 \Rightarrow p_2)^f$ iff for some $\beta \geq \alpha$, $\boxed{\beta}\,p_1^t$ and $\boxed{\beta}\,p_2^f$. By distributivity, the formula $\bigvee_{\beta \geq \alpha}(\boxed{\beta}\,p_1^t \wedge \boxed{\beta}\,p_2^f)$ can be written as $\bigwedge_{\substack{S_1, S_2, S_1 \cap S_2 = \emptyset \\ S_1 \cup S_2 = \{\beta \mid \beta \geq \alpha\}}} \left(\bigvee_{\beta_1 \in S_1} \boxed{\beta_1}\,p_1^t \vee \bigvee_{\beta_2 \in S_2} \boxed{\beta_2}\,p_2^f\right)$.

We then use the fact that, for every $\alpha \in D(S_{n^2})$, the set $\{\beta \mid \beta \geq \alpha\}$ is finite and totally ordered, hence every non-empty set $S_1 \subseteq \{\beta \mid \beta \geq \alpha\}$, contains a maximal element $\mathsf{max}(\mathsf{S}_1)$ and every non-empty $S_2 \subseteq \{\beta \mid \beta \geq \alpha\}$ contains a minimal element $\mathsf{min}(\mathsf{S}_2)$, and $\bigvee_{\beta_1 \in S_1} \boxed{\beta_1}\,p_1^t$ iff $\boxed{\mathsf{max}(S_1)}\,p_1^t$, resp. $\bigvee_{\beta_2 \in S_2} \boxed{\beta_2}\,p_2^f$ iff $\boxed{\mathsf{min}(S_2)}\,p_2^f$. Additionally, since $S_1 \cap S_2 = \emptyset$, $\mathsf{max}(\mathsf{S}_1) \neq \mathsf{min}(\mathsf{S}_2)$. Hence,

$$
\begin{array}{ll}
\boxed{\alpha}\,(p_1 \Rightarrow p_2)^f \quad \text{iff} & \boxed{\mathsf{max}\{\beta \mid \beta \geq \alpha\}}\,p_1^t \wedge \bigwedge_{\substack{S_1, S_2 \neq \emptyset, S_1 \cap S_2 = \emptyset \\ S_1 \cup S_2 = \{\beta \mid \beta \geq \alpha\}}} \left(\boxed{\mathsf{max}(S_1)}\,p_1^t \vee \boxed{\mathsf{min}(S_2)}\,p_2^f\right) \wedge \boxed{\alpha}\,p_2^f, \\
\text{iff} & \boxed{\mathsf{max}\{\beta \mid \beta \geq \alpha\}}\,p_1^t \wedge \bigwedge_{\substack{\alpha_1, \alpha_2 \geq \alpha, \\ \alpha_1 \neq \alpha_2}} \left(\boxed{\alpha_1}\,p_1^t \vee \boxed{\alpha_2}\,p_2^f\right) \wedge \boxed{\alpha}\,p_2^f.
\end{array}
$$

□

For quantifiers we have the following results (for related results we refer to [6]).

Lemma 4 *Let $\mathcal{I} = (D, I, d)$ be an interpretation of L in $\mathcal{O}(D(A))$. The following hold:*

($\forall$) $\boxed{\alpha}\,(\forall x L(x, x_1, \ldots, x_n))^t$ *in $\mathcal{I}$ if and only if* $\boxed{\alpha}\,(L(x, x_1, \ldots, x_n))^t$ *in $\mathcal{I}_{x/d}$ for every instantiation $d \in D$ of x.*

($\exists$) $\boxed{\alpha}\,(\exists x L(x, x_1, \ldots, x_n))^t$ *in $\mathcal{I}$ if and only if* $\boxed{\alpha}\,(L(f_\phi(x_1, \ldots, x_n), x_1, \ldots, x_n))^t$ *in $\mathcal{I}$ (where f_ϕ is a new function symbol).*

It is easy to see that for every $\alpha \in D(A)$ the branching factor induced by the rules above is 3 in case of $\vee$ and $\wedge$; 2 for $S_1, \ldots, S_{n-1}$, $\sim$, and $\forall, \exists$; $\mathcal{O}(n)$ for $\neg$; and $\mathcal{O}(n^2)$ (i.e. linear in the size of the set of truth values) for $\Rightarrow$.

After this translation, from any formula ϕ we obtain a formula Φ in clause form, containing literals of the form $\boxed{\alpha}\,L^t$ and $\boxed{\alpha}\,L^f$ where L is an atom and $\alpha \in D(S_{n^2})$, such that ϕ is a theorem iff Φ is unsatisfiable.

Proposition 5 *The number of clauses generated from a given formula ϕ is $O(n^3 l)$, where l is the number of subformulae of ϕ, and n^2 the number of truth values. If the formula ϕ does not contain the connective $\Rightarrow$, then the number of clauses generated from ϕ is $O(n^2 l)$. If the formula ϕ does not contain the connectives $\Rightarrow$ and $\neg$, then the number of clauses generated from ϕ is $O(nl)$.*

Proof: The maximal number of clauses is generated by the subformulae of the form $\psi = \psi_1 \Rightarrow \psi_2$. In this case, for every $\alpha \in D(S_{n^2})$ the number of clauses generated by

($\boxed{\alpha}\, p^f_\psi \vee \boxed{\alpha}\, \psi^t$) is $|\{\beta \mid \beta \geq \alpha\}|$, and the number of clauses generated by ($\boxed{\alpha}\, p^t_\psi \vee \boxed{\alpha}\, \psi^f$) is $2+|\{(\alpha_1, \alpha_2) \mid \alpha_1 \neq \alpha_2, \alpha_1, \alpha_2 \geq \alpha\}|$. Thus, the number of clauses generated from a given formula ϕ has as upper bound $1 + 2l\Sigma_{i=1}^{n-1}(i + i(i-1) + 2) = 1 + 4l(n-1) + l\frac{(n-1)n(2n-1)}{3}$. Hence, the number of clauses generated from a given formula ϕ is $O(n^3 l)$. If ϕ does not contain the operator $\Rightarrow$, then the maximal number of clauses is generated by the subformulae of the form $\psi = \neg\psi_1$. In this case, for every $\alpha \in D(S_{n^2})$ the number of clauses generated by ($\boxed{\alpha}\, p^f_\psi \vee \boxed{\alpha}\, \psi^t$) is equal to $|\{\beta \mid \beta \geq \alpha\}|$; ($\boxed{\alpha}\, p^t_\psi \vee \boxed{\alpha}\, \psi^f$) gives rise to only one clause. The number of clauses generated from ϕ is thus bounded by $1 + 2l\Sigma_{i=1}^{n-1}(i+1) = 1 + l(n^2 + n - 2)$. The estimations for formulae not containing $\Rightarrow$ or $\neg$ follow from the fact that the branching factor is constant for $\sim$, $S_1, \ldots, S_{n-1}$, $\vee, \wedge$, $\exists, \forall$. $\Box$

For comparison, we point out that the very general structure-preserving methods for translation to clause form that use the algebra of truth values given in [2], the split degree of rules of the type $(\Box, v)^+$ induced by a k-ary operator $\Box$, given a truth value v, is at most $|W|^{k-1}$, if W is the set of truth values. Since in a structure-preserving translation to clause form such rules have to be considered for every truth value v, the clause form of a formula with n occurrences of at most r-ary operators and m occurrences of quantifiers contains no more than $n|W|^r + m2^{|W|} + 1$ clauses if optimal rules are used (cf. [2]). In the presence of operators that are at most binary, and if the only quantifiers are $\forall$ and $\exists$, the number of clauses generated this way is in the worst case quadratic in the number of truth values, and generate $\mathcal{O}(ln^4)$ clauses for formulae containing at least one binary operator. This upper bound is actually reached in the case of SHn-logics if the very general rules in [2] are used as presented there. Additional improvements may be achieved by exploiting the particular properties of the algebra of truth values. For linearly-ordered sets of truth values, for instance, it can be shown that by using more refined methods, such as that of Salzer [10], the number of clauses generated by conjunction, disjunction, and Heyting implication is linear in the number of truth values. However, it is not clear to us whether the same holds for sets of truth values which are not linearly ordered.

One of the main advantages of the method we present here is that only elements of the Priestley dual of the set of truth values are used as signs, thus fewer signs have to be taken into account in the process of translation to clause form. Our method proves to be especially efficient in situations when the difference between the number of elements of the algebra of truth values and the number of elements of its Priestley dual is large (this is not the case, for instance, if the set of truth values is linearly ordered).

4.1 Further improvements

The structure of $D(S_{n^2})$ can be used in order to further reduce the number of clauses[2]. For this, we use the fact that $D(S_{n^2})$ consists of two branches and that the transformation rules for the operations in $\{\vee, \wedge, \neg, \Rightarrow\}$ preserve the branch of $D(S_{n^2})$. For formulae that do not contain the De Morgan negation $\sim$, all the clauses generated by the renaming of subformulae only contain signs in one of the branches of $D(S_{n^2})$. It is sufficient to give a refutation for the clauses corresponding to one of the branches of $D(S_{n^2})$. For the other branch a similar refutation can be constructed by simply renaming the nodes, and they can be then combined to a refutation by resolution for the clause form of ϕ.

[2]We thank L. Iturrioz who suggested that it may be possible to further improve the efficiency of automated theorem proving for SHn-logics by exploiting the structure of $D(S_{n^2})$; the remarks below show that this is possible; a more thorough research, also for more general logics, is planned for future work.

The set of signed clauses that would be generated in this case corresponds to the following conjunction of formulae:

$$\left\{\begin{array}{c}
\boxed{\uparrow(1,0)}\ P_\phi^f \text{ in } \mathcal{I}, \\
(\boxed{\uparrow(i,0)}\ P_\psi(x_1,\dots,x_n)^t \vee \boxed{\uparrow(i,0)}\ \sigma(P_{\psi_1},\dots,P_{\psi_m})^f) \\
(\boxed{\uparrow(i,0)}\ P_\psi(x_1,\dots,x_n)^f \vee \boxed{\uparrow(i,0)}\ \sigma(P_{\psi_1},\dots,P_{\psi_m})^t) \\
\text{in } \mathcal{I};\ \text{for every } i=1,\dots,n-1, \text{ for all subformulae } \psi=\sigma(\psi_1,\dots,\psi_m) \\
\text{of } \phi \text{ and all instantiations in } D \text{ of the free variables } x_1,\dots,x_n \text{ of } \psi, \\
(\boxed{\uparrow(i,0)}\ P_\psi(x_1,\dots,x_n)^t \vee \boxed{\uparrow(i,0)}\ (Qx)P_{\psi_1}(x,x_1,\dots,x_n)^f) \\
(\boxed{\uparrow(i,0)}\ P_\psi(x_1,\dots,x_n)^f \vee \boxed{\uparrow(i,0)}\ (Qx)P_{\psi_1}(x,x_1,\dots,x_n)^t) \\
\text{in } \mathcal{I};\ \text{for every } i=1,\dots,n-1, \text{ for all subformulae } \psi=(Qx)\psi_1(x,x_1,\dots,x_n) \\
\text{of } \phi \text{ and all instantiations in } D \text{ of the free variables } x_1,\dots,x_n \text{ of } \psi.
\end{array}\right.$$

In [13], where we focus on theorem proving for sets of signed clauses, we note that the translation to clause form described above is a translation to (many-sorted) classical logic ($\boxed{\alpha}\ L^t$ stands for $\mathsf{holds}(L,\alpha)$, and $\boxed{\alpha}\ L^f$ for $\neg\mathsf{holds}(L,\alpha)$). This also explains the fact that the polarity of formulae can be used to further reduce the number of clauses that are generated during the translation to clause form. We think that the number of clauses generated (e.g. by ($\Rightarrow$, f)) can be further reduced by using redundancy elimination rules.

5 Resolution

Satisfiability of sets of signed clauses can be checked by using a signed resolution procedure:

$$\frac{C \vee \boxed{x}\ L^t \qquad D \vee \boxed{y}\ L^f}{C, D} \text{ provided that } x \leq y.$$

A version of negative hyperresolution for signed clauses, inspired by the method for regular clauses in [5], is presented below. Soundness and completeness can be proved as in [1, 5].

Negative Hyperresolution

$$\frac{\left\{\boxed{x_1}\ p_1^f\right\} \cup D_1, \dots, \left\{\boxed{x_n}\ p_n^f\right\} \cup D_n, \left\{\boxed{y_1}\ p_1^t, \dots, \boxed{y_n}\ p_n^t\right\} \cup E}{D_1 \cup \dots \cup D_n \cup E}$$

provided that $n \geq 1$, $y_i \leq x_i$ for all $i = 1, \dots, n$ and $D_1, \dots, D_n, E$ are negative.

Alternatively, as shown in [13], the validity of ϕ can be checked by applying classical resolution to the set Φ^c of classical clauses obtained from the signed CNF of ϕ by replacing $\boxed{\alpha}\ L^t$ with $\mathsf{holds}(L,\alpha)$, and $\boxed{\alpha}\ L^f$ with $\neg\mathsf{holds}(L,\alpha)$, to which the set of clauses $\mathsf{Her} = \{\{\neg\mathsf{holds}(x,\alpha), \mathsf{holds}(x,\beta)\} \mid \alpha \leq \beta\}$, expressing the heredity of truth, is adjoined.

6 Conclusions

In this paper we illustrated by one example, namely SHn-logics, an efficient transformation procedure to a signed clause form, and a refutation procedure based on negative hyperresolution for many-valued logics based on distributive lattices with operators. The signed formulae we use, namely $\boxed{\alpha}\ \phi^t$ and $\boxed{\alpha}\ \phi^f$, are very similar to the "positive and negative regular formulae" of the form $\boxed{\geq i}\ \phi$ resp. $\boxed{\leq i}\ \phi$ introduced in [5] for regular logics. The only difference is that in [5] totally ordered sets are considered, whereas we consider duals of finite distributive lattices[3]. In the particular case of totally ordered

[3]For the linearly-ordered case, the negation of "$\boxed{\geq i}\ \phi$ is true" is "$\boxed{\leq i-1}\ \phi$ is true", in cases when the set of truth values is not linearly ordered this does not necessarily hold.

lattices of truth values, Hähnle's notions of positive and negative literal are recovered: $\boxed{\geq i}\ \phi$ corresponds to $\boxed{\alpha}\ \phi^t$ and $\boxed{\leq i-1}\ \phi$ to $\boxed{\alpha}\ \phi^f$, where $\alpha = \uparrow i$, which justifies the terminology "literal with positive (negative) polarity" in [5]. A detailed presentation of these ideas for SHn-logics, an extension to a more general framework, and the description of a Prolog implementation can be found in [12]. The detailed presentation of a general method based on these ideas is the subject of an extended paper, currently in preparation [14].

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