An equational re-engineering of set theories *

Andrea Formisano† Eugenio Omodeo‡

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Abstract

New successes in dealing with set theories by means of state-of-the-art theorem-provers may ensue from terse and concise axiomatizations, such as can be moulded in the framework of the (fully equational) Tarski-Givant map calculus. In this paper we carry out this task in detail, setting the ground for a number of experiments.

Key words: Set theory, relation algebras, first-order theorem-proving, algebraic logic.

1 Introduction

Like other mature fields of mathematics, Set Theory deserves sustained efforts that bring to light richer and richer decidable fragments of it [5], general inference rules for reasoning in it [23, 2], effective proof strategies based on its domain-knowledge, and so forth.

Advances in this specialized area of automated reasoning tend, in spite of their steadiness, to be slow compared to the overall progress in the field. Many experiments with set theories have hence been carried out with standard theorem-proving systems. Still today such experiments pose considerable stress on state-of-the-art theorem provers, or demand man to give much guidance to proof assistants; they therefore constitute ideal benchmarks. Moreover, in view of the pervasiveness of Set Theory, they are likely —when successful in something tough— to have a strong echo amidst computer scientists and mathematicians. Even for those who are striving to develop something entirely *ad hoc* in the challenging arena of set theories, it is important to assess what can today be achieved by unspecialized proof methods and where the context-specific bottlenecks of Set Theory precisely reside.

In its most popular first-order version, namely the Zermelo-Fraenkel-Skolem axiomatic system ZF, set theory (very much like Peano arithmetic) presents an immediate obstacle: it does not admit a finite axiomatization. This is why the von Neumann-Gödel-Bernays theory GB of sets and classes is sometimes preferred to it as a basis for experimentation [3, 22, 16]. Various authors (e.g., [10, 13, 14]) have been able to retain the traits of ZF, by resorting to higher-order features of specific theorem-provers such as Isabelle.

In this paper we will pursue a minimalist approach, to propose a purely equational formulation of both ZF and finite set theory. Our approach heavily relies on [21], but we go into much finer detail with the axioms, ending in such a concise formulation as to offer a good starting point for experimentation (with Otter [9], say, or with a more markedly equational theorem-prover). Our formulation of the axioms is based on the formalism $\mathcal{L}^{\times}$ of [21], which is equational and devoid of variables, but somewhat out of standards.

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†University “La Sapienza” of Rome, Dept. of Computer Science. `formisan@dimi.uniud.it`

‡University of L'Aquila, Dept. of Pure and Applied Mathematics. `omodeo@univaq.it`

Luckily, $\mathcal{L}^{\times}$ can easily be emulated through a first-order system, simply by treating the meta-variables that occur in the schematic formulation of its axioms (both the logical ones and those endowed with a genuinely set-theoretic content) as if they were first-order variables. In practice, this means treating ZF as if it were an extension of the theory of relation algebras; we can express it through a finite number of axioms, because variables are not supposed to range over sets but over the dyadic relations on the universe of sets.

Taken in its entirety, Set Theory offers a *panorama of alternatives* (cf. [18], p.x); that is, it consists of axiomatic systems not equivalent (and sometimes antithetic, cf. [11]) to one another. This is why we will not produce the axioms of just one theory and will also touch the theme of 'individuals' (ultimate entities entering in the formation of sets). Future work will expand the material of this paper into a toolkit for assembling set theories of all kinds—after we have singled out, through experiments, formulations of the axioms that work decidedly better than others.

2 Syntax and semantics of $\mathcal{L}^{\times}$

$\mathcal{L}^{\times}$ is a ground equational language where one can state properties of dyadic relations —MAPS, as we will call them— over an unspecified, yet fixed, *domain $\mathcal{U}$ of discourse*. In this paper, the map whose properties we intend to specify is the membership relation $\in$ over the class $\mathcal{U}$ of all sets. The language $\mathcal{L}^{\times}$ consists of *map equalities* $Q{=}R$, where Q and R are *map expressions*:

Definition. MAP EXPRESSIONS *are all terms of the following signature:*

symbol :	$\emptyset$	$1\!\!1$	ι	$\in$	$\cap$	$\triangle$	$\circ$	$^{-1}$	$^{-}$	$\setminus$	$\cup$	$\dagger$
degree :	0	0	0	0	2	2	2	1	1	2	2	2
priority :					5	3	6	7		2	2	4

(Of these, $\cap, \triangle, \circ, \setminus, \cup, \dagger$ will be used as left-associative infix operators, $^{-1}$ as a postfix operator, and $^{-}$ as a line topping its argument.) $\square$

For an *interpretation* of $\mathcal{L}^{\times}$, one must fix, along with a nonempty $\mathcal{U}$, a subset $\in^{\Im}$ of $\mathcal{U}^2 =_{\text{Def}} \mathcal{U} \times \mathcal{U}$. Then each map expression P comes to designate a specific map $P^{\Im}$ (and, accordingly, any equality $Q{=}R$ between map expressions turns out to be either true or false), on the basis of the following evaluation rules:

$$\emptyset^{\Im} =_{\text{Def}} \emptyset, \qquad 1\!\!1^{\Im} =_{\text{Def}} \mathcal{U}^2, \qquad \iota^{\Im} =_{\text{Def}} \{[a,a] \,:\, a \text{ in } \mathcal{U}\};$$
$$(Q{\cap}R)^{\Im} =_{\text{Def}} \{\, [a,b] \in Q^{\Im} \,:\, [a,b] \in R^{\Im} \,\};$$
$$(Q{\triangle}R)^{\Im} =_{\text{Def}} \{\, [a,b] \in \mathcal{U}^2 \,:\, [a,b] \in Q^{\Im} \text{ if and only if } [a,b] \notin R^{\Im} \,\};$$
$$(Q{\circ}R)^{\Im} =_{\text{Def}} \{\, [a,b] \in \mathcal{U}^2 \,:\, \text{there are } cs \text{ in } \mathcal{U} \text{ for which } [a,c] \in Q^{\Im} \text{ and } [c,b] \in R^{\Im} \,\};$$
$$(Q^{-1})^{\Im} =_{\text{Def}} \{\, [b,a] \,:\, [a,b] \in Q^{\Im} \,\}.$$

Of the operators and constants in the signature of $\mathcal{L}^{\times}$, only a few deserve being regarded as *primitive* constructs; indeed, we choose to regard as *derived* constructs the ones for which we gave no evaluation rule, as well as others that we will tacitly add to the signature:

$\overline{P}$	$\equiv_{\text{Def}}$	$P{\triangle}1\!\!1$	$P{\dagger}Q$	$\equiv_{\text{Def}}$	$\overline{\overline{P}{\circ}\overline{Q}}$
$P{\setminus}Q$	$\equiv_{\text{Def}}$	$P{\cap}\overline{Q}$	$\mathsf{funPart}(P)$	$\equiv_{\text{Def}}$	$P{\setminus}P{\circ}\overline{\iota}$
$P{\cup}Q$	$\equiv_{\text{Def}}$	$\overline{\overline{P}{\setminus}Q}$	etc.		

The interpretation of $\mathcal{L}^{\times}$ obviously extends to the new constructs; e.g.,

$$(P{\dagger}Q)^{\Im} =_{\text{Def}} \{\, [a,b] \in \mathcal{U}^2 \,:\, \text{for all } c \text{ in } \mathcal{U}, \text{ either } [a,c] \in P^{\Im} \text{ or } [c,b] \in Q^{\Im} \,\},$$
$$\mathsf{funPart}(P)^{\Im} =_{\text{Def}} \{[a,b] \in P^{\Im} : [a,c] \notin P^{\Im} \text{ for any } c \neq b\},$$

so that $\mathsf{funPart}(P){=}P$ will mean "P is a partial function", very much like $\mathsf{Fun}(P)$ to be seen below.

Through abbreviating definitions, we can also define shortening notation for map equalities that follow certain patterns, e.g.,

$\mathsf{Fun}(P) \equiv_{\text{Def}} P^{-1} \circ P \backslash \iota = \emptyset$	$\mathsf{Total}(P) \equiv_{\text{Def}} P \circ 1\!\!1 = 1\!\!1$

so that $\mathsf{Total}(P)$ states that for all a in $\mathcal{U}$ there is at least one pair $[a, b]$ in $P^{\Im}$.

One often strives to specify the class $\mathcal{C}$ of interpretations that are of interest in some application through a collection of equalities that must be true in every $\Im$ of $\mathcal{C}$. The task we are undertaking here is of this nature; our aim is to capture through simple map equalities the interpretations of $\in$ that comply with

- standard Zermelo-Fraenkel theory, on the one hand;
- a theory of finite sets ultimately based on individuals, on the other hand.

In part, the game consists in expressing in $\mathcal{L}^{\times}$ common set-theoretic notions. To start with something obvious,

$$\notin \equiv_{\text{Def}} \overline{\in}, \qquad \ni \equiv_{\text{Def}} \in^{-1}, \qquad \not\ni \equiv_{\text{Def}} \overline{\ni};$$

$\varepsilon_0\varepsilon_1 \cdots \varepsilon_n \equiv_{\text{Def}} \varepsilon_0 \circ \varepsilon_1 \circ \cdots \circ \varepsilon_n$, where each ε_i stands for one of $\in, \notin, \ni, \not\ni, 1\!\!1$.

To see something slightly more sophisticated:

Examples. With respect to an interpretation $\Im$, one says that a *intersects* b if a and b have some element in common, i.e., there is a c for which $c \in^{\Im} a$ and $c \in^{\Im} b$. A map expression P such that $P^{\Im} = \{ [a, b] \in \mathcal{U}^2 : a \text{ intersects } b \}$ is $\ni\in$.

Likewise, one can define in $\mathcal{L}^{\times}$ the relation a *includes* b (i.e., 'no element of b fails to belong to a'), by the map expression $\overline{\not\ni\in}$. The expression $\overline{\ni\notin\cup\iota}$ translates the relation a *is strictly included in* b, and so on.

Let a *splits* b mean that every element of a intersects b and that no two elements of a intersect each other. These conditions translate into the map expression defined as follows:

$$\mathsf{splits} \equiv_{\text{Def}} (\not\ni \dagger \ni \in) \cap \overline{(\ni \cap \ni \circ (\ni \in \cap \overline{\iota}))\circ 1\!\!1}.$$

□

Secondly, the reconstruction of set theory within $\mathcal{L}^{\times}$ consists in restating ordinary axioms (and, subsequently, theorems), through map equalities.

Example. One of the many ways of stating the much-debated AXIOM OF CHOICE (under adequately strong remaining axioms) is by claiming that *when a splits some b, there is a c which is also split by a and which does not strictly include any other set split by a.* Formally:

(Ch) $\qquad \mathsf{Total}(\overline{\mathsf{splits}\circ 1\!\!1} \cup \mathsf{splits} \setminus \mathsf{splits} \circ \overline{\ni\notin\cup\iota})$,

where the second and third occurrence of splits could be replaced by $\not\ni\dagger\ni\in$.

To relate the original version of this axiom in [24] with ours,[1] notice that a set a splits some b if and only if a consists of pairwise disjoint sets (and, accordingly, a splits $\bigcup a$). Moreover, an inclusion-minimal c split by a must have a singleton intersection with each d in a (otherwise, of two elements in $c \cap d$, either one could be removed from c); conversely, if c is included in $\bigcup a$ and has a singleton intersection with each d in a, then none of its elements e can be removed (else $c \setminus \{e\}$ would no longer intersect the d in a to which e belongs). □

In the third place, we are to prove theorems about sets by equational reasoning, moving from the equational specification of the set axioms. To discuss this point we must refer to an inferential apparatus for $\mathcal{L}^{\times}$; we hence delay this discussion to much later (cf. Sec.7).

[1] For 19 alternative versions of this axiom, cf. [15], p.309.

3 Extensionality, subset, sum-set, and power-set axioms

Two derived constructs, ∂ and ∇, will be of great help in stating the properties of membership simply:

$$\partial(\,P\,) \equiv_{\mathrm{Def}} \overline{P\circ\notin}, \qquad \nabla(\,P\,) \equiv_{\mathrm{Def}} \partial(\,P\,)\backslash\overline{P}\circ\in.$$

Plainly, $a\partial(\,Q\,)^{\Im}b$ and $a\nabla(\,R\,)^{\Im}b$ will hold in an interpretation $\Im$ if and only if, respectively,

- all those c in $\mathcal{U}$ for which $aQ^{\Im}c$ holds, are 'elements' of b (in the sense that $c{\in}^{\Im}b$);
- the elements of b are precisely those c in $\mathcal{U}$ for which $aR^{\Im}c$ holds.

Our first axiom, EXTENSIONALITY, states that *sets are the same whose elements are the same*:

(E) $\qquad\qquad \nabla(\,\ni\,){=}\boldsymbol{\iota}.$

A useful strenghtening of this axiom is the scheme $\mathsf{Fun}(\,\nabla(\,P\,)\,)$, where P ranges over all map expressions.

Two rather elementary postulates, the POWER-SET axiom and the SUM-SET axiom, state that for any set a there is a set whose elements comprise all sets included in a, and there is one whose elements comprise all elements of elements of a:

($\mathcal{P}ow$) $\qquad\qquad \mathsf{Total}(\,\partial(\,\overline{\not\ni\in}\,)\,),$

($\mathcal{U}n$) $\qquad\qquad \mathsf{Total}(\,\partial(\,\ni\ni\,)\,).$

A customary strenghtening of the sum-set axiom is the TRANSITIVE EMBEDDING axiom, stating that every a belongs to a set b which is transitively closed w.r.t. membership, in the sense specified by trans here below:

(T) $\qquad \mathsf{Total}(\,\in\circ\mathsf{trans}\,),\qquad$ where $\mathsf{trans} \equiv_{\mathrm{Def}} \boldsymbol{\iota}\cap\partial(\,\ni\ni\,).$

The SUBSET axioms enable one to extract from any given a the set b consisting of those elements of a that meet a condition specified by means of a predicate expression P. In this form, still overly naïve, this 'separation' principle could be stated as simply as: $\mathsf{Total}(\,\nabla(\,\ni\cap P\,)\,)$. This would suffice (taking $\emptyset$ as P) to ensure the existence of a *null* set, devoid of elements. We need the following more general form of separation (whence the previous one is obtained by taking $\boldsymbol{\iota}$ as Q):

(S) $\qquad \mathsf{Total}\big(\,\nabla(\,\mathsf{funPart}(\,Q\,)\circ\ni\cap P\,)\,\big).$

Example. Plainly, $\mathsf{funPart}(\,\ni\,)^{\Im}$ is the map holding between c and d in $\mathcal{U}$ iff $c=\{d\}$, i.e. d is the sole element of c; moreover $\mathsf{funPart}(\,\ni\circ\mathsf{funPart}(\,\ni\,)\,)^{\Im}$ is the map holding between a and d iff there is exactly one singleton c in a and d is the element of that particular c. Thus, the instance $\mathsf{Total}\big(\,\nabla(\,\mathsf{funPart}(\,\ni\circ\mathsf{funPart}(\,\ni\,)\,)\circ\ni\cap\not\ni\,)\,\big)$ of **(S)** states that to every set a there corresponds a set b which is null unless there is exactly one singleton $c=\{d\}$ in a, and which in the latter case consists of all elements of d that do not belong to a. $\square$

4 Pairing and finiteness axioms

A list $\pi_0, \pi_1, \ldots, \pi_n$ of maps are said to be CONJUGATED QUASI-PROJECTION if they are (partial) functions and they are, collectively, *surjective*, in the sense that for any list $a_0, \ldots, a_n$ of entities in $\mathcal{U}$ there is a b in $\mathcal{U}$ such that $\pi_i(b) = a_i$ for $i = 0, 1, \ldots, n$. We assume in what follows that $\boldsymbol{\pi}_0, \boldsymbol{\pi}_1$ are map expressions designating a pair of conjugated quasi-projections. It is immaterial whether they are added as primitive constants to $\mathcal{L}^{\times}$, or they are map expressions suitably chosen so as to reflect one of the various notions of ordered pair available around, and subject to axioms that are adequate to ensure that the desired conditions, namely

(Pair) $\qquad \boldsymbol{\pi}_0^{-1}\circ\boldsymbol{\pi}_1 = 1\!\!1, \qquad \mathsf{Fun}(\boldsymbol{\pi}_0), \qquad \mathsf{Fun}(\boldsymbol{\pi}_1), \qquad \in\ni = 1\!\!1,$
hold (cf. [21], pp.127–135). Notice that the clause $\mathbf{(Pair)}_4$ of this PAIRING AXIOM will become superfluous when the replacement axiom scheme will enter into play (cf. [8], pp.9–10).

Examples. A use of the $\boldsymbol{\pi}_b$s is that they enable one to represent set-theoretic functions by means of entities f of $\mathcal{U}$ such that no two elements b, c of f for which $\boldsymbol{\pi}_0^{\Im}$ yields a value have $\boldsymbol{\pi}_0^{\Im}(b) = \boldsymbol{\pi}_0^{\Im}(c)$. Symbolically, we can define the class of these *single-valued sets* as

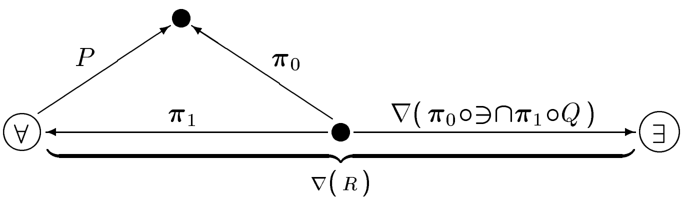
$$\mathsf{sval} \equiv_{\mathrm{Def}} \boldsymbol{\iota} \cap \overline{\boldsymbol{\sigma}\circ\in}, \qquad \text{where } \boldsymbol{\sigma} \equiv_{\mathrm{Def}} \ni\circ(\boldsymbol{\pi}_0\circ\boldsymbol{\pi}_0^{-1}\cap\overline{\boldsymbol{\iota}}).$$

Cantor's classical theorem that the power-set of a set has more elements than the set itself can be phrased (cf. [2], p.410) as follows: *for every set a and for every function f, there is a subset b of a which is not 'hit' by the function f* (restricted to the set a in question).[2] A rendering of this theorem in $\mathcal{L}^{\times}$ could be $\mathsf{Total}(\overline{\overline{\not\ni\in}\cap\ni\circ\mathsf{funPart}(P)})$, but this would not faithfully reflect the idea that the theorem concerns set-theoretic functions rather than functions, $\mathsf{funPart}(P)$, of $\mathcal{L}^{\times}$. The typical use of $\boldsymbol{\pi}_0$ and $\boldsymbol{\pi}_1$ is illustrated by a more faithful translation, which exploits the possibility to encode the pair a, f by an entity c with $\boldsymbol{\pi}_0^{\Im}(c) = a$ and $\boldsymbol{\pi}_1^{\Im}(c) = f$:

$$\mathsf{Total}\big(\overline{\boldsymbol{\pi}_0\circ\overline{\not\ni\in}\cap(\boldsymbol{\pi}_0\circ\ni\circ\boldsymbol{\pi}_0^{-1}\cap\boldsymbol{\pi}_1\circ(\overline{\boldsymbol{\sigma}}\cap\ni))\circ\boldsymbol{\pi}_1}\big) \qquad (\boldsymbol{\sigma} \text{ as before}).$$

The latter states that to every c there corresponds a b such that
- if it exists, $\boldsymbol{\pi}_0^{\Im}(c)$ includes b;
- if $\boldsymbol{\pi}_0^{\Im}(c) = a$ and $\boldsymbol{\pi}_1^{\Im}(c) = f$ both exist, then $b \neq \boldsymbol{\pi}_1^{\Im}(d)$ for any d in f such that $\boldsymbol{\pi}_0^{\Im}(d) = e$ exists and belongs to a and no d' in f other than d fulfills $\boldsymbol{\pi}_0^{\Im}(d') = e$. □

A standard technique used to derive statements of the form $\mathsf{Total}(\nabla(R))$, which are often very useful, is by breaking $\nabla(R)$ into an equivalent expression of the form $(P\circ\boldsymbol{\pi}_0^{-1}\cap\boldsymbol{\pi}_1^{-1})\circ\nabla(\boldsymbol{\pi}_0\circ\ni\cap\boldsymbol{\pi}_1\circ Q)$, where $\mathsf{Total}(P)$ is easier to prove. Exploiting the same graph representation of map expressions utilized in [4], this situation can be depicted as follows:



The desired totality of $\nabla(R)$ will then follow, in view of $\mathbf{(Pair)}_1$ and of **(S)**, $\mathbf{(Pair)}_2$. For example, by means of the instantiation $P \mapsto \in\circ\mathsf{trans}$, $Q \mapsto \boldsymbol{\iota}$ of this proof scheme, we obtain $\mathsf{Total}(\nabla(\boldsymbol{\iota}))$, where $\nabla(\boldsymbol{\iota})$ designates the *singleton* operation $a \mapsto \{a\}$ on $\mathcal{U}$; then, by taking $P \mapsto ((\boldsymbol{\pi}_0\cup\overline{\boldsymbol{\pi}_0\circ 1\!\!1})\circ\in\cap(\boldsymbol{\pi}_1\cup\overline{\boldsymbol{\pi}_1\circ 1\!\!1})\circ\nabla(\boldsymbol{\iota})\circ\in)\circ\partial(\ni\ni)$, $Q \mapsto \boldsymbol{\pi}_0\circ\ni\cup\boldsymbol{\pi}_1$, we obtain the totality of $\nabla(\boldsymbol{\pi}_0\circ\ni\cup\boldsymbol{\pi}_1)$, which designates the *adjunction* operation $[a, b] \mapsto a \cup \{b\}$. Similarly, one gets the totality of $\nabla(\overline{\not\ni\in})$, $\nabla(\ni\ni)$, $\nabla(\boldsymbol{\pi}_0\cup\boldsymbol{\pi}_1)$, of any $\nabla(R)$ such that both $R\backslash Q=\emptyset$ and $\mathsf{Total}(\partial(Q))$ are known for some Q, etc. Even the full **(S)** could be derived with this approach from its restrained version $\mathsf{Total}\big(\nabla(\boldsymbol{\pi}_0\circ\ni\cap\boldsymbol{\pi}_1\circ P)\big)$.

Under the set axioms **(E)**, **($\mathcal{P}ow$)**, **(S)**, **(Pair)** introduced so far, it is reasonable to characterize a set a as being *finite* if and only if every set b of which a is an element has an element which is minimal w.r.t. inclusion (cf. [20], p.49). Intuitively speaking, in fact, the set formed by all infinite cs in the power-set $\wp(a)$ of a has no minimal elements when a is

[2]This was one of the first major theorems whose proof was automatically found by a theorem prover, cf. [1]. This achievement originally took place in the framework of typed lambda-calculus.

infinite, because every such c remains infinite after a single-element removal. Conversely, if a belongs to some b which has no minimal elements, then the intersection of b with $\wp(\,a\,)$ has no minimal elements either, and hence a is infinite. In conclusion, to instruct a theory concerned exclusively with finite sets, one can adopt the following FINITENESS AXIOM:

(F) $\quad \mathsf{finite}{=}\boldsymbol{\iota}, \qquad$ where $\mathsf{finite} \equiv_{\mathrm{Def}} \boldsymbol{\iota}\cap\big(\,\mathbb{1}\circ(\,\in\cap(\,(\,\boldsymbol{\iota}\cup\not\ni\in\,)\dagger\not\in\,)\,)\dagger\not\ni\,\big).$

5 Bringing individuals into set theory: Foundation and plenitude axioms

Taken together with the *foundation* axiom to be seen below, the axioms **(E)**, ($\mathcal{P}ow$), **(T)**, **(S)**, **(Pair)**, and **(F)** discussed above constitute a full-blown theory of finite sets. However, they do not say anything about *individuals* (or 'urelements', cf. [7]), entities that common sense places at the bottom of the formation of sets. These are not essential for theoretical development, but useful to model practical situations. To avoid a revision of **(E)** —necessary, if we wanted to treat individuals as entities devoid of elements but different from the null set— let us agree that individuals are self-singletons $a = \{a\}$ (cf. [18], pp.30–32). Moreover, to bring plenty of individuals into $\mathcal{U}$ (at least as many individuals as there are sets, hence infinitely many individuals), we require that there are individuals outside the sum-set of any set. Here comes the PLENITUDE AXIOM:

(Ur) $\quad \mathsf{Total}(\,\overline{\ni\ni}\circ\mathsf{ur}\,), \qquad$ where $\mathsf{ur} \equiv_{\mathrm{Def}} \boldsymbol{\iota}\cap\nabla(\,\boldsymbol{\iota}\,).$

To develop a theory of *pure* sets, one will postulate 'lack' of individuals, by adopting the axiom $\mathsf{ur}{=}\emptyset$ instead of plenitude.

When individuals are lacking, the FOUNDATION (or 'regularity') axiom ensures that the membership relation $\in^{\Im}$ is well-founded on $\mathcal{U}$, and can be stated as follows: *when some b belongs to a, there is a c also belonging to a that does not intersect a.* On the surface, this statement has the same structure as the version of the axiom of choice seen at the end of Sec.2; in $\mathcal{L}^{\times}$ it can hence be rendered by $\mathsf{Total}(\,\overline{\ni\mathbb{1}}\cup\ni\backslash\ni\in\,)$. To reconcile this statement with individuals, we can recast it as

(R) $\quad \mathsf{Total}(\,(\,\not\ni\cup\mathbb{1}\circ\mathsf{ur}\,)\dagger\emptyset\cup\ni\backslash\ni\circ(\,\boldsymbol{\iota}\backslash\mathsf{ur}\,)\circ\in\backslash\mathbb{1}\circ\mathsf{ur}\,),$

which means: *unless every b in a is an individual, there is a c in a such that every element of $a\cap c$ is an individual and c itself is not an individual.*

As is well-known (cf. [8], p.35), foundation helps one in making the definitions of basic mathematical notions very simple. In our framework, we propose to adopt the following definition of the class of *natural numbers*:[3]

$$\mathsf{nat} \equiv_{\mathrm{Def}} \boldsymbol{\iota}\cap\big(\,\ni\circ(\,\nabla(\,\ni\cup\boldsymbol{\iota}\,)\setminus\boldsymbol{\iota}\,)\dagger\overline{(\,\boldsymbol{\iota}\cup\in\,)\cap\ni\mathbb{1}}\,\big),$$

which means: *a is a natural number if for every b in $a\cup\{a\}$ other than the null set, there is a c in a such that $b=c\cup\{c\}$ and $b\neq c$.*

6 An infinity axiom, and the replacement axioms

Similarly, under the foundation axiom, the definition of *ordinal numbers* becomes

$$\mathsf{ord} \equiv_{\mathrm{Def}} (\,\mathsf{trans}\backslash\ni\circ\mathsf{ur}\circ\mathbb{1}\,)\cap(\,\not\ni\dagger(\,\in\cup\boldsymbol{\iota}\cup\ni\,)\dagger\not\in\,),$$

where trans is the same as in **(T)**, hence $\mathsf{trans}\backslash\boldsymbol{\iota}{=}\emptyset$ holds, and hence (thanks to **(R)**) $\not\ni\dagger(\,\in\cup\boldsymbol{\iota}\cup\ni\,)\dagger\not\in$ requires that an ordinal be totally ordered by membership.

[3]From this simple start one can rapidly reach the definition of important data structures, e.g., ordered and oriented finite trees.

The existence of infinite sets is often postulated by claiming that $\mathsf{ord}\backslash\mathsf{nat}$ is not empty: $\mathbb{1}\circ(\mathsf{ord}\backslash\mathsf{nat})\circ\mathbb{1}=\mathbb{1}$, or equivalently $\mathsf{Total}(\mathbb{1}\circ(\mathsf{ord}\backslash\mathsf{nat}))$. The following more essential formulation of the INFINITY axiom, based on [12] and presupposing **(R)**, seems preferable to us:[4]

(I) $$\mathsf{Total}\big(\mathbb{1}\circ(\partial(\ni\ni)\cap\partial(\ni\ni)^{-1}\backslash\in\backslash\ni\backslash\iota\backslash\ni\circ\overline{\in\triangle\ni}\circ\in)\big).$$

What **(I)** means is: *There are distinct sets a_0, a_1 such that the sum-set of either one is included in the other, neither one belongs to the other, and for any pair c_0, c_1 with c_0 in a_0 and c_1 in a_1, either c_0 belongs to c_1 or c_1 belongs to c_0.*[5] [6] Of course this axiom is antithetic to the axiom **(F)** seen earlier: one can adopt either one, but only one of the two.

In a theory with infinite sets, the REPLACEMENT AXIOM SCHEME plays a fundamental rôle. Two simple-minded versions of it are:

$$\mathsf{Total}\big(\partial(\ni\circ\mathsf{funPart}(Q))\big), \qquad \mathsf{Total}\big(\partial(\ni\circ\nabla(Q)\circ\ni)\big).$$

Both of these state —under different conditions on a certain map P— that to every c there corresponds a (superset of a) set of the form $P[c] =_{\mathrm{Def}} \{u : vP^{\Im}u \text{ for some } v\in^{\Im}c\}$. The former applies when P ($= \mathsf{funPart}(Q)$) designates a function, the latter when $\nabla(P)$ (with $P = \nabla(Q)\circ\ni$) designates a total map. [19] adopts a formulation of replacement closer in spirit to the latter, but it is the former that we generalize in what follows.

Parameter-less replacement, like a parameter-less subset axiom scheme, would be of little use. Given an entity d of $\mathcal{U}$, we can think that $\boldsymbol{\pi}_0^{\Im}(d)$ represents the domain to which one wants to restrict a function, and $\boldsymbol{\pi}_1^{\Im}(d)$ represents a list of parameters. To state replacement simply, it is convenient to add to the conditions on the $\boldsymbol{\pi}_i$s a new one. Specifically, we impose that distinct entities never encode the same pair:[7]

$\mathbf{(Pair)}_5$ $$\boldsymbol{\pi}_0\circ\boldsymbol{\pi}_0^{-1}\cap\boldsymbol{\pi}_1\circ\boldsymbol{\pi}_1^{-1}\backslash\iota=\emptyset.$$

The simplest formulation of replacement we could find in $\mathcal{L}^{\times}$, so far, is:

(Repl) $$\mathsf{Total}\big(\partial((\boldsymbol{\pi}_0\circ\ni\circ\boldsymbol{\pi}_0^{-1}\cap\boldsymbol{\pi}_1\circ\boldsymbol{\pi}_1^{-1})\circ\mathsf{funPart}(Q))\big).$$

This means: *To every pair d there corresponds a set comprising the images, under the functional part of Q, of all pairs e that fulfill the conditions $\boldsymbol{\pi}_0^{\Im}(e)\in^{\Im}\boldsymbol{\pi}_0^{\Im}(d)$, $\boldsymbol{\pi}_1^{\Im}(e) = \boldsymbol{\pi}_1^{\Im}(d)$.*

Example. To see that $\mathbf{(Pair)}_4$ can be derived from $\mathbf{(Pair)}_{1,2,3}$, **(T)**, **(S)**, and **(Repl)**, one can argue as follows. Thanks to **(S)**, a null set $\{\}$ exists: $\overline{\mathbb{1}\in}\circ\mathbb{1}=\mathbb{1}$. Then, by virtue of **(T)**, a set to which this null set belongs exists too: $\overline{\mathbb{1}\in}\circ\in\circ\mathbb{1}=\mathbb{1}$. Again through **(T)**, we obtain a set c to which both of the preceding sets belong: $\overline{\mathbb{1}\in}\circ(\in\cap\in\in)\circ\mathbb{1}=\mathbb{1}$. The latter c can be combined with any given a to form a pair d fulfilling both $\boldsymbol{\pi}_0^{\Im}(d) = c$ and $\boldsymbol{\pi}_1^{\Im}(d) = a$, by $\mathbf{(Pair)}_1$. Two uses of **(Repl)**, referring to the single-valued maps

$$Q_\ell \equiv_{\mathrm{Def}} \boldsymbol{\pi}_0\circ\overline{\ni\mathbb{1}} \cap \boldsymbol{\pi}_\ell \triangle \boldsymbol{\pi}_0\circ\ni\mathbb{1} \cap \boldsymbol{\pi}_{1-\ell}$$

with $\ell = 0$ and $\ell = 1$ respectively, will complete the job. Indeed, the first use of **(Repl)** will form from d a set c_a comprising a and $\{\}$ as elements, while the second will form

[4]Here, like in the case of **(Ur)** (which could have been stated more simply as $\mathsf{Total}(\overline{\ni}\circ\mathsf{ur})$), our preference goes to a formulation whose import is as little dependent as possible from the remaining axioms.

[5]Notice that when c belongs to a_ℓ ($\ell = 0, 1$), then $c \subsetneq a_{1-\ell}$; hence there is a c' in $a_{1-\ell} \setminus c$, so that c belongs to c'. Then $c' \subsetneq a_\ell$, and so on. Starting w.l.o.g. with c_0 in a_0, one finds distinct sets $c_0, c_1, c_2, \ldots$ with $c_{\ell+2\cdot i}$ in a_ℓ for $\ell = 0, 1$ and $i = 0, 1, 2, \ldots$.

[6]For the sake of completeness, let us mention here that for a statement not relying on **(R)**, the following cumbersome expression should be subtracted from the argument of Total in **(I)**:

$$(\ni\circ\boldsymbol{\pi}_0^{-1}\cap\ni\circ\boldsymbol{\pi}_1^{-1})\circ(\boldsymbol{\pi}_0\circ\in\circ\boldsymbol{\pi}_0^{-1}\cap\boldsymbol{\pi}_1\circ\in\circ\boldsymbol{\pi}_1^{-1}\cap\boldsymbol{\pi}_1\circ\ni\circ\boldsymbol{\pi}_0^{-1}\cap\overline{\boldsymbol{\pi}_0\circ\in\circ\boldsymbol{\pi}_1^{-1}})\circ(\boldsymbol{\pi}_0\circ\in\cap\boldsymbol{\pi}_1\circ\in).$$

[7]Notice that $\mathbf{(Pair)}_{2,3,4,5}$ can be superseded (retaining $\mathbf{(Pair)}_1$) by the definitions

$$\boldsymbol{\pi}_0 \equiv_{\mathrm{Def}} \mathsf{funPart}(\ni\circ\mathsf{funPart}(\ni)), \qquad \boldsymbol{\pi}_1 \equiv_{\mathrm{Def}} \ni\ni\cap\big((\overline{\ni\ni}\cup\boldsymbol{\pi}_0)\dagger\iota\big)\cap(\not\ni\dagger\ni\mathbb{1}).$$

from a pair d_a with components c_a and b a set c_{ab} comprising a and b as elements, for any given b.

Notice that either use of **(Repl)** in the above argument has exploited a single parameter, which was a and b respectively. □

7 Setting up experiments on a theorem-prover

A MAP CALCULUS, i.e, an inferential apparatus for $\mathcal{L}^{\times}$ is defined in [21], pp.45–47, along the following lines:

- A certain number of equality schemes are chosen as *logical axioms*. Each scheme comprises infinitely many map equalities $P{=}Q$ such that $P^{\Im} = Q^{\Im}$ holds in every interpretation $\Im$; syntactically it differs from an ordinary map equality in that *meta-variables*, which stand for arbitrary map expressions, may occur in it.
- *Inference rules* are singled out for deriving new map equalities $V{=}W$ from two equalities $P{=}Q$, $R{=}S$ (either assumed or derived earlier). Of course $V^{\Im} = W^{\Im}$ must hold in any interpretation $\Im$ fulfilling both $P^{\Im} = Q^{\Im}$ and $R^{\Im} = S^{\Im}$. The smallest collection $\Theta^{\times}(\mathtt{E})$ of map equalities that comprises a given collection $\mathtt{E}$ (of *proper* axioms) together with all instances of the logical axioms, and which is closed w.r.t. application of the inference rules, is regarded as the *theory* generated by $\mathtt{E}$.

A variant of this formalism, which differs in the choice of the logical axioms (because $\cap$ and $\triangle$ seem preferable to $\cup$ and $^{-}$ as primitive constructs), has been proposed in [6]. We omit the details here, although we think that the choice of the logical axioms can critically affect the performance of automatic deduction within our theories. Ideally, only minor changes in the formulation $\mathtt{E}$ of the set axioms should be necessary if the logical axioms are properly chosen. Similarly, in the automation of GB, one has to bestow some care to the treatment of Boolean constructs (cf. [16], pp.107–109).

To follow [21] orthodoxly, we should treat $\mathcal{L}^{\times}$ as an autonomous formalism, on a par with first-order predicate calculus. This, however, would pose us two problems: we should develop from scratch a theorem-prover for $\mathcal{L}^{\times}$, and we should cope with the infinitely many instances of **(S)** and of **(Repl)**. Luckily, this is unnecessary if we treat as first-order variables the meta-variables that occur in the logical axioms or in **(S)**, **(Repl)** (as well as in induction schemes, should any enter into play either as additional axioms or as theses to be proved). Within the framework of first-order logic, the logical axioms lose their status and become just axioms on *relation algebras*, conceptually forming a chapter of axiomatic set theory interesting *per se*, richer than Boolean algebra and more fundamental and stable than the rest of the axiomatic system.

Attempts (some of which rather challenging) that one might carry out with any first-order theorem prover have the following flavor:

- Under the axioms **(E)**, $(\mathcal{P}ow)$, **(T)**, **(S)**, $\mathbf{(Pair)}_{1,2,3,4}$, **(R)**, $\mathsf{ur}{=}\emptyset$, and **(F)**, prove $(\mathcal{U}n)$, **(Repl)**, **(Ch)**, $\mathsf{Fun}(\,\nabla(\,P\,)\,)$, and $\mathsf{trans}{\circ}\overline{\iota}\backslash\mathbb{1}{\in}\backslash{\ni}{=}\emptyset$,[8] as theorems.
- Under **(E)**, $(\mathcal{U}n)$, **(S)**, **(Pair)**, **(R)**, and $\underline{\mathbf{(Ur)}}$, prove that ${\in}{\circ}(\,\in\cdots\in\,){\cap}\iota{=}\mathsf{ur}$, $\mathsf{nat}{\cap}\mathsf{ur}{=}\emptyset$, $\mathsf{ord}{\cap}\mathsf{ur}{=}\emptyset$, ${\in}{\circ}\mathsf{ord}\backslash\mathsf{ord}{\circ}\mathbb{1}{=}\emptyset$, $\emptyset{\dagger}{\notin}{\triangle}\mathbb{1}{\circ}\overline{\mathsf{ord}}{\circ}{\in}{\cup}\mathbb{1}{\circ}\big(\,{\in}{\cap}(\,(\,{\ni}{\triangle}\iota\,){\dagger}{\notin}\,)\,\big){=}\mathbb{1}$,[9] etc.

[8]The last of these states that the null set belongs to every transitively closed non-null set.

[9]The last two of these state that elements of ordinals are ordinals and that every non-null set of ordinals has a minimum w.r.t. $\in$.

- Under **(E)**, **($\mathcal{P}ow$)**, **($\mathcal{U}n$)**, **(S)**, **(Pair)**$_{1,2,3,5}$, and **(Repl)**, prove **(Pair)**$_4$, Cantor's theorem, and the totality of $\nabla(\emptyset)$, $\nabla(\boldsymbol{\iota})$, $\nabla(\overline{\not\ni\in})$, $\nabla(\ni\ni)$, $\nabla(\ni\cup\boldsymbol{\iota})$, $\nabla(\boldsymbol{\pi}_0\cup\boldsymbol{\pi}_1)$, $\nabla(\boldsymbol{\pi}_0\circ\ni\cup\boldsymbol{\pi}_1)$, $\nabla(\mathsf{funPart}(Q)\circ\ni\cup\mathsf{setPart}(P))$ where $\mathsf{setPart}(P) \equiv_{\mathrm{Def}} P\cap\partial(P)\circ\mathbb{1}$, etc.

We count on the opportunity to soon start a systematic series of experiments of this nature, for which we are inclined to using Otter. The latter is not specifically oriented to equational logic, and it is conceivable that a system based on term rewriting might fit our needs better. However, we plan to perform extensive experimentation with theories of number and sets specified in $\mathcal{L}^{\times}$, and we are eager to compare the results of our experiments with the work of others. Otter is attractive in this respect, because it has been the system underlying experiments of the kind we have in mind, as reported in [16, 17]. Moreover, the fact that Otter encompasses full first-order logic paves the way to combined reasoning tactics that, e.g., perform resolution of $\mathbb{1}\circ P\circ\mathbb{1}=\mathbb{1}$ against $P=\emptyset$.

8 Conclusions

The language $\mathcal{L}^{\times}$ may look distasteful to reading, but it ought to be clear that techniques for moving back and forth between first-order logic and map logic exist and are partly implemented (cf. [21, 4, 6]); moreover they can be ameliorated, and can easily be extended to meet the specific needs of set theories. Thanks to these, the automatic crunching of set axioms of the kind discussed in this paper can be hidden inside the back-end of an automated reasoner.

Anyhow, we think that it is worthwhile to riddle through experiments our expectation that a basic machine reasoning layer designed for $\mathcal{L}^{\times}$ may significantly raise the degree of automatizability of set-theoretic proofs. This expectation relies on the merely equational character of $\mathcal{L}^{\times}$ and on the good properties of the map constructs; moreover, when the calculus of $\mathcal{L}^{\times}$ gets emulated by means of first-order predicate calculus, we see an advantage in the finiteness of the axiomatization of ZF.

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