Backdoors for SAT

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Content

Outline

Introduction (1/2)

A few auxiliary definitions:

- Class $\mathcal{C}$: a set of formulas sharing some property.
- A CNF formula F is in *Horn* iff each of its clauses has *at most* one positive literal.
- F is in Horn:

$$F = (x \vee \neg y) \wedge (y \vee \neg z \vee \neg w) \wedge (\neg w \vee \neg z)$$

- Horn can be solved in polynomial time (e.g., by unit propagation)

Introduction (2/2)

Assume I give you a CNF like this (1000 clauses, >100 variables, 3-SAT):

$$
\begin{array}{ll}
(x_1 \vee \neg x_2 \vee \neg x_3) & \vee \\
\text{... 997 Horn clauses...} & \vee \\
(x_{n_1} \vee \neg x_{n_2} \vee \neg x_{n_3}) & \vee \\
(x_{m_1} \vee x_{m_2} \vee x_{m_3}) &
\end{array}
$$

- How difficult is this instance? (E.g., could you solve it by hand?)
- Solving instances of this kind gives me an NP-hard problem?
- This is FPT !

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Motivation

- A backdoor is a set of variables. Once we assign a value to the backdoor variables, the problem becomes tractable (in P)
- Introduced by Williams et al. ([8]) to try to explain the good performances of modern SAT solvers.
- Their claim: modern SAT solver can find backdoors easily.

→ But solvers are NOT designed for this!

! Why don't we try to pro-actively find these backdoors?

- Finding backdoors is an NP-Hard problem!

Previous work

Theoretical work:

- Defining several types of backdoors,
- Complexity of finding them (especially parameterized complexity)

Empirical work:

- Showing that backdoor sets are "small", for different types of backdoors
- Results are mostly based on local search algorithms! (Incompleteness)
- Little information on *runtime* required to find them
- One work ([5]) shows that using Horn-backdoors improves SAT solving speed. However, no information on how long it takes to find them!

Questions:

→ How can we find backdoors *efficiently*?

→ Can we "predict" backdoors by using additional domain knowledge?

Outline

Deletion Backdoors (1/2)

$F - B$**:** Replacing in each clause of F the occurrences of x and $\neg x$ with $\bot$ for each variable $x \in B$ (with $B \subseteq var(F)$) and simplifies the clause. Basically, remove the occurrences (positive and negative) of the variables in B from F

Definition: Deletion C-backdoor ([4])

A non-empty subset B of the variables of the formula F ($B \subseteq var(F)$) is a *deletion backdoor* w.r.t. a class $\mathcal{C}$ for F iff $F - B \in C$.

Example

$$F = (x \vee y \vee z) \wedge (\neg x \vee z \vee w) \wedge (y \vee w) \qquad B = \{z, y\}$$

$$F - B \equiv (x \vee \not{y} \vee \not{z}) \wedge (\neg x \vee \not{z} \vee w) \wedge (\not{y} \vee w)$$

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Deletion Backdoors (2/2)

From a deletion C-backdoor we can generate $2^{|B|}$ formulas, each with a different assignment to the backdoor variables. Once the backdoor is decided, we can solve these instances in polynomial time!

Vertex Cover (1/2)

Definition: Vertex Cover

Given a graph $G = (V, E)$, we call $R = \{v_1, .., v_n\} \subseteq V$ a *vertex cover* of G iff for all $e \in E$ there exists a $v_i \in R$ s.t. $v_i \in e$.

In other words, we have a vertex $v_i \in R$ as *representative* of each edge in E. We call $|R|$ the size of the vertex cover.

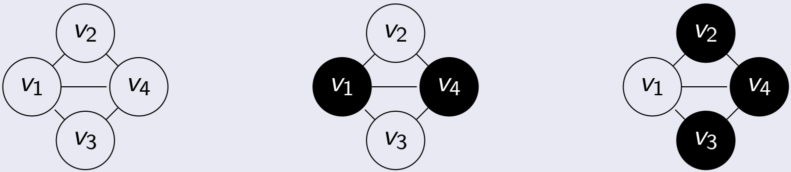
Definition: Vertex Cover Problem

Given a graph $G = (V, E)$ and an integer $k > 0$, is there a vertex cover R for G s.t. $|R| \leq k$?

Vertex Cover (2/2)

Example

Consider $G = (V, E)$:



Then $C_1 = \{v_1, v_4\}$ and $C_2 = \{v_2, v_3, v_4\}$ are vertex covers for G but only C_1 is a solution for the Vertex cover instance $(G, 2)$.

Reduction to Vertex Cover (1/2)

Samer and Szeider ([7]) propose a reduction from deletion Horn-backdoor detection to the vertex cover problem.

Definition

G_F is the graph composed by the variables of the CNF formula F in which two variables v, u are adjacent iff v and u appear positively in a clause from F.

Lemma

A set $B \subseteq var(F)$ is a deletion Horn-backdoor for F iff B is a vertex cover of G_F

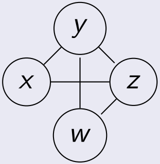
This relation extends also to Minimum Vertex Cover, in which we are interested in the vertex cover/backdoor of minimal size (smallest backdoor).

Reduction to Vertex Cover (2/2)

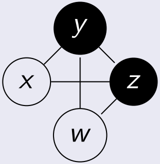
Example

$$F = (x \vee y \vee z) \wedge (\neg x \vee z \vee w) \wedge (y \vee w)$$

$$F - C \equiv x \wedge (\neg x \vee w) \wedge w$$



G_F



Cover/Deletion Horn-backdoor

We can use existing results from Vertex Cover (including FPT results) to solve deletion Horn-backdoor detection!

Goals & Challenges

- Study deletion Horn-backdoors in SAT instances

→ Build a dataset *efficiently*, e.g. FPT

– No available implementation of parameterized Vertex Cover

→ Implement algorithms that are a good trade-off between performance and implementation complexity

- Test whether we can use local search to efficiently find smallest deletion Horn-backdoors
- Study the relation of the backdoor size with features of the instances

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Solving Vertex Cover

- Local search algorithm: COVER ([6])
- FPT algorithms: Kernelization and Bounded search
- Reduction to SAT

Outline

Methodology

1. Benchmark with 3239 instances from various sources;
2. Generate the associated vertex cover instances;
3. Run a modified version of COVER [6] to obtain an upper-bound on the size of the smallest deletion Horn-backdoor;
4. For instances with small backdoors ($k \leq 150$) verify the minimality of the backdoor;
5. For instances with bigger backdoors confirm that the lower-bound is bigger than 150;
6. Considering only the instances for which we have the exact value of the smallest backdoor, compute the quality of the solution provided by a fast version of COVER.

Results

2418 (74%) instances have upper-bound on the size of the deletion Horn-backdoor up to 150:

- 2357 (97.5%) verified by the FPT algorithms,
- 8 more with CryptoMinisat,
- 53 remain unverified

Runtime[1]

- Time-out: 90 minutes
- Average: 25 second; 87% in less than 5 seconds, 93% in under a minute (thanks to kernelization!)

The generated vertex cover instances were in most of the cases easy, but a few were really hard (2%).

[1] Timings based on Intel Centrino 1.7Ghz, 1GB RAM

COVER Results

We define two configurations for COVER.

Full computation:

- Runtime: 30-90 minutes
- Solution quality: Always finds the optimum

Fast computation:

- Runtime: 115 ms (avg), 97 ms (avg) for $k \leq 150$
- Solution quality: 98% of the times optimum

Average error among all the instances is 0.11%.

COVER is a good method to compute smallest deletion Horn-backdoors.

Correlations

We are interested in the relation between smallest deletion Horn-backdoor (sdH-bd) size and other properties of the instances:

- **Flat** Colouring on flat graphs; sdH-bd size is exactly two times the number of vertices in the graph.
- **Random** Uniform random formulas (uf/uuf): correlation between number of variables and sdH-bd size is 0.99
- **CarConf** Verify some consistency properties of requested car configuration. Correlation between number of variables and upper-bound of the sdH-bd size for the *same* configuration is high:

Base configuration	Correlation (r)	# Instances
C168	0.99	58
C170	0.99	6
C202	0.83	23
C208	0.99	16
C210	0.89	32
C220	0.95	348
C638	0.73	84

Conclusions

We studied the relation between deletion Horn-backdoor detection and the vertex cover problem.

- COVER is a good way of computing quickly (and with an excellent quality) smallest deletion Horn-backdoors
- Kernelization can play a key role. Even with a simple kernelization, we solved many instances without search.
- In some cases, features of an instance can be related with the size of its smallest deletion Horn backdoor.

Further work

- Implement COVER/deletion Horn-backdoor detection in a solver and allow branching only on backdoors variables;
- Use backdoors to explore different solver architectures, and not only DPLL;
- Influence of preprocessing on different classes of backdoors;
- Build predictive models that can find backdoors!

Questions?

Datasets, tools and slides are available online: http://marco.gario.org/work/

References I

Faisal N. Abu-Khzam, R.L. Collins, M.R. Fellows, M.A. Langston, W.H. Suters, and C.T. Symons.
Kernelization algorithms for the vertex cover problem: Theory and experiments.
In *Proceedings of the 6th Workshop on Algorithm Engineering and Experiments (ALENEX)*, pages 62–69, 2004.

Y. Crama, O. Ekin, and P.L. Hammer.
Variable and term removal from Boolean formulae.
Discrete Applied Mathematics, 75(3):217–230, 1997.

F. Hüffner, R. Niedermeier, and S. Wernicke.
Techniques for Practical Fixed-Parameter Algorithms.
The Computer Journal, 51(1):7–25, March 2007.

References II

Naomi Nishimura and Prabhakar Ragde.
Solving #SAT using vertex covers.
Acta Informatica, 44(7):509–523, 2007.

Lionel Paris, Richard Ostrowski, Pierre Siegel, and Lakhdar Sais.
Computing Horn Strong Backdoor Sets Thanks to Local Search.
2006 18th IEEE International Conference on Tools with Artificial Intelligence (ICTAI'06), pages 139–143, November 2006.

Silvia Richter, Malte Helmert, and Charles Gretton.
A stochastic local search approach to vertex cover.
KI 2007: Advances in Artificial Intelligence, pages 412–426, 2007.

Marko Samer and Stefan Szeider.
Backdoor trees.
Proceedings of the 23rd Conference on Artificial, pages 363–368, 2008.

References III

Ryan Williams, C.P. Gomes, and Bart Selman.
Backdoors to typical case complexity.
In *Proceeding of IJCAI-03*, volume 18, pages 1173–1178, 2003.

Fixed-parameter tractable

- For an NP-Hard problem we have an exponential worst-case runtime in the size of the problem: e.g. SAT $O(2^n)$
- We can do better by defining a *parameter* k on which to confine the exponential explosion: e.g. p-SAT $O(2^k * n^c)$, with k number of variables and some constant c
- Problems for which we can identify such parameters are called *Fixed-parameter tractable* (FPT)
- Note that SAT parameterized by the size of a backdoor is FPT ($O(2^{|B|} * n^c)$)

Bounded search

- We implement a simple bounded search by Hüffner ([3]),
- The *trivial* solution for vertex cover has complexity $O(2^k)$
- The *best* $O(1.2738^k)$
- This one $O(1.47^k)$

Results (3/3)

Kernelization is important:

- 42% of all instances were solved by kernelization (51.5% in the group with $k \leq 150$);
- Otherwise, parameter reduction of 17.8% (avg)
- Parameter has exponential influence on runtime
- Extreme case from $k = 109$ to $k' = 6$

219 instances might have a solution of size ≤ 150, but remain unverified.

SAT Notation (1/2)

- F: a propositional formula in CNF
- $var(F)$: the set of variables occurring in F.
- l, $\bar{l}$: a positive variable or its negation
- J : (partial) interpretation. Partial mapping from $var(F)$ to the boolean values $\{\top, \bot\}$.
- Class $\mathcal{C}$: a set of formulas sharing some property.
- Horn class: F is in Horn iff each of its clauses has *at most* one positive literal.

SAT Notation (2/2)

Example

- $F = a \wedge \neg b \wedge (\neg d \vee c) \wedge (\neg c \vee d)$
- $var(F) = \{a, b, c, d\}$
- $J = \{\overline{b}, c\}$
- $F|_J \equiv a \wedge \cancel{\neg b} \wedge \cancel{(\neg d \vee c)} \wedge (\cancel{\neg c} \vee d) \equiv a \wedge d$
- For $V = \{c\}$,
  $F - V \equiv a \wedge \neg b \wedge (\neg d \vee \cancel{c}) \wedge (\cancel{\neg c} \vee d) \equiv a \wedge \neg b \wedge \neg d \wedge d$
- $F \in Horn$

Subsolvers

Definition: Subsolver [8]

We call an algorithm C a subsolver if, given an input formula F:

- Tricotomy: C either rejects the input F, or "determines" F correctly (as unsatisfiable or satisfiable, returning a solution if satisfiable),
- Efficiency: C runs in polynomial time,
- Trivial solvability: C can determine if F is trivially true (has no constraints) or trivially false (has contradictory constraint),
- Self-reducibility: if C determines F, then for any assignment J of the variable x C determines $F|_J$

There exists a subsolver for Horn! And also for other classes: e.g. RHorn, 2SAT, UP+PL.

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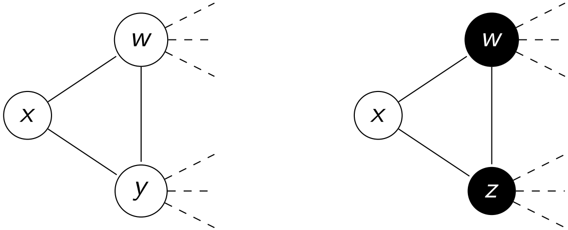
Simple Preprocessing I

For preprocessing we use the following four rules:

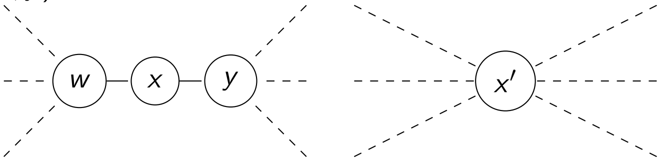
- P1 A vertex of degree 0 cannot be part of any cover, therefore we obtain G' by removing it from G.
- P2 If there is a vertex x of degree 1, then there is an optimal vertex cover in which its neighbour y is in the cover.

Simple Preprocessing II

P3 If there is a vertex of degree 2 with two adjacent neighbours, then there is an optimal vertex cover containing both these neighbours.



P4 If there is a vertex x of degree 2 with two non-adjacent neighbours y and w, then x can be removed by contracting the edges (w, x) and (x, y).



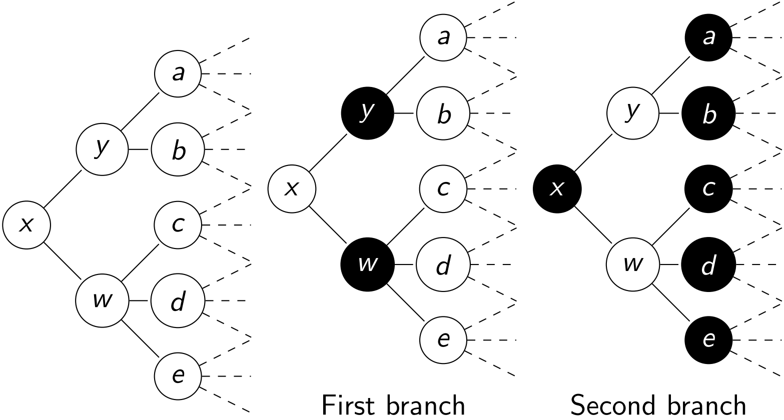
Bounded search I

We use this simple bounded search by Hüffner ([3])

S1 If there is a vertex of degree one, put its neighbour into the cover.

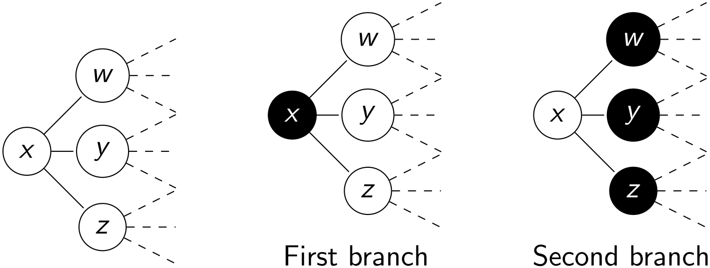
S2 If there is a vertex x of degree two, then either i) both neighbours of x are in an optimal vertex cover, or ii) x is in an optimal cover together with all neighbours of its neighbours.

Bounded search II



Bounded search III

S3 If there is a vertex x of degree at least three, then either x or all its neighbours are in the cover.



First branch

Second branch

Strong/Deletion Backdoors (1/2)

$F|_J$**:** reduct of F w.r.t. the (partial) interpretation J; it is obtained by replacing each variable v in F with $J(v)$ and simplifying.

Definition: Strong C-Backdoor ([8])

A non-empty subset B of the variables of the formula F ($B \subseteq var(F)$) is a *strong backdoor* w.r.t. the subsolver C for F iff **for all** interpretations $J : B \rightarrow \{\top, \bot\}$, C returns a satisfying assignment or concludes unsatisfiability of $F|_J$.

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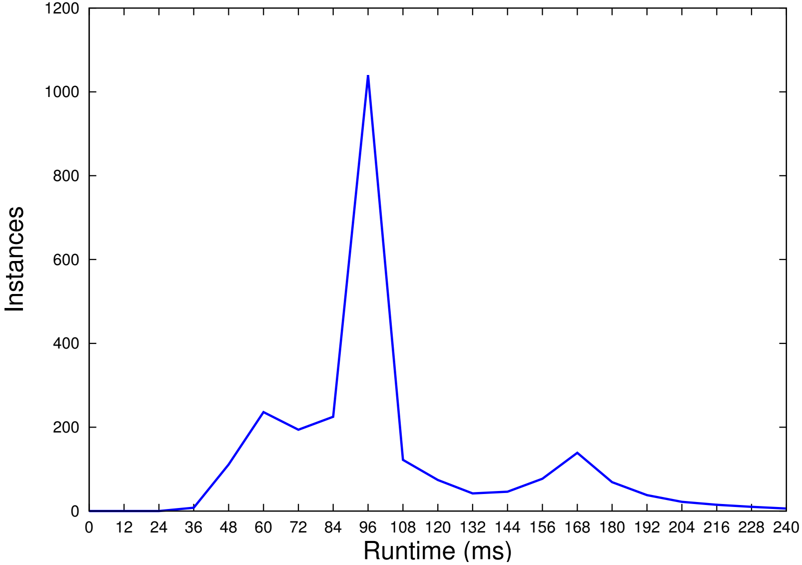
Strong/Deletion Backdoors (1/2)

For some classes, like Horn, deletion backdoor and strong backdoor are equivalent. ([2])

Example

This is good! To verify whether B is a deletion C-backdoor we just need to check whether $F - B$ is in $\mathcal{C}$ and not whether all the $2^{|B|}$ reducts of F are.

Local Search: COVER



Kernelization

Kernelization

Given a parameterized problem (x, k), a kernelization is a polynomial time preprocessing technique that either:

- returns a new equisatisfiable instance (x', k') (with $|x'| \leq g(k)$ and $k' \leq k$),
- rejects the instance as unsatisfiable

The high degree kernelization is simple but powerful:

HdK A vertex of degree $> k$ must be in any cover of size $\leq k$.

By applying some other pre-processing to our graph, we can apply the following result:

Property ([1])

If G is a graph with a vertex cover of size k and there is no vertex of G with degree $> k$ or degree < 3, then $|V| \leq \frac{k^2}{3} + k$.