Non-classical Logics: Theory, Applications and Tools

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Joint work with
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Non-classical logics

- are logics different from classical logic
- provide adequate languages for reasoning, e.g., about computer programs, dynamic data structures, resources, algebraic varieties, natural language, vague or inconsistent information, ...

E.g. logics for reasoning with inconsistencies

- Within classical logic, inconsistency leads to the trivialization of the knowledge base, as everything becomes derivable:

$$A, \neg A \vdash B$$

- Paraconsistent logics are logics which allow contradictory but non-trivial theories.

Definition

A propositional logic L is *paraconsistent* (with respect to $\neg$) if there are L-formulas A, B, such that $A, \neg A \nvdash B$.

How many interesting and useful logics?

Describing Logics

Non-classical logics are often described/introduced by adding suitable properties to known systems:

- Hilbert axioms
- Semantic conditions

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- Semantic conditions

Example: Gödel logic is obtained from intuitionistic logic

- by adding the Hilbert axiom $(\phi \to \psi) \lor (\psi \to \phi)$, or
- by adding the algebraic equation $1 \leq (x \to y) \lor (y \to x)$ to Heyting algebras

Why Proof Theory?

The applicability/usefulness of non-classical logics strongly depends on the availability of analytic calculi.

"*A logic without an analytic calculus is like a car without an engine*" (J.Y. Girard)

Why Proof Theory? II

Analytic calculi

- are key for developing automated reasoning methods
- are useful for establishing various properties of logics

(if *uniform*) also facilitate the switch from one logic to another, deepening the understanding of the relations between them.

Sequent Calculus

Sequents (Gentzen 1934)

$$A_1, \ldots, A_n \vdash B_1, \ldots, B_m$$

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$$A_1, \ldots, A_n \vdash B_1, \ldots, B_m$$

Axioms: E.g., $A \vdash A$, $\bot \vdash A$

Rules (left and right):

- Structural

  E.g.

  $$\frac{\Gamma, A, A \Rightarrow \Pi}{\Gamma, A \Rightarrow \Pi}\ (c, l) \quad \frac{\Gamma, A \Rightarrow \Pi}{\Gamma, A, B \Rightarrow \Pi}\ (w, l) \quad \frac{\Gamma, B, A \Rightarrow \Pi}{\Gamma, A, B \Rightarrow \Pi}\ (e, l)$$

- Logical
- Cut

Sequent Calculus – the cut rule

$$\frac{\Gamma \Rightarrow A \quad A \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Pi}\ \mathit{Cut}$$

- key to prove completeness w.r.t. Hilbert systems

$$\text{modus ponens} \quad \frac{A \quad A \rightarrow B}{B}$$

- corresponds to transitivity in algebra: from $x \leq a$ and $a \leq y$ follows $x \leq y$
- bad for proof search

Cut-elimination theorem

Each proof using Cut can be transformed into a proof without Cut.

Sequent Calculus – state of the art

Cut-free sequent calculi have been successfully used

- to prove consistency, decidability, interpolation, ...
- as bases for automated theorem proving
- to give syntactic proofs of algebraic properties for which (in particular cases) semantic methods are not known

Many useful and interesting logics have no cut-free sequent calculus

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Many useful and interesting logics have no cut-free sequent calculus

A large range of generalizations of sequent calculus have been introduced

State of the art

The definition of analytic calculi is usually logic-tailored.

Steps:

(i) choosing a framework

(ii) looking for the "right" inference rule(s)

(iii) proving cut-elimination

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Our Dream

Uniform procedures and automated support to

- define analytic calculi for non-classical logics 
- use the introduced calculi for proving interesting properties about the formalized logics in a uniform and systematic way

This talk

- General method to define analytic calculi and their exploitation
- Case Studies:

Substructural logics

analytic calculi : sequent and hypersequent

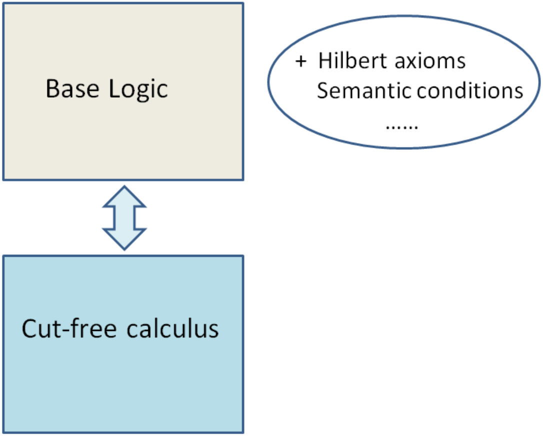
applications : order-theoretic completions, standard completeness

Paraconsistent logics

analytic calculi : sequent

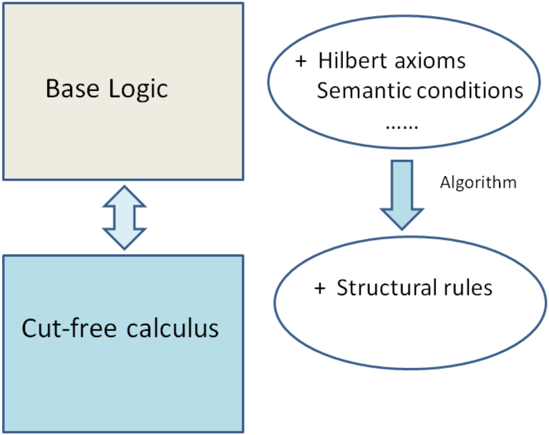
applications : non-deterministic matrices, decidability

The idea



Base Logic
+ Hilbert axioms
Semantic conditions
......
Cut-free calculus

The idea



Base Logic
Cut-free calculus
+ Hilbert axioms
Semantic conditions
......
Algorithm
+ Structural rules

Case study: Substructural Logics

- encompass e.g., intuitionistic logic, linear logic, fuzzy logics, the logic of Bunched Implications ...
- defined by adding Hilbert axioms to Full Lambek calculus **FL** or algebraic equations to residuated lattices

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Algebraic semantics for **FLe** = **FL** + exchange

(bounded pointed) commutative residuated lattice

$$\mathbf{P} = \langle P, \wedge, \vee, \otimes, \rightarrow, \top, \mathbf{0}, \mathbf{1}, \bot \rangle$$

1. $\langle P, \wedge, \vee, \top, \mathbf{0} \rangle$ is a lattice with $\top$ greatest and $\bot$ least
2. $\langle P, \otimes, \mathbf{1} \rangle$ is a commutative monoid.
3. For any $x, y, z \in P$, $x \otimes y \leq z \Longleftrightarrow y \leq x \rightarrow z$
4. $\mathbf{0} \in P$.

The sequent calculus for **FLe**

$$\frac{A, B, \Gamma \vdash \Pi}{A \otimes B, \Gamma \vdash \Pi}\ \otimes l \qquad \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\ \otimes r$$

$$\frac{\Gamma \vdash A \quad B, \Delta \vdash \Pi}{\Gamma, A \to B, \Delta \vdash \Pi} \to l \qquad \frac{A, \Gamma \vdash B}{\Gamma \vdash A \to B} \to r$$

$$\frac{A, \Gamma \vdash \Pi \quad B, \Gamma \vdash \Pi}{A \vee B, \Gamma \vdash \Pi}\ \vee l \qquad \frac{\Gamma \vdash A_i}{\Gamma \vdash A_1 \vee A_2}\ \vee r \qquad \frac{}{\mathbf{0} \vdash}\ \mathbf{0}l$$

$$\frac{A_i, \Gamma \vdash \Pi}{A_1 \wedge A_2, \Gamma \vdash \Pi}\ \wedge l \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ \wedge r \qquad \frac{}{\Gamma \vdash \top}\ \top r$$

$$\frac{\Gamma \vdash}{\Gamma \vdash \mathbf{0}}\ \mathbf{0}r \qquad \frac{}{\vdash \mathbf{1}}\ \mathbf{1}r \qquad \frac{}{\bot, \Gamma \vdash \Pi}\ \bot l \qquad \frac{\Gamma \vdash \Pi}{\mathbf{1}, \Gamma \vdash \Pi}\ \mathbf{1}l$$

Π contains at most one formula

Case study: Substructural Logics

They lack the properties expressed by sequent calculus structural rules

- Contraction: $\alpha \to \alpha \wedge \alpha$ $$\frac{A, A, \Gamma \vdash \Pi}{A, \Gamma \vdash \Pi}\ (c)$$
- Exchange: $\alpha \wedge \beta \to \beta \wedge \alpha$ $$\frac{\Gamma, B, A, \Pi \vdash \Delta}{\Gamma, A, B, \Pi \vdash \Delta}\ (e)$$
- Weakening: $\alpha \wedge \beta \to \alpha$ $$\frac{\Gamma \vdash \Pi}{\Gamma, A \vdash \Pi}\ (w)$$

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$$\frac{A, A, \Gamma \vdash \Pi}{A, \Gamma \vdash \Pi}\ (c)$$

- Exchange: $\alpha \wedge \beta \to \beta \wedge \alpha$

$$\frac{\Gamma, B, A, \Pi \vdash \Delta}{\Gamma, A, B, \Pi \vdash \Delta}\ (e)$$

- Weakening: $\alpha \wedge \beta \to \alpha$

$$\frac{\Gamma \vdash \Pi}{\Gamma, A \vdash \Pi}\ (w)$$

$\vdash_{FLe+axiom}$ 

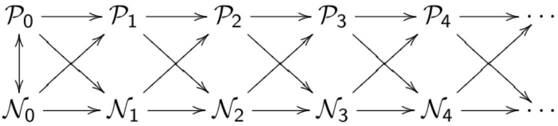
$=$ $\vdash_{FLe+rule}$ 

Classification – substructural logics

Definition (Classification; AC, Galatos and Terui, LICS 2008)

The classes $\mathcal{P}_n, \mathcal{N}_n$ of positive and negative axioms/equations are:

- $\mathcal{P}_0 ::= \mathcal{N}_0 ::=$ atomic formulas
- $\mathcal{P}_{n+1} ::= \mathcal{N}_n \mid \mathcal{P}_{n+1} \vee \mathcal{P}_{n+1} \mid \mathcal{P}_{n+1} \otimes \mathcal{P}_{n+1} \mid 1 \mid \bot$
- $\mathcal{N}_{n+1} ::= \mathcal{P}_n \mid \mathcal{P}_{n+1} \to \mathcal{N}_{n+1} \mid \mathcal{N}_{n+1} \wedge \mathcal{N}_{n+1} \mid 0 \mid \top$



Examples

Class	Axiom	Name
$\mathcal{N}_2$	$\alpha \to \mathbf{1}$, $\bot \to \alpha$	weakening
	$\alpha \to \alpha \otimes \alpha$	contraction
	$\alpha \otimes \alpha \to \alpha$	expansion
	$\otimes\alpha^n \to \otimes\alpha^m$	knotted axioms
	$\neg(\alpha \wedge \neg\alpha)$	weak contraction
$\mathcal{P}_2$	$\alpha \vee \neg\alpha$	excluded middle
	$(\alpha \to \beta) \vee (\beta \to \alpha)$	prelinearity
$\mathcal{P}_3$	$\neg\alpha \vee \neg\neg\alpha$	weak excluded middle
	$\neg(\alpha \otimes \beta) \vee (\alpha \wedge \beta \to \alpha \otimes \beta)$	(wnm)
$\mathcal{N}_3$	$((\alpha \to \beta) \to \beta) \to ((\beta \to \alpha) \to \alpha)$	Łukasiewicz axiom
	$(\alpha \wedge \beta) \to \alpha \otimes (\alpha \to \beta)$	divisibility

Transformation

Given a base cut-free calculus $\mathcal{C}$. The algorithm is based on:

Ingredient 1

The use of the invertible logical rules of $\mathcal{C}$

Ingredient 2: Ackermann Lemma

An algebraic equation $t \leq u$ is equivalent to a quasiequation $u \leq x \Rightarrow t \leq x$, and also to $x \leq t \Rightarrow x \leq u$, where x is a fresh variable not occurring in t, u.

Example: the sequent $A \vdash B$ is equivalent to

$$\frac{\Gamma \vdash A}{\Gamma \vdash B} \qquad\qquad \frac{\Gamma, B \vdash \Delta}{\Gamma, A \vdash \Delta}$$

(Γ, Δ fresh metavariables for multisets of formulas)

From axioms to sequent rules: an example

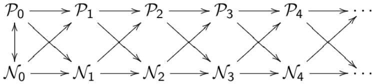
Axiom $\neg(\neg A \wedge A)$

Equivalent to	$\dfrac{}{\Rightarrow \neg(\neg A \wedge A)}$
Invertibility	$\dfrac{}{\neg A \wedge A \Rightarrow}$
Ackermann Lemma	$\dfrac{\Gamma \Rightarrow \neg A \wedge A}{\Gamma \Rightarrow}$
Invertibility	$\dfrac{\Gamma \Rightarrow \neg A \qquad \Gamma \Rightarrow A}{\Gamma \Rightarrow}$
Invertibility	$\dfrac{\Gamma, A \Rightarrow \qquad \Gamma \Rightarrow A}{\Gamma \Rightarrow}$
Equivalent rule	$\dfrac{\Gamma, \Gamma \Rightarrow}{\Gamma \Rightarrow}$

Preliminary results

Algorithm to transform:

- axioms/equations up to the class $\mathcal{N}_2$ into "good" structural rules in sequent calculus



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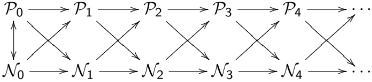

An application to universal algebra

Analytic sequent calculi iff Dedekind-MacNeille completion

(AC, N. Galatos and K. Terui. LICS 2008 and APAL 2012)

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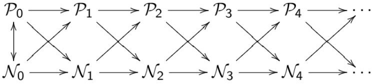

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Behond $\mathcal{N}_2$?

$$\text{Ex.} \quad (\alpha \to \beta) \vee (\beta \to \alpha)$$

Hypersequent calculus

It is obtained embedding sequents into hypersequents

$$\Gamma_1 \vdash \Pi_1 \mid \ldots \mid \Gamma_n \vdash \Pi_n$$

where for all $i = 1, \ldots n$, $\Gamma_i \vdash \Pi_i$ is a sequent.

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where for all $i = 1, \ldots n$, $\Gamma_i \vdash \Pi_i$ is a sequent.
Axioms within the class $\mathcal{P}_3$ have the form

$$\mathcal{N}_2 \vee \mathcal{N}_2 \vee \cdots \vee \mathcal{N}_2$$

Hypersequent calculus

It is obtained embedding sequents into hypersequents

$$\Gamma_1 \vdash \Pi_1 \mid \dots \mid \Gamma_n \vdash \Pi_n$$

where for all $i = 1, \dots n$, $\Gamma_i \vdash \Pi_i$ is a sequent.

$$\frac{G \mid \Gamma \vdash A \quad G \mid A, \Delta \vdash \Pi}{G \mid \Gamma, \Delta \vdash \Pi}\ \mathit{Cut} \qquad \frac{}{G \mid A \vdash A}\ \mathit{Identity}$$

$$\frac{G \mid \Gamma \vdash A \quad G \mid B, \Delta \vdash \Pi}{G \mid \Gamma, A \rightarrow B, \Delta \vdash \Pi} \rightarrow l \qquad \frac{G \mid A, \Gamma \vdash B}{G \mid \Gamma \vdash A \rightarrow B} \rightarrow r$$

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$$\frac{G \mid \Gamma \vdash A \quad G \mid B, \Delta \vdash \Pi}{G \mid \Gamma, A \to B, \Delta \vdash \Pi} \to l \qquad \frac{G \mid A, \Gamma \vdash B}{G \mid \Gamma \vdash A \to B} \to r$$

and adding suitable rules to manipulate the additional layer of structure.

$$\frac{G}{G \mid \Gamma \vdash A} \; (\mathrm{ew}) \qquad \frac{G \mid \Gamma \vdash A \mid \Gamma \vdash A}{G \mid \Gamma \vdash A} \; (\mathrm{ec})$$

Structural hypersequent rules: an example

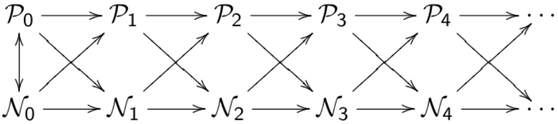
Gödel logic = Intuitionistic logic + $(\alpha \to \beta) \vee (\beta \to \alpha)$

$$\frac{G \mid \Gamma, \Sigma' \vdash \Delta' \quad G \mid \Gamma', \Sigma \vdash \Delta}{G \mid \Gamma, \Sigma \vdash \Delta \mid \Gamma', \Sigma' \vdash \Delta'} \ (com)$$

(Avron, Annals of Math and art. Intell. 1991)

$$\dfrac{\dfrac{\dfrac{\beta \vdash \beta \qquad \alpha \vdash \alpha}{\alpha \vdash \beta \mid \beta \vdash \alpha}\ (\text{com})}{\vdash \alpha \to \beta \mid \vdash \beta \to \alpha}\ (\to,\text{r}),(\to,\text{r})}{\dfrac{\vdash (\alpha \to \beta) \vee (\beta \to \alpha) \mid \vdash (\alpha \to \beta) \vee (\beta \to \alpha)}{\vdash (\alpha \to \beta) \vee (\beta \to \alpha)}\ (\text{EC})}\ (\vee_i,\text{r}),(\vee_i,\text{r})$$

Climbing up the hierarchy



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From axioms to rules – our tool

http://www.logic.at/people/lara/axiomcalc.html

Input: Hilbert axioms

TINC - AxiomCalc

General Information | AxiomCalc Web Interface | Obtaining AxiomCalc | Using AxiomCalc | References | Contact

› Input Syntax

› Examples

Input Axiom: ☐ Check for Standard Completeness

✓ Check

Output ×

Output AxiomCalc for: (a -> b) v (b -> a)

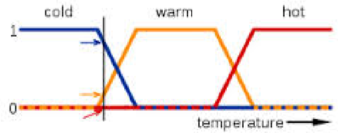
```
This axiom is in the class: p(2)

Equivalent Analytic Rule:

G|G+1,D+2 => P+2  G|G+2,D+1 => P+1
-----------------------------
 G|  G+2,D+2 => P+2|  G+1,D+1 => P+1
```

Get a paper containing an analytic calculus for FLe/HFLew extended with this axiom!

An application to standard completeness

Completeness of axiomatic systems with respect to algebras whose lattice reduct is the real unit interval $[0, 1]$.

(Hajek 1998) Formalizations of *Fuzzy Logic*



Some standard complete logics

T-norm based logics

Example: Gödel logic

$$v : \text{Propositions} \rightarrow [0,1]$$

$v(A \wedge B) = \min\{v(A), v(B)\}$ $\qquad$ $v(\bot) = 0$
$v(A \vee B) = \max\{v(A), v(B)\}$
$v(A \rightarrow B) = 1$ if $v(A) \leq v(B)$, and $v(B)$ otherwise

Monoidal T-norm based logic MTL (Godo, Esteva, FSS 2001)

$\text{MTL} = \text{FLew} + (\alpha \rightarrow \beta) \vee (\beta \rightarrow \alpha))$
$v(A \otimes B) = v(A) * v(B)$, $\quad *$ left continous t-norm
$v(A \vee B) = \max\{v(A), v(B)\}$
$v(A \rightarrow B) = v(A) \Rightarrow_* v(B)$
$v(\bot) = 0$

Standard Completeness?

Question Given a logic $\mathcal{L}$ obtained by extending MTL with

- $A \vee \neg A$ (excluded middle)?
- $A^{n-1} \to A^n$ (n-contraction)?
- $\neg(A \otimes B) \vee (A \wedge B \to A \otimes B)$ (weak nilpotent minimum)?
- ....

Is $\mathcal{L}$ standard complete? (*is it a formalization of Fuzzy Logic?*)

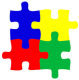
Case-by-case answer

A uniform proof of standard completeness

Theorem (Baldi and AC, TCS 2014, ISMVL 2015)

Given any set *Ax* of Hilbert axioms satisfying "suitable conditions" $MTL + Ax$ is standard complete.

Proof



- Define an analytic hypersequent calculus
- Elimination of the density rule (p eigenvariable):

$$\frac{G \mid \Gamma \vdash p \mid \Sigma, p \vdash \Delta}{G \mid \Gamma, \Sigma \vdash \Delta} \ (\textit{density})$$

- Dedekind-MacNeille style completion

Substructural logics – summary

- automated introduction of sequent and hypersequent calculi for large classes of logics
- preservation of various order-theoretic completions of the corresponding algebras
- uniform proofs of standard completeness for many (well known or new) logics

Case study: Paraconsistent logics

E.g. Da Costa's axioms defining the C-systems:

$(\mathbf{n_1})$	$\alpha \vee \neg\alpha$	$(\mathbf{n_2})$	$\alpha \to (\neg\alpha \to \beta)$
$(\mathbf{c})$	$\neg\neg\alpha \to \alpha$	$(\mathbf{e})$	$\alpha \to \neg\neg\alpha$
$(\mathbf{n}^{\mathbf{l}}_{\wedge})$	$\neg(\alpha \wedge \beta) \to (\neg\alpha \vee \neg\beta)$	$(\mathbf{n}^{\mathbf{r}}_{\wedge})$	$(\neg\alpha \vee \neg\beta) \to \neg(\alpha \wedge \beta)$
$(\mathbf{n}^{\mathbf{l}}_{\vee})$	$\neg(\alpha \vee \beta) \to (\neg\alpha \wedge \neg\beta)$	$(\mathbf{n}^{\mathbf{r}}_{\vee})$	$(\neg\alpha \wedge \neg\beta) \to \neg(\alpha \vee \beta)$
$(\mathbf{n}^{\mathbf{l}}_{\to})$	$\neg(\alpha \to \beta) \to (\alpha \wedge \neg\beta)$	$(\mathbf{n}^{\mathbf{r}}_{\to})$	$(\alpha \wedge \neg\beta) \to \neg(\alpha \to \beta)$
$(\mathbf{b})$	$\alpha \to (\neg\alpha \to (\circ\alpha \to \beta))$	$(\mathbf{r}_{\diamond})$	$\circ(\alpha \diamond \beta) \to (\circ\alpha \vee \circ\beta)$
$(\mathbf{k})$	$\circ\alpha \vee (\alpha \wedge \neg\alpha)$	$(\mathbf{k_2})$	$\circ\alpha \vee \neg\alpha$
$(\mathbf{o}^{\mathbf{1}}_{\diamond})$	$\circ\alpha \to \circ(\alpha \diamond \beta)$	$(\mathbf{o}^{\mathbf{2}}_{\diamond})$	$\circ\beta \to \circ(\alpha \diamond \beta)$
$(\mathbf{a}_{\diamond})$	$(\circ\alpha \wedge \circ\beta) \to \circ(\alpha \diamond \beta)$	$(\mathbf{a}_{\neg})$	$\circ\alpha \to \circ\neg\alpha$

.....

- Divide propositions into consistent and inconsistent ones.
- Reflect this classification within the language.
- Employ a special (primitive or defined) connective $\circ$ with intuitive meaning $\circ\alpha$: "α is consistent".

Case study: Paraconsistent logics

We identified a formal grammar generating (infinitely many) axioms in the language of CL^+ with *new unary connectives*.

For any set of axioms generated by this grammar we provided algorithms (and a PROLOG program):

(Step 1) to extract a corresponding **sequent calculus**

(Step 2) exploit the calculus to define **suitable semantics** for the logic, which is used:
$\Rightarrow$ to show the decidability of the logic

(AC, O. Lahav, L. Spendier, A. Zamansky. LFCS 2013 and TOCL 2014)

Useful semantics

Definition

A partial non-deterministic matrix (PNmatrix) $\mathcal{M}$ consists of:

(i) a set $\mathcal{V}_{\mathcal{M}}$ of truth values,

(ii) a subset of $\mathcal{V}_{\mathcal{M}}$ of designated truth values, and

(iii) a truth-table $\diamond_{\mathcal{M}} : \mathcal{V}_{\mathcal{M}}^{n} \to P(\mathcal{V}_{\mathcal{M}})$ for every n-ary connective $\diamond$.

PNmatrices generalise the notion of non-deterministic matrices (A. Avron, 2001) by allowing *empty sets* in the truth tables.

Why non-determinism?

Standard rules for classical negation and conjunction:

$$\frac{\Gamma \vdash \Delta, \psi}{\Gamma, \neg\psi \vdash \Delta} \qquad \frac{\Gamma, \psi \vdash \Delta}{\Gamma \vdash \Delta, \neg\psi}$$

$$\frac{\Gamma, \psi, \varphi \vdash \Delta}{\Gamma, \psi \wedge \varphi \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta, \psi \quad \Gamma \vdash \Delta, \varphi}{\Gamma \vdash \Delta, \psi \wedge \varphi}$$

	$\neg$
1	**0**
0	**1**

		$\wedge$
1	**1**	**1**
1	**0**	**0**
0	**1**	**0**
0	**0**	**0**

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$$\frac{\Gamma, \psi, \varphi \vdash \Delta}{\Gamma, \psi \wedge \varphi \vdash \Delta} \qquad \frac{\quad}{\quad}$$

	$\neg$
1	**0**
0	**???**

		$\wedge$
1	**1**	???
1	**0**	**0**
0	**1**	**0**
0	**0**	**0**

Why non-determinism?

Standard rules for classical negation and conjunction:

$$\frac{\Gamma \vdash \Delta, \psi}{\Gamma, \neg\psi \vdash \Delta} \qquad \frac{}{}$$

$$\frac{\Gamma, \psi, \varphi \vdash \Delta}{\Gamma, \psi \wedge \varphi \vdash \Delta} \qquad \frac{}{}$$

	$\neg$
1	$\{\mathbf{0}\}$
0	$\{\mathbf{1},\mathbf{0}\}$

		$\wedge$
1	**1**	$\{\mathbf{1}, \mathbf{0}\}$
1	**0**	$\{\mathbf{0}\}$
0	**1**	$\{\mathbf{0}\}$
0	**0**	$\{\mathbf{0}\}$

Step 2: Extracting PNmatrices

- Truth values $\mathcal{V}_{\mathcal{M}}$: tuples of size $=$ $\#$ of unary connectives $+1$
- New rules reduce the level of non-determinism

Type 3: $\dfrac{\mathcal{P}}{\Gamma, \star\psi \Rightarrow \Delta}$ reduce the set of truth values $\mathcal{V}_{\mathcal{M}}$

Type 2: $\dfrac{\mathcal{P}}{\Gamma, \star_i \star_j \psi \Rightarrow \Delta}$ determine truth tables for $\star_j$

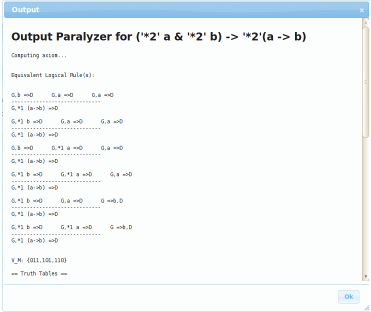
Type 1: $\dfrac{\mathcal{Q}}{\Gamma, \star(\psi \diamond \varphi) \Rightarrow \Delta}$ determine truth tables for $\diamond$

Our Tool : Paralyzer (PARAconsistent logic anaLYZER)

Input: Set of axioms $\mathcal{A}$ according to our grammar.

Output:

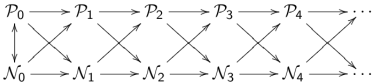
- Proof Theory: sequent calculus for CL^+ with $\mathcal{A}$
- Semantics: truth tables (using PNmatrices)
- Encoding of the calculus into ISABELLE



http://www.logic.at/staff/lara/tinc/webparalyzer/paralyzer.html

Open problems and work in progress

- Systematic introduction of analytic calculi
  - enlarge the set of axioms we can capture



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- Their exploitation
  - automated deduction methods
  - algebraic completions
  - Curry Howard hysomorphism
  - ...
- Applications: e.g. analysis of the Indian sacred texts (Vedas)

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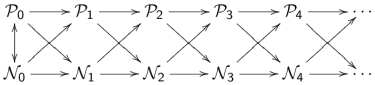

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"Non-classical Proofs: Theory, Applications and Tools", research project 2012-2017 (START prize – Austrian Research Fund)