Higher-order Unification (a personal perspective)

Tomer Libal

Inria, France

EMCL Workshop 2016
Vienna, February 12

History

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$$\vdash \neg Pa, Pffa, \exists x.Px \wedge \neg Pfx$$

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In 1962 J.A. Robinson has read Prawitz's paper and
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  - Martelli and Montanari (1982) - linear, relatively efficient

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Where x does not occur in v and $\sigma = [v/x]$.

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  - imitation $y \mapsto \lambda z.b$
  - projection $y \mapsto \lambda z.z$

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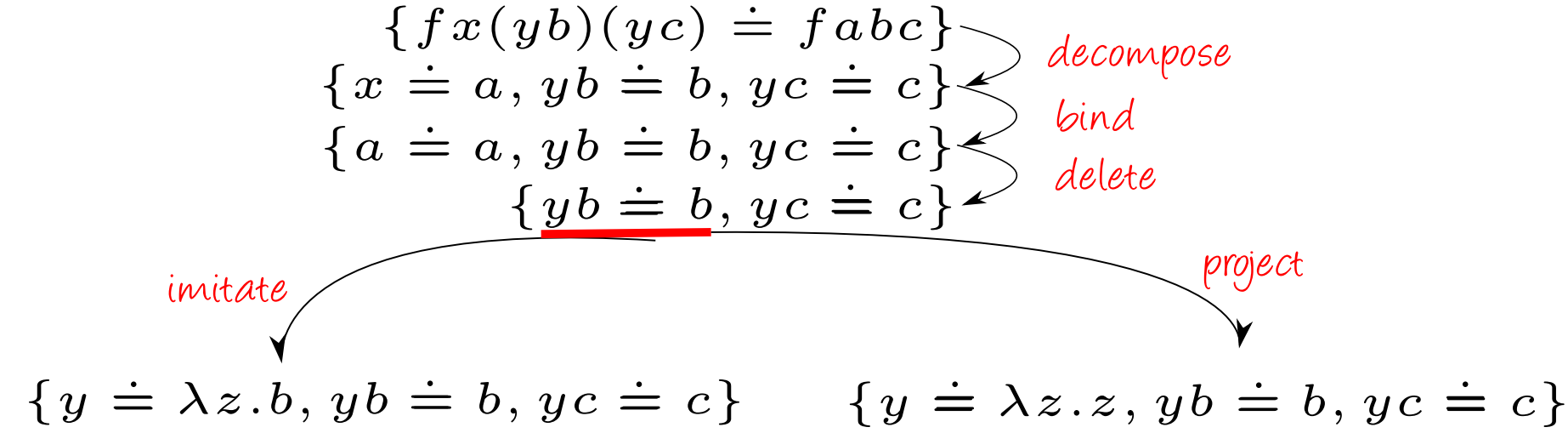

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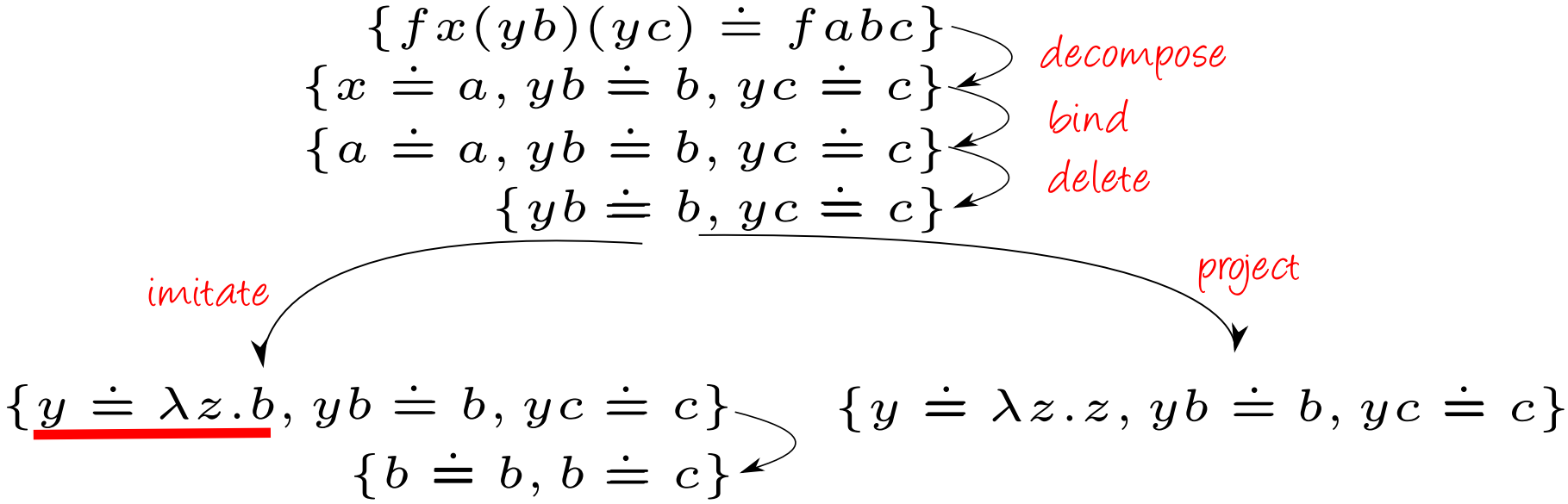

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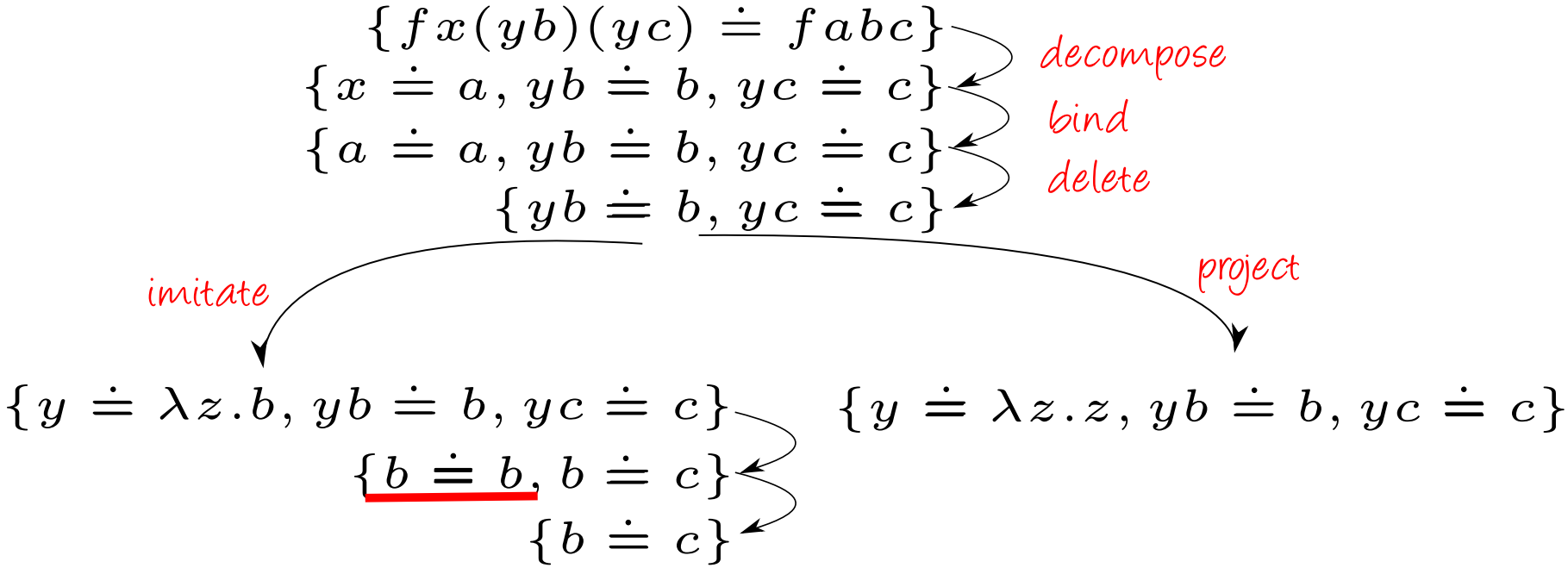

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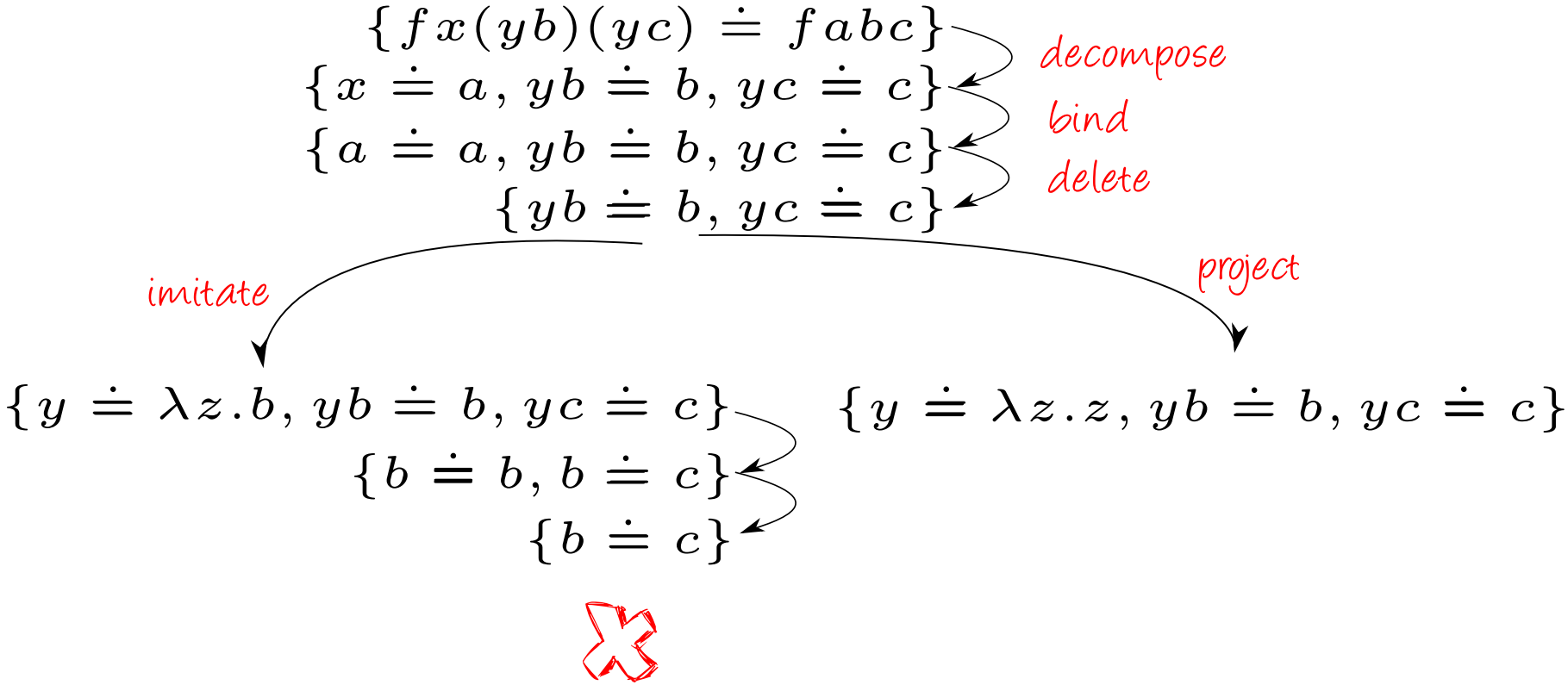

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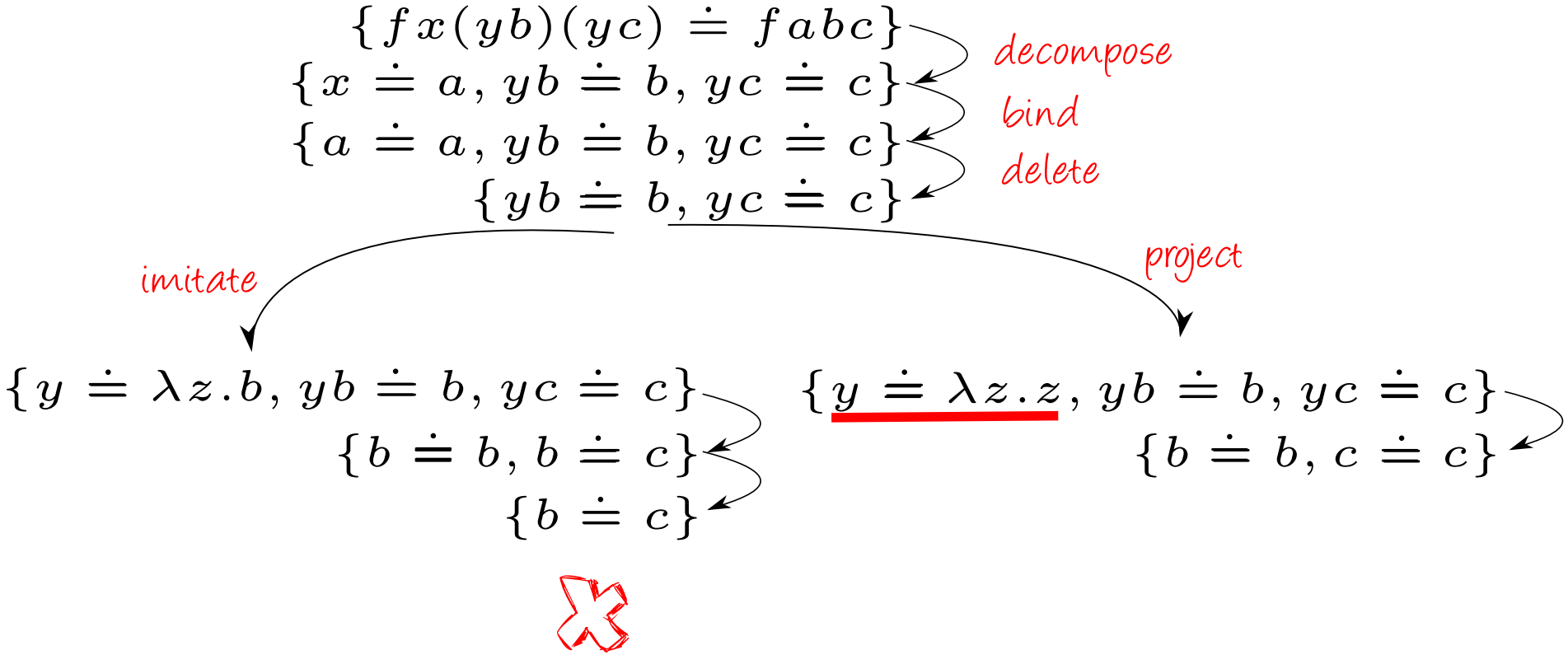

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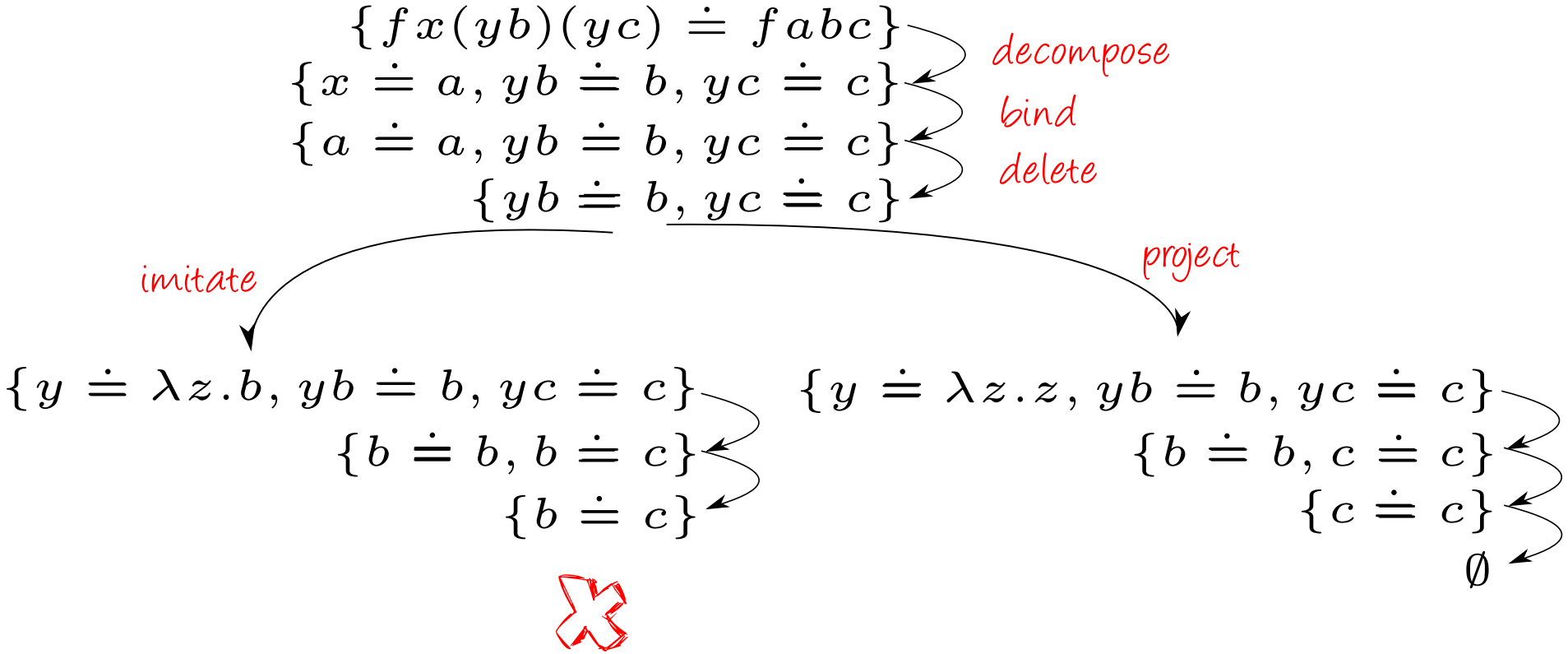

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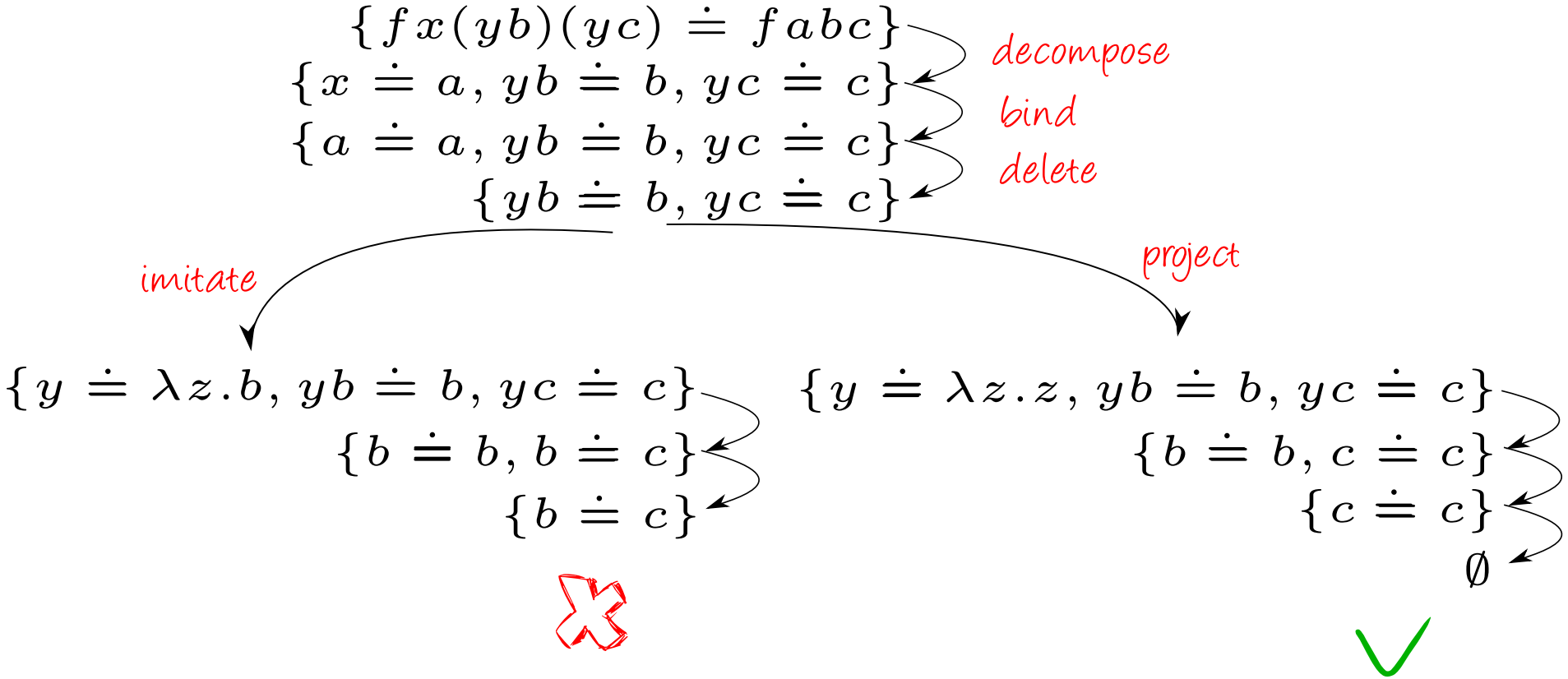

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$$\frac{\{u \doteq u\} \cup S}{S}\ \textit{delete} \qquad \frac{\{\lambda\overline{x_k}.z(\overline{x_k}) \doteq \lambda\overline{x_k}.v\} \cup S}{\{\langle\langle z, \lambda\overline{x_k}.v\rangle\rangle\} \cup \sigma(S)\downarrow_\beta}\ \textit{bind}$$

$$\frac{\{\lambda\overline{x_k}.a(\overline{v_n}) \doteq \lambda\overline{x_k}.a(\overline{u_n}\} \cup S}{\{\lambda\overline{x_k}.v_1 \doteq \lambda\overline{x_k}.u_1, \ldots, \lambda\overline{x_k}.v_n \doteq \lambda\overline{x_k}.u_n\} \cup S}\ \textit{decomp}$$

$$\frac{\{\lambda\overline{x_k}.y(\overline{u_n}) \doteq \lambda\overline{x_k}.b(\overline{v_m})\} \cup S}{\{y \uparrow_\eta \doteq t \uparrow_\eta, \lambda\overline{x_k}.y(\overline{u_n}) \doteq \lambda\overline{x_k}.b(\overline{v_m})\} \cup S}\ \textit{imitate} \qquad \frac{\{\lambda\overline{x_k}.y(\overline{u_n}) \doteq \lambda\overline{x_k}.a(\overline{v_m})\} \cup S}{\{y \uparrow_\eta \doteq s \uparrow_\eta, \lambda\overline{x_k}.y(\overline{u_n}) \doteq \lambda\overline{x_k}.a(\overline{v_m})\} \cup S}\ \textit{project}$$

Where $a \in \Sigma$ or $a \in \overline{x_k}$; $b \in \Sigma$; z does not occur in v;
$\sigma = [\lambda\overline{x_k}.v/x]$; $t = \lambda\overline{x_n}.b(\overline{y_m(\overline{x_n})})$;
$s = \lambda\overline{x_n}.x_i(\overline{y_l(\overline{x_n})})$ for $0 < i \leq n$ and $l = \mathtt{ty}(x_i)$.

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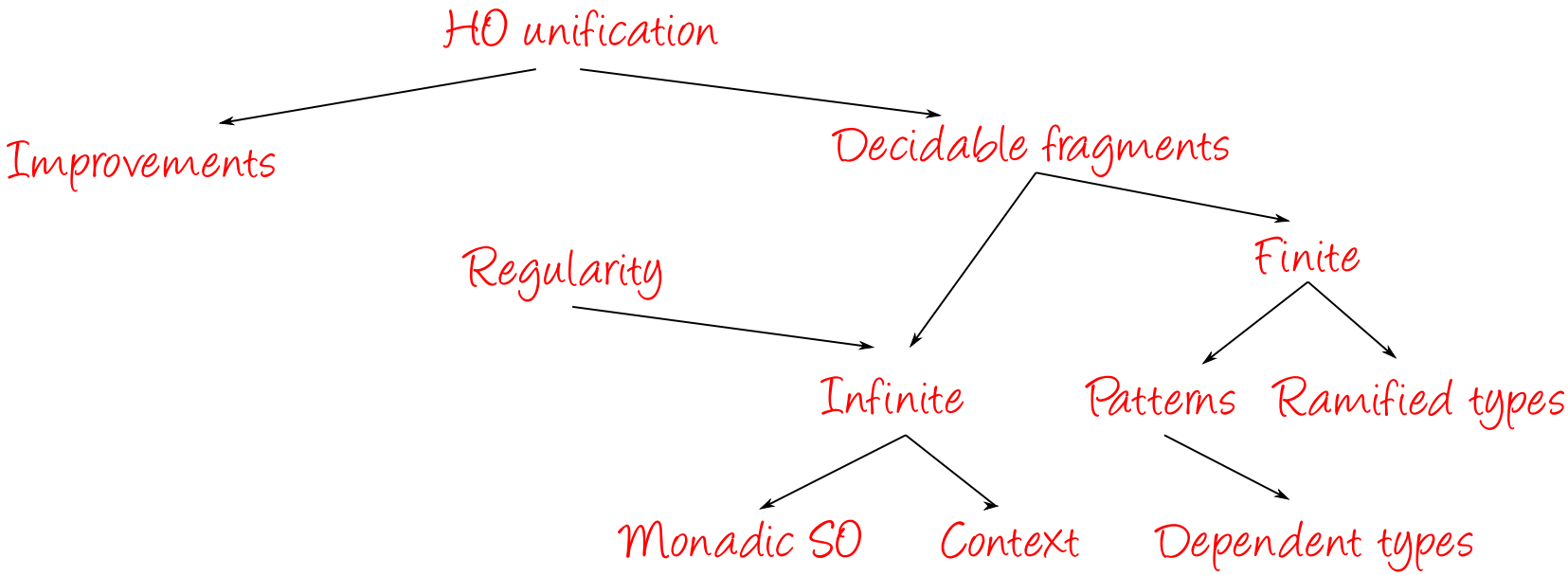
- Higher-order unification problem is semi-decidable

Higher-order Unification - Overview



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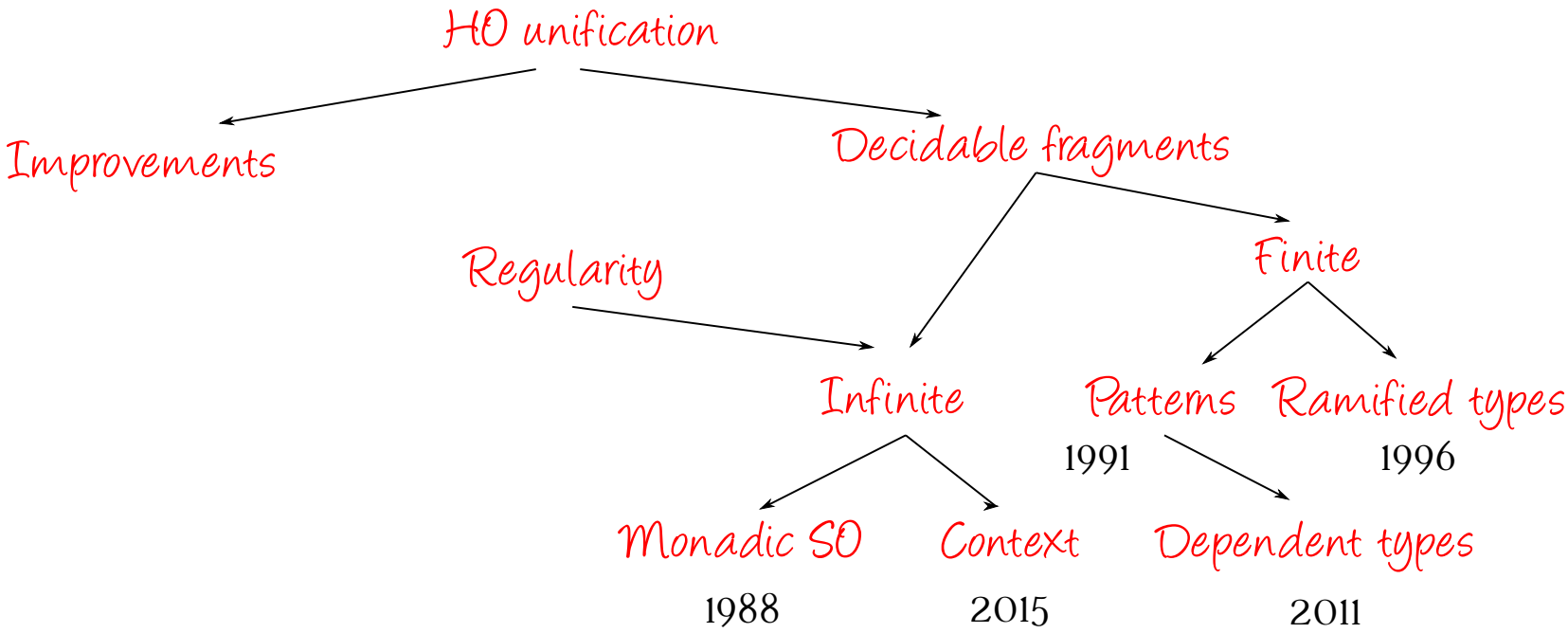

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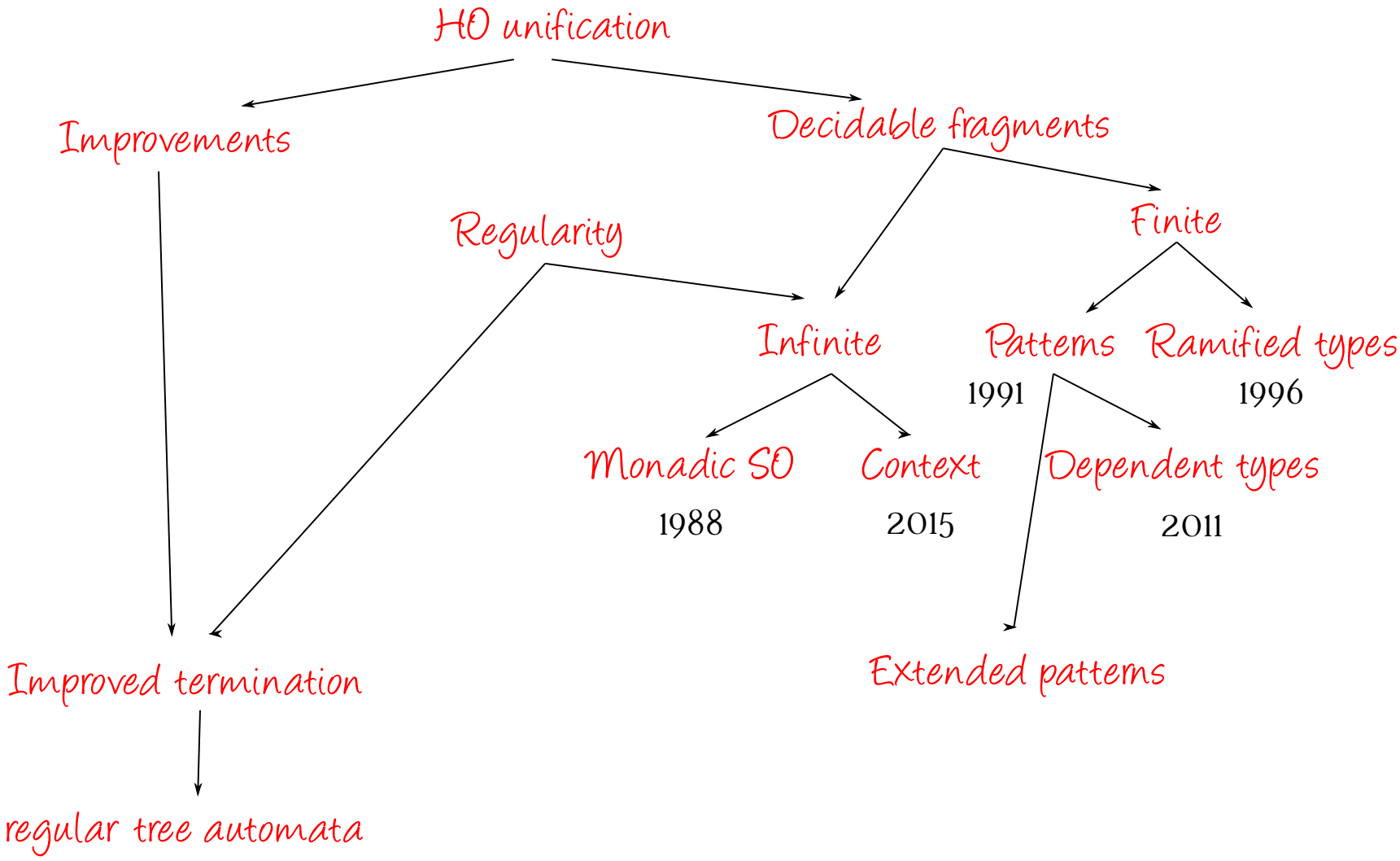

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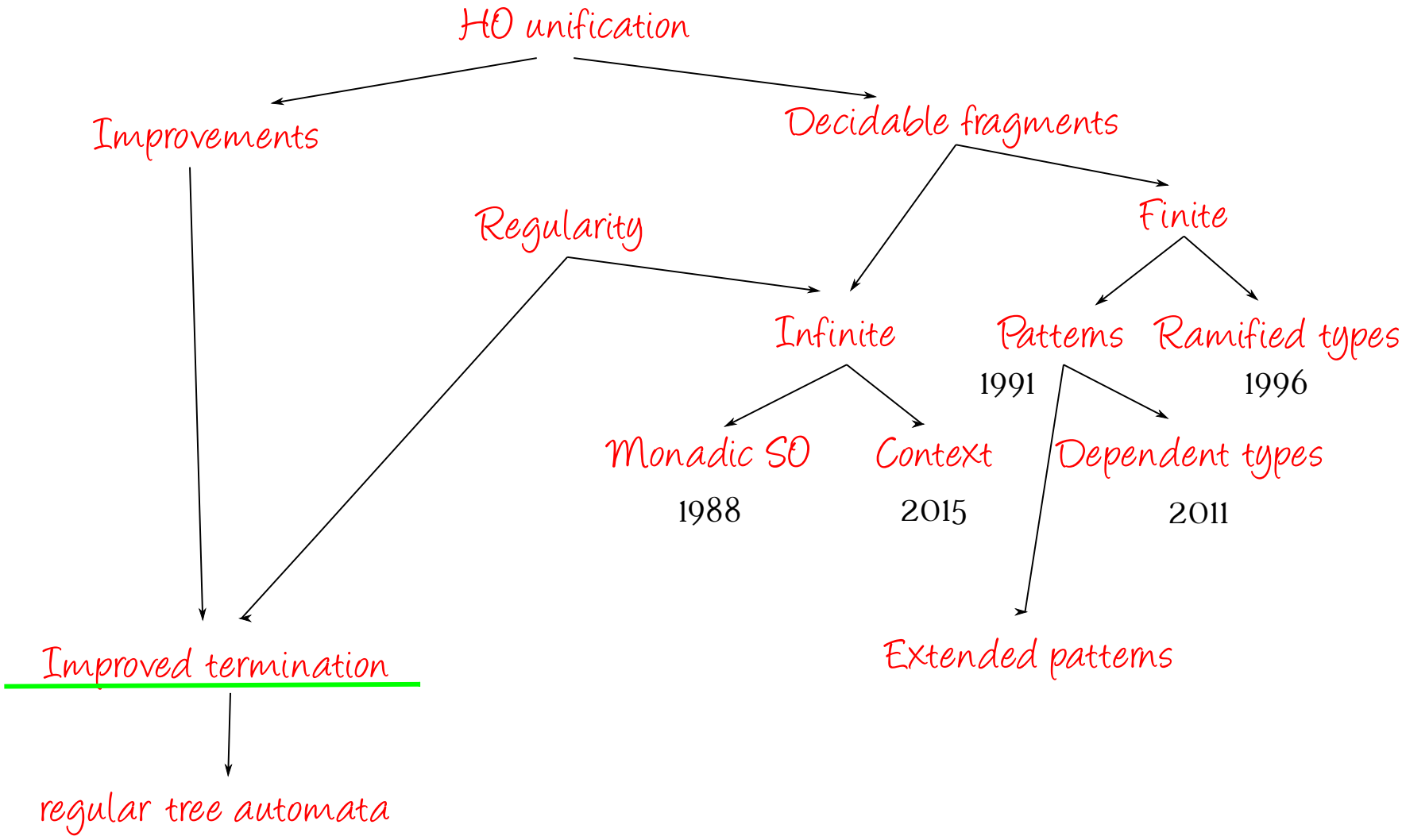
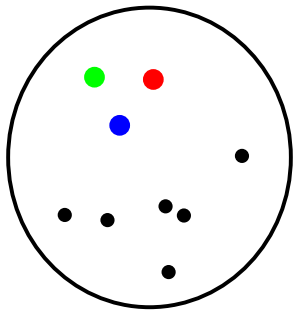
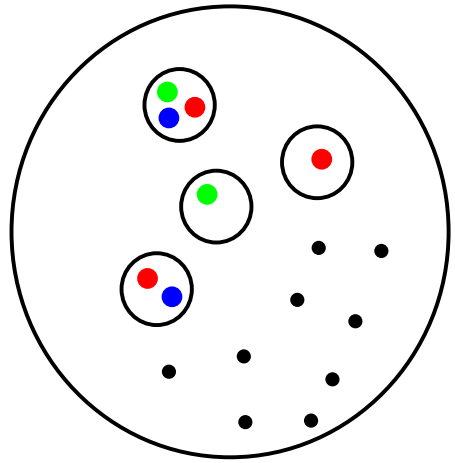


Cantor's Theorem





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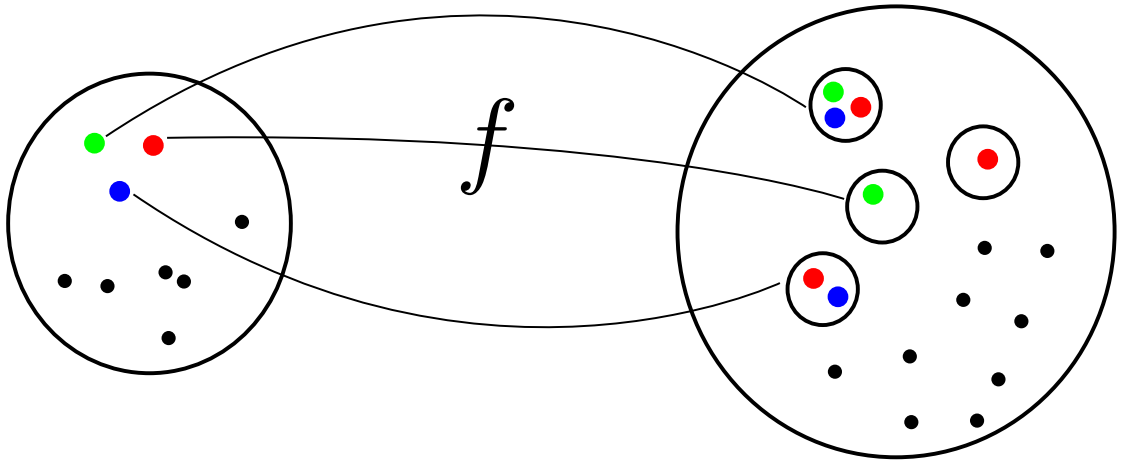

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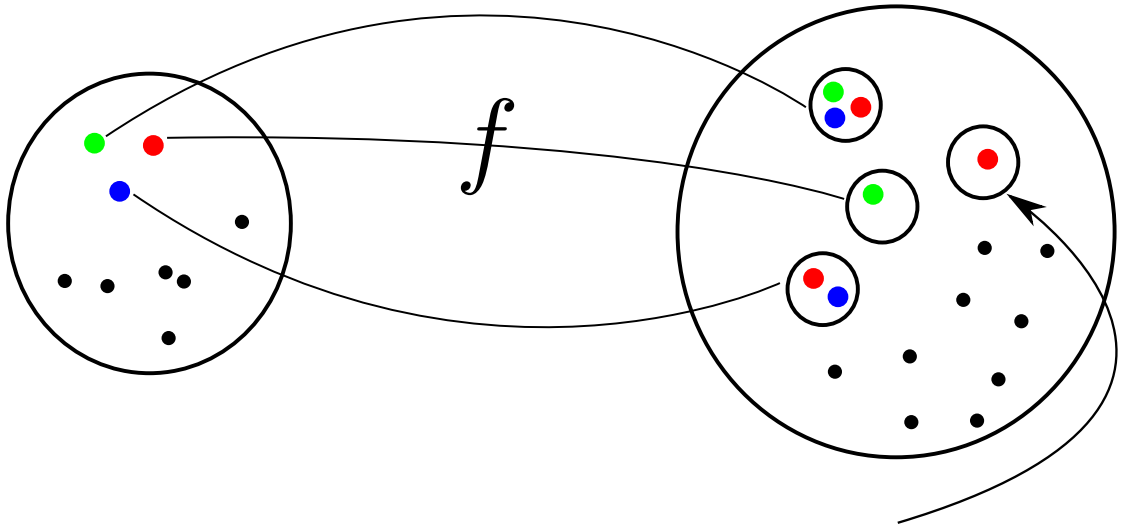
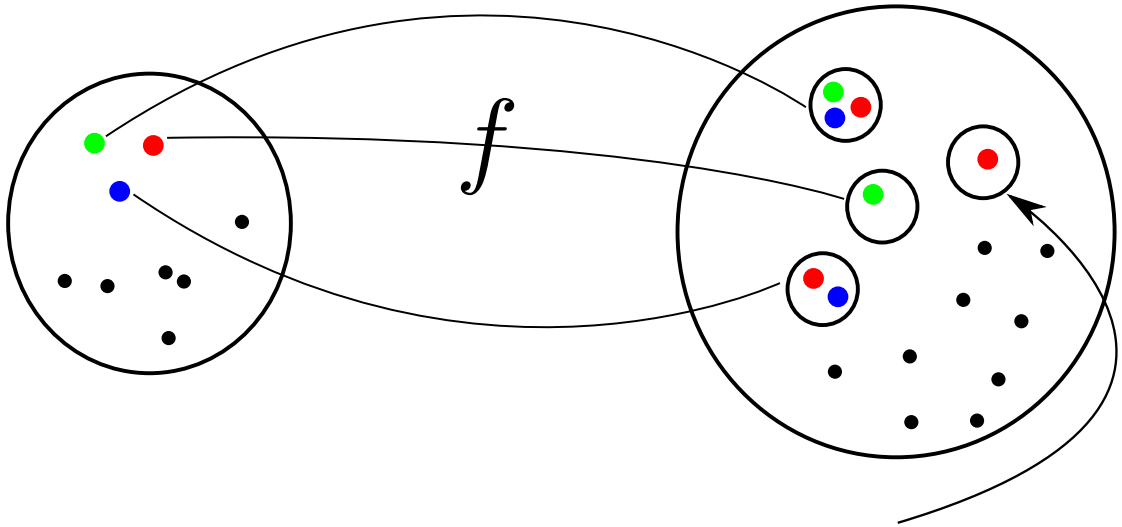


$\{x : x \notin f(x)\}$

Cantor's diagonal set

Cantor's Theorem



Cantor's diagonal set

Andrews-Miller-Cohen-Pfenning ('84)

$$\neg \exists f_{i \to i \to o} \forall b_{i \to o} \exists a_i : f(a) = b$$

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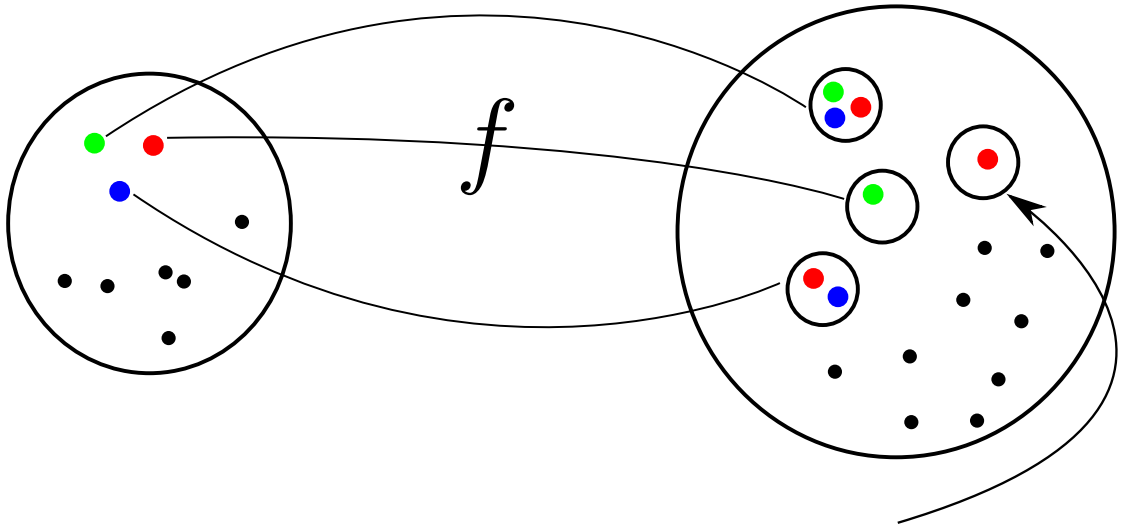

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LEO-III HOL Theorem Prover

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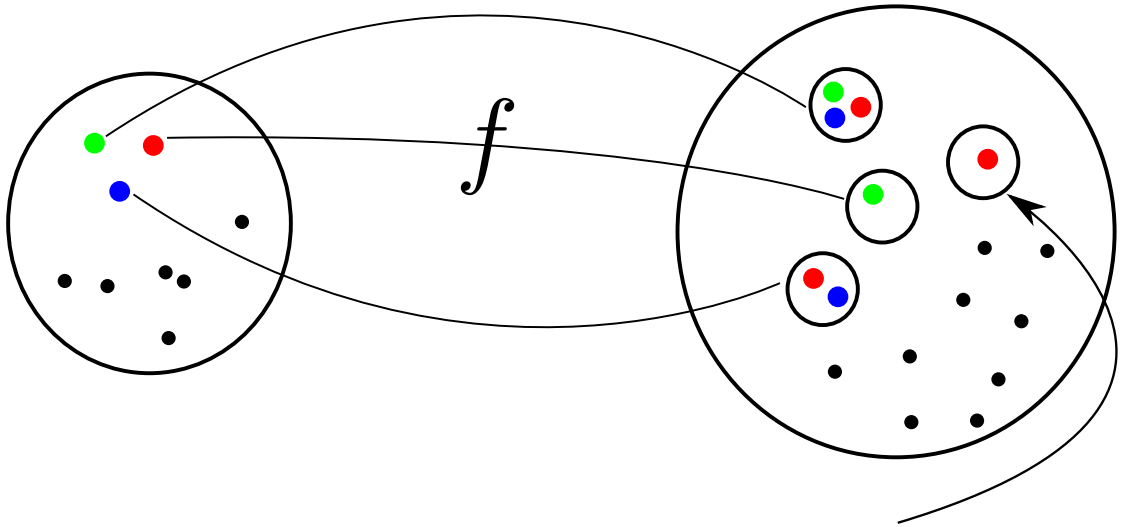

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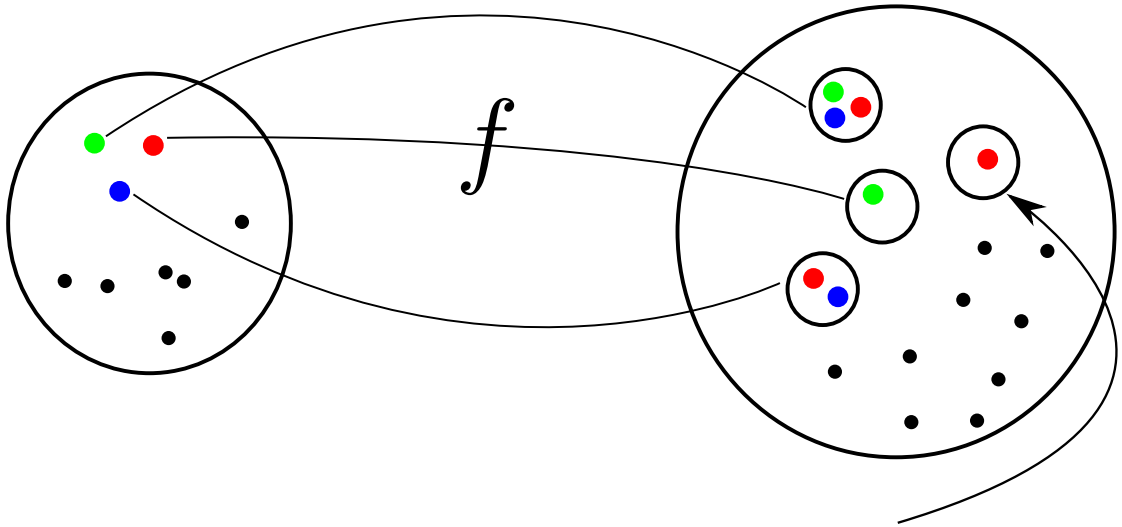
$\lambda z . \neg f(z, z)$ ← LEO-III HOL Theorem Prover

Cantor's Theorem



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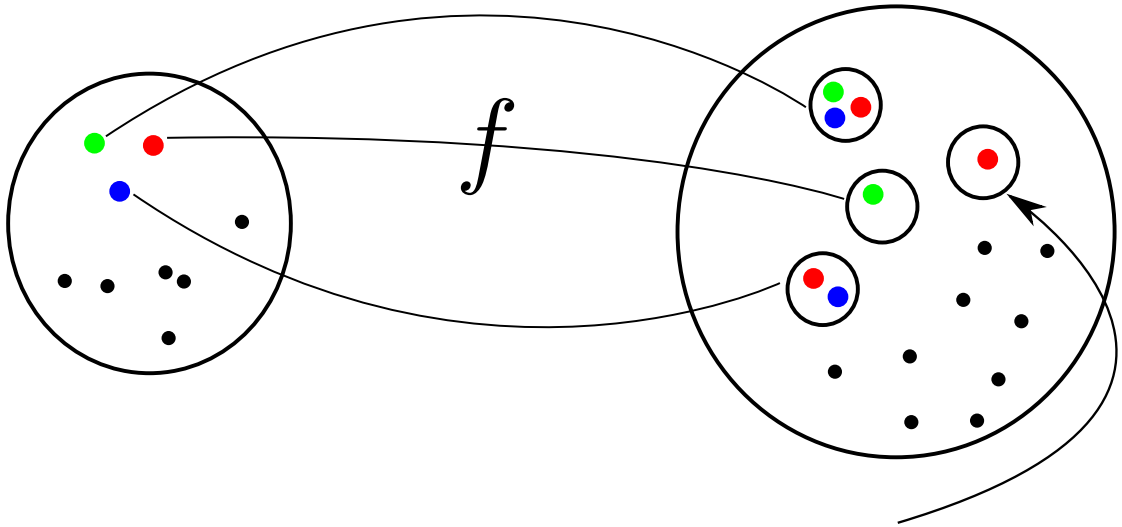

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$$\uparrow \qquad \downarrow$$

Huet's pre-unification procedure ('75)

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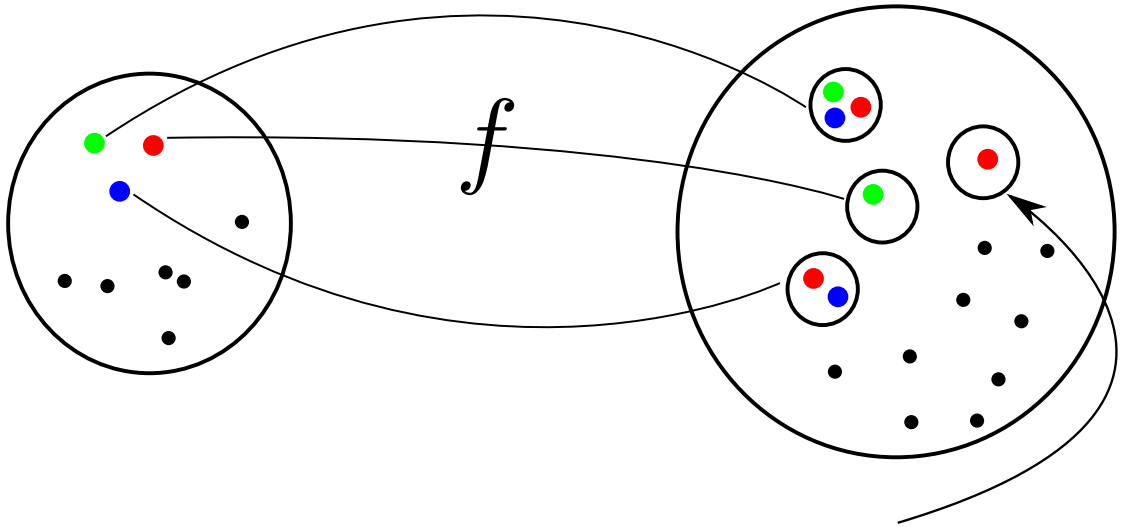

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- Semi-decidable

Forcing termination

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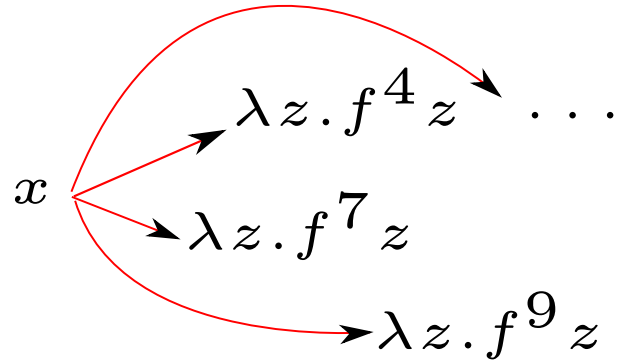

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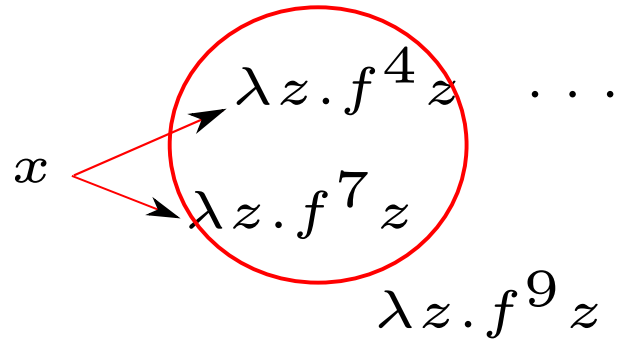

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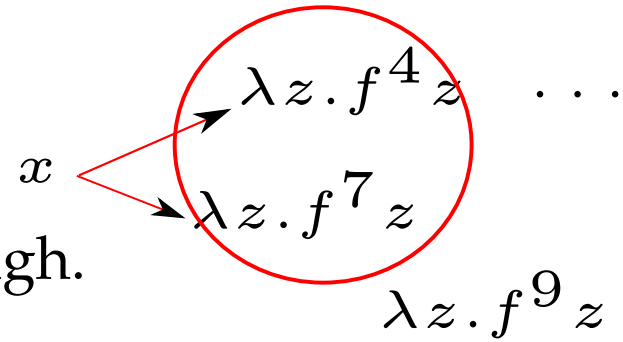

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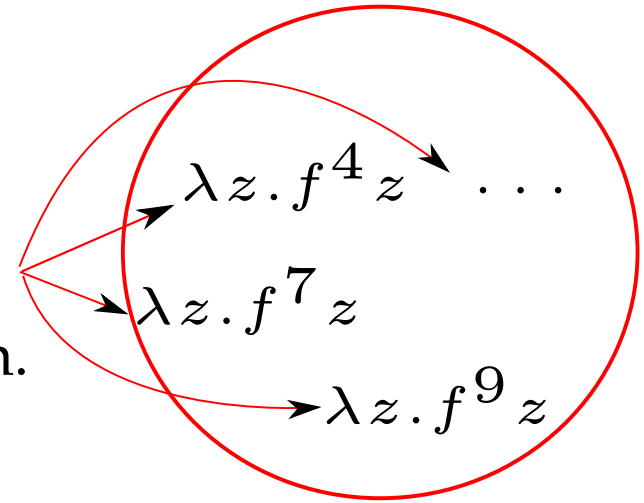

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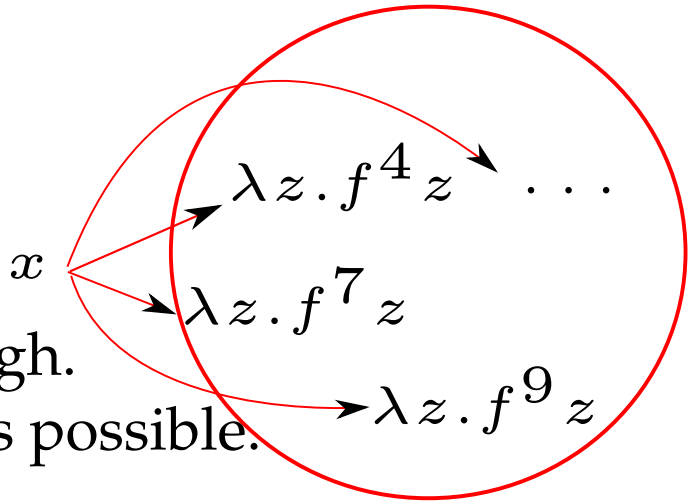

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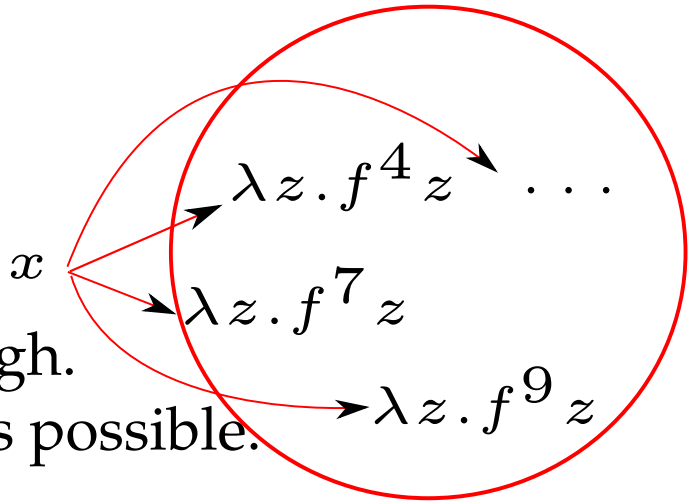

Forcing termination

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- This part: a second-order pre-unification procedure.
  - Sound and complete.
  - Same complexity as Huet's pre-unification procedure.
  - Terminates on more problems than Huet's.
    - including all problems generated by LEO-III for Cantor's theorem.

Non-termination

$$x\,0\,y \doteq \neg\, x\,0\,y$$

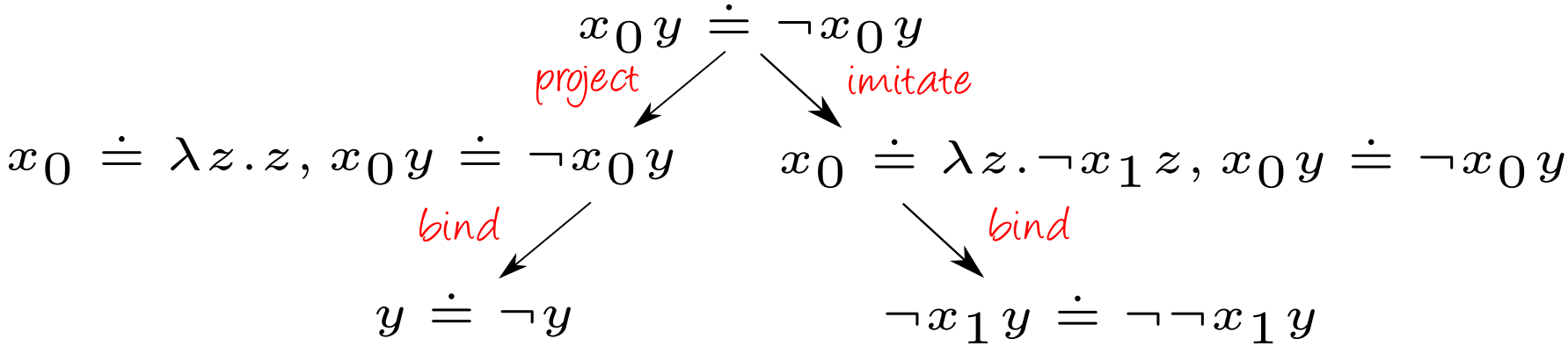
Non-termination

$$x_0\, y \doteq \neg x_0\, y$$

project: $x_0 \doteq \lambda z.\, z,\ x_0\, y \doteq \neg x_0\, y$

imitate: $x_0 \doteq \lambda z.\, \neg x_1\, z,\ x_0\, y \doteq \neg x_0\, y$

Non-termination



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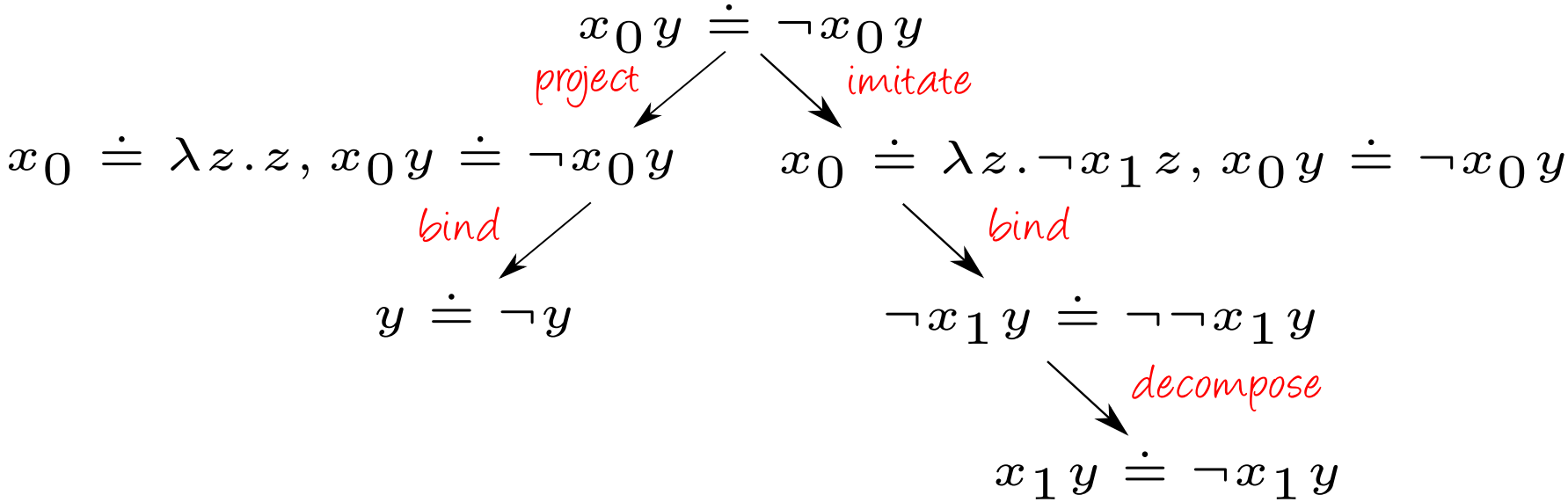

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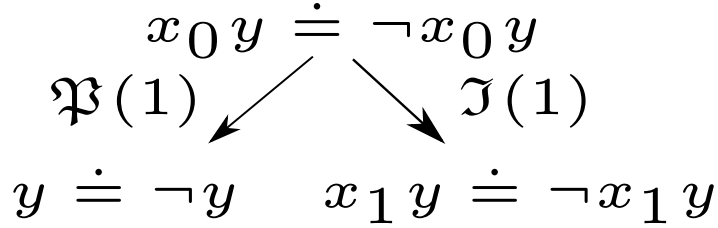


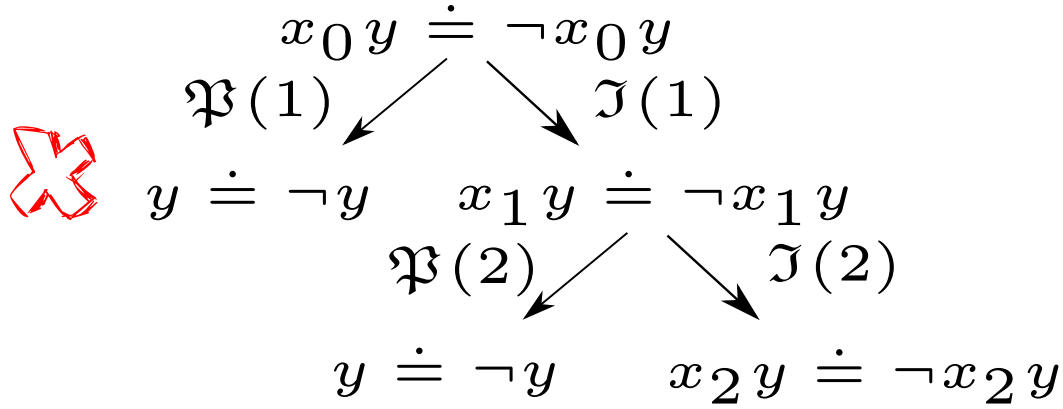
Non-termination

$$x_0\, y \doteq \neg x_0\, y$$

$\mathfrak{P}(1)$ ↙ ↘ $\mathfrak{I}(1)$

$$y \doteq \neg y \qquad x_1\, y \doteq \neg x_1\, y$$

Non-termination



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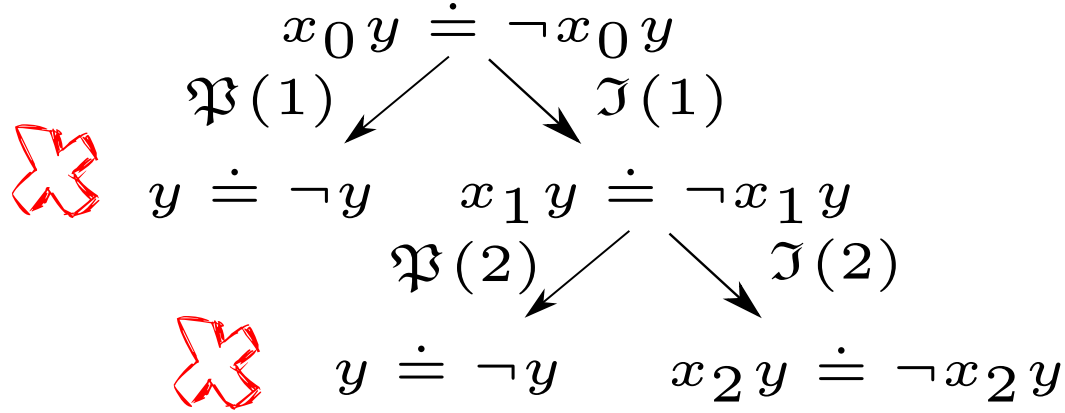
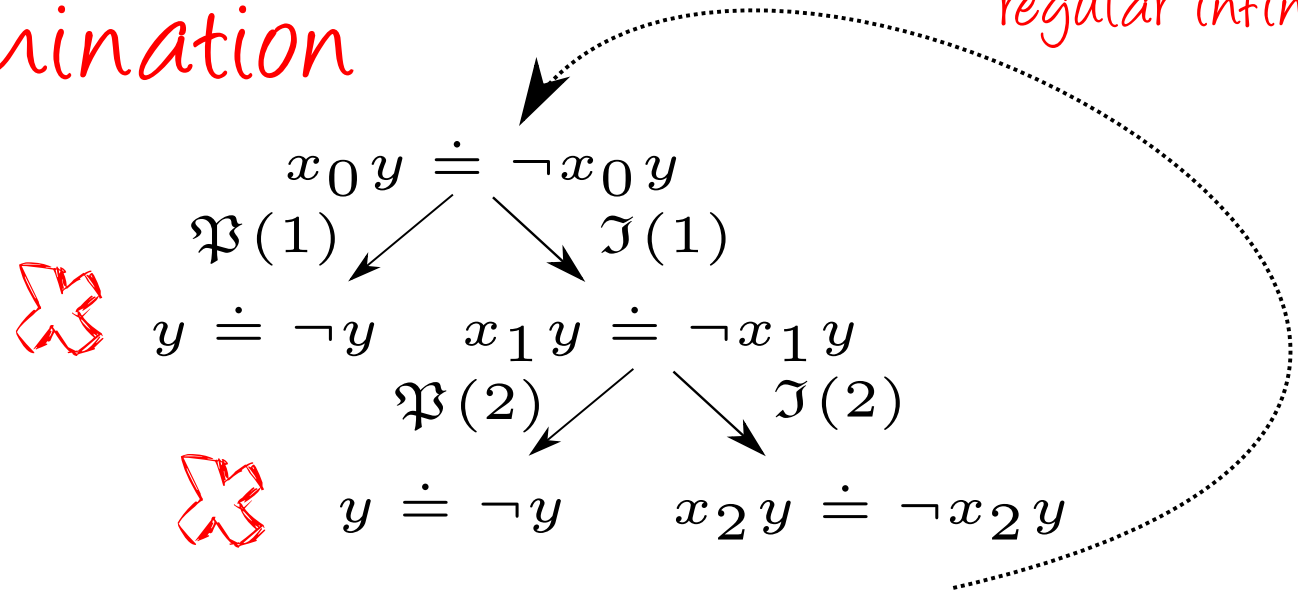


Non-termination

regular infinite tree



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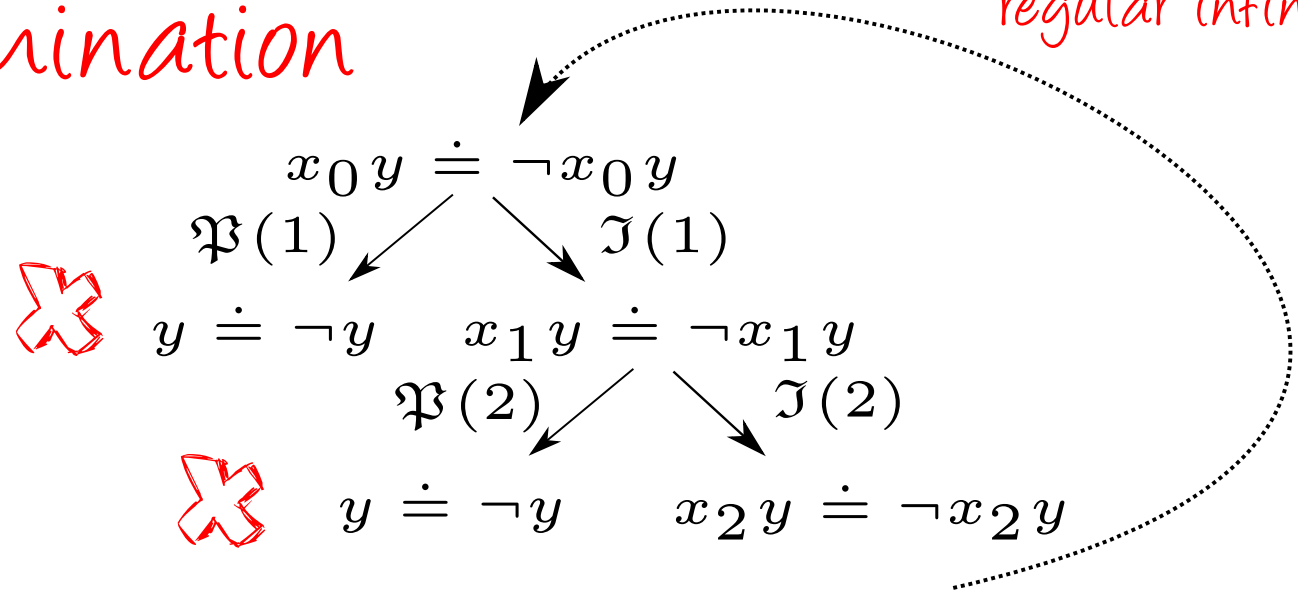
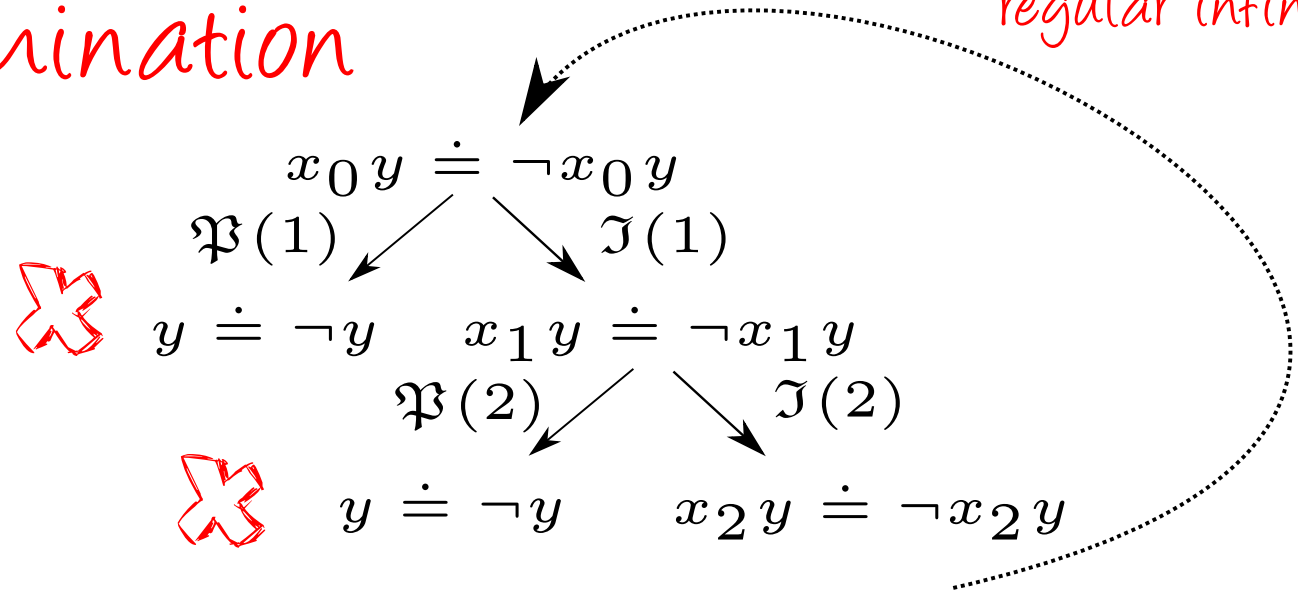


- Semi-decidable: possible non-termination only if not unifiable.

Non-termination



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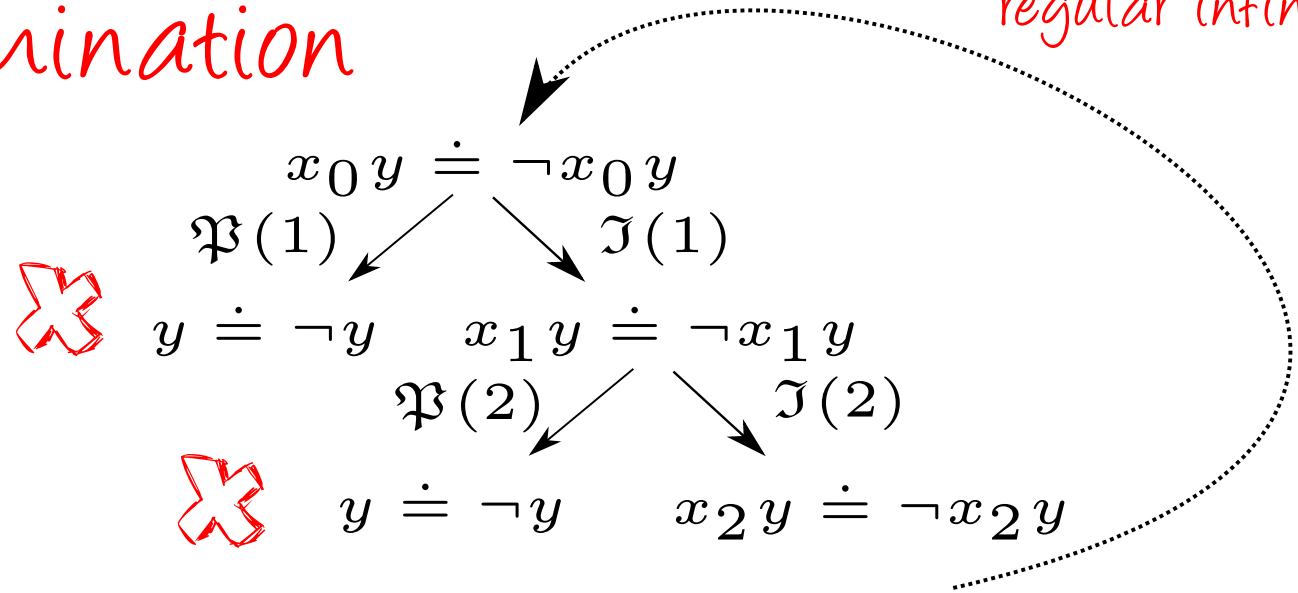
Non-termination



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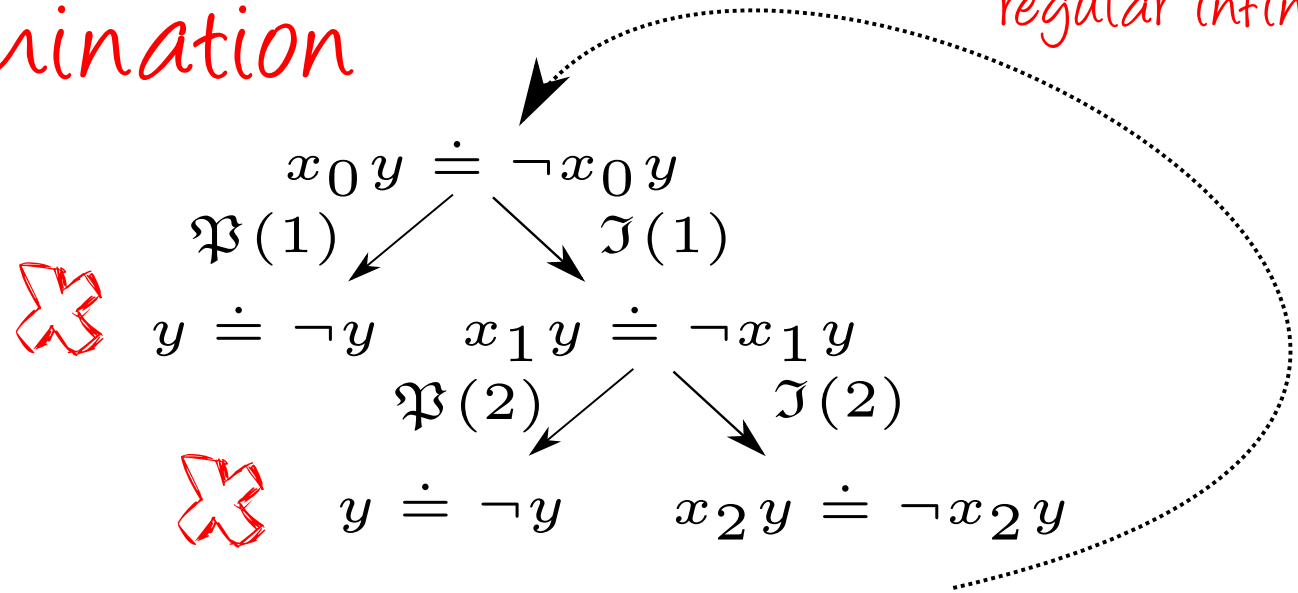

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- Lemma 2: $\forall i, j > 0\ \mathfrak{P}(i)$ is unifiable if $\mathfrak{P}(j)$ is.

Cyclic equations - monadic signature

$$x_0\, t \doteq C(x_0\, s)$$

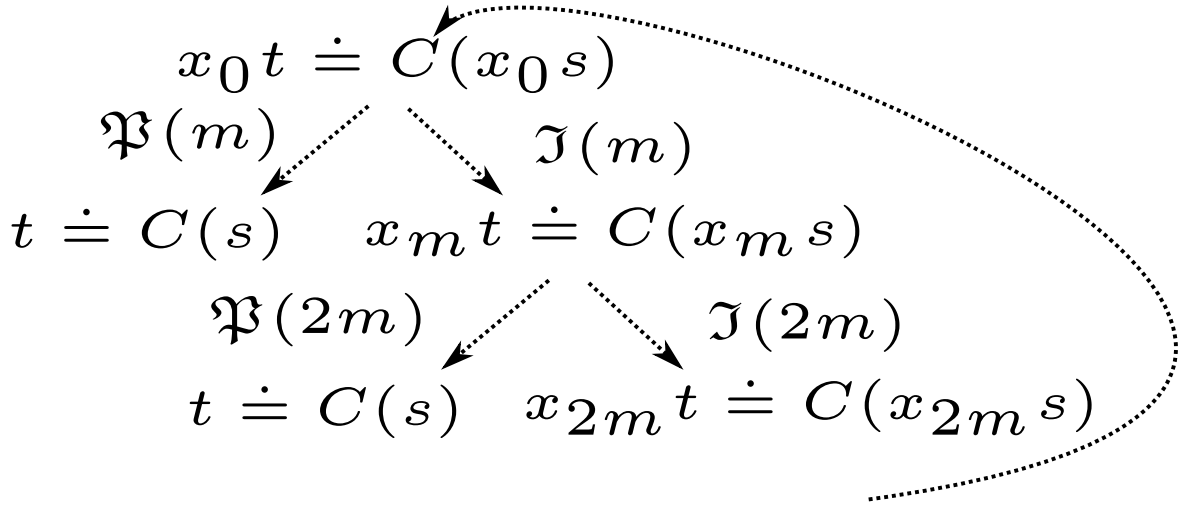
Cyclic equations - monadic signature

$$x_0\, t \doteq C(x_0\, s)$$

$$x_0 \notin \mathtt{FV}(t, C, s)$$

Cyclic equations - monadic signature

$$x_0 t \doteq C(x_0 s)$$



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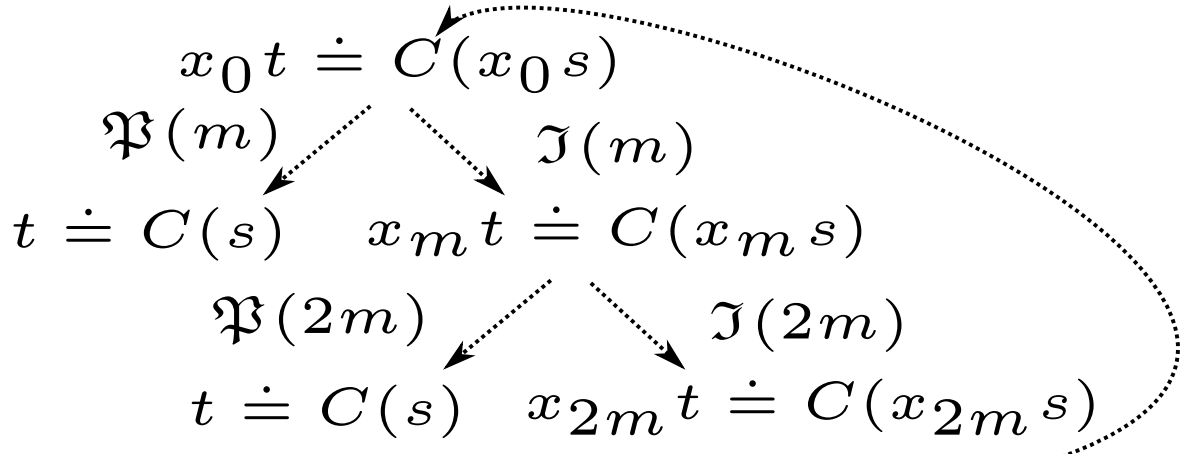

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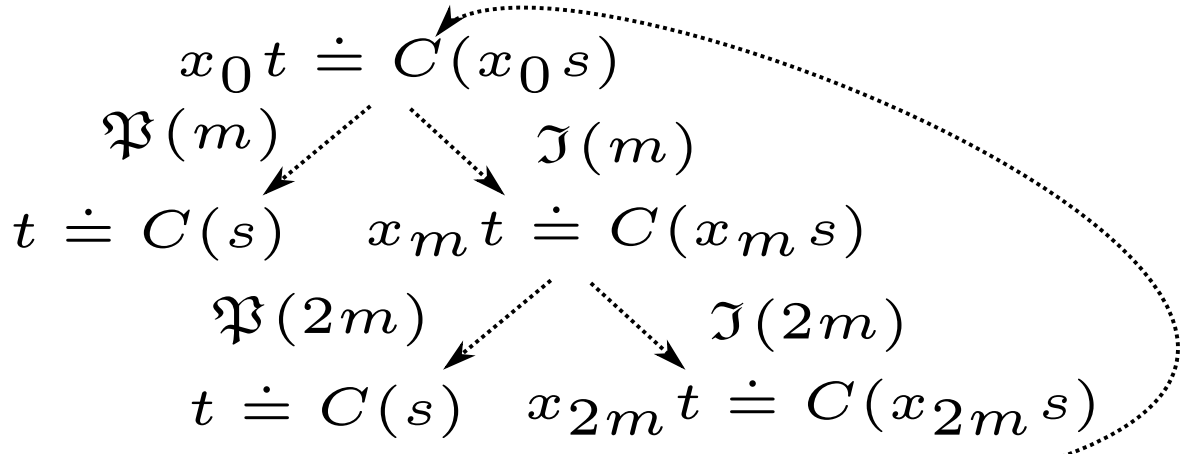

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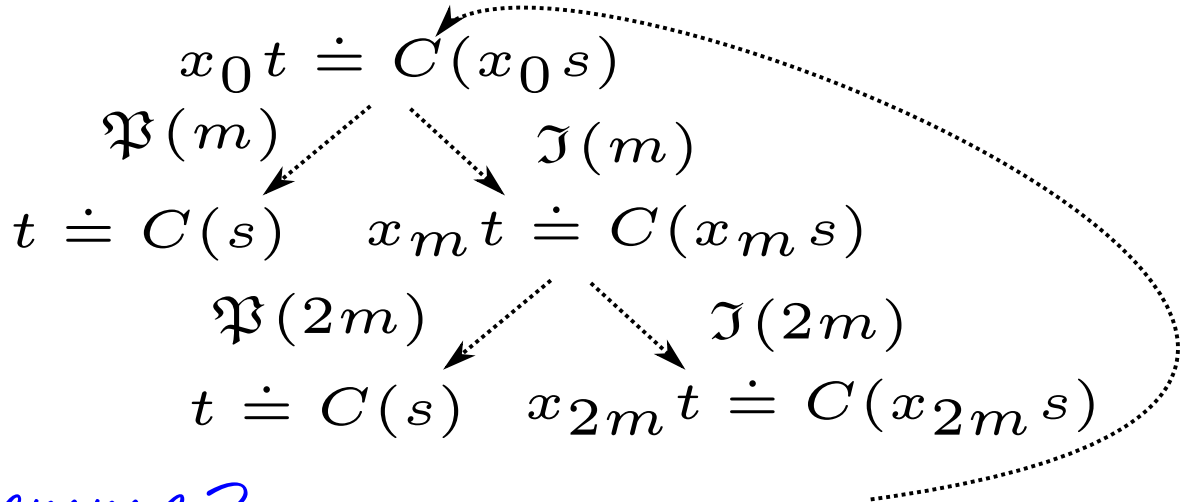

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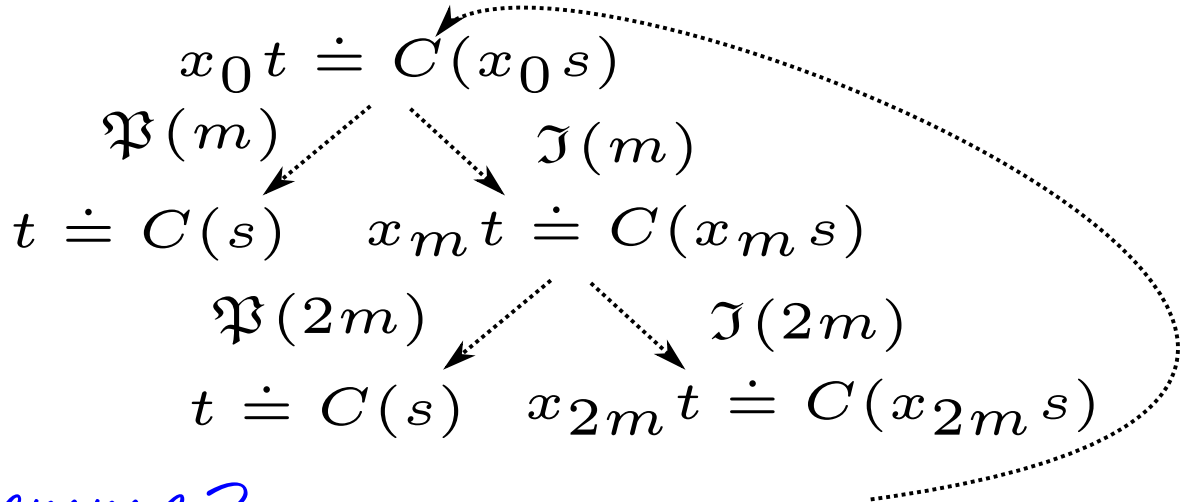

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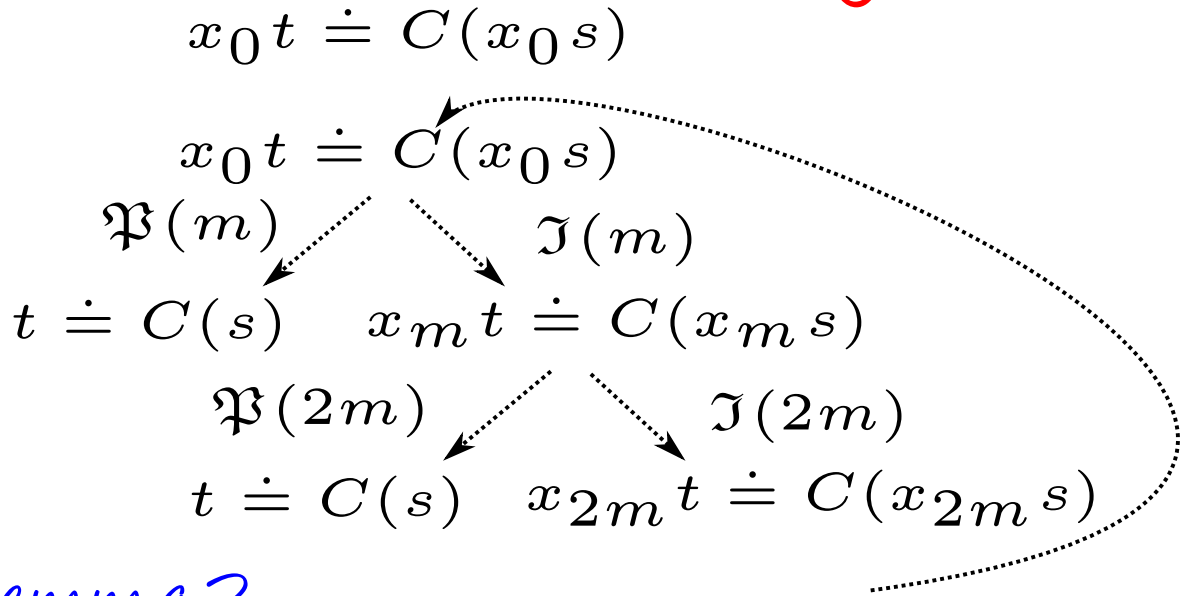

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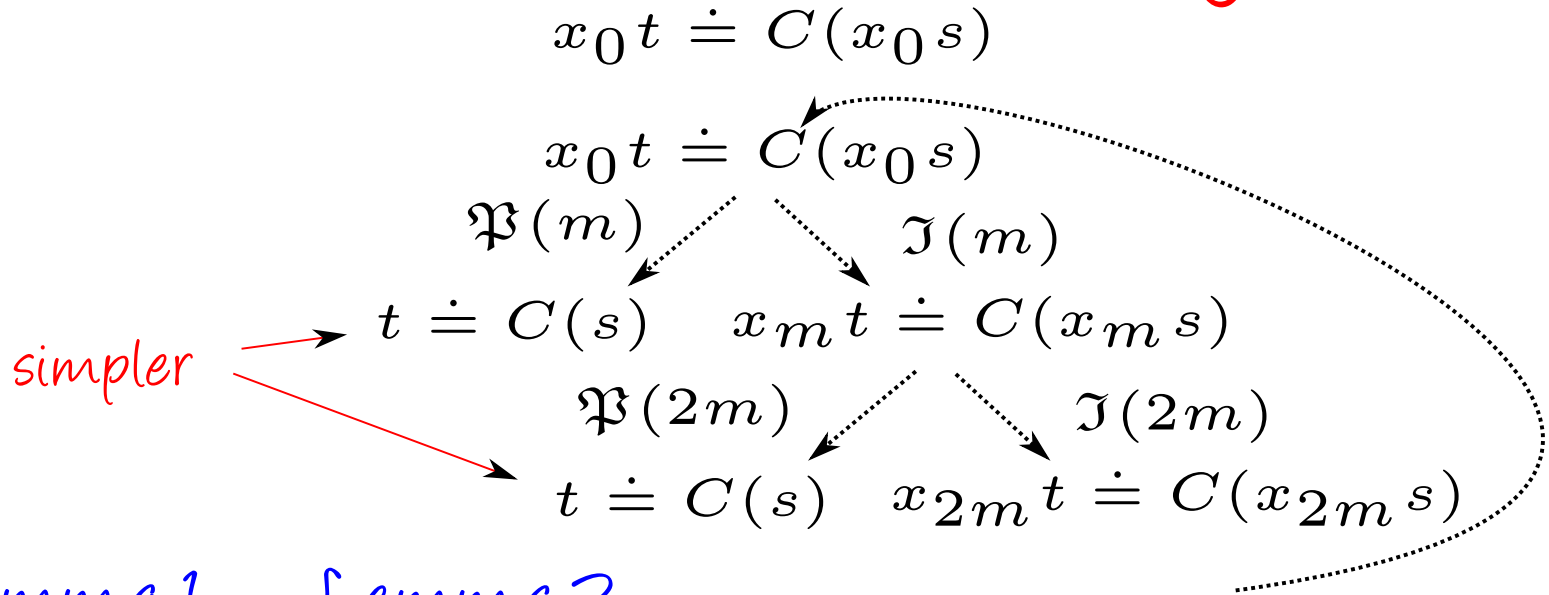

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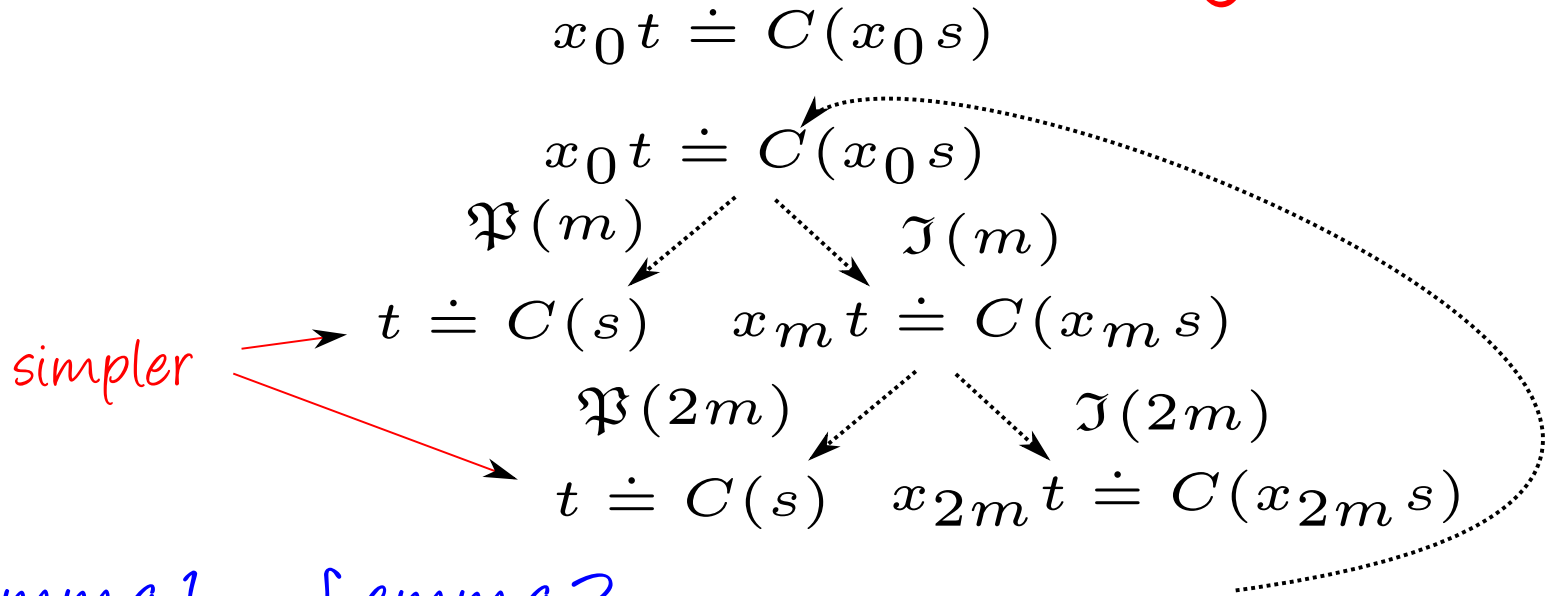

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- Corollary:Unification over monadic "cyclic equations" is decidable. (Farmer '88 for full monadic SOU)

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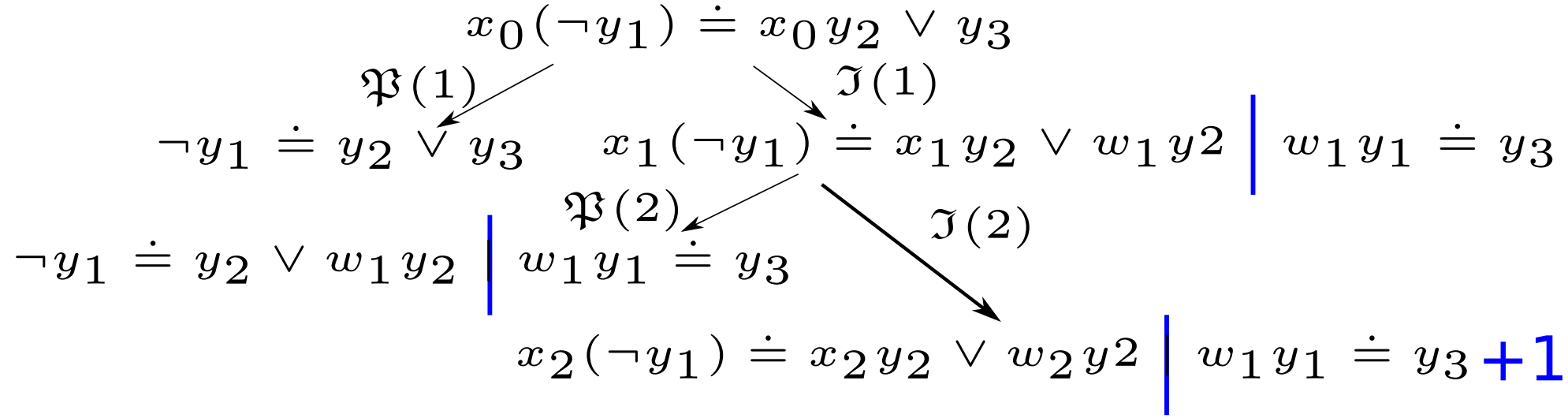

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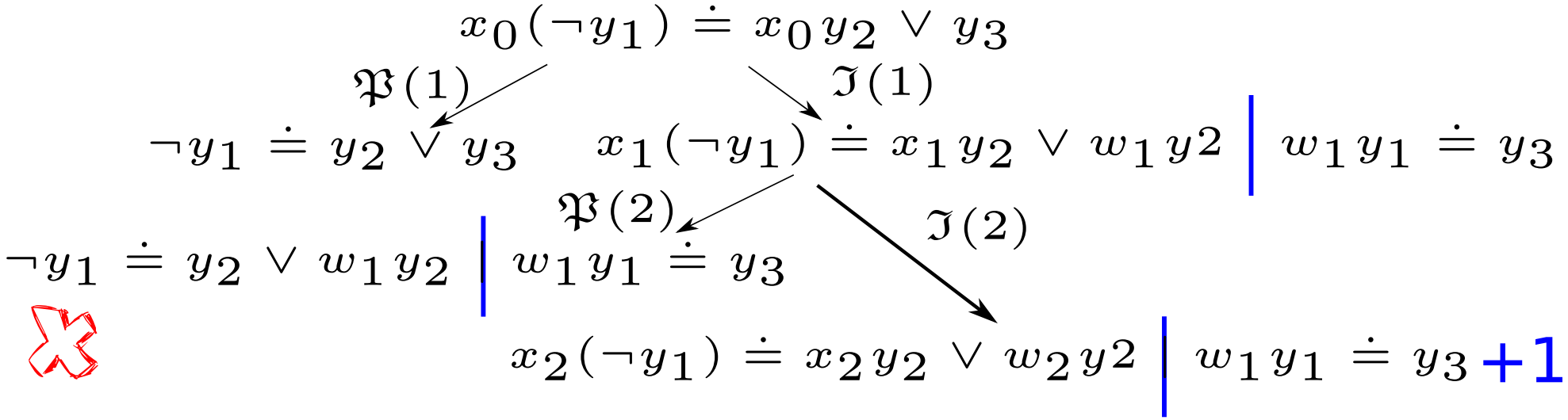

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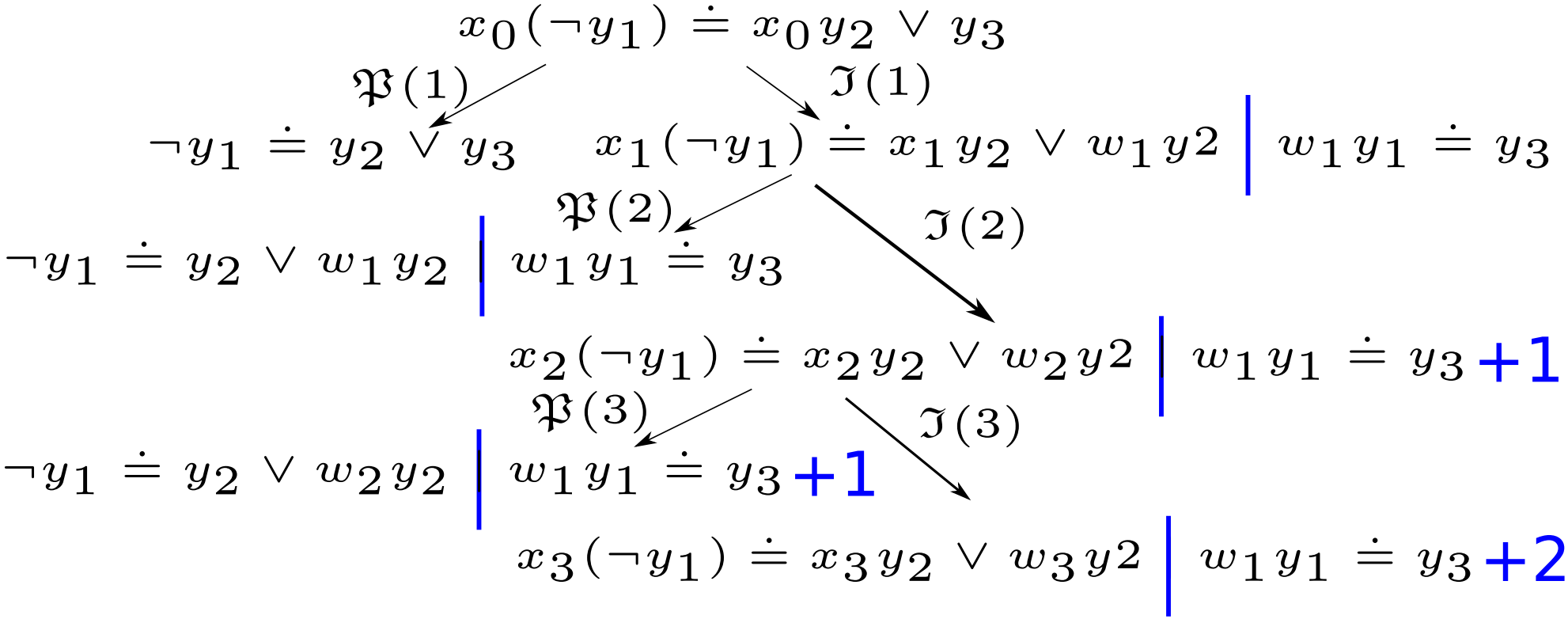

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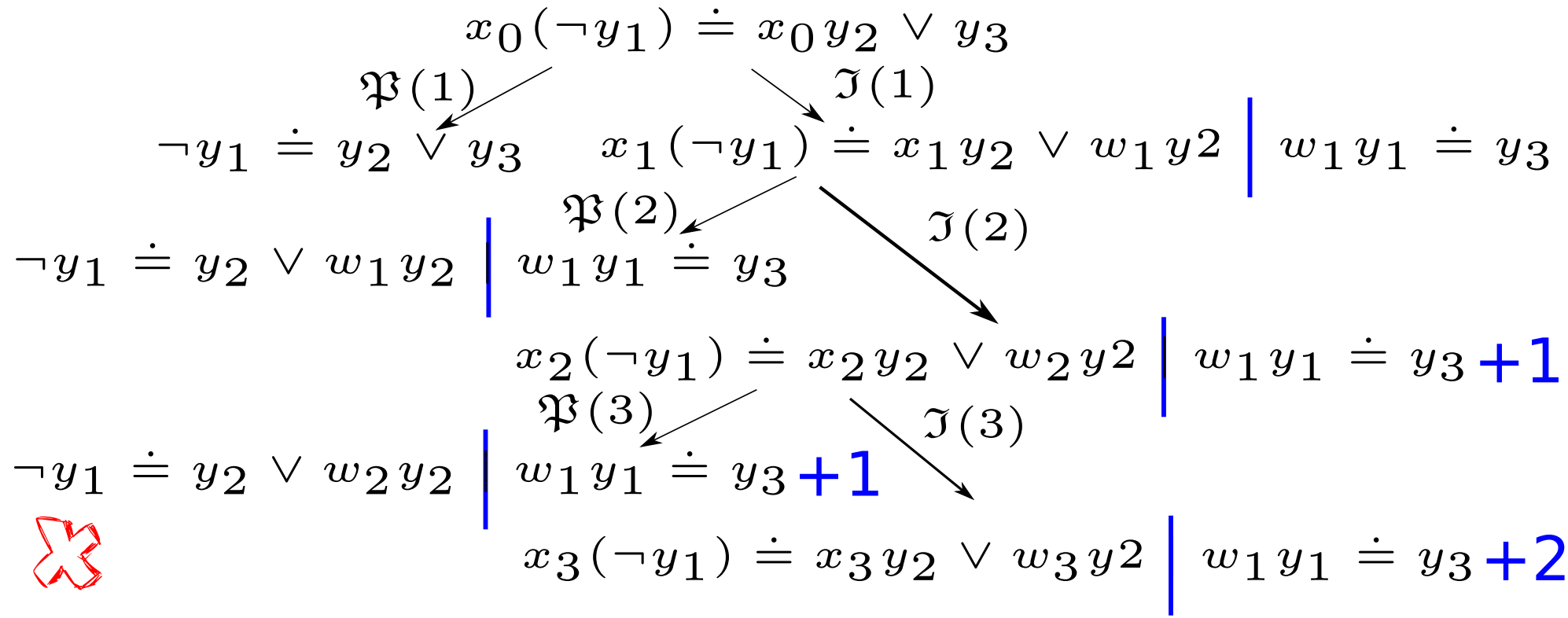

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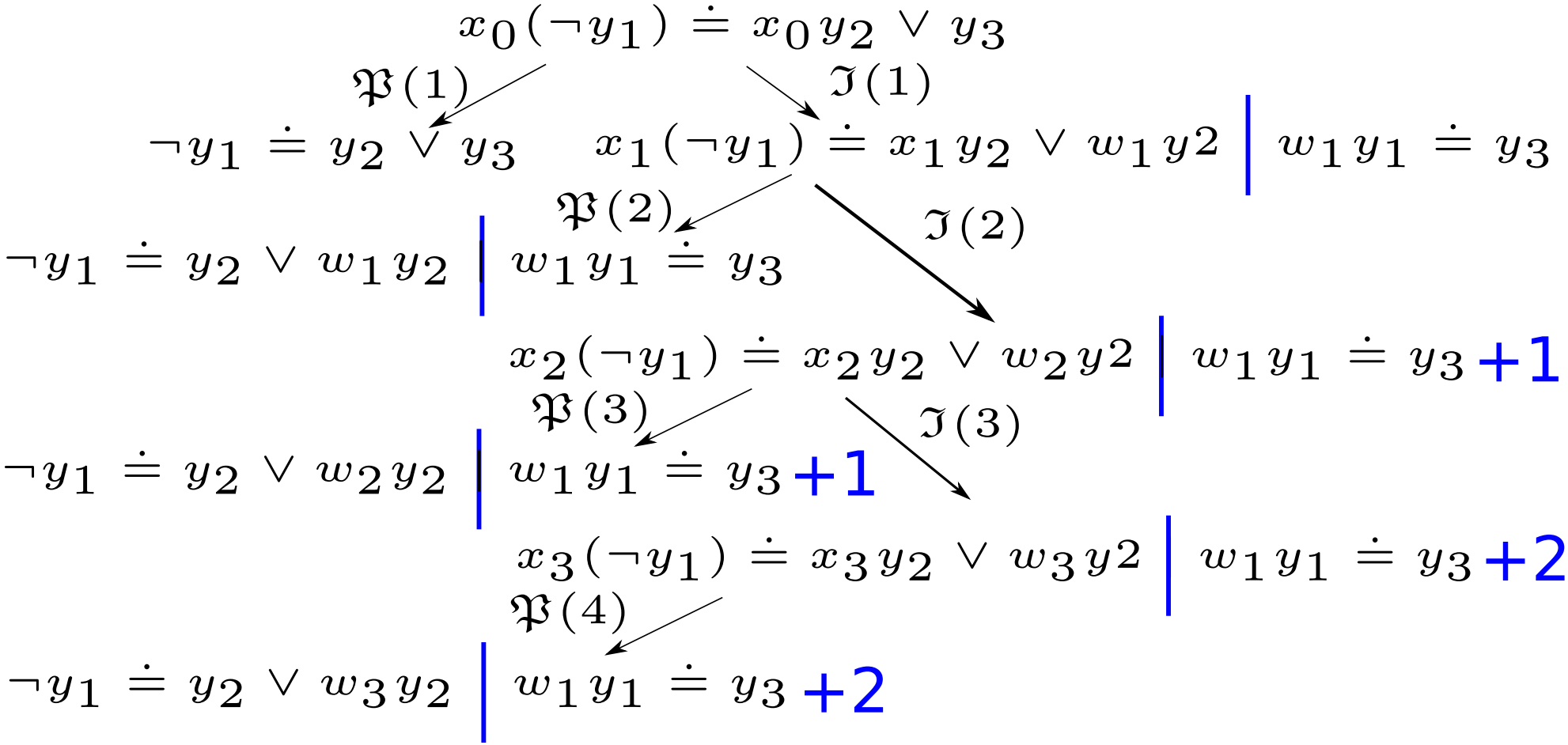

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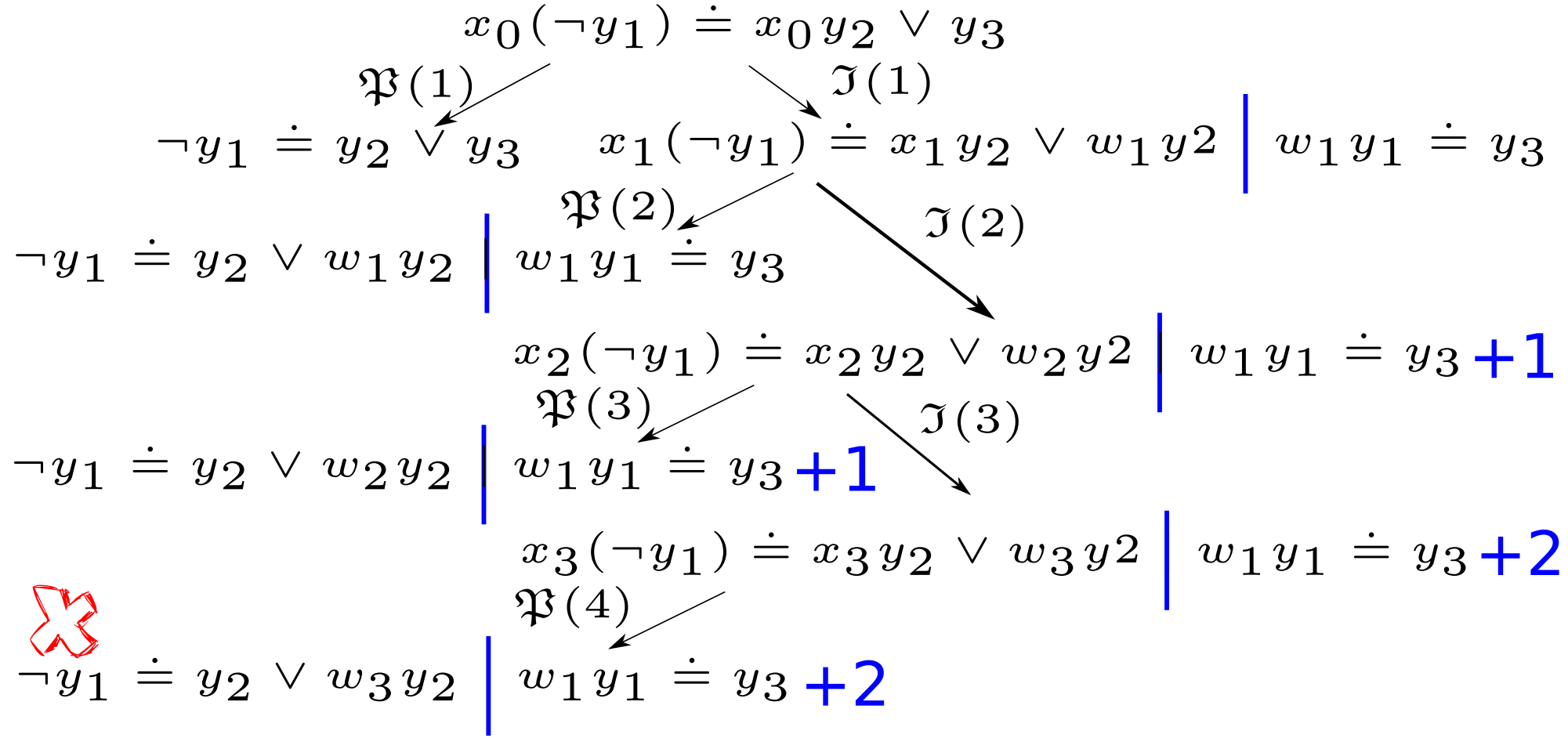

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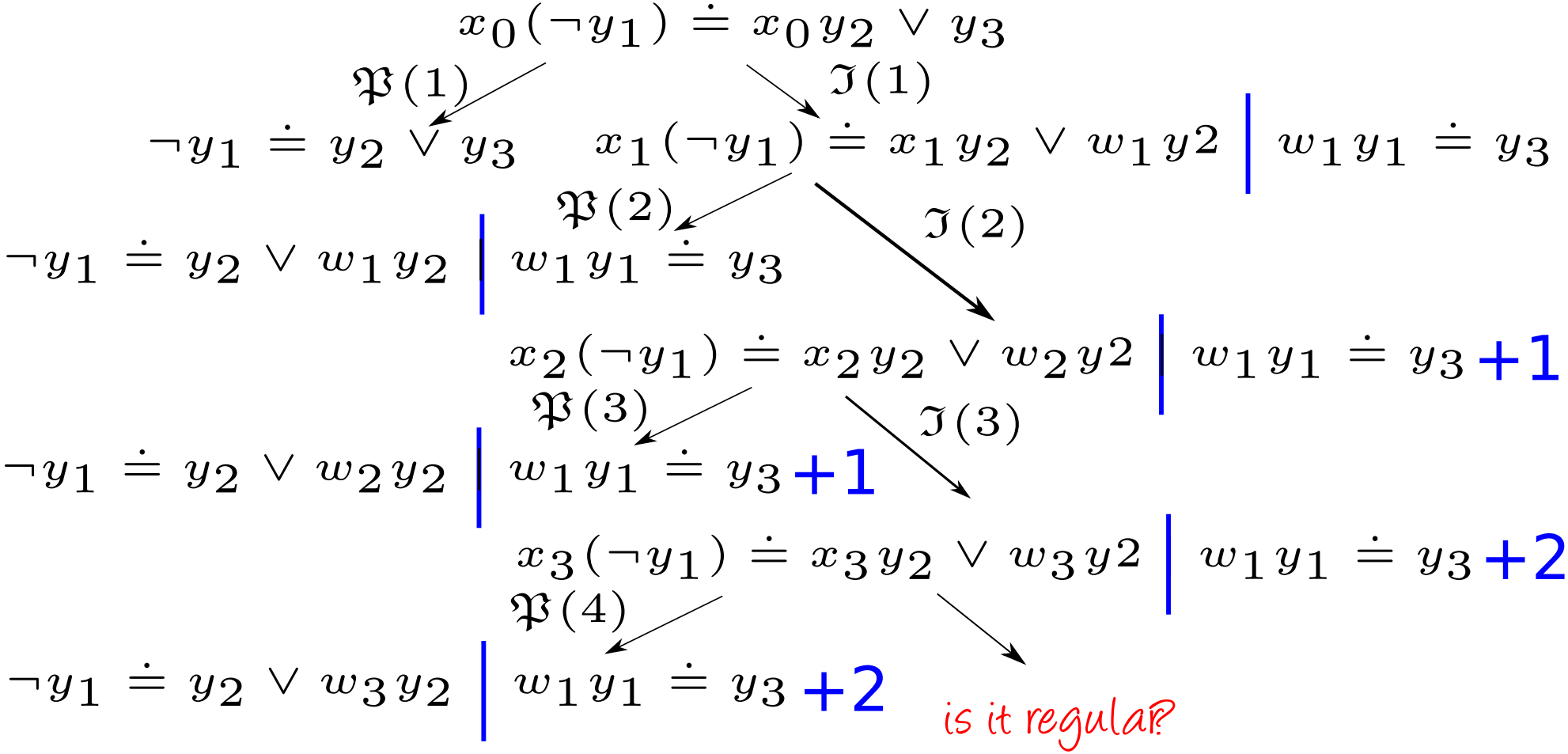

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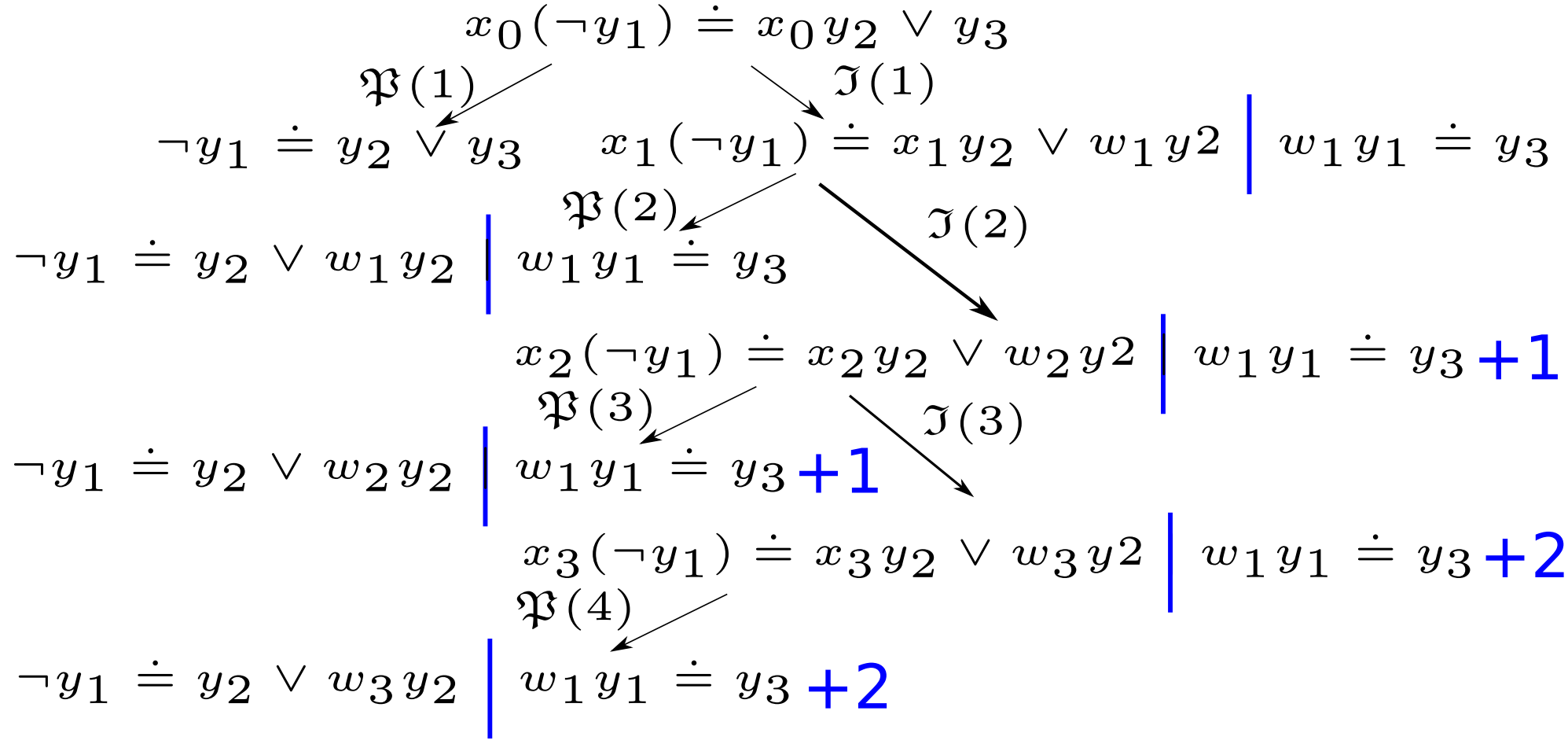

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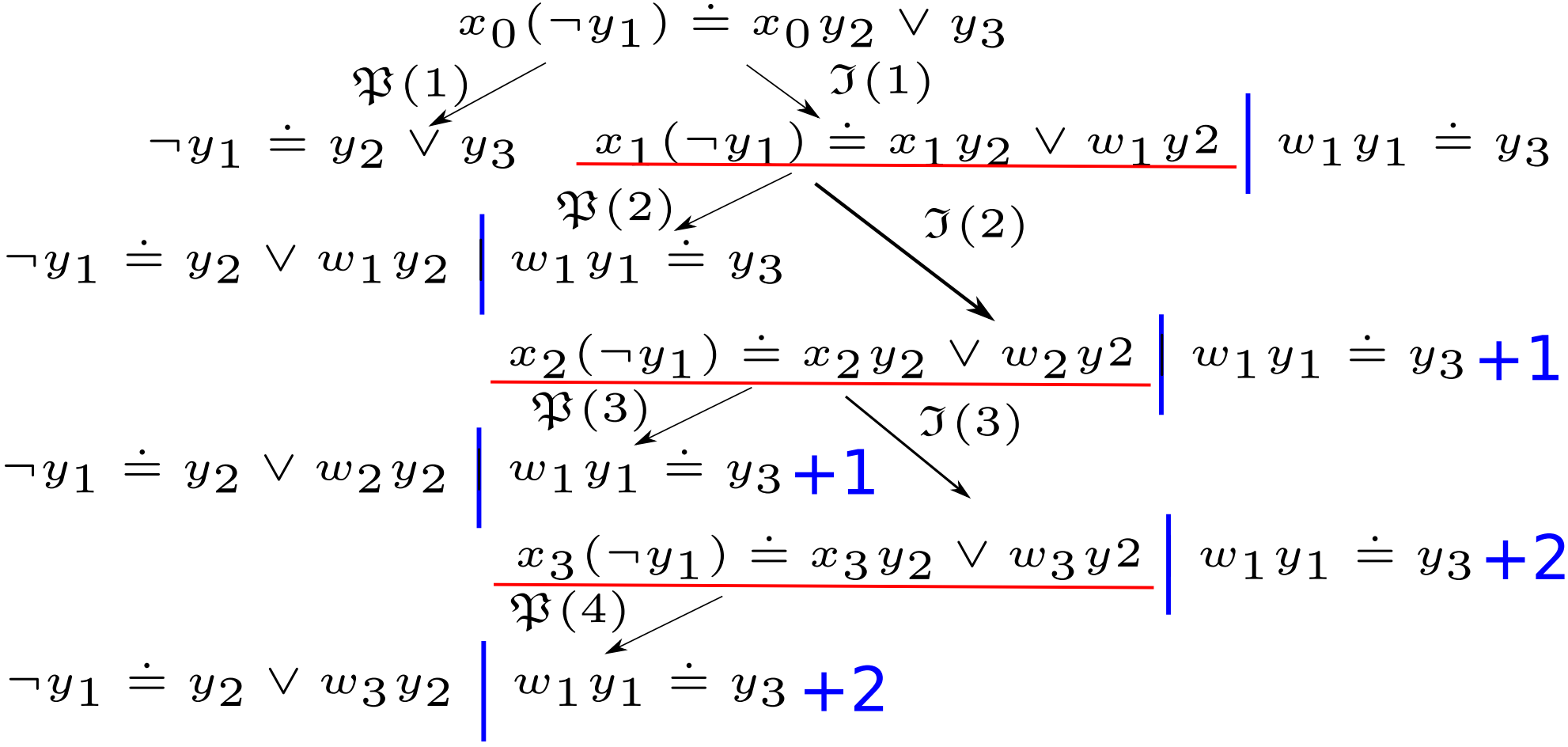

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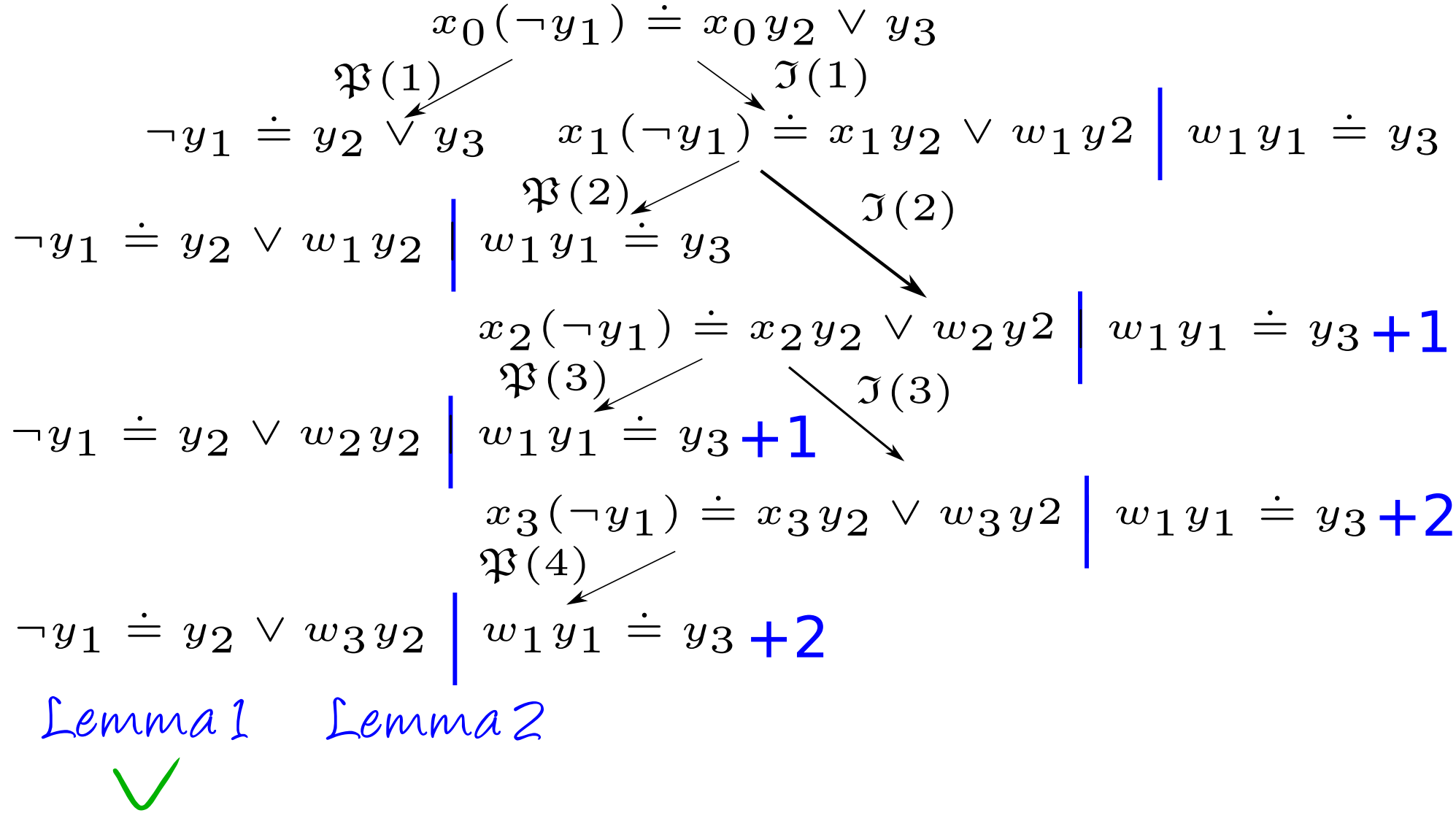

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$\mathfrak{P}(4)$

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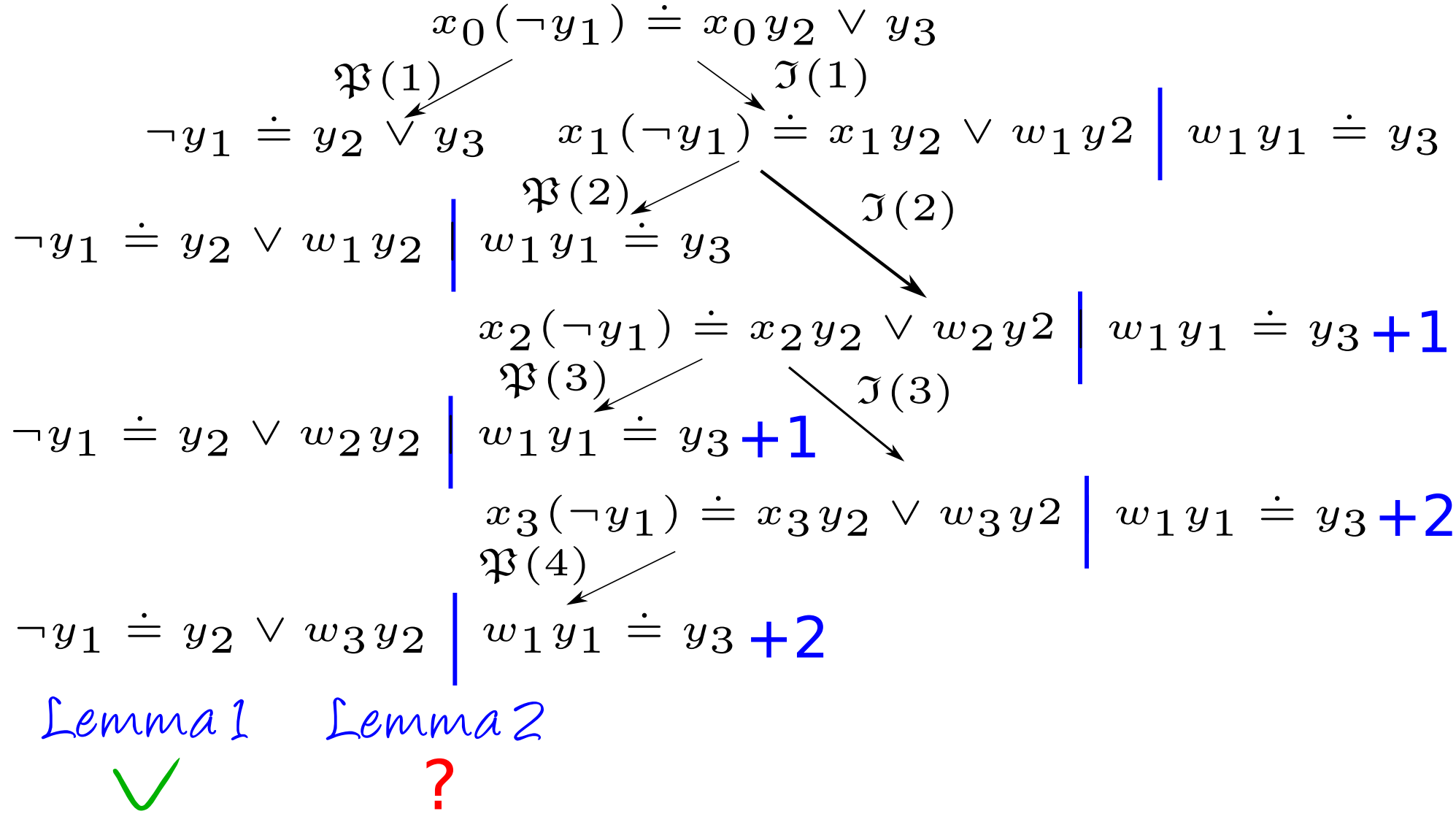

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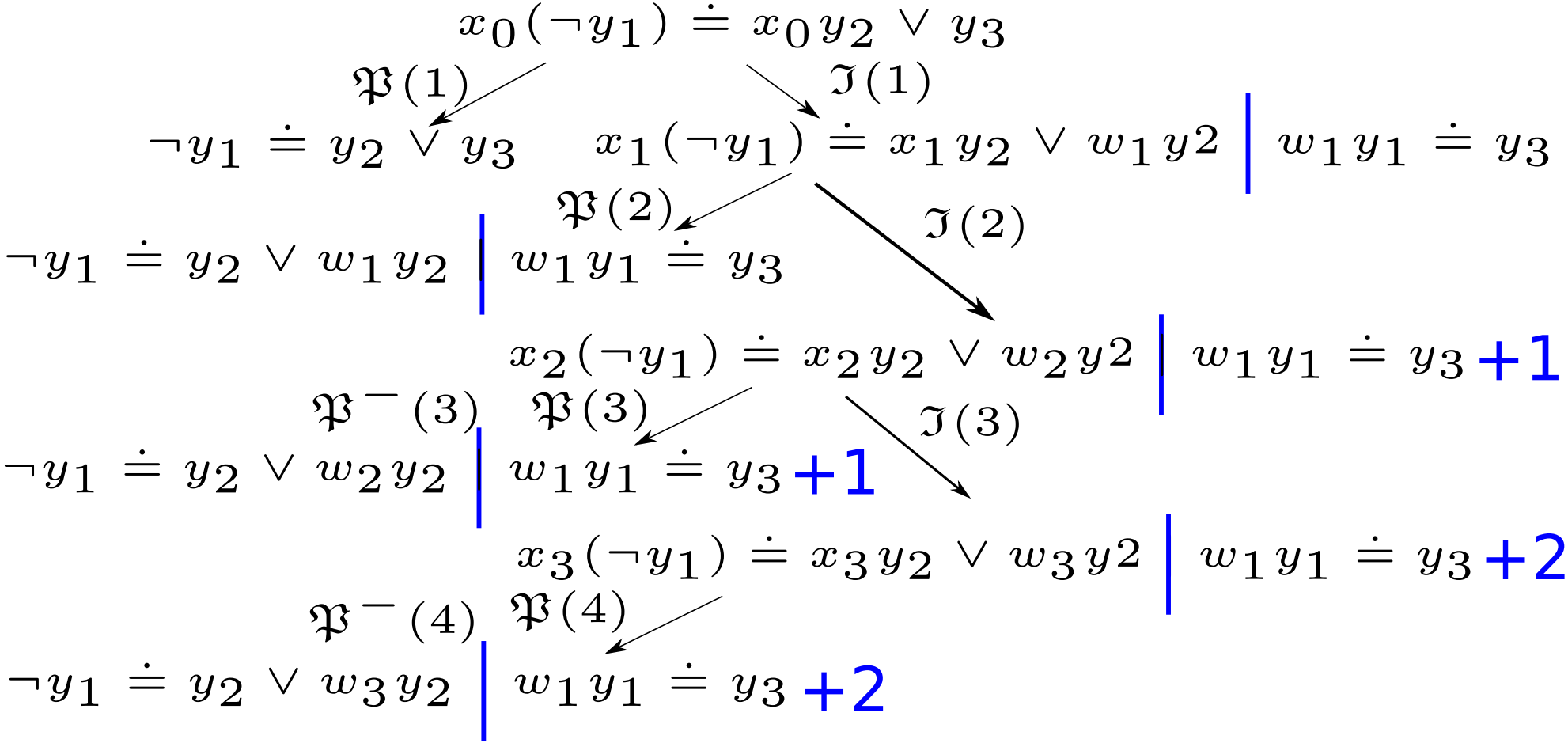

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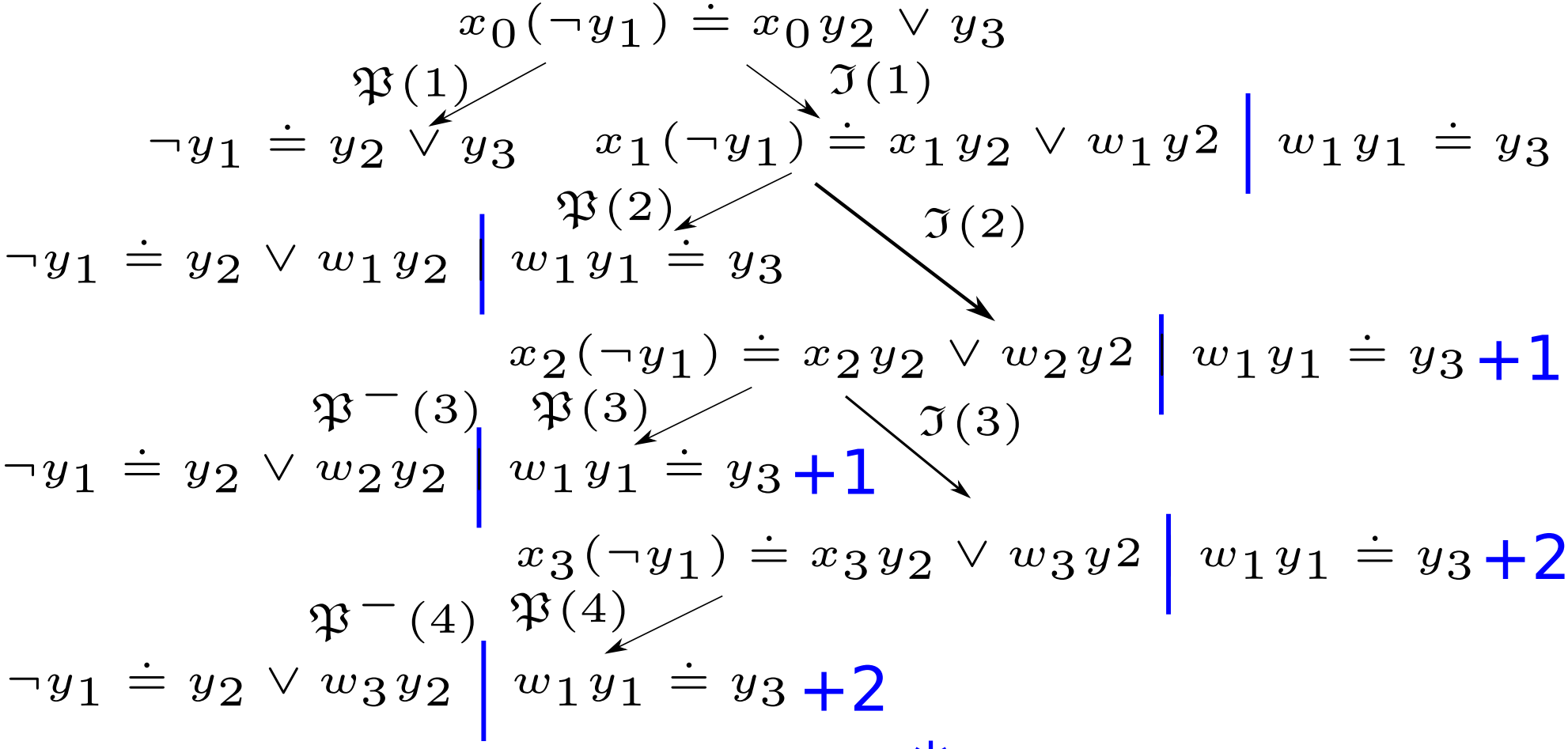

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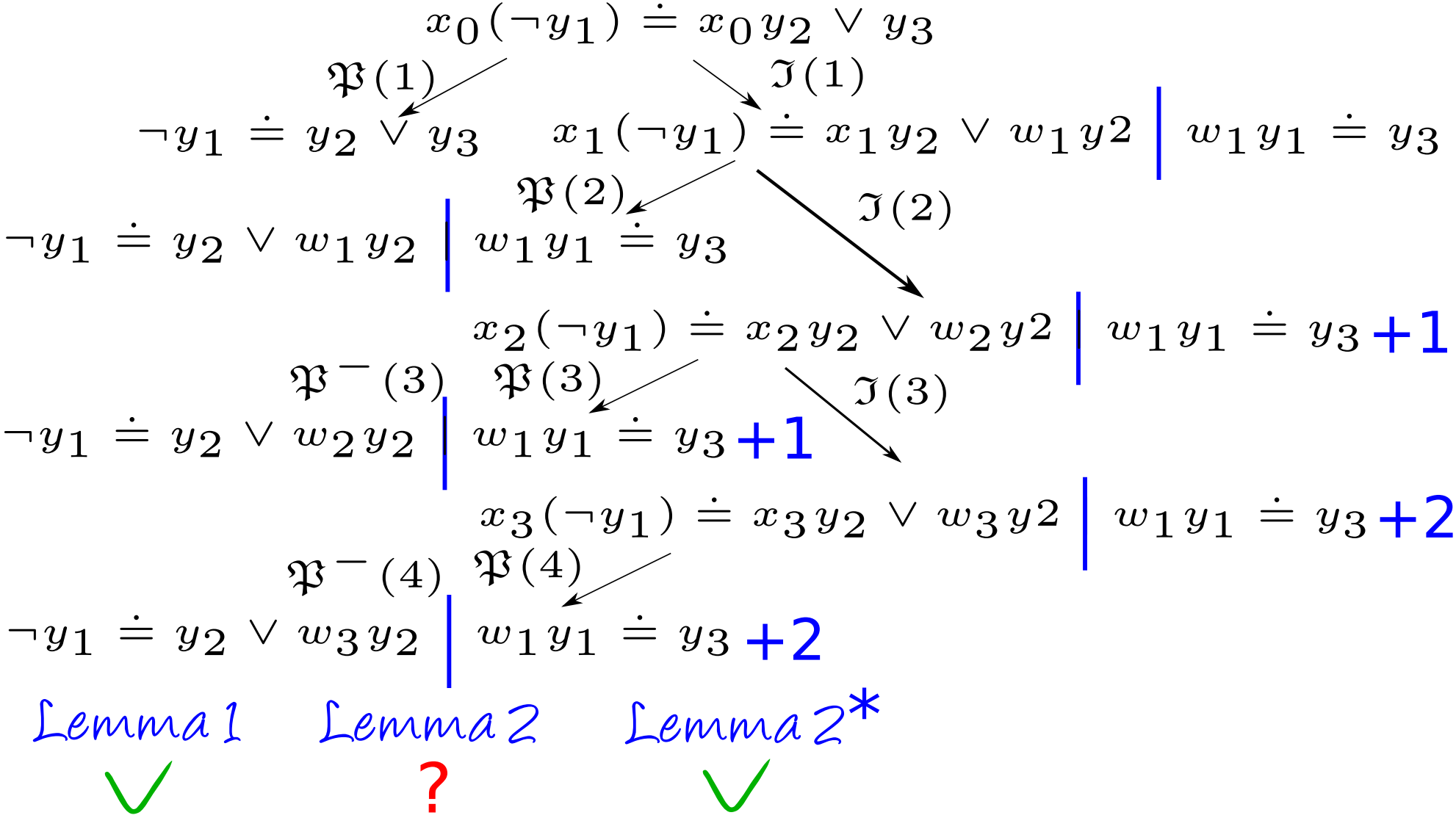

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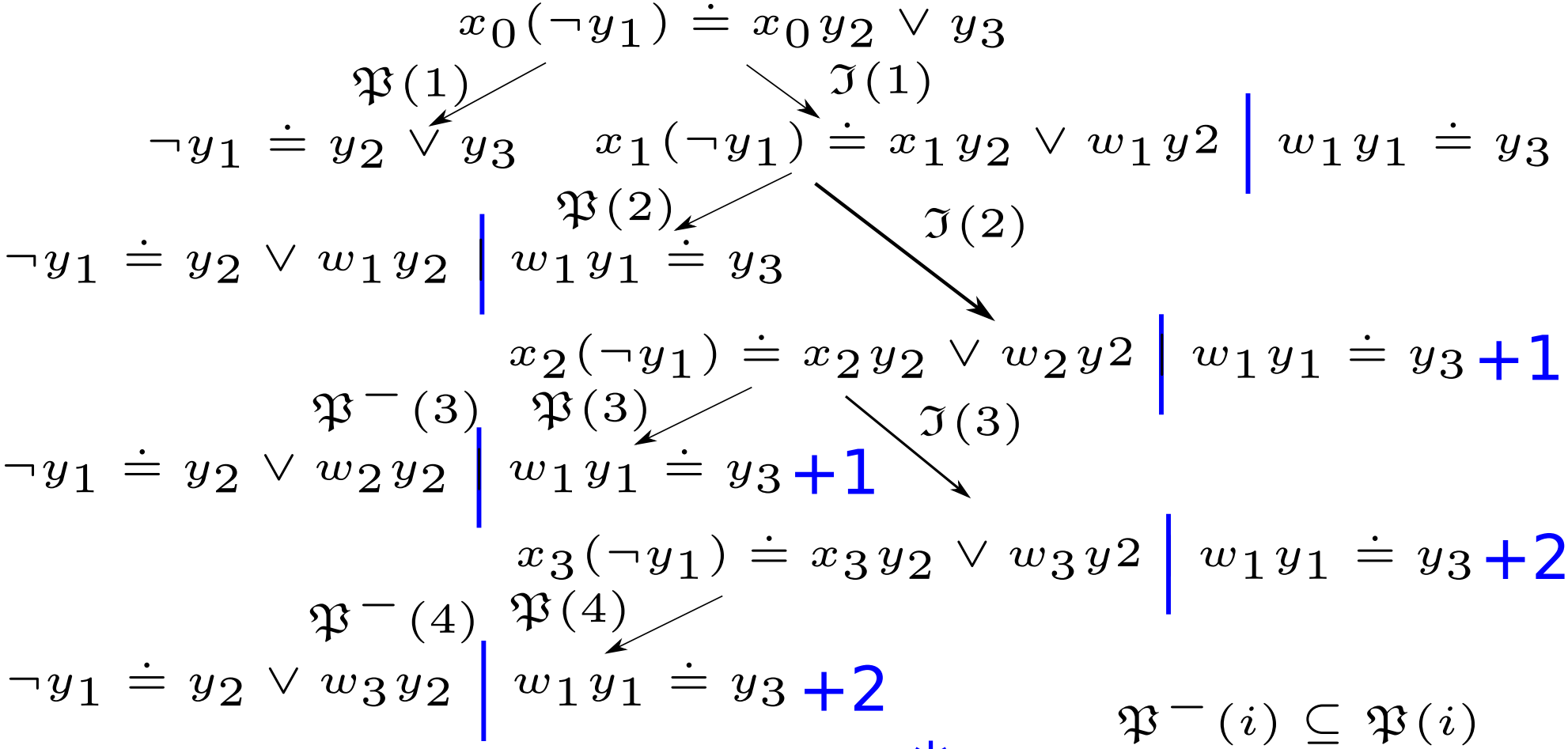

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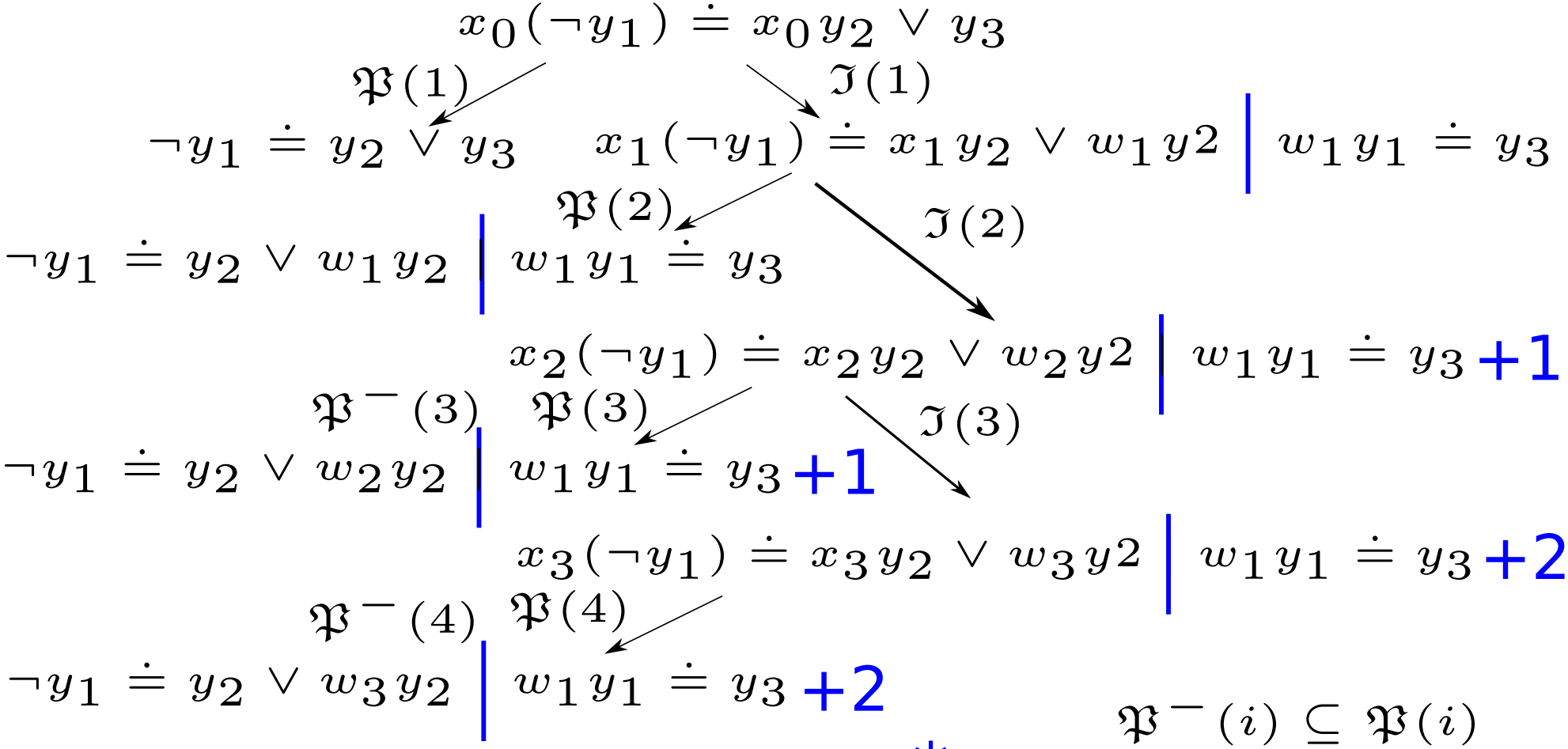

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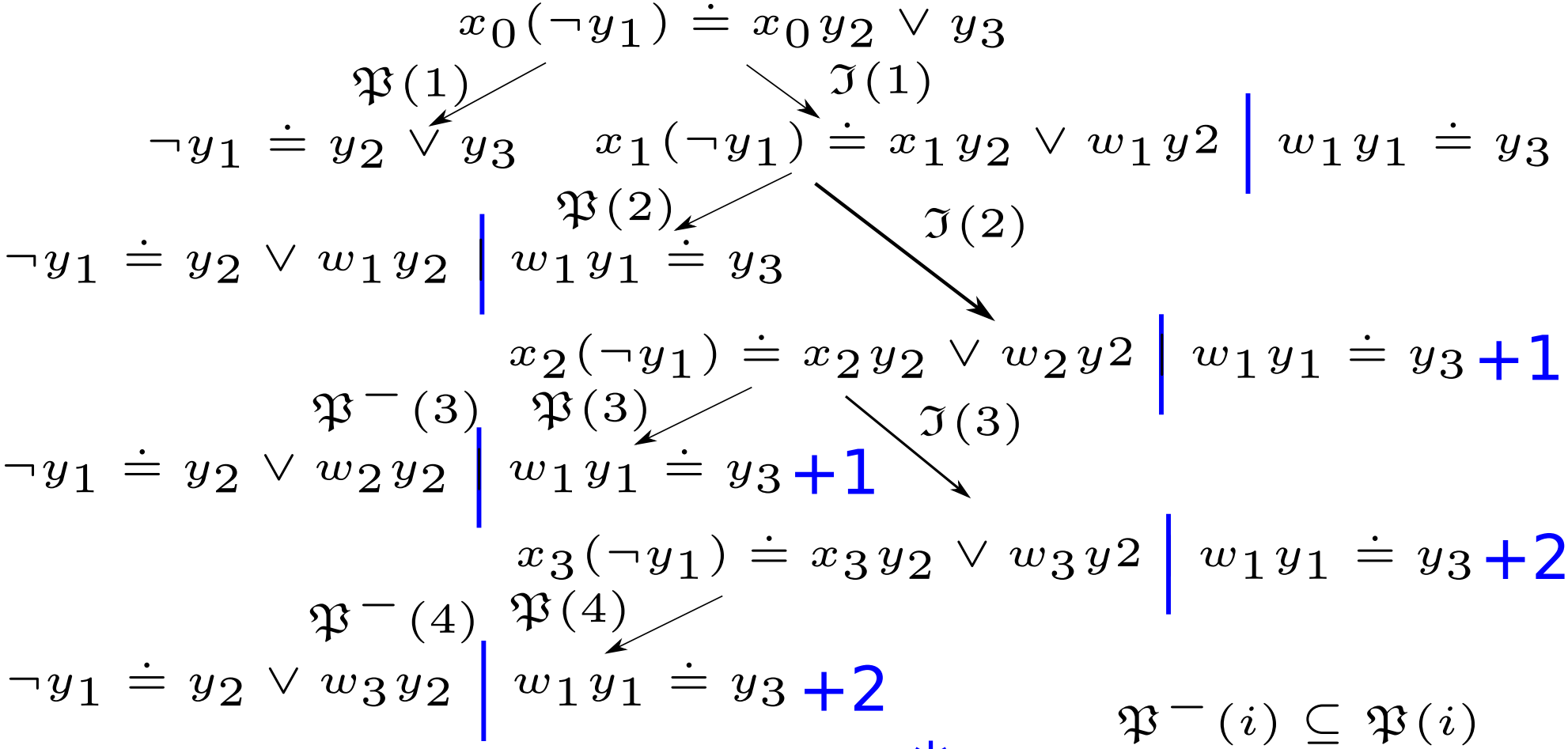

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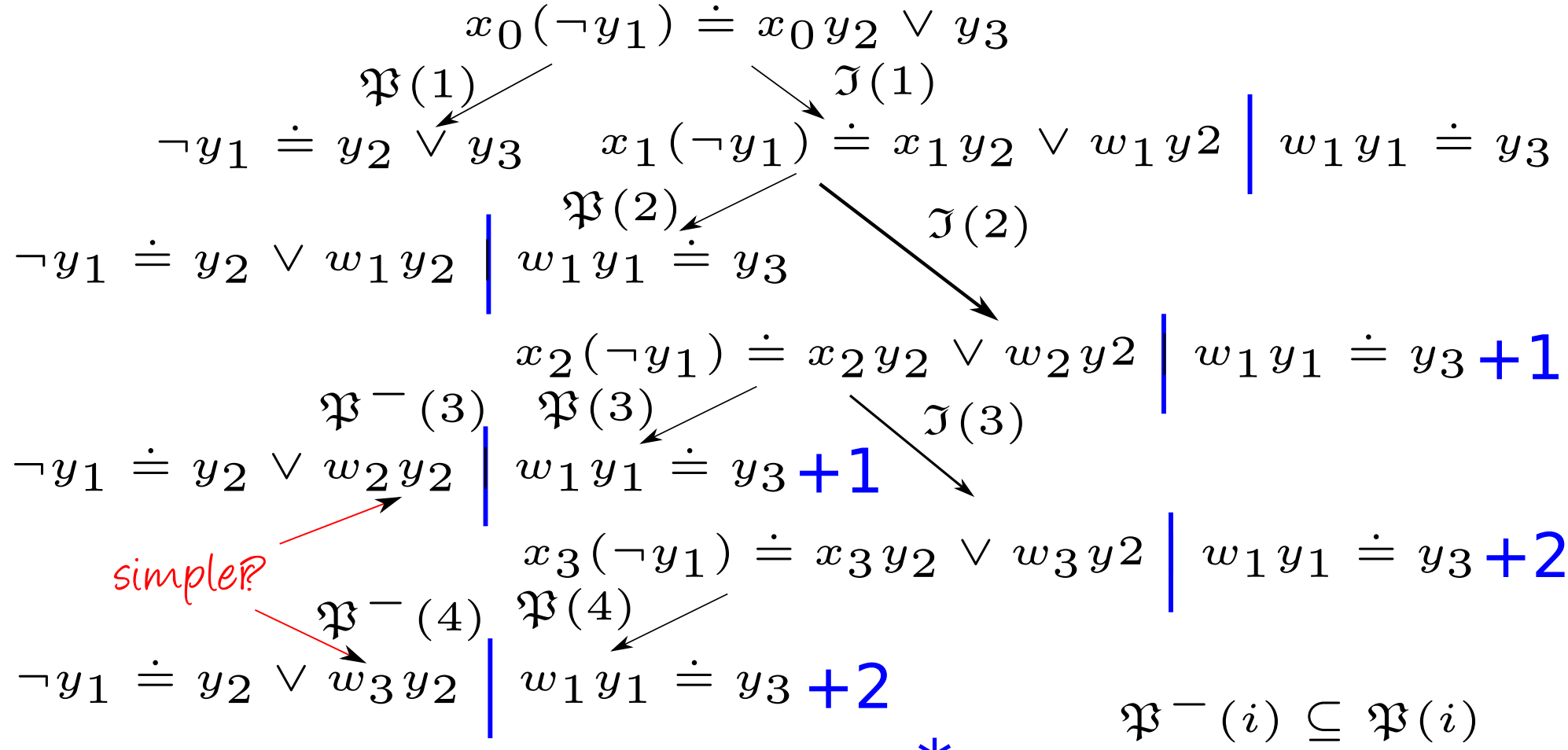

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