Abstract

FUNCTIONAL SYSTEMS OF BL-ALGEBRAS

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This is a contribution to the study of functional systems of BL-algebras on $[0, 1]$. In more details, we deal with functions defined on $[0, 1]$ and represented by a BL-algebra formulas. Our aim is to characterize a class of those functions or, in other words, a functional system of a BL-algebra. This means that we will find properties which characterize these functions as mappings and which are common for all functions represented by a BL-algebra formulas. To achieve this goal we will consider operations of a BL-algebra on $]0, 1$ as functions and then construct a Post algebra on these objects.

Let us briefly overview the known results in this area of investigation. They concern functional systems of some particular BL-algebras: boolean algebra, Łukasiewicz algebra, product algebra and Gödel algebra.

(i) Each boolean function, i.e. a function which is defined on $\{0, 1\}$ and takes values also from $\{0, 1\}$ can be represented by some formula of the boolean algebra on $\{0, 1\}$. At the same time, each function different from a constant may be represented in both disjunctive as well as conjunctive normal forms.

(ii) Each piecewise linear function with integer coefficients which is defined on $[0, 1]$ and takes values from $[0, 1]$ can be represented by some formula of Łukasiewicz algebra on $[0, 1]$. In several proofs (see e.g. [1, 6, 7]) of this statement (known as Mc Naughton theorem), various representing formulas have been suggested.

(iii) The functional system of a product algebra has been investigated in [2, 3] where it has been proved that after a certain transformation it consists of either constants or homogeneous piecewise linear functions on $[0, +\infty]$ with integer coefficients.

For an arbitrary BL-algebra on $[0, 1]$, we will present a unified functional

representation of its operations and then a unified functional representation of an arbitrary formula of Łukasiewicz or product algebra.

We will extend the signature of an arbitrary BL-algebra on $[0,1]$ by constants from $[0,1]$. Then we define the notion of a BL-algebra formula over the extended signature with the permission to have infinite sup and inf. For this case, we will present representation theorems which characterize the respective class of functions as extensional functions, or functions which are Lipschitz continuous. As a technical means, we will use generalized normal forms to be able to work with uniform representation of functions in the mentioned class.

We will also focus on canonical representations of each function from this class with the help of generalized boolean normal forms [10, 11]. We will show that in BL-algebras where the operation $*$ is a continuous Archimedean t-norm with the continuous additive generator, the normal forms may be restricted to those of countable length. This means that in this case, a BL-algebra with infinite formulas of an arbitrary length has the same functional system as a BL-algebra with infinite formulas of at most countable length.

Finally, we will consider an approximate representation of functions from a functional system of a BL-algebra by discrete normal forms [8, 9]. We will formulate this problem as a classical problem of approximation, and present some theoretical results useful to its solution. The optimal approximate representation by the discrete disjunctive normal form will be also discussed and the technique of genetic algorithms will be employed.

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