From Semantics to Deduction with Rasiowa-Sikorski Methodology

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Abstract

In case of logics stemming from real life applications, we typically have as a starting point a certain semantics, for which we then try to develop a sound and complete deduction system. An ideal tool for this purpose is a decompositional methodology for deriving a deduction system out of the semantics of a logic in a systematic way, based on the concept of a Rasiowa-Sikorski (R-S) deduction system, and hence referred to as R-S methodology.

An R-S deduction system operates on sequences of formulae and consists of:

- decomposition rules for sequences of formulae (two-way sound "inference rules"),
- fundamental sequences (simple valid sequences constituting "axioms" of the system).

Decomposition rules divide in: **replacement rules** "breaking down" each composed formula into some simpler ones, whose validity is equivalent to the validity of the original formula, and **expansion rules** adding some formulae to the sequence to close it e.g. under a symmetry or transitivity principle for some relation underlying the semantics. They are used for constructing a decomposition tree of a formula (sequence of formulae) Ω; if the tree is finite and all its leaves are labelled by fundamental sequences, then Ω is said to be provable. The cornerstone of the standard completeness proof for an R-S system is showing that each non-fundamental sequence of "primitive", non-decomposable formulas has a counter-model.

Given the language and semantics of a logic, the R-S methodology can be outlined as follows:

- First, we choose a suitable set of simple valid sequences as fundamental sequences.
- Second, to ensure that each complex formula can be decomposed into simpler ones, for each complex formula α, and each logical value k, we must have a two-way sound decomposition rule expressing validity of $value(\alpha) = k$ in terms of validity of some combination of $value(\alpha_i) = k_i$, where the α_i's are formulae simpler than α, and k_i are some logical values. This is effected by developing such decomposition rules for each constructor C of the logic, or, if this is not possible, for its combinations with all other constructors. As a result, we get a set of replacement rules capable of decomposing any complex formula.
- After generating all the replacement rules we need, we now add enough expansion rules to ensure that every non-fundamental sequence of non-decomposable formulas has a counter-model, which guarantees completeness of the system.

In this way we obtain a sound and complete R-S deduction system for the given logic derived directly from its semantics. Since R-S systems are proof-theoretically equivalent to Gentzen calculi, this system can be easily transformed to a sound and complete Gentzen calculus for the logic.

This methodology has been succesfully used to develop deduction systems for numerous non-classical logic, including Gödel-Dummett logic, one of the three fundamental fuzzy logic. The deduction system for this logic will serve as illustration for application of the R-S methodology.