Refutation of the characteristic clause set of the Prime-Divisor Proof

November 2, 2010

The following table gives an $\mathcal{R}_{al}$-refutation of

$$
\begin{array}{lll}
\mathrm{CS}(\pi) = & \{ & \\
& C_1: & \langle z_0 < s_0(\lambda x.\neg X_0(x))\rangle^{\lambda x.\neg X_0(x), z_0} \vdash \langle X_0(y_0)\rangle^{\lambda x.\neg X_0(x), y_0}, \\
& & \langle X_0(z_0)\rangle^{\lambda x.\neg X_0(x), z_0}; \\
& C_2: & \langle X_0(s_0(\lambda x.\neg X_0(x)))\rangle^{\lambda x.\neg X_0(x)} \vdash \langle X_0(y_0)\rangle^{\lambda x.\neg X_0(x), y_0}; \\
& C_3: & \vdash y_0 * 1 = y_0; \\
& C_4: & z_0 * z_1 = y_0 \vdash z_0 = 1, z_0 = y_0, z_0 < y_0; \\
& C_5: & z_0 * z_1 = y_0, y_0 > 1 \vdash z_0 = 1, z_0 > 1; \\
& C_6: & \vdash w_0 * (z_1 * z_2) = (w_0 * z_1) * z_2; \\
& C_7: & \vdash s_3 > 1; \\
& C_8: & x_0 > 1, x_0 * y_0 = s_3 \vdash s_2(x_0) * s_1(x_0) = x_0; \\
& C_9: & x_0 > 1, s_2(x_0) = 1, x_0 * y_0 = s_3 \vdash; \\
& C_{10}: & x_0 > 1, s_2(x_0) = x_0, x_0 * y_0 = s_3 \vdash \\
& \} &
\end{array}
$$

We use the notation $n \times \mathbf{F}$ to denote n occurrences of $\mathbf{F}$ in a sequent. We define $F(x) \equiv \exists z(\mathrm{D}(z, s_3) \wedge z + x < s_3 \wedge z > 1)$, where $\mathrm{D}(x, y) \equiv \exists z\ x * z = y$, further we define $t \equiv \lambda x.\neg F(x)$.

1	$z_0 < s_0(t) \vdash \langle F(z_0)\rangle^{t,z_0}, \langle F(y_0)\rangle^{t,y_0}$	By Sub, $[X_0 \leftarrow \lambda x.F(x)]$ from C_1
2	$\langle F(s_0(t))\rangle^{t} \vdash \langle F(y_0)\rangle^{t,y_0}$	By Sub, $[X_0 \leftarrow \lambda x.F(x)]$ from C_2
3	$s_2(x_0) * s_1(x_0) = x_0, x_0 * y_0 = s_3 \vdash s_2(x_0) * (s_1(x_0) * y_0) = s_3$	equality axiom
4	$x_0 > 1, 2 \times (x_0 * y_0 = s_3) \vdash s_2(x_0) * (s_1(x_0) * y_0) = s_3$	By Cut from C_8, 3
5	$s_2(x_0) * z_1 = s_3, s_3 > 1 \vdash s_2(x_0) = 1, s_2(x_0) > 1$	By Sub, $[z_0 \leftarrow s_2(x_0), y_0 \leftarrow s_3]$ from C_5
6	$s_2(x_0) * z_1 = s_3, \vdash s_2(x_0) = 1, s_2(x_0) > 1$	By Cut from 5, C_7
7	$x_0 > 1, x_0 * y_0 = s_3, s_2(x_0) * z_1 = s_3 \vdash s_2(x_0) > 1$	By Cut from 6, C_9
8	$x_0 > 1, x_0 * y_0 = s_3, s_2(x_0) * (s_1(x_0) * y_0) = s_3 \vdash s_2(x_0) > 1$	By Sub, $[z_1 \leftarrow s_1(x_0) * y_0]$, from 7
9	$2 \times (x_0 > 1), 3 \times (x_0 * y_0 = s_3) \vdash s_2(x_0) > 1$	By Cut from 8, 4
10	$s_2(x_0) * z_1 = x_0 \vdash s_2(x_0) = 1, s_2(x_0) = x_0, s_2(x_0) < x_0$	By Sub, $[z_0 \leftarrow s_2(x_0), y_0 \leftarrow x_0]$ from C_4
11	$x_0 > 1, x_0 * y_0 = s_3, s_2(x_0) * z_1 = x_0 \vdash s_2(x_0) = 1, s_2(x_0) < x_0$	By Cut from 10, C_{10}
12	$x_0 > 1, x_0 * y_0 = s_3, s_2(x_0) * s_1(x_0) = x_0 \vdash s_2(x_0) = 1, s_2(x_0) < x_0$	By Sub, $[z_1 \leftarrow s_1(x_0)]$ from 11
13	$2 \times (x_0 > 1), 2 \times (x_0 * y_0 = s_3) \vdash s_2(x_0) = 1, s_2(x_0) < x_0$	By Cut from 12, C_8
14	$x_0 + z_0 < s_3, s_2(x_0) < x_0 \vdash s_2(x_0) + (z_0 + 1) < s_3$	axiom of $<$
15	$2 \times (x_0 > 1), 2 \times (x_0 * y_0 = s_3), x_0 + z_0 < s_3 \vdash s_2(x_0) = 1, s_2(x_0) + (z_0 + 1) < s_3$	By Cut from 13, 14
16	$3 \times (x_0 > 1), 3 \times (x_0 * y_0 = s_3), x_0 + z_0 < s_3 \vdash s_2(x_0) + (z_0 + 1) < s_3$	By Cut from 15, C_9
17	$z_0 < s_0(t) \vdash \langle D(s_4(t,z_0), s_3) \wedge s_4(t,z_0) + z_0 < s_3 \wedge s_4(t,z_0) > 1\rangle^{t,z_0}, \langle F(y_0)\rangle^{t,y_0}$	By CNF rules from 1 (s_4 a new Skolem sym.)
18	$z_0 < s_0(t) \vdash \langle D(s_4(t,z_0), s_3)\rangle^{t,z_0}, \langle F(y_0)\rangle^{t,y_0}$	By CNF rules from 17
19	$z_0 < s_0(t) \vdash s_4(t,z_0) + z_0 < s_3, \langle F(y_0)\rangle^{t,y_0}$	By CNF rules from 17
20	$z_0 < s_0(t) \vdash s_4(t,z_0) > 1, \langle F(y_0)\rangle^{t,y_0}$	By CNF rules from 17
21	$z_0 < s_0(t) \vdash s_4(t,z_0) * s_5(t,z_0) = s_3, \langle F(y_0)\rangle^{t,y_0}$	By CNF rules from 18 (s_5 a new Skolem sym.)
22	$s_4(t,z_0) > 1, 2 \times (s_4(t,z_0) * s_5(t,z_0) = s_3) \vdash s_2(s_4(t,z_0)) * (s_1(s_4(t,z_0)) * s_5(t,z_0)) = s_3$	By Sub, $[x_0 \leftarrow s_4(t,z_0), y_0 \leftarrow s_5(t,z_0)]$ from 4
23	$z_0 < s_0(t), s_4(t,z_0) > 1 \vdash s_2(s_4(t,z_0)) * (s_1(s_4(t,z_0)) * s_5(t,z_0)) = s_3, \langle F(y_0)\rangle^{t,y_0}$	By Cut from 21, 22
24	$2 \times (z_0 < s_0(t)) \vdash s_2(s_4(t,z_0)) * (s_1(s_4(t,z_0)) * s_5(t,z_0)) = s_3, 2 \times \langle F(y_0)\rangle^{t,y_0}$	By Cut from 23, 20
25	$2 \times (s_4(t,z_0) > 1), 3 \times (s_4(t,z_0) * s_5(t,z_0) = s_3) \vdash s_2(s_4(t,z_0)) > 1$	By Sub, $[x_0 \leftarrow s_4(t,z_0), y_0 \leftarrow s_5(t,z_0)]$ from 9
26	$z_0 < s_0(t), 2 \times (s_4(t,z_0) > 1) \vdash s_2(s_4(t,z_0)) > 1, \langle F(y_0)\rangle^{t,y_0}$	By Cut from 25, 21
27	$2 \times (z_0 < s_0(t)) \vdash s_2(s_4(t,z_0)) > 1, 2 \times \langle F(y_0)\rangle^{t,y_0}$	By Cut from 26, 20
28	$3 \times (s_4(t,z_0) > 1), 3 \times (s_4(t,z_0) * s_5(t,z_0) = s_3), s_4(t,z_0) + z_0 < s_3 \vdash s_2(s_4(t,z_0)) + (z_0 + 1) < s_3$	By Sub, $[x_0 \leftarrow s_4(t,z_0), y_0 \leftarrow s_5(t,z_0)]$ from 16
29	$z_0 < s_0(t), 3 \times (s_4(t,z_0) > 1), s_4(t,z_0) + z_0 < s_3 \vdash s_2(s_4(t,z_0)) + (z_0 + 1) < s_3, \langle F(y_0)\rangle^{t,y_0}$	By Cut from 28, 21
30	$2 \times (z_0 < s_0(t)), 3 \times (s_4(t,z_0) > 1) \vdash s_2(s_4(t,z_0)) + (z_0 + 1) < s_3, 2 \times \langle F(y_0)\rangle^{t,y_0}$	By Cut from 29, 19
31	$3 \times (z_0 < s_0(t)) \vdash s_2(s_4(t,z_0)) + (z_0 + 1) < s_3, 3 \times \langle F(y_0)\rangle^{t,y_0}$	By Cut from 30, 20
32	$z_0 + 1 \leq s_0(t) \vdash z_0 < s_0(t)$	axiom for $\leq$
33	$z_0 + 1 \leq s_0(t) \vdash s_2(s_4(t,z_0)) * (s_1(s_4(t,z_0)) * s_5(t,z_0)) = s_3, 2 \times \langle F(y_0)\rangle^{t,y_0}$	By Cut from 24, 32
34	$z_0 + 1 \leq s_0(t) \vdash s_2(s_4(t,z_0)) > 1, 2 \times \langle F(y_0)\rangle^{t,y_0}$	By Cut from 27, 32
35	$z_0 + 1 \leq s_0(t) \vdash s_2(s_4(t,z_0)) + (z_0 + 1) < s_3, 3 \times \langle F(y_0)\rangle^{t,y_0}$	By Cut from 31, 32

36	$z_1 * z_2 = s_3, z_1 + s_0(t) < s_3, z_1 > 1 \vdash \langle F(y_0)\rangle^{t,y_0}$	By CNF rules from 2
37	$s_2(s_3) * s_1(s_3) = s_3, s_2(s_3) + s_0(t) < s_3, s_2(s_3) > 1 \vdash \langle F(y_0)\rangle^{t,y_0}$	By Sub, $[z_1 \leftarrow s_2(s_3), z_2 \leftarrow s_1(s_3)]$ from 36
38	$s_3 > 1, s_3 * 1 = s_3 \vdash s_2(s_3) * s_1(s_3) = s_3$	By Sub, $[x_0 \leftarrow s_3, y_0 \leftarrow 1]$ from C_8
39	$\vdash s_2(s_3) * s_1(s_3) = s_3$	By Cut from 38, C_7, C_3
40	$s_2(s_3) * s_1(s_3) = s_3, s_3 > 1 \vdash s_2(s_3) = 1, s_2(s_3) > 1$	By Sub, $[z_0 \leftarrow s_2(s_3), z_1 \leftarrow s_1(s_3), y_0 \leftarrow s_3]$ from C_5
41	$\vdash s_2(s_3) = 1, s_2(s_3) > 1$	By Cut from 40, 39, C_7
42	$s_3 > 1, s_2(s_3) = 1, s_2(s_3) * s_1(s_3) = s_3 \vdash$	By Sub, $[x_0 \leftarrow s_3, y_0 \leftarrow s_1(s_3)]$ from C_9
43	$s_2(s_3) = 1 \vdash$	By Cut from 42, C_7, 39
44	$\vdash s_2(s_3) > 1$	By Cut from 41, 43
45	$s_2(s_3) * s_1(s_3) = s_3 \vdash s_2(s_3) = 1, s_2(s_3) = s_3, s_2(s_3) < s_3$	By Sub, $[z_0 \leftarrow s_2(s_3), z_1 \leftarrow s_1(s_3), y_0 \leftarrow s_3]$ from C_4
46	$\vdash s_2(s_3) = s_3, s_2(s_3) < s_3$	By Cut from 45, 39, 43
47	$s_3 > 1, s_2(s_3) = s_3, s_3 * 1 = s_3 \vdash$	By Sub, $[x_0 \leftarrow s_3, y_0 \leftarrow 1]$ from C_{10}
48	$s_2(s_3) = s_3 \vdash$	By Cut from 47, C_7, C_3
49	$\vdash s_2(s_3) < s_3$	By Cut from 48, 46
50	$s_2(s_3) + s_0(t) < s_3 \vdash \langle F(y_0)\rangle^{t,y_0}$	By Cut from 37, 39, 44
51	$s_2(s_3) < s_3 \vdash s_2(s_3) + s_0(t) < s_3, s_0(t) > 0$	axiom of $<$ and $>$.
52	$\vdash s_0(t) > 0, \langle F(y_0)\rangle^{t,y_0}$	By Cut from 49, 50, 51
53	$s_0(t) > 0 \vdash s_0(t) = p(s_0(t)) + 1$	Predecessor axiom
54	$\vdash \langle F(y_0)\rangle^{t,y_0}, s_0(t) = p(s_0(t)) + 1$	By Cut from 52, 53
55	$z_0 + 1 \leq p(s_0(t)) + 1 \vdash s_2(s_4(t, z_0)) * (s_1(s_4(t, z_0)) * s_5(t, z_0)) = s_3, 2 \times \langle F(y_0)\rangle^{t,y_0}$	By paramodulation from 33, 54
56	$z_0 + 1 \leq p(s_0(t)) + 1 \vdash s_2(s_4(t, z_0)) > 1, 2 \times \langle F(y_0)\rangle^{t,y_0}$	By paramodulation from 34, 54
57	$p(s_0(t)) + 1 \leq p(s_0(t)) + 1 \vdash s_2(s_4(t, p(s_0(t)))) + s_0(t) < s_3, 3 \times \langle F(y_0)\rangle^{t,y_0}$	By Sub and paramodulation from 35, 54
58	$p(s_0(t)) + 1 \leq p(s_0(t)) + 1$	axiom of $\leq$
59	$\vdash s_2(s_4(t, p(s_0(t)))) * (s_1(s_4(t, p(s_0(t)))) * s_5(t, p(s_0(t)))) = s_3, 3 \times \langle F(y_0)\rangle^{t,y_0}$	By Sub, Cut from 58, 55
60	$\vdash s_2(s_4(t, p(s_0(t)))) > 1, 3 \times \langle F(y_0)\rangle^{t,y_0}$	By Sub, Cut from 58, 56
61	$\vdash s_2(s_4(t, p(s_0(t)))) + s_0(t) < s_3, 4 \times \langle F(y_0)\rangle^{t,y_0}$	By Cut from 58, 57
62	$s_2(s_4(t, p(s_0(t)))) * (s_1(s_4(t, p(s_0(t)))) * s_5(t, p(s_0(t)))) = s_3,$ $s_2(s_4(t, p(s_0(t)))) + s_0(t) < s_3, s_2(s_4(t, p(s_0(t)))) > 1 \vdash \langle F(y_0)\rangle^{t,y_0}$	By Sub, $[z_1 \leftarrow s_2(s_4(t, p(s_0(t)))),$ $z_2 \leftarrow s_1(s_4(t, p(s_0(t)))) * s_5(t, p(s_0(t)))]$ from 36
63	$\vdash 10 \times \langle F(y_0)\rangle^{t,y_0}$	By Cut from 62, 59, 61, 60
64	$\vdash s_6(t, y_0) + y_0 < s_3, \ldots, s_{15}(t, y_0) + y_0 < s_3$	By CNF rules from 63 ($s_6, \ldots, s_{15}$ new Skolem symbols)
65	$x_0 + s_3 < s_3 \vdash$	axiom of $<$
66	$\vdash$	By $10 \times$ Sub and Cut, using $[x_0 \leftarrow s_i(t, s_3), y_0 \leftarrow s_3]$ for $6 \leq i \leq 15$,from 65, 64