Building Projections in CERES

This document contains a theoretical description of the basic idea underlying the algorithm used in CERES to build the characteristic clause set and the projections on these clauses.

The characteristic set of clauses is defined as follows:

Definition 1. *We define a set of clauses $\mathcal{C}_\nu$ for every node ν in φ inductively:*

- *If ν is an occurrence of an axiom sequent $S(\nu)$, and S' is the subsequent of $S(\nu)$ containing only the ancestors of Ω then $\mathcal{C}_\nu = \{S'\}$.*
- *Let ν' be the predecessor of ν in a unary inference then $\mathcal{C}_\nu = \mathcal{C}_{\nu'}$.*
- *Let ν_1, ν_2 be the predecessors of ν in a binary inference. We distinguish two cases*

  (a) The auxiliary formulas of ν_1, ν_2 are ancestors of Ω. Then

  $$\mathcal{C}_\nu = \mathcal{C}_{\nu_1} \cup \mathcal{C}_{\nu_2}.$$

  (b) The auxiliary formulas of ν_1, ν_2 are not ancestors of Ω. Then

  $$\mathcal{C}_\nu = \mathcal{C}_{\nu_1} \times \mathcal{C}_{\nu_2}.$$

  where $\mathcal{C} \times \mathcal{D} = \{C \circ D \mid C \in \mathcal{C}, D \in D\}$ and $C \circ D$ is the merge of the clauses C and D.

The characteristic clause set $\mathrm{CL}(\varphi)$ *of φ is defined as $\mathcal{C}_{\nu_0}$, where ν_0 is the root.*

For effectively calculating the set of clauses and the proof projections it is convenient to modify the above definition in such a way that it not only computes the characteristic set of clauses $\{c_1, \ldots, c_n\}$ but a set of pairs $\{\langle c_1, \psi_1\rangle, \ldots, \langle c_n, \psi_n\rangle\}$ where the i-th pair contains in its first component the clause c_i and in its second component the projection to the clause c_i. Generally, in a pair $\langle c, \psi\rangle \in \mathcal{C}_\nu$, c is a clause and ψ is a cut-free derivation of $\Gamma \vdash \Delta \circ c$ where $\Gamma \vdash \Delta$ is the part of the conclusion sequent of the rule at ν which contains no Ω-ancestors.

The nicely implementable definition is as follows:

Definition 2. *We define a set of pairs (with a clause in the first component and a projection (to this clause) in the second component) $\mathcal{C}_\nu$ for every node ν in φ inductively:*

- *If ν is an occurrence of an axiom sequent $S(\nu)$, S' is the subsequent of $S(\nu)$ containing only the ancestors of Ω and S is the whole axiom, then $\mathcal{C}_\nu = \{\langle S', S\rangle\}$.*

- *Let ν' be the predecessor of ν in a unary inference ρ. Let $\mathcal{C}_{\nu'} = \{\langle c_1, \psi_1\rangle, \ldots, \langle c_n, \psi_n\rangle\}$.*
  - *(a) The auxiliary formulas of ν' are ancestors of Ω. Then*
  $$\mathcal{C}_\nu = \mathcal{C}_{\nu'}$$
  - *(b) The auxiliary formulas of ν' are not ancestors of Ω. Then*
  $$\mathcal{C}_\nu = \{\langle c_1, \rho(\psi_1)\rangle, \ldots, \langle c_n, \rho(\psi_n)\rangle\}$$
  where $\rho(\psi)$ denotes the derivation that is obtained from ψ by applying ρ to its end-sequent.
- *Let ν_1, ν_2 be the predecessors of ν in a binary inference ρ.*
  - *(a) The auxiliary formulas of ν_1, ν_2 are ancestors of Ω. Then*
  $$\mathcal{C}_\nu = \mathcal{C}_{\nu_1} \cup \mathcal{C}_{\nu_2}.$$
  - *(b) The auxiliary formulas of ν_1, ν_2 are not ancestors of Ω. Then*
  $$\mathcal{C}_\nu = \mathcal{C}_{\nu_1} \times \mathcal{C}_{\nu_2}.$$
  where
  $$\mathcal{C} \times \mathcal{D} = \{\langle c \circ d, \rho(\psi, \chi)\rangle \mid \langle c, \psi\rangle \in C, \langle d, \chi\rangle \in D\}$$
  where $c \circ d$ is the merge of clauses and $\rho(\psi, \chi)$ denotes the derivation that is obtained from the derivations ψ and χ by applying the binary inference ρ.

The characteristic clause set $\mathrm{CL}(\varphi)$ *of φ is defined as $\mathcal{C}_{\nu_0}$, where ν_0 is the root.*

Note that - dependent on the variant of the calculus - one has to take some care about structural rules, for example permutation rules have to be projected in the sense that the only permute the formulas occurring in the projection. Similar remarks hold for weakening and contraction.

Proposition 1 (correctness). *Let φ be a proof, let ν be a node in φ, let $\Gamma \vdash \Delta$ be the part of the conclusion sequent at ν which contains no Ω-ancestors and let $\langle c, \psi\rangle \in \mathcal{C}_\nu$. Then ψ is a cut-free derivation of $\Gamma \vdash \Delta \circ c$*

Proof. induction on φ □

In fact, the modified definition is a "conservative extension" of the previous one in the sense that (as can easily be verified) at each point if we regard the first component of the pairs only we have exactly the clause sets. The big advantage of the synchronuous computation of clause set and projections is that the connection between a clause and its projection is clear per constructionem. There is no need to establish this connection afterwards by analysing end-sequents of projections or tracing the clause term in the derivation.