A logic for approximate reasoning with a comparative connective

Thomas Vetterlein[1]

Abstract. The Logic of Approximate Entailment (LAE), introduced in R. Rodríguez's Ph.D. Thesis, uses a graded version of the classical consequence relation. In LAE, reasoning about facts is possible even if relationships between them hold only approximately.

Here, we consider a modification of LAE. Namely, we introduce an additional binary connective $\nearrow$ expressing the relative proximity of a proposition when compared to another one. We propose a proof system for the new logic and show finite strong completeness. Certain common problems with the axiomatisation of logics for approximate reasoning are shown to be avoidable in the extended language.

1 Introduction

Approximate reasoning, proposed originally by E. Ruspini in his seminal paper [9], aims at a formalisation of implicative relationships between facts for the case that these relationships do not necessarily hold strictly. The framework that he proposed is as simple as convincing. To model the statement that a proposition α implies another one β to a possibly non-one degree d, a set of worlds W is endowed with a similarity relation s; α and β being interpreted by $A \subseteq W$ and $B \subseteq W$, respectively, the statement $\alpha \overset{d}{\Rightarrow} \beta$ is satisfied if $A \subseteq U_d(B)$. Here, $U_d(B)$ contains all worlds similar to B to the degree at least d.

Logics for metric spaces have been studied in the past in various contexts. Among the more recent examples, we may mention the papers [11, 10]. Here, we follow the lines of research on logics that are associated with approximate reasoning. For an overview over the field to which we intend to contribute, we refer to [6]. Among the proposed formalisms we find, for instance, logics that use a graded modal operator to express similarity [4, 3]. An alternative possibility is to use a graded entailment relation; this idea appears in [2, 3] and was systematically developped in R. Rodríguez Thesis [8]. The Logic of Approximate Entailment, or LAE for short, is in the centre of our own interest.

The expressive power of LAE is lower than in case of the modal logics. Here, we even go one step further and restrict the expressiveness of the language once more. Our motivation is the following. Our ultimate aim is to develop logics for the automatic generation of arguments as done by expert systems; the medical expert system CADIAG-2 [1] is an example. System like CADIAG-2 are not based on probability theory; they are rather designed to produce a chain of arguments which could originate from a human expert. Here, the inference relation appears exclusively at the outermost level; implications do not occur as proper subformulas. In fact, to allow the nesting of relational implication would significantly complicate the interpretation of automatically generated arguments. In the present work, we are interested to avoid this complication as well. This is why we deal only with statements of the form "fact A suggests fact B (to a possibly restricted extent)".

The completeness proof does not become easier by the restriction of the language. The typical technical difficulties arise also in the present framework. Recall that completeness theorems exist for LAE [8, 5]. For the "pure" version of LAE, however, based on a countable number of propositions and an arbitrary similarity space, an axiomatisation has not yet been found. By now, certain additional conditions have been used, most remarkably finiteness of the language and of the model. This restriction cannot easily be removed. A conjunction of all variables, each of which can be negated, has been called a m.e.c.; in the presence of an infinite number of variables, axioms containing m.e.c.'s are not usable.

To find an axiomatisation for LAE requires in fact a solution for two problems. When, in the completeness proof, we construct a model of a theory of LAE we must (1) ensure the symmetry of the similarity relation, and (2) achieve that the degree of provability of one proposition from another one leads to a Hausdorff similarity. Both problems can be overcome by means of m.e.c.'s.

The present contribution is meant as a step towards an axiomatisation of LAE in a more general framework. That is, the two axiom schemes of the proof system in [8] that contain m.e.c.'s is no longer used. However, we offer a progress only in case of one of these axiom schemes. The second one is avoided by a simple generalisation of the model and a more elegant solution would requires surely not less of an effort than in the present case.

We tackle problem (2). The key idea of the present approach is to use a new connective, in addition to conjunction, disjunction, and negation. The connective has a comparative character and is denoted by $\nearrow$; a proposition $\alpha \nearrow \beta$ holds in all worlds that are similar to α at least to the degree to which they are similar to β. Problem (1), in contrast, remains unsolved. To overcome it, we simply give up the requirement of the symmetry of the similarity relation; we work with a quasisimilarity relation.

A connective of a similar type like $\nearrow$ can be found in other areas of logic as well. A comparative connective is present, for instance, in logics of preference; see, e.g., von Wright's monograph [12].

Furthermore, the connective $\nearrow$ might be found to have some resemblance with the implication connective $\rightarrow$ in fuzzy logic. However, this resemblance mainly exists on the formal level; otherwise the two concepts are not comparable, simply because the settings are different. Our setting uses a notion of proximity and $\alpha \nearrow \beta$ holds whenever α is closer than β. In fuzzy logic, $\alpha \rightarrow \beta$ is the weakest proposition implying β when combined with α. We note that, in par-

[1] Johannes Kepler University Linz, Austria, email: Thomas.Vetterlein@jku.at

ticular, that the problem of interpreting the implication in fuzzy logic in an intuitively satisfactory manner is not inherited.

2 The logic LAEC

Our setting for approximate reasoning follows the lines of the papers [2, 4, 8, 6]. The basic framework consists, first of all, of a non-empty set W, called the set of possible worlds. Second, W is endowed with a quasisimilarity relation, which reflects the assumption that a given world may more or less resemble to another one.

In contrast to earlier papers on the topic, we allow the similarity to be non-symmetric. In spite of the afore mentioned proof-technical background, we can say that this choice is in line with applications where similarity models an agent's subjective estimations. In this case, indeed, it is reasonable to have a degree telling how close a property w is when seen from v, and a second one for the converse viewpoint.

Definition 2.1. Let W be any non-empty set; let $[0, 1]$ be the real unit interval; and let $\odot$ be the Łukasiewicz t-norm. A function $s\colon W \times W \to [0, 1]$ is called a *quasisimilarity relation* on s w.r.t. $\odot$ if, for any $u, v, w \in W$,

(S1) $s(u, u) = 1$ (*reflexivity*),

(S2) $s(u, v) = 1$ implies $u = v$ (*separability*),

(S3) $s(u, v) \odot s(v, w) \leq s(u, w)$ ($\odot$-*transitivity*).

In this case, we call (W, s) a *quasisimilarity space*. The similarity of a world $w \in W$ with a set $A \subseteq W$ of worlds is then defined by

$$k(w, A) \;=\; \sup_{a \in A} s(w, a).$$

Finally, for $A \subseteq W$ and $d \in [0, 1]$ we put

$$U_d(A) \;=\; \{w \in W \colon k(w, A) \geq d\}.$$

In what follows, we will use the following well-known notion. Given a quasisimilarity $s\colon W \times W \to [0, 1]$, there is a natural way to measure the similarity between two subsets of W. The *Hausdorff quasisimilarity* induced by s is given by

$$\begin{aligned} h(A, B) \;&=\; \inf_{a \in A} k(a, B) \\ &=\; \inf_{a \in A} \sup_{b \in B} s(a, b) \end{aligned}$$

for $A, B \subseteq W$. Note that this measure of the difference between two sets was also used by Ruspini in his influential paper [9].

Definition 2.2. Let (W, s) be a quasisimilarity space. For any pair $A, B \subseteq W$, we define

$$A \nearrow B \;=\; \{w \in W \colon k(w, A) \geq k(w, B)\}.$$

We define the Logic of Approximate Entailment with Comparison, or LAEC for short, model-theoretically as follows.

Definition 2.3. The *propositional formulas* of LAEC are built up from a countable set of *variables* $\varphi_0, \varphi_1, \ldots$ and the *constants* $\bot, \top$ by means of the binary operators $\wedge$, $\vee$, and $\nearrow$, and the unary operator $\neg$. The set of propositional formulas is denoted by $\mathcal{F}$. A *conditional formula* of LAEC is a triple consisting of two propositional formulas α and β as well as a value $d \in [0, 1]$, denoted

$$\alpha \overset{d}{\Rightarrow} \beta.$$

Let (W, s) be a quasisimilarity space. An *evaluation* for LAEC is a structure-preserving mapping v from $\mathcal{F}$ to (W, s). A conditional formula $\alpha \overset{d}{\Rightarrow} \beta$ is *satisfied* by an evaluation v if

$$v(\alpha) \subseteq U_d(v(\beta)).$$

A *theory* of LAEC is a set of conditional formulas. We say that a theory $\mathcal{T}$ *semantically entails* a conditional formula $\alpha \overset{d}{\Rightarrow} \beta$ if all evaluations satisfying all elements of $\mathcal{T}$ also satisfy $\alpha \overset{d}{\Rightarrow} \beta$.

We present now a calculus for LAEC. Whereas the content of the rules (at least those that do not involve $\nearrow$) reflects the content of the axioms used in earlier papers on LAE, the chosen style of the syntax is inspired by the Gentzen-style proof systems that have been developped in fuzzy logic during the last years [7].

In what follows, a CPL tautology is meant to be a formula that arises from a tautology of classical propositional logic by uniform replacement of its atoms by propositional formulas of LAEC.

We note furthermore that, for $c \in [0, 1]$, $c \odot c$ is abbreviated as c^2.

Definition 2.4. The rules and axioms of LAEC are, for any $\alpha, \gamma, \beta \in \mathcal{F}$, for any finite set $\Gamma \subseteq \mathcal{F}$, and for any $c, d \in [0, 1]$, the following:

$$\frac{\Gamma, \alpha, \beta \overset{d}{\Rightarrow} \gamma}{\Gamma, \alpha \wedge \beta \overset{d}{\Rightarrow} \gamma} \qquad \frac{\Gamma \overset{d}{\Rightarrow} \beta}{\Gamma, \alpha \overset{d}{\Rightarrow} \beta}$$

$$\frac{\Gamma, \alpha \overset{d}{\Rightarrow} \gamma \quad \Gamma, \beta \overset{d}{\Rightarrow} \gamma}{\Gamma, \alpha \vee \beta \overset{d}{\Rightarrow} \gamma} \qquad \frac{\Gamma \overset{d}{\Rightarrow} \alpha}{\Gamma \overset{d}{\Rightarrow} \alpha \vee \beta}$$

$$\frac{\Gamma \overset{c}{\Rightarrow} \beta \nearrow \alpha \quad \Gamma \overset{d}{\Rightarrow} \alpha}{\Gamma \overset{c^2 \odot d}{\Rightarrow} \beta} \qquad \frac{\alpha \overset{1}{\Rightarrow} \beta}{\top \overset{1}{\Rightarrow} \beta \nearrow \alpha}$$

$$\alpha \overset{1}{\Rightarrow} \alpha \nearrow \beta \qquad \alpha \nearrow \beta,\ \beta \nearrow \gamma \overset{1}{\Rightarrow} \alpha \nearrow \gamma$$

$$\frac{\Gamma, \alpha \nearrow \beta \overset{d}{\Rightarrow} \gamma \quad \Gamma, (\neg\alpha \wedge \beta) \nearrow \alpha \overset{d}{\Rightarrow} \gamma}{\Gamma \overset{d}{\Rightarrow} \gamma}$$

$$\frac{\Gamma \overset{c}{\Rightarrow} \alpha \quad \alpha \overset{d}{\Rightarrow} \gamma}{\Gamma \overset{c \odot d}{\Rightarrow} \gamma}$$

$$\frac{\Gamma \overset{c}{\Rightarrow} \alpha}{\Gamma \overset{d}{\Rightarrow} \alpha},\ \text{where } d \leq c \qquad \frac{\Gamma \overset{d}{\Rightarrow} \bot}{\Gamma \overset{1}{\Rightarrow} \bot},\ \text{where } d > 0$$

$$\alpha \overset{0}{\Rightarrow} \beta \qquad \frac{\alpha \overset{1}{\Rightarrow} \beta}{\alpha \wedge \neg\beta \overset{1}{\Rightarrow} \bot}$$

$$\alpha \overset{1}{\Rightarrow} \beta,\ \text{ where } \neg\alpha \vee \beta \text{ is a CPL tautology}$$

The notion of proof of a conditional formula $\alpha \overset{d}{\Rightarrow} \beta$ from a theory $\mathcal{T}$ is defined as usual; we write $\mathcal{T} \vdash \alpha \overset{d}{\Rightarrow} \beta$ if it exists.

A theory $\mathcal{T}$ is called *consistent* if $\mathcal{T}$ does not prove $\top \overset{1}{\Rightarrow} \bot$.

To illustrate how statements in LAEC read, we consider the following example:

Lemma 1. *The following rule is derivable in* LAEC*:*

$$\frac{\Gamma \overset{d}{\Rightarrow} \alpha \vee \beta}{\Gamma, \alpha \nearrow \beta \overset{d}{\Rightarrow} \alpha}$$

Proof. We just note that both $\alpha \nearrow \beta, \alpha \overset{1}{\Rightarrow} \alpha$ and $\alpha \nearrow \beta, \beta \overset{1}{\Rightarrow} \alpha$ are provable in LAEC. ☐

In words, we can express Lemma 1 as follows: If some world w has a similarity $\geq d$ to α or β and w has a greater similarity to α than to β, then w has the similarity $\geq d$ to α.

3 Completeness for LAEC

The proof of the completeness theorem requires some preparations.

Lemma 2. *The following rules are derivable in* LAEC*:*

$$\frac{\alpha \stackrel{1}{\Rightarrow} \beta}{\alpha \nearrow \gamma \stackrel{1}{\Rightarrow} \beta \nearrow \gamma} \qquad \frac{\beta \stackrel{1}{\Rightarrow} \alpha}{\gamma \nearrow \alpha \stackrel{1}{\Rightarrow} \gamma \nearrow \beta}$$

$$\frac{\Gamma \stackrel{d}{\Rightarrow} \beta \nearrow \alpha}{\Gamma, \alpha \stackrel{d^2}{\Rightarrow} \beta}$$

$$\frac{\Gamma, \beta \nearrow \alpha \stackrel{d}{\Rightarrow} \gamma \quad \Gamma, \alpha \nearrow \beta \stackrel{d}{\Rightarrow} \gamma}{\Gamma \stackrel{d}{\Rightarrow} \gamma}$$

$$\frac{\Gamma \stackrel{d}{\Rightarrow} \alpha \vee \beta}{\Gamma, \alpha \nearrow \beta \stackrel{d}{\Rightarrow} \alpha}$$

In what follows, $\lceil r \rceil$, where $r \in \mathbb{R}^+$, denotes the smallest natural number greater than or equal to r.

Definition 3.1. Let $\mathcal{T}$ be a theory of LAEC, and let α, β be propositional formulas. We define the *provability degree* of the pair α, β w.r.t. $\mathcal{T}$ by

$$p_{\mathcal{T}}(\alpha, \beta) \;=\; \sup\,\{t \in [0,1] \colon \mathcal{T} \vdash \alpha \stackrel{t}{\Rightarrow} \beta\}.$$

Furthermore, by the *density* of $p_{\mathcal{T}}$, denoted by $\mathrm{density}(p_{\mathcal{T}})$, we mean the infimum of all differences between distinct elements of the range of $p_{\mathcal{T}}$.

If the theory $\mathcal{T}$ is understood, we will write p instead of $p_{\mathcal{T}}$.

We note that, in the following proofs, we consider $[0,1]$ as a lattice and write $\wedge$, $\vee$ for the minimum and maximum operations, respectively.

Lemma 3. *Let $\mathcal{T}$ be a consistent finite theory of* LAEC *such that $\mathcal{T}$ does not prove the conditional formula $\zeta \stackrel{e}{\Rightarrow} \eta$. Then there is a consistent theory $\mathcal{T}' \supseteq \mathcal{T}$ such that the following holds:*

(E1) *$\mathcal{T}'$ does not prove $\zeta \stackrel{e}{\Rightarrow} \eta$.*

(E2) *For any sequence $(\varepsilon_i)_{i \in \mathbb{N}}$ in $\mathcal{F}$ such that $\mathcal{T}$ proves $\varepsilon_1 \stackrel{1}{\Rightarrow} \varepsilon_0$, $\varepsilon_2 \stackrel{1}{\Rightarrow} \varepsilon_1$, …, and for any pair $\alpha, \beta \in \mathcal{F}$ such that $\mathcal{T}$ proves $\alpha \wedge \beta \stackrel{1}{\Rightarrow} \bot$ and $\varepsilon \wedge (\alpha \vee \beta) \stackrel{1}{\Rightarrow} \bot$, $\bigwedge_i p(\varepsilon_i, \alpha) \neq \bigwedge_i p(\varepsilon_i, \beta)$.*

(E3) *There is an $l \in [0,1)$ such that, for any pair $\alpha, \beta \in \mathcal{F}$, either $p(\alpha, \beta) = 1$ or $p(\alpha, \beta) \leq l$.*

Proof. Note first that $e > 0$. Let $\bar{e} \in [0,1]$ the largest value $< e$ such that $\mathcal{T} \vdash \zeta \stackrel{\bar{e}}{\Rightarrow} \eta$. Such a value exists because $\mathcal{T} \vdash \zeta \stackrel{0}{\Rightarrow} \eta$ and because $\mathcal{T}$, and consequently the range of $p_{\mathcal{T}}$, is finite. Put $\vartheta = (\bar{e} - e) \wedge \mathrm{density}(p_{\mathcal{T}})$.

Let (α_i, β_i), $i < \omega$, be all pairs of formulas α and β such that $\mathcal{T}$ proves $\alpha \wedge \beta \stackrel{1}{\Rightarrow} \bot$. We will define a sequence of consistent finite theories

$$\begin{aligned} \mathcal{T} \;=\; \mathcal{T}_1 \;&=\; \mathcal{T}_1^0 \subseteq \ldots \subseteq \mathcal{T}_1^{k_1} \;= \\ =\; \mathcal{T}_2 \;&=\; \mathcal{T}_2^0 \subseteq \ldots \subseteq \mathcal{T}_2^{k_2} \;= \\ =\; &\quad \ldots \end{aligned}$$

and along with each theory $\mathcal{T}_i^j$, we will define values ϑ_i^j with the following properties:

(1) $\vartheta_i^j \leq \frac{1}{4}\mathrm{density}(p_{\mathcal{T}_i^{j-1}})$;

(2) $\vartheta_i^j \leq \frac{1}{4}\vartheta_i^{j-1}$;

(3) $|p_{\mathcal{T}_i^j}(\gamma, \delta) - p_{\mathcal{T}_i^{j-1}}(\gamma, \delta)| \leq \vartheta_i^{j-1}$ for any $\gamma, \delta \in \mathcal{F}$,

where $1 \leq j \leq k_i$.

Let $\mathcal{T}_1 = \mathcal{T}_1^0 = \mathcal{T}$ and $\vartheta_1 = \vartheta_1^0 = \vartheta$. Assume that, for $i \geq 1$, $\mathcal{T}_i = \mathcal{T}_i^0$ and $\vartheta_i = \vartheta_i^0$ are already defined. Let $V_i = \{v_i^0, \ldots, v_i^{k_i}\}$ be the range of $p_{\mathcal{T}_i}$, where $v_i^1 < \ldots < v_i^{k_i} = 1$. Let $\mathcal{T}_i^{k_i+1} = \mathcal{T}_i$ and $\vartheta_i^{k_i+1} = \vartheta_i$. For $j = 1, \ldots, k_i$, let

$$\begin{aligned} \mathcal{G}_i^j \;=\; \{\varepsilon \in \mathcal{F} \colon \mathcal{T} \vdash \varepsilon \wedge (\alpha_i \vee \beta_i) \stackrel{1}{\Rightarrow} \bot \text{ and} \\ p_{\mathcal{T}_i}(\varepsilon, \alpha_i) = p_{\mathcal{T}_i}(\varepsilon, \beta_i) = v_i^j\} \end{aligned}$$

and

$$\vartheta_i^j \;=\; \frac{1}{4\lceil \frac{1}{1 - v_i^j} \rceil}(\vartheta_i^{j-1} \wedge \mathrm{density}(p_{\mathcal{T}_i^{j-1}})).$$

$$\mathcal{T}_i^j \;=\; \mathcal{T}_i^{j-1} \cup \{\varepsilon \stackrel{v_i^j + \vartheta_i^j}{\Rightarrow} \alpha_i \colon \varepsilon \in \mathcal{G}_i^j\}.$$

Properties (1) and (2) are obviously fulfilled. Let furthermore $\gamma, \delta \in \mathcal{F}$ and consider a proof of $\gamma \stackrel{c}{\Rightarrow} \delta$ from $\mathcal{T}_i^j$, where $c = p_{\mathcal{T}_i^j}(\gamma, \delta)$. Then there is a proof of $\gamma \stackrel{c'}{\Rightarrow} \delta$ from $\mathcal{T}_i^{j+1}$, where $c' = (c - n\vartheta_i^j) \vee 0$, where $0 \leq n \leq \lceil \frac{1}{1 - v_i^j} \rceil$. Then c' is the largest element $\leq c$ in the range of $p_{\mathcal{T}_i^{j-1}}$, hence $c' = p_{\mathcal{T}_i^{j-1}}(\gamma, \delta)$, and (3) follows.

Let $\mathcal{T}' = \bigcup_i \mathcal{T}_i$. Let $\gamma, \delta \in \mathcal{F}$; then for any i, j, we have

$$|p_{\mathcal{T}_i^j}(\gamma, \delta) - p_{\mathcal{T}'}(\gamma, \delta)| \leq \tfrac{1}{3}\mathrm{density}(p_{\mathcal{T}_i^{j+1}}).$$

In particular, $|p_{\mathcal{T}}(\gamma, \delta) - p_{\mathcal{T}'}(\gamma, \delta) \leq \frac{1}{3}\vartheta|$. Claim (E1) and (E3) follow as well as the consistency of $\mathcal{T}'$.

To show (E2), let $\varepsilon_0, \varepsilon_1, \ldots$ and $\alpha, \beta \in \mathcal{F}$ be as indicated. Note that, since all $\mathcal{T}_i$ are finite, $p_{\mathcal{T}_i}(\varepsilon_l, \alpha)$, $l = 0, 1, \ldots$, is eventually constant. There are two possibilities:

Case 1. For some i and j and some $m \geq 1$, $|p_{\mathcal{T}_i}(\varepsilon_l, \alpha) - p_{\mathcal{T}_i}(\varepsilon_l, \beta)| = d > 0$ for all $l \geq m$. Then $|p_{\mathcal{T}'}(\varepsilon_l, \alpha) - p_{\mathcal{T}'}(\varepsilon_l, \beta)| \geq \frac{1}{3}d$ for all $l \geq m$, and claim (E2) follows.

Case 2. For all i, $p_{\mathcal{T}_i}(\varepsilon_l, \alpha) = p_{\mathcal{T}_i}(\varepsilon_l, \beta)$ eventually. This is then in particular the case for the i that indexes the pair (α, β). Let m and j be such that $p_{\mathcal{T}_i}(\varepsilon_l, \alpha) = p_{\mathcal{T}_i}(\varepsilon_l, \beta) = v_i^j$ for all $l \geq m$. Then $\varepsilon_m \in \mathcal{G}_i^j$ for all $m \geq l$. It follows $p_{\mathcal{T}_i^j}(\varepsilon_m, \beta) = v_j^i$ and $p_{\mathcal{T}_i^j}(\varepsilon_m, \alpha) = v_i^j - \vartheta_i^j$. But this implies that the difference remains strictly positive for all extensions of $\mathcal{T}_i^j$; a contradiction. Thus Case 2 never occurs. □

Theorem 3.1. *Let $\mathcal{T}$ be a consistent finite theory of* LAEC*. Then $\mathcal{T}$ proves a conditional formula $\zeta \stackrel{e}{\Rightarrow} \eta$ if and only if $\mathcal{T}$ semantically entails $\zeta \stackrel{e}{\Rightarrow} \eta$.*

Proof. It is not difficult to check the soundness. To prove the completeness, assume that $\mathcal{T}$ does not prove $\zeta \stackrel{e}{\Rightarrow} \eta$. By Lemma 3, we can assume that $\mathcal{T}$ fulfills the following conditions instead of the indicated ones: $\mathcal{T}$ is consistent, does not prove $\zeta \stackrel{e}{\Rightarrow} \eta$, and has properties (E2) and (E3).

For $\alpha, \beta \in \mathcal{F}$, let $\alpha \preccurlyeq \beta$ if $\mathcal{T} \vdash \alpha \stackrel{1}{\Rightarrow} \beta$, and let $\alpha \approx \beta$ if $\alpha \preccurlyeq \beta$ and $\beta \preccurlyeq \alpha$. Then $\approx$ is an equivalence relation, and it is not difficult to see that $\approx$ is compatible with $\wedge$, $\vee$, and $\neg$. By Lemma 2, $\approx$ is also compatible with $\nearrow$. Endowed with the induced operations and the classes of $\bot$ and $\top$, the quotient $(\langle \mathcal{F} \rangle; \wedge, \vee, \neg, \nearrow, \langle \bot \rangle, \langle \top \rangle)$, is a Boolean algebra endowed with the additional operation $\nearrow$. Note that $\mathcal{F}$ and thus also $\langle\mathcal{F}\rangle$ is countable.

As our first step, we establish some facts about the provability degree p. Clearly, for any $\alpha, \beta \in \mathcal{F}$ and $d \in [0,1]$, $\mathcal{T} \vdash \alpha \stackrel{d}{\Rightarrow} \beta$ implies $d \leq p_{\mathcal{T}}(\alpha,\beta)$, and $d < p_{\mathcal{T}}(\alpha,\beta)$ implies $\mathcal{T} \vdash \alpha \stackrel{d}{\Rightarrow} \beta$.

It is furthermore easily seen that, for any $\alpha_1, \alpha_2, \beta \in \mathcal{F}$,

$$p(\alpha_1 \vee \alpha_2, \beta) \;=\; p(\alpha_1,\beta) \wedge p(\alpha_2,\beta).$$

Furthermore, for any α, β_1, β_2, there are α_1, α_2 such that $\alpha \approx \alpha_1 \vee \alpha_2$ and

$$p(\alpha, \beta_1 \vee \beta_2) \;=\; p(\alpha_1,\beta_1) \wedge p(\alpha_2,\beta_2).$$

Indeed, we may choose $\alpha \wedge (\beta_1 \nearrow \beta_2)$ for α_1 and $\alpha \wedge (\beta_2 \nearrow \beta_1)$ for α_2.

Let W be the set of prime filters of $\langle\mathcal{F}\rangle$. Due to the consistency of $\mathcal{T}$, W is non-empty. For $w \in W$ and $\alpha \in \mathcal{F}$, we write $w \lhd \alpha$ for $\langle\alpha\rangle \in w$. Then $\iota\colon \langle\mathcal{F}\rangle \to \mathcal{P}W,\ \langle\alpha\rangle \mapsto \{w \in W\colon w \lhd \alpha\}$ is an injective homomorphism of Boolean algebras.

For $w \in W$ and $\alpha \in \mathcal{F}$, we put

$$k(w,\alpha) \;=\; \sup_{w \lhd \varepsilon} p(\varepsilon,\alpha),$$

and for $v, w \in W$, put

$$s(v,w) \;=\; \inf_{w \lhd \delta} k(v,\delta).$$

It is not difficult to check that $s\colon W \times W \to [0,1]$ is reflexive and $\odot$-transitive. To see that also separability holds for s, that is, to see that s is actually a quasisimilarity, assume $s(v,w) = 1$, but $v \neq w$, for some $v, w \in W$. Then $k(v,\delta) = 1$ for some $w \lhd \delta$ such that $v \not\lhd \delta$. Consequently, for any $\vartheta < 1$, there is an ε such that $\delta \wedge \varepsilon \approx \perp$ and $p(\varepsilon,\delta) > \vartheta$. But $p(\varepsilon,\delta) < 1$ then, and a contradiction to property (E3) arises.

Note that p can be viewed as a function on $\langle\mathcal{F}\rangle$ instead of $\mathcal{F}$, and consequently also as a function on $\iota(\langle\mathcal{F}\rangle)$, a Boolean subalgebra of $\mathcal{P}W$. Adopting the latter view, we claim that p coincides with the Hausdorff quasisimilarity induced by s. To see this, we first show

$$k(w, \alpha \vee \beta) \;=\; k(w,\alpha) \vee k(w,\beta)$$

for any $w \in W$ and $\alpha, \beta \in \mathcal{F}$. Clearly, $k(w,\alpha \vee \beta) \geq k(w,\alpha) \vee k(w,\beta)$. Furthermore, by definition $k(w,\alpha\vee\beta) = \sup_{w\lhd\varepsilon} p(\varepsilon,\alpha\vee\beta)$, hence for any $\vartheta > 0$ there is a particular ε' such that $w \lhd \varepsilon'$ and $k(w,\alpha\vee\beta) - \vartheta \leq p(\varepsilon',\alpha\vee\beta)$. Then $p(\varepsilon',\alpha\vee\beta) = p(\varepsilon'_1,\alpha) \wedge p(\varepsilon'_2,\beta)$, where $\varepsilon'_1 \vee \varepsilon'_2 \approx \varepsilon'$. We assume, w.l.o.g., that $w \lhd \varepsilon'_1$, and we conclude $k(w,\alpha\vee\beta) - \vartheta \leq p(\varepsilon'_1,\alpha) \leq \sup_{w\lhd\varepsilon} p(\varepsilon,\alpha) = k(w,\alpha) \leq k(p,\alpha) \vee k(w,\beta)$, that is, $k(w,\alpha\vee\beta) \leq k(p,\alpha) \wedge k(w,\beta)$.

We next show

$$k(v,\alpha) \;=\; \sup_{w \lhd \alpha} s(v,w)$$

for $v \in W$ and $\alpha \in \mathcal{F}$. Assume first that $\alpha \approx \perp$. Then $k(v,\alpha) = k(v,\perp) = \sup_{w\lhd\varepsilon} p(\varepsilon,\perp) = 0$ because $\varepsilon \in w$ for some $w \in W$ implies $\varepsilon \not\approx \perp$, hence $\mathcal{T} \nvdash \varepsilon \stackrel{d}{\Rightarrow} \perp$ for any $d > 0$. Furthermore, there is no prime filter $w \in W$ containing $\langle\alpha\rangle = \langle\perp\rangle$; hence the claim follows.

Assume that $\alpha \not\approx \perp$. Then we obviously have $k(v,\alpha) \geq \inf_{w\lhd\delta} k(v,\delta) = s(v,w)$ for all $w \lhd \alpha$. Now, note that for any $\chi \in \mathcal{F}$, $k(p,\alpha) = k(p,(\alpha\wedge\chi)\vee(\alpha\wedge\neg\chi)) = k(p,\alpha\wedge\chi) \vee k(p,\alpha\wedge\neg\chi)$; it follows that there is a sequence $\alpha = \alpha_0 \succcurlyeq \alpha_1 \succcurlyeq \ldots$ that is a basis of a filter $w \lhd \alpha$ such that $k(v,\alpha_i) = k(v,\alpha)$ for all i, in particular $k(v,\alpha) = s(v,w)$.

The last step to show that p is induced by s is the proof of

$$p(\alpha,\beta) \;=\; \inf_{w \lhd \alpha} k(w,\beta).$$

In case that $\alpha \approx \perp$, there is no $w \in W$ such that $w \lhd \alpha$, and the claim is verified noting that $p(\perp,\beta) = 1$. Assume that $\alpha \not\approx \perp$. Obviously, $p(\alpha,\beta) \leq \max_{w\lhd\varepsilon} p(\varepsilon,\beta) = k(w,\beta)$ for all $w \lhd \alpha$. Similarly as above, we choose a sequence $\alpha = \alpha_0 \succcurlyeq \alpha_1 \succcurlyeq \ldots$ that is a basis of a filter $w \lhd \alpha$ such that $p(\alpha,\beta) = p(\alpha_i,\beta)$ for all i. Then $p(\alpha,\beta) = k(w,\beta)$.

Consider now again the Boolean homomorphism ι. We have to show that

$$\iota(\alpha \nearrow \beta) \;=\; \{w \in W\colon k(w,\alpha) \geq k(w,\beta)\}.$$

Indeed, $w \lhd \alpha \nearrow \beta$ implies $k(w,\alpha) \geq k(w,\beta)$. Furthermore, from $k(w,\alpha) > k(w,\beta)$ it follows $w \lhd \alpha \nearrow \beta$. In case that $k(w,\alpha) = k(w,\beta) = 1$, we have seen above that $w \lhd \alpha$ and $w \lhd \beta$ and thus $w \lhd \alpha \nearrow \beta$. Finally, $k(w,\alpha) = k(w,\beta) < 1$ contradicts condition (E2) of Lemma 3 above.

The proof is complete that (W,s) provides a model for LAEC. Furthermore, it is easily verified that all elements of $\mathcal{T}$ are satisfied and that $\zeta \stackrel{e}{\Rightarrow} \eta$ is not satisfied. □

4 Conclusion

We have presented a logic for approximate reasoning – LAEC, the Logic of Approximate Entailment with Comparison. LAEC differs from LAE, the Logic of Approximate Entailment, in that it contains a connective that is non-standard in approximate reasoning: the comparative connective $\nearrow$. A further difference between LAEC and LAE is that our models are quasisimilarity spaces rather than similarity spaces. We have presented a Gentzen-type proof system for LAEC and have proven its completeness for finite theories.

The rules are transparent and allow a straightforward interpretation, the new ones for $\nearrow$ included. Formulas of special syntactical form are not required.

There is a lot of room for further research. Most desirably, it should be examined if the possibly non-symmetric similarity spaces, allowed in the present approach, can be excluded.

In fact, we do not know if the symmetry of the similarity relation would actually matter. That is, we are not sure if the calculus presented here is not already complete also for the symmetric case. We are not able to provide an example to show the difference.

Another topic concerns proof-theory. This is an aspect that, according to our impression, has been largely neglected for logics of the type discussed here. However, if such logics are to be used for expert systems, the question of an automatic proof search, decidability and the like should be examined as well.

ACKNOWLEDGEMENTS

The author was partially supported by the Vienna Science and Technology Fund (WWTF) Grant MA07-016.

REFERENCES

[1] K.-P. Adlassnig and G. Kolarz, ‘Cadiag-2: Computer-assisted medical diagnosis using fuzzy subsets’, in *Approximate Reasoning in Decision Analysis*, eds., M. M. Gupta and E. Sanchez, 219–247, North-Holland Publ. Comp., Amsterdam, (1982).

[2] Didier Dubois, Francesc Esteva, Pere Garcia, Lluís Godo, and Henri Prade, ‘A logical approach to interpolation based on similarity relations’, *Int. J. Approx. Reasoning*, **17**(1), 1–36, (1997).

[3] Francesc Esteva, Pere Garcia, and Lluís Godo, 'Similarity-based reasoning', in *Discovering the World with Fuzzy Logic: Perspectives and Approaches to Formalization of Human-Consistent Logical Systems*, eds., Vilém Novák and Irina Perfilieva, 367–393, Springer (Physica-Verlag), Heidelberg, (2000).
[4] Francesc Esteva, Pere Garcia, Lluís Godo, and Ricardo Rodríguez, 'A modal account of similarity-based reasoning', *Int. J. Approx. Reasoning*, **16**(3-4), 235–260, (1997).
[5] Francesc Esteva, Lluís Godo, Ricardo O. Rodríguez, and Thomas Vetterlein, 'On the logics of similarity-based approximate and strong entailment', in *Proceedings of ESTYLF 2010*, (2010).
[6] Lluís Godo and Ricardo O. Rodríguez. Logical approaches to fuzzy similarity-based reasoning: an overview. Della Riccia, Giacomo (ed.) et al., Preferences and similarities. Lectures from the 8th international workshop of the international school for the synthesis of expert knowledge (ISSEK), Udine, Italy, October 5–7, 2006; Wien: Springer. CISM Courses and Lectures 504, 75-128 (2008), 2008.
[7] George Metcalfe, Nicola Olivetti, and Dov Gabbay, *Proof theory for fuzzy logics*, Applied Logic Series 36. Dordrecht: Springer, 2009.
[8] Ricardo O. Rodríguez, *Aspectos formales en el Razonamiento basado en Relaciones de Similitud Borrosas*, Ph.D. dissertation, Technical University of Catalonia (UPC), 2002.
[9] Enrique H. Ruspini, 'On the semantics of fuzzy logic', *Int. J. Approx. Reasoning*, **5**(1), 45–88, (1991).
[10] M. Sheremet, D. Tishkovsky, F. Wolter, and M. Zakharyaschev, 'A logic for concepts and similarity.', *J. Log. Comput.*, **17**(3), 415–452, (2007).
[11] Frank Wolter and Michael Zakharyaschev, 'A logic for metric and topology.', *J. Symb. Log.*, **70**(3), 795–828, (2005).
[12] Georg Henrik von Wright, *The Logic of Preference*, Edinburgh University Press, Edinburgh, 1963.