On the logics of similarity-based approximate and strong entailments

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Abstract

We consider two kinds of logics for approximate reasoning: one is weaker than classical logic and the other is stronger. In the first case, we are led by the principle that from given premises we can jump to conclusions which are only approximately (or possibly) correct. In the second case, which was not considered so far, in contrast, we follow the principle that conclusions must remain (necessarily) correct even if the premises are slightly changed. In this paper we recall the definitions and characterizations of the first logic, and we investigate the basic properties of the second logic, as well as its soundness and completeness with respect to Ruspini's semantics based on fuzzy similarity relations.

1 Introduction

In order to open up a topic for systematic investigation in a formal way, we are required to fix a framework of reference; we need to choose a model to which our considerations refer. Roughly speaking, we often proceed as follows. To discuss a topic means to discuss the relative or absolute properties of a manifold of different situations. To make the subject under consideration precise a particular set of characteristic properties needs to be selected. We are then led to a collection of situations which vary with respect to certain properties and are identified otherwise. The properties are in turn identified with the set of situations in which they hold.

To reason in this framework we may use classical propositional logic. In this logic we find what we need: yes-no propositions modeled by subsets, and connectives which are interpreted by set-theoretic operations like meet, join, and complement. Furthermore, implication corresponds to set-theoretic inclusion.

When using classical propositional logic for practical purposes the problem however might occur that different properties might not always be delimitable from each other in practice. This is in particular always the case if the universe of discourse results from an infinite process of stepwise refinement, in which case it has a continuous character. In fact, often we work in metric spaces like the reals. But then it might no longer be convenient to model implicational relationships as strict set-theoretic inclusion; the relation of inclusion is sensitive to arbitrary small changes. This is undesirable if bounds of propositions are given only roughly.

A possible solution is to make the relationship between properties and the pool of possible interpretations in a model more flexible. We may require for instance that inferences should not break down, but be tolerant, with regard to small changes. In this paper, we develop this idea in a specific way, continuing a series of earlier works.

We will actually propose two different ways to proceed and in order to illustrate why we do so, consider the following example, which is actually inappropriate because it refers to first-order logic but might serve for the illustrative purpose. Let a set of objects be distinguished by their sizes. We may consider two different kinds of relations, $\leq$ meaning "smaller or equal than" and $<$ meaning "strictly smaller than", in order to infer information about one object, say Q, from information about another one, say P. However, a statement like $a \leq b$, where a and b are the sizes of P and Q, respectively, might be practically undecidable if P and Q are similar. The principle of *approximate reasoning* introduces tolerance in the following way: letting $a \leq b$ still hold if a is slightly larger than b, but endowing this statement with a truth-like degree which is 1 if $a \leq b$ actually holds but decreases continuously to 0 when a becomes larger than b. The case of $<$ is, in a sense, dual. Here we may require that a is by a sufficient amount smaller than b so that we are able to make a clear statement. The principle of *strong reasoning* says that a statement must be tolerant, that is, remain valid, under small changes. It is assigned a truth degree as well, which is 1 if there is no tolerance, and the smaller the greater the tolerance is.

Logics of approximate reasoning have been studied quite intensively. The initial approach is due to Ruspini [6]. Approaches in a logical style have been developed in several papers of three of the authors and others [2, 3, 5, 4]. The most flexible approach is provided by multi-modal logics [3]. Here, the universe of discourse W is a similarity space and a modal operator $\Diamond_\alpha$ is added to the language of classical propositional logic (CPL) for each α from a fixed set C of truth degrees. $\Diamond_\alpha$ is interpreted by the α-neighborhoods U_α in W. Then the implication "from φ it follows ψ to the degree α" holds according to the principles of approximate reasoning if $[\varphi] \subseteq U_\alpha([\psi])$ where $[\varphi]$ and $[\psi]$ are the sets of interpretations satisfying φ and ψ respectively. In a similar way, this implication holds according to the principles of strong reasoning if $U_\alpha([\varphi]) \subseteq [\psi]$.

We see in particular that both our above concerns can be fulfilled in the multi-modal approach. Here however we first consider what is called the logic of approximate entailment, LAE, developed in [5]. This logic considers graded implications like $\varphi >_\alpha \psi$, which correspond to inclusions $[\varphi] \subseteq U_\alpha([\psi])$ when interpreted in the above logic. Furthermore, in complete analogy we develop the logic of strong entailment, denoted LSE. This logic deals with graded implications $\varphi \succ_\alpha \psi$,which correspond to inclusions $U_\alpha([\varphi]) \subseteq [\psi]$.

This paper is organized as follows. Section 2 contains necessary background on fuzzy similarity measures and their associated implication and consistency measures proposed by Ruspini [6]. In Section 3 we describe the approximate and strong similarity-based entailments and show their basic properties. In Section 4 we recall the logic LAE of approximate entailment, providing its syntax and semantics. Finally, in Section 6, we introduce a new logic for the strong entailment LSE and we propose an axiomatization. We finish with some remarks about future work.

2 Preliminaries: fuzzy similarity relations and Ruspini's measures

Let us consider a propositional language L built up from a finite set of propositional variables $p_1, \dots p_n$ and the *constants* $\top$ and $\bot$ by means of the binary operators $\wedge, \vee$ and the unary operator $\neg$. Propositions will be denoted by greek letters φ, ψ, ... The set of propositions will be denoted by $\mathcal{P}$ and the set of classical interpretations of $\mathcal{P}$ will be denoted by Ω, we will also call them possible worlds. We will use the expressions $w \models \varphi$ and $\varphi \models \psi$ to respectively denote that w satisfies φ (or that ω is a model φ) and that ψ is a logical consequence of φ in classical propositional logic. We will also denote by $[\varphi]$ the subset of interpretations of Ω that satisfy φ, i.e. $[\varphi] = \{\omega \in \Omega \mid \omega \models \varphi\}$.

Following Ruspini [6], the starting point of our framework is to assume that a possible world or state of a system may resemble more to some worlds than to another ones, and this basic fact may help us to evaluate to what extent a partial description (a proposition) may be close or similar to some other. This is modelled by assuming a fuzzy binary relation $S : \Omega \times \Omega \to [0, 1]$ on the set Ω of classical interpretations of $\mathcal{P}$ capturing a suitable notion of similarity is given. Usual properties that are considered in the literature for such fuzzy binary relations are:

Reflexivity: $S(u, u) = 1$ for all $u \in \Omega$
Separability: $S(u, v) = 1$ iff $u = v$
Symmetry: $S(u, v) = S(v, u)$, for all $u, v \in \Omega$
$\otimes$-Transivity: $S(u, v) \otimes S(v, w) \le S(u, w)$, for all $u, v, w \in \Omega$

where $\otimes$ is a t-norm. The reflexivity property establishes that the similarity degree of any world with itself has the highest value. Separability is a bit stronger since it forbids to have $S(u, v) = 1$ for $u \neq v$. Symmetry has a clear meaning, and $\otimes$-Transitivity is a relaxed form of transitivity since it establishes $S(u, v) \otimes S(v, w)$ as a lower bound for $S(u, w)$. Note that $S(u, v) = S(v, w) = 1$ implies $S(u, w) = 1$. Reflexive and symmetric fuzzy relations are often called *closeness* or *proximity* relations, while those further satisfying $\otimes$-transitivity are usually called *$\otimes$-transitive similarity relations* or *$\otimes$-indistinguishability operators*. In this paper we will use the term $\otimes$-similarity relations to refer to these class of relations, and sometimes we will also require them to be separable. Note that Zadeh called simply *similarity relations* to min-transitive similarity relations in the previous sense.

Dually, one can think of $1 - S$ as a kind of metric on worlds, indeed, when $\otimes$ is Łukasiewicz t-norm and S is a separating $\otimes$-transitive similarity relation, then $1 - S$ is a metric, c.f. [7].

When trying to extend the similarity on worlds to propositions, Ruspini defined in [6] the following two measures

$$I_S(\varphi \mid \psi) = \inf_{\omega \models \psi} \sup_{\omega' \models \varphi} S(\omega, \omega')$$

$$C_S(\varphi \mid \psi) = \sup_{\omega \models \psi} \sup_{\omega' \models \varphi} S(\omega, \omega')$$

which are the lower and upper bounds respectively of the resemblance or proximity degree between φ and ψ. Indeed, I_S is an implication (i.e. inclusion-like) measure, while C_S is a consistency (i.e. intersection-like) measure and thus it is symmetric.

The value of $I_S(\varphi \mid \psi)$ provides the measure to what extent φ is close to be true given ψ for granted and the similarity between worlds represented by S. In particular, when S is separating and the set of worlds is finite then, $I_S(\varphi \mid \psi) = 1$ iff $\psi \models \varphi$. Moreover, if S is $\otimes$-transitive then I_S is $\otimes$-transitive as well [6], i.e. the inequality

$$I_S(\chi \mid \varphi) \otimes I_S(\varphi \mid \psi) \leq I_S(\chi \mid \psi)$$

holds for any propositions φ, ψ and χ, capturing a form of generalised modus ponens.

On the other hand, the value of $C_S(\varphi \mid \psi)$ provides the measure of what extent φ can be considered compatible with the available knowledge ψ. In particular, in the finite case and with S satisfying separation property, $C_S(\varphi \mid \psi) = 1$ iff $\psi \not\models \neg\varphi$. Observe that, when the propositional language is finitely generated and ψ is equivalent to a maximal consistent set of propositions, both measures coincide because there is a unique world w such that $w \models \psi$. In addition, it is easy to show that, given a fixed χ, the measure $C_S(\cdot \mid \chi)$ is a possibility measure [2] since the following identities hold true:

1. $C_S(\top \mid \chi) = 1$
2. $C_S(\bot \mid \chi) = 0$
3. $C_S(\varphi \vee \psi \mid \chi) = \max(C_S(\varphi \mid \chi), C_S(\psi \mid \chi))$.

By duality, one can define the following similarity-based measure N_S as follows:

$$N_S(\psi \mid \varphi) = 1 - C_S(\neg\psi \mid \varphi) = \inf_{w \models \neg\psi} \inf_{w' \models \varphi} 1 - S(w, w').$$

The following are relevant properties of N_S measures which will be used in the next section and which can be easily verified:

1. $N_S(\cdot \mid \varphi)$ is a necessity measure for each p such that $\varphi \not\models \bot$, i.e.
   - $N_S(\top \mid \varphi) = 1$
   - $N_S(\bot \mid \varphi) = 0$
   - $N_S(\psi \wedge \chi \mid \varphi) = \min(N_S(\psi \mid \varphi), N_S(\chi \mid \varphi))$
2. $N_S(\psi \mid \varphi \vee \chi) = \min(N_S(\psi \mid \varphi), N_S(\psi \mid \chi))$
3. $N_S(\psi \mid \varphi) = N_S(\neg\varphi \mid \neg\psi)$
4. if $\varphi \models \psi$ then $N_S(\varphi \mid \chi) \leq N_S(\psi \mid \chi)$ and $N_S(\chi \mid \psi) \leq N_S(\chi \mid \varphi)$
5. if $N_S(\psi \mid \varphi) > 0$ then $\varphi \models \psi$

3 Approximate and strong entailment relations

Given a $\otimes$-similarity relation on Ω, in this section we will first recall from [5] a corresponding family of graded approximate entailment relations $\models_S^\alpha$ and then we will introduce a family of strong entailment relations $|\!\approx_S^\alpha$, both indexed by values $\alpha \in [0, 1]$, and being respectively weaker and stronger notions than the classical logical entailment $\models$.

A (graded) approximate satisfaction relation $\models_S^\alpha \subset \Omega \times \mathcal{P}$, for each $\alpha \in [0, 1]$ by

$\omega \models_S^\alpha \varphi$ iff there exists a model ω' of φ which is α-similar to ω, i.e. $S(\omega, \omega') \geq \alpha$

The approximate satisfaction relation can be extended in the natural way over to an *approximate entailment* relation $\models_S^\alpha \subseteq \mathcal{P} \times \mathcal{P}$: a proposition φ entails a proposition ψ at degree α, written $\varphi \models^\alpha \psi$, if each model of φ is an approximate model of ψ at level α, that is,

$$\varphi \models_S^\alpha \psi \quad \text{iff } w \models_S^\alpha \psi \text{ for all model } w \text{ of } \varphi$$

It is easy to check that $\varphi \models_S^\alpha \psi$ holds iff any of the following equivalent conditions hold:

- $I_S(\psi \mid \varphi) \geq \alpha$
- $[\varphi] \subseteq U_\alpha([\psi])$

where $U_\alpha([\psi]) \subseteq \Omega$ is the neighborhood of radius α of the set of models of ψ, that is, $U_\alpha([\psi]) = \{w \in \Omega \mid$ there exists w' s.t. $w' \models \psi$ and $S(w', w) \geq \alpha\}$.

The characterizing properties of this graded entailment relation are (the reader is referred to [3] for further details):

(1) **Nestedness:** if $\varphi \models_S^\alpha \psi$ and $\beta \leq \alpha$ then $\varphi \models_S^\beta \psi$;

(2) **$\otimes$-Transitivity:** if $\varphi \models_S^\alpha \chi$ and $\chi \models_S^\beta \psi$ then $\varphi \models_S^{\alpha \otimes \beta} \psi$;

(3) **Reflexivity:** $\varphi \models_S^1 \varphi$;

(4) **Right weakening:** if $\varphi \models_S^\alpha \psi$ and $\psi \models \chi$ then $\varphi \models_S^\alpha \chi$;

(5) **Left strengthening:** if $\varphi \models \chi$ and $\chi \models_S^\alpha \psi$ then $\varphi \models_S^\alpha \psi$;

(6) **Left OR:** $\varphi \vee \chi \models_S^\alpha \psi$ iff $\varphi \models_S^\alpha \psi$ and $\chi \models_S^\alpha \psi$;

(7) **Right OR:** if χ has a single model, $\chi \models_S^\alpha \varphi \vee \psi$ iff $\chi \models_S^\alpha \varphi$ or $\chi \models_S^\alpha \psi$.

Now we are interested in another kind of graded entailment, a *strong entailment*, which is in a sense dual to the above approximate entailment. We define a (graded) satisfaction relation $\approx\!\!\!|_S^\alpha \subseteq \Omega \times \mathcal{P}$, for each $\alpha \in [0,1]$ by

$$\omega \approx\!\!\!|_S^\alpha \varphi \quad \text{iff } \omega' \models \varphi \text{ for each } \omega' \text{ s.t. } S(\omega, \omega') \geq \alpha$$

that is, if any model w' in the neighborhood (of radius α) of w is a model of φ. If $\omega \approx\!\!\!|_S^\alpha \varphi$ we say that w is an *strong model* (at level α) of φ. The natural corresponding entailment relation comes defined in the following way: a proposition φ strongly entails a proposition ψ at degree α, written $\varphi \mid\!\approx_S^\alpha \psi$, if each approximate model of φ at level α is a model of ψ that is,

$$\varphi \approx\!\!\!|_S^\alpha \psi \text{ iff, for all } w,\ w \models_S^\alpha \varphi \text{ implies } w \models \psi$$

An equivalent definition, making use of the neighborhoods, would simply be

$$\varphi \approx\!\!\!|_S^\alpha \psi \text{ iff } U_\alpha([\varphi]) \subseteq [\psi]$$

Moreover, in a similar way the approximate entailment was linked to the implication measure I_S, this strong graded entailment is related to the consistency measure C_S, or equivalently, to the necessity measure N_S. Indeed, the following proposition relate in a precise way the strong entailment $\approx\!\!\!|_S^\alpha$ and the N_S measure.

Proposition 1 *Assume the language is finitely generated and let $\alpha > 0$. Then $\varphi \approx\!\!\!|_S^\alpha \psi$ iff $N_S(\psi \mid \varphi) > 1 - \alpha$.*

Proof: $\rightarrow$) If $\varphi \mid\!\approx_S^\alpha \psi$, it means that if w' is such that $S(w, w') \geq \alpha$ for some $w \models \varphi$, then $w' \models \psi$. Therefore, if $w' \not\models \psi$ then $S(w, w') < \alpha$ for all $w \models \varphi$, i.e. $\max_{w \models \varphi} S(w, w') < \alpha$ for all $w' \models \neg\psi$, i.e. $\max_{w' \models \neg\psi} \max_{w \models \varphi} S(w, w') < \alpha$, i.e. $C(\neg\psi \mid \varphi) < \alpha$.

$\leftarrow$) $N_S(\psi \mid \varphi) > 1 - \alpha$ amounts to $C(\neg\psi \mid \varphi) < \alpha$, i.e. for all $w \not\models \psi$ and for all $w' \models \varphi$ we have $S(w, w') < \alpha$. So, if $w \models \varphi$ is such that $S(w, w') \geq \alpha$ for some other world w' it must necessarily be $w' \models \psi$, hence $\varphi \approx\!\!\!|_S^\alpha \psi$. $\square$

Main properties of this strong graded entailment relation induced by a $\otimes$-similarity measure S are:

(1) **Nestedness:**
if $\varphi \approx\!\!\!|_S^\alpha \psi$ and $\beta \geq \alpha$ then $\varphi \approx\!\!\!|_S^\beta \psi$;
(2) $\varphi \approx\!\!\!|_S^0 \psi$ iff either $\models \neg\varphi$ or $\models \psi$
(3) $\varphi \approx\!\!\!|_S^1 \psi$ iff $\varphi \models \psi$
(4) **min-Transitivity:**
if $\varphi \approx\!\!\!|_S^\alpha \psi$ and $\psi \approx\!\!\!|_S^\beta \chi$ then $\varphi \approx\!\!\!|_S^{\min(\alpha,\beta)} \chi$;
(5) **Left OR:**
$\varphi \vee \chi \approx\!\!\!|_S^\alpha \psi$ iff $\varphi \approx\!\!\!|_S^\alpha \psi$ and $\chi \approx\!\!\!|_S^\alpha \psi$;
(6) **Right AND:**
$\chi \approx\!\!\!|_S^\alpha \varphi \wedge \psi$ iff $\chi \approx\!\!\!|_S^\alpha \varphi$ and $\chi \approx\!\!\!|_S^\alpha \psi$.
(7) **Contraposition:**
If $\varphi \approx\!\!\!|_S^\alpha \psi$ then $\neg\psi \approx\!\!\!|_S^\alpha \neg\varphi$
(8) **Restricted $\otimes$-Transitivity:**
If φ, ψ, χ have a single model then
if $\varphi \approx\!\!\!|_S^{\alpha\otimes\beta} \neg\psi$ then either $\varphi \approx\!\!\!|_S^\alpha \neg\chi$ or $\chi \approx\!\!\!|_S^\beta \neg\psi$

Two interesting derived properties from (3) and (4) are:

Right weakening:
if $\varphi \approx\!\!\!|_S^\alpha \psi$ and $\psi \models \chi$ then $\varphi \approx\!\!\!|_S^\alpha \chi$;
Left strengthening:
if $\varphi \models \chi$ and $\chi \approx\!\!\!|_S^\alpha \psi$ then $\varphi \approx\!\!\!|_S^\alpha \psi$;

Indeed one can show that these properties characterize these graded strong entailments, but the proof is not included for space reasons.

4 A logic of approximate entailment

In [5], it has been developed a graded conditional logic approach, that we will call LAE, to encode in the language syntactical objects representing approximate entailments $\varphi \models_S^\alpha \psi$. To do so, binary (graded) modal operators are introduced (under some restrictions, e. g. nested modal formulas are not allowed, and the language is finitely generated) and appropriate semantics in terms of similarity Kripke structures are given.

Through the rest of the paper, let us fix a countable set $C \subset [0,1]$ of *similarity degrees*; we require that C contains 0 and 1 and that for any non-zero $\alpha \in C$ there is a largest $\beta < \alpha$ in C. Furthermore, let us fix an operation $\otimes : C \times C \rightarrow C$ which is commutative, associative, in both arguments isotone, and has 1 as its neutral element.

The propositional language of LAE results from extending the propositional language $\mathcal{P}$ introduced in Section 2 with a family $\{>_\alpha\}_{\alpha \in C}$ of binary operators. *Conditional formulas* are built as follows:

- If $\varphi \in \mathcal{P}$ then φ is also a conditional formula.
- If $\varphi, \psi \in \mathcal{P}$ then, for every $\alpha \in C$, $\varphi >_\alpha \psi$ is an *atomic* conditional formula.
- If φ and ψ are conditional formulas then $\neg\varphi$ and $\varphi \rightarrow \psi$ are conditional formulas

Note that in this language, nested conditional formulas are not allowed.

Semantics are specified by $\otimes$-similarity Kripke models $\mathcal{M} = (W, S, e)$, where W ia s set of worlds, $S : W \times W \rightarrow C$ is a $\otimes$-similarity on W and $e : W \times \mathcal{P} \rightarrow \{0,1\}$ is such that $e(w, \cdot)$ is a usual Boolean interpretation of propositions of $\mathcal{P}$, with the extra condition that $e(w, \cdot) \neq e(w', \cdot)$ when $w \neq w'$. Then we define the satisfaction relation by stipulating $(\mathcal{M}, \omega) \models \varphi$ if

$e(w,\varphi) = 1$ for $\varphi \in \mathcal{P}$, and is extended to atomic conditional formulas by defining

$$(\mathcal{M},\omega) \models \varphi >_\alpha \psi \quad \text{if} \quad [\varphi] \subseteq U_\alpha([\psi]) \ ,$$

or equivalently, if $I_S(\psi \mid \varphi) \geq \alpha$, where now U_α refers to neighborhoods in W (i.e. $[\psi] = \{w \in W \mid e(w,\varphi) = 1\}$). The rest of the conditions for compound conditional formulas are the usual ones. Note that the notion of satisfiability for $>_\alpha$ is independent of any particular world, i. e. it is a global notion. The condition of satisfiability makes clear that in the object language $\varphi >_\alpha \psi$ represents lower bounds of $I_S(\psi \mid \varphi)$.

The axioms and rules of LAE are those of Classical Propositional logic (CPL) plus the following ones, where α and β represent any values of C: [1]

(N)	$\varphi >_\alpha \psi \to \varphi >_\beta \psi$ if $\beta \leq \alpha$
(CS)	$\varphi >_1 \psi \to (\varphi \to \psi)$
(EX)	$\varphi >_0 \psi$
(B)	$\chi >_\alpha \chi' \to \chi' >_\alpha \chi$, if χ and χ' are m.e.c.'s
(4)	$(\varphi >_\alpha \psi) \wedge (\psi >_\beta \chi) \to \varphi >_{\alpha\otimes\beta} \chi$
(LO)	$(\varphi \vee \psi >_\alpha \chi) \leftrightarrow (\varphi >_\alpha \chi) \wedge (\psi >_\alpha \chi)$
(RO)	$(\chi >_\alpha \varphi \vee \psi) \leftrightarrow (\chi >_\alpha \varphi) \vee (\chi >_\alpha \psi)$, if χ is a m.e.c.

and the following inference rule:

(RK) From $\varphi \to \psi$ infer $\varphi >_1 \psi$

In [5], the author proves, among other results, that LAE is complete with respect to the class of similarity models (W,S) where S is a separating $\otimes$-similarity, when C is finite. Moreover, if the axiom (CS) is dropped, then one gets completeness w.r.t. similarity models where the similarity relation is not necessarily separating.

Notice that a very related approach using metrics instead of similarities has been proposed in [8].

5 Logic of strong entailment

Let us define the logic LSE in a similar way, but not exactly, as we did for the logic LAE. Formulas of LSE are built from propositions of $\mathcal{P}$ by introducing new *graded implications* which are a triples consisting of two propositions $\varphi,\psi \in \mathcal{P}$ and a value $\alpha \in C$, denoted

$$\varphi \succ_\alpha \psi.$$

[1] In these axioms we use m.e.c. to denote a maximal elementary conjunction, that is, a conjunction where each propositional variable appears either in positive or negative form (remember that we are assuming a finite set of propositional variables).

A *statement* of LSE is built up from graded implications of LSE by means of the binary operators $\wedge, \vee$ and the unary operator $\neg$. The additional operator $\to$ will be used as an abbreviation for $\neg\cdot\vee\cdot$.

Semantics for LSE are given by Kripke models $\mathcal{M} = (W,S,e)$, where W is a set of worlds, $e : W \times \mathcal{P} \to \{0,1\}$ is such that, for all $w \in W$, $e(w,\cdot)$ is a Boolean interpretation of propositional formulas of $\mathcal{P}$, and $S : W \times W \to C$ is a *separating symmetric* fuzzy relation over possible worlds valued on C. The satisfaction of a graded implication $\varphi \succ_\alpha \psi$ in a model $\mathcal{M} = (W,S,e)$, is defined as

$$\mathcal{M} \models \varphi \succ_\alpha \psi \quad \text{if} \quad U_\alpha([\varphi]) \subseteq [\psi] \ ,$$

where again $U_\alpha([\varphi]) = \{w \in W \mid S(w',w) \geq \alpha \text{ for some } w' \in W \text{ s.t. } e(w',\varphi) = 1\}$. According to Proposition 1, when $\alpha > 0$, $\mathcal{M} \models \varphi \succ_\alpha \psi$ iff $N_S(\psi \mid \varphi) > 1-\alpha$. The satisfaction of statements is defined classically from the satisfaction of graded implications.

Finally, a *theory* of LSE is a finite set of statements of LSE. We say that a theory $\mathcal{T}$ *semantically entails* a statement Φ, written $\mathcal{T} \models_{\mathrm{LSE}} \Phi$, if Φ is satisfied by any model satisfying every element of $\mathcal{T}$.

We shall axiomatise the logic LSE in the following way.

Definition 1 *The following graded implications are axioms of* LSE *for any $\varphi,\psi,\chi \in \mathcal{P}$ and $\alpha,\beta \in C$:*

(A1) $\bot \succ_0 \varphi$
(A2) $\varphi \succ_1 \psi$ *where φ,ψ are such that $\varphi \to \psi$ is a tautology of* CPL
(A3) $\varphi \succ_0 \top$
(A4) $(\varphi \succ_\alpha \psi) \wedge (\varphi \succ_\alpha \chi) \to (\varphi \succ_\alpha \psi \wedge \chi)$
(A5) $(\varphi \succ_\alpha \chi) \wedge (\psi \succ_\alpha \chi) \to (\varphi \vee \psi \succ_\alpha \chi)$
(A6) $(\varphi \succ_\alpha \psi) \to (\neg\psi \succ_\alpha \neg\varphi)$
(A7) $(\varphi \succ_\beta \psi) \wedge (\psi \succ_\alpha \chi) \to (\varphi \succ_{\min(\beta,\alpha)} \chi)$
(A8) $(\varphi \succ_\beta \psi) \to (\varphi \succ_\alpha \psi)$ *where $\alpha \geq \beta$*
(A9) $(\varphi \succ_0 \psi) \to (\varphi \succ_1 \bot) \vee (\top \succ_1 \psi)$

In addition, given a tautology of CPL*, the statement resulting from a uniform replacement of the atoms by graded implications of* LSE *is an axiom. Moreover, the following is a rule of* LSE *for any statements Φ,Ψ:*

(MP) *from Φ and $\Phi \to \Psi$ derive Ψ*

A proof of a statement Φ from a theory $\mathcal{T}$ is defined as usual. If it exists, we write $\mathcal{T} \vdash_{\mathrm{LSE}} \Phi$.

Theorem 1 (Completeness) *Let $\mathcal{T}$ be a consistent theory of* LSE*, and let Φ be a statement. Then $\mathcal{T} \vdash_{\mathrm{LSE}} \Phi$ if and only if $\mathcal{T} \models_{\mathrm{LSE}} \Phi$.*

Proof: The "only if" part is evident. As for the "if" part, notice first that it is easy to check that $\mathcal{T} \vdash_{\text{LSE}} \Phi$ iff $\mathcal{T} \cup AX^*$ proves Φ just using the axioms and rules of classical propositional logic (using graded implications as propositional variables), written $AX^* \cup \mathcal{T} \vdash_{\mathsf{CPL}} \Phi$, where AX^* is the set of instances of the LSE axioms.

Assume that $\mathcal{T} \not\vdash_{\text{LSE}} \Phi$. Then, there is a $\{0,1\}$-evaluation which is a model of $AX^* \cup \mathcal{T}$ and $v(\Phi) = 0$. We are going to define a similarity-Kripke model $M = (W, S, e)$ such that M is a model of $\mathcal{T}$ but not of Φ. We take $W = \Omega$, the set of propositional models of $\mathcal{P}$ and we define $S : \Omega \times \Omega \to [0,1]$ as

$$S(w, w') = \max\{t \in C \mid v(\underline{w} \succ_t \neg \underline{w'}) = 0\}$$

where, for each $w \in \Omega$, $\underline{w}$ denotes the proposition whose only model is w. Actually, $\underline{w}$ always exists since we are assuming that the propositional language $\mathcal{P}$ is generated by a finite set of variables. For each $\varphi \in \mathcal{P}$, define $[\varphi] = \{w \in \Omega \mid w \models \varphi\}$.

Claim 1: Let $t > 0$. Then, $v(\varphi \succ_t \psi) = 1$ iff for all $w, w' \in W$ if $w \models \varphi$ and $w' \models \neg\psi$ then $S(w, w') < t$.

Proof: Since φ and ψ are logically equivalent to $\bigvee_{w \models \varphi} \underline{w}$ and to $\bigwedge_{w' \models \neg\psi} \neg \underline{w'}$ and v satisfies all the axiom instances in AX^*, we have that $v(\varphi \succ_t \psi) = 1$ iff $v(\bigvee_{w \models \varphi} \underline{w} \succ_t \bigwedge_{w' \models \neg\psi} \neg \underline{w'}) = 1$ iff $v(\underline{w} \succ_t \neg \underline{w'}) = 1$ for all $w, w' \in W$ such that $w \models \varphi$ and $w' \models \neg\psi$. But according to the definition of S above, $v(\underline{w} \succ_t \neg \underline{w'}) = 1$ iff $S(w, w') < t$. □

Claim 2: $v(\varphi \succ_t \psi) = 1$ iff $U_t([\varphi]) \subseteq [\psi]$, where U_t is defined w.r.t. S.

Proof: Assume $t > 0$. Then, by Claim 1, $v(\varphi \succ_t \psi) = 1$ iff for all $w, w' \in W$ if $w \models \varphi$ and $w' \models \neg\psi$ then $S(w, w') < t$, which is equivalent to the condition that for all $w, w' \in W$ if $w \models \varphi$ and $S(w, w') \geq t$ then $w' \models \psi$, that is, $U_t([\varphi]) \subseteq [\psi]$.

If $t = 0$, by axiom (A9), $v(\varphi \succ_0 \psi) = 1$ iff either $v(\varphi \succ_1 \bot) = 1$ or $v(\top \succ_1 \psi) = 1$, that is, iff either $[\varphi] = U_1([\varphi]) = \emptyset$ or $\Omega = U_1([\top]) \subseteq [\psi]$. So, noticing that $U_0(\emptyset) = \emptyset$, we have that in both cases $U_0([\varphi]) \subseteq [\psi]$. Conversely, if $U_0([\varphi]) \subseteq [\psi]$ and $[\varphi] \neq \emptyset$, then $\Omega = U_0([\varphi]) \subseteq [\psi]$. □

Finally, define $M = (\Omega, S, e)$ where e is defined as $e(w, \varphi) = 1$ if $w \models \varphi$, $e(w, \varphi) = 0$ otherwise, for each $\varphi \in \mathcal{P}$, and the satisfaction condition for graded implications in M as expected:

$$M \models \varphi \succ_t \psi \text{ iff } U_t([\varphi]) \subseteq [\psi] \text{ iff } v(\varphi \succ_t \psi) = 1$$

Therefore, from Claim 2, it is clear that $M \models \Psi$ for all $\Psi \in \mathcal{T}$ but $M \not\models \Phi$, hence $\mathcal{T} \not\models_{LSE} \Phi$. □

6 Conclusions and future work

In this paper we have been concerned with two notions of graded similarity-based entailment arising in the framework of practical reasoning when looking for robust inferences in the presence of small variations either in the premise or in the conclusion[2]. The main contribution of the paper is the study and axiomatization of the so-called logic of strong entailment LSE, which incorporates at the object level constructs as $\varphi \succ_\alpha \psi$ capturing the notion that ψ is a strong consequence (at degree α) of φ. We have shown soundness and completeness results for special kinds of derivations in such a logic but a lot of work remains to be done to fully exploit this logical framework.

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[2] Analogous notions of approximate and strong entailments correspond in the field of mathematical morphology to the so-called dilation and erosion operators, see e.g. [1].