A Resolution Mechanism for Prenex Gödel Logic

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Abstract. First order Gödel logic $\mathbf{G}_\infty^\triangle$, enriched with the projection operator $\triangle$—in contrast to other important t-norm based fuzzy logics, like Łukasiewicz and Product logic—is well known to be recursively axiomatizable. However, unlike in classical logic, testing (1-)unsatisfiability, i.e., checking whether a formula has no interpretation that assigns the designated truth value 1 to it, cannot be straightforwardly reduced to testing validity. We investigate the prenex fragment of $\mathbf{G}_\infty^\triangle$ and show that, although standard Skolemization does not preserve 1-satisfiability, a specific Skolem form for satisfiability can be computed nevertheless. In a further step an efficient translation to a particular structural clause form is introduced. Finally, an adaption of a chaining calculus is shown to provide a basis for efficient, resolution style theorem proving.

1 Introduction

Gödel logic is a prominent example of a t-norm based fuzzy logic [12], distinguished by the fact that validity and satisfiability depend only on the relative order of truth values of atomic formulas. It is sometimes also called intuitionistic fuzzy logic, following [18]. The importance of this logic is emphasized by the fact that one may arrive at it by different routes. Already Gödel [11] had introduced the truth tables for what is now called the family of propositional finite valued Gödel logics. Dummett [8] later generalized these to an infinite set of truth values and demonstrated that the set of corresponding tautologies is axiomatized by intuitionistic logic extended by the linearity axiom $(A \to B) \vee (B \to A)$; hence the alternative name Dummett's **LC** or Gödel-Dummett logic. On the first order level, different Gödel logics arise from differently ordered set of truth values (see, e.g., [2]). Here, we will deal with *standard first order Gödel logic* $\mathbf{G}_\infty$, where the truth values set is the real closed unit interval $[0,1]$ in its natural order. In fact we will focus on the natural extension $\mathbf{G}_\infty^\triangle$ that arises from $\mathbf{G}_\infty$ by adding the unary propositional connective $\triangle$ that maps all formulas to 0 that do not receive the designated value 1. Unlike other important fuzzy logics defined over $[0,1]$, including Łukasiewicz logic and Product logic (see, e.g., [12]), that are not recursively axiomatizable, validity for $\mathbf{G}_\infty$ and $\mathbf{G}_\infty^\triangle$ is Σ_1^0-complete.

While, besides Hilbert type systems, also cut-free Gentzen type systems are complete for $\mathbf{G}_\infty^\triangle$, none of these systems provides a suitable basis for automated deduction. In [5] it has been shown that the *prenex fragment* of $\mathbf{G}_\infty^\triangle$ admits

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a Herbrand theorem and thus Skolemization. This in turn allows to translate prenex formulas of $\mathbf{G}_\infty^\triangle$ into a special clause form, based on the natural order relations $<$ and $\leq$. By adapting a so-called chaining calculus [7, 6] an efficient resolution style mechanism for testing *validity* of prenex $\mathbf{G}_\infty^\triangle$-formulas is obtained. We call a formula A valid if $\|A\|_\mathcal{I} = 1$ for all interpretations $\mathcal{I}$. If $\|A\|_\mathcal{I} = 1$ for some interpretation $\mathcal{I}$ then A is satisfiable, or more precisely 1-satisfiable, otherwise A is called 1-unsatisfiable. It is important to keep in mind that, unlike in classical logic, *validity* and *satisfiability* are not dual to each other in most fuzzy logics. (Compare the case of monadic Łukasiwicz logic, for which the satisfiability problem is known to be Π_1^0-complete, but where the decidability of the validity is a well known open problem, see [16, 17, 15].) In fact, for Gödel logic *without* $\triangle$ satisfiability of a formula A *is* equivalent to the non-validity of $\neg A$ (see [13]). However for $\mathbf{G}_\infty^\triangle$ the duality vanishes: e.g., $\neg(B \wedge \neg\triangle B)$ is not valid, although $B \wedge \neg\triangle B$ is unsatisfiable. Therefore, if we are interested in testing whether a given formula A with occurrences of $\triangle$ has a model, we cannot rely on a mechanism for testing whether $\neg A$ is valid. Indeed, quite differently from classical logic, no direct and efficient reduction of the satisfiability problem to the validity problem, or *vice versa*, is known for $\mathbf{G}_\infty^\triangle$ and for other standard fuzzy logics.

We argue that it is in fact more important to be able to test whether a given specification expressed by a $\mathbf{G}_\infty^\triangle$-formula A is coherent in the sense of admitting a model, i.e., an interpretation that assigns 1 to A, than to have a procedure for testing whether $(\neg)A$ is valid. In this paper we address the former problem. Like in [5] we focus on the prenex fragment of $\mathbf{G}_\infty^\triangle$. It is shown in [5] that Skolemization preserves validity for prenex $\mathbf{G}_\infty^\triangle$-formulas. In contrast, we will observe that satisfiability is not preserved by standard Skolemization. We overcome this problem by defining a special alternative form of Skolemization that, in addition to Skolem terms, introduces a fresh monadic predicate symbol.

The central contribution of the paper consists in showing that any conjunction A of prenex formulas of $\mathbf{G}_\infty^\triangle$ can indeed be translated into a purely universal form A' that is equivalent to A with respect to 1-satisfiability. Similarly to [5] we then translate A' into a clausal form that is based on the underlying order relation. Once more we do not simply re-use results from [5], but rather present a more efficient version of a definitional clause form. The final part of the suggested deduction mechanism can then be directly transfered from [5]: a so-called chaining calculus can be straightforwardly adapted to test unsatisfiability. To achieve a self contained presentation, that does not rely on familiarity with Gödel logic or with chaining calculi we will explicitly specify all relevant schemes and inference rules.

The paper is organized as follows. After clarifying basic notions about Gödel logic in Section 2, a satisfiability preserving Skolemization operator is defined and investigated in Section 3. In Section 4 we present so-called chain normal forms. In particular we show how to translate arbitrary universal formulas into sets of clauses where the literals express basic order relations between atomic formulas. Section 5 describes a refutationally complete set of inference rules that can be applied to test unsatisfiability efficiently. Section 6 summarizes the results indicates related problems.

2 Basic notions and facts

Kurt Gödel [11] has introduced the following truth functions for conjunction, disjunction, and implication

$$\|A \wedge B\|_{\mathcal{I}} = \min(\|A\|_{\mathcal{I}}, \|B\|_{\mathcal{I}}), \qquad \|A \vee B\|_{\mathcal{I}} = \max(\|A\|_{\mathcal{I}}, \|B\|_{\mathcal{I}}),$$

$$\|A \to B\|_{\mathcal{I}} = \begin{cases} 1 & \text{if } \|A\|_{\mathcal{I}} \leq \|B\|_{\mathcal{I}} \\ \|B\|_{\mathcal{I}} & \text{otherwise.} \end{cases}$$

While Gödel referred to a finite set of values, we consider so-called standard Gödel logic $\mathbf{G}_\infty$ [13, 14], where the set of truth values is the real unit interval $[0, 1]$. The propositional constants $\bot$ and $\top$ are specified by $\|\bot\|_{\mathcal{I}} = 0$ and $\|\top\|_{\mathcal{I}} = 1$, respectively; $\neg A$ abbreviates $A \to \bot$ and $A \leftrightarrow B$ abbreviates $(A \to B) \wedge (B \to A)$. Therefore

$$\|\neg A\|_{\mathcal{I}} = \begin{cases} 1 & \text{if } \|A\|_{\mathcal{I}} = 0 \\ 0 & \text{otherwise} \end{cases} \quad \|A \leftrightarrow B\|_{\mathcal{I}} = \begin{cases} 1 & \text{if } \|A\|_{\mathcal{I}} = \|B\|_{\mathcal{I}} \\ \min(\|A\|_{\mathcal{I}}, \|B\|_{\mathcal{I}}) & \text{otherwise.} \end{cases}$$

Following [1] we enrich this set of connectives by adding the unary operator $\triangle$ with the following meaning:

$$\|\triangle A\|_{\mathcal{I}} = \begin{cases} 1 & \text{if } \|A\|_{I} = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\triangle$ allows to embed classical logic immediately. For propositional logic a *(standard) interpretation* $\mathcal{I}$ is simply an assignment of values in $[0, 1]$ to propositional variables. In first order logic atomic formulas (rather than $\top$ and $\bot$) are of the form $P(t_1, \dots, t_n)$, where P is a predicate symbol and $t_1, \dots, t_n$ are terms, where terms are built up from (object) variables and constant symbols using function symbols, as usual. An interpretation $\mathcal{I}$ now consists of a nonempty *domain* D and a *signature interpretation* $v_{\mathcal{I}}$ that maps constant symbols and object variables to elements of D, as well as every n-ary predicate symbol f to a function $v_{\mathcal{I}}(f)$ of type $D^n \mapsto D$. $v_{\mathcal{I}}$ homomorphically extends to arbitrary terms, as usual. Moreover, $v_{\mathcal{I}}$ maps every n-ary predicate symbol P to a function $v_{\mathcal{I}}(P)$ of type $D^n \mapsto V$. The truth value of an atomic formula $P(t_1, \dots, t_n)$ is thus defined as

$$\|P(t_1, \dots, t_n)\|_{\mathcal{I}} = v_{\mathcal{I}}(P)(v_{\mathcal{I}}(t_1), \dots, v_{\mathcal{I}}(t_n)).$$

For quantification we define the *distribution* of a formula A with respect to a free variable x in an interpretation $\mathcal{I}$ as $\mathrm{distr}_{\mathcal{I}}(A(x)) = \{\|A(x)\|_{\mathcal{I}[d/x]} \mid d \in D\}$, where $\mathcal{I}[d/x]$ denotes the interpretation that is exactly as $\mathcal{I}$, except for insisting on $v_{\mathcal{I}[d/x]}(x) = d$. (Similarly we will use $\mathcal{I}[\overline{d}/\overline{x}]$ for the interpretation arising form $\mathcal{I}$ by assigning for all $1 \leq i \leq n$ the domain element d_i in $\overline{d} = d_1, \ldots, d_n$ to the variable x_i in $\overline{x} = x_1, \ldots, x_n$.) The universal and existential quantifiers correspond to the infimum and supremum, respectively, in the following sense:

$$\|\forall x A(x)\|_{\mathcal{I}} = \inf \mathrm{distr}_{\mathcal{I}}(A(x)) \qquad \|\exists x A(x)\|_{\mathcal{I}} = \sup \mathrm{distr}_{\mathcal{I}}(A(x)).$$

By $\mathbf{G}_\infty^\triangle$ we mean the just defined *(standard) first order Gödel logic with* $\triangle$. A $\mathbf{G}_\infty^\triangle$-formula A is *valid* if $\|A\|_{\mathcal{I}} = 1$ for all interpretations $\mathcal{I}$; A is *1-satisfiable* if there exists an interpretation $\mathcal{I}$ such that $\|A\|_{\mathcal{I}} = 1$. The set of all 1-satisfiable formulas will be denoted by 1SAT. As already mentioned in the introduction, 1-satisfiability is not dual to non-validity in $\mathbf{G}_\infty^\triangle$. But as for $\mathbf{G}_\infty$, both problems are Π_1^0-complete according to [13, 14].

Like in intuitionistic logic, also in Gödel logic (with or without $\triangle$) quantifiers cannot be shifted arbitrarily. While we have

- $\models_{\mathbf{G}_\infty^\triangle} \exists x(A \to B) \to (\forall x A \to B)$ and
- $\models_{\mathbf{G}_\infty^\triangle} \exists x(B \to A) \to (B \to \exists x A)$,

the converse implications are not valid. As a consequence arbitrary formulas are not equivalent to prenex formulas, in general. Nevertheless the prenex fragment of $\mathbf{G}_\infty^\triangle$ is quite expressive. E.g., classical logic, where formulas can be reduced to prenex form without loss of generality, can straightforwardly be embedded into prenex $\mathbf{G}_\infty^\triangle$ using $\triangle$ as indicated above. We list a few further valid schemes of $\mathbf{G}_\infty^\triangle$ that will be used in later sections.

- $\models_{\mathbf{G}_\infty^\triangle} \triangle A \to A$
- $\models_{\mathbf{G}_\infty^\triangle} ((A \wedge B) \vee C) \leftrightarrow (A \vee C) \wedge (B \vee C))$
- $\models_{\mathbf{G}_\infty^\triangle} \triangle(A \to B) \vee \neg\triangle(A \to B)$
- $\models_{\mathbf{G}_\infty^\triangle} A(t) \to \exists x A(x)$, where x does not occur in $A(t)$
- $\models_{\mathbf{G}_\infty^\triangle} \triangle\forall x A \leftrightarrow \forall x \triangle A$

Since $\mathbf{G}_\infty^\triangle$ does not contain the identity predicate, the following version of the Löwenheim-Skolem theorem is straightforwardly obtained, just like for classical or intuitionistic logic.

Proposition 1. *Every 1-satisfiable formula of* $\mathbf{G}_\infty^\triangle$ *has a model with countably infinite domain.*

3 A non-standard form of Skolemization

In [5] it is shown that the prenex fragment of $\mathbf{G}_\infty^\triangle$ admits Skolemization *with respect to validity*: we have $\models_{\mathbf{G}_\infty^\triangle} A \iff \models_{\mathbf{G}_\infty^\triangle} \mathrm{sko}(A)$, where the operator $\mathrm{sko}(\cdot)$ replaces universally quantified variables by Skolem terms in the usual manner. However, standard Skolemization for satisfiability (where existentially quantified variables are replaced by Skolem terms) does *not preserve 1-satisfiability*, as seen in the following simple example. A model $\mathcal{I}$ with domain $\{d_1, d_2, \ldots\}$ for

$\exists x A(x) \wedge \forall x \neg \triangle A(x)$ is obtained by setting $\sup_{i\in\omega} \|A(x)\|_{\mathcal{I}[d_i/x]} = 1$, without admitting $\|A(x)\|_{\mathcal{I}[d_i/x]} = 1$ for any domain element d_i. But the Skolemized form $A(c) \wedge \forall x \neg \triangle A(x)$ is not 1-satisfiable. The example can obviously be made prenex by moving the second quantifier to the front. However, we will slightly widen our focus to *conjunctions* of prenex formulas, anyway.

We demonstrate that Skolem forms *with respect to satisfiability* can nevertheless be achieved by a nice trick, involving an additional monadic predicate symbol.

Definition 1. *Let E be a new monadic predicate symbol. The operator $\Psi_E(\cdot)$, to be applied to prenex formulas from outside to inside, is defined by*

- $\Psi_E(\forall x A(x)) = \forall x \Psi_E(A(x))$;
- $\Psi_E(\exists x A(x)) = \forall x(E(x) \to \Psi_E(A(f(x,\overline{y}))))$, *where f is a new (Skolem) function symbol and $\overline{y}$ are the free variables in $\exists x A(x)$;*
- $\Psi_E(A) = A$, *if A is quantifier free.*

The Skolem form $\mathrm{SKO}_E(A)$ *of A is obtained by moving all (universal) quantifiers in $\Psi_E(A)$ to the front and inserting an occurrence of $\triangle$ immediately after the quantifiers. More precisely,*

$$\mathrm{SKO}_E(A) = \forall \overline{x} \triangle (\Psi_E(A)^-)$$

where $\overline{x}$ are the variables in $\Psi_E(A)$ and $\Psi_E(A)^-$ denotes $\Psi_E(A)^-$ after removal of all quantifier occurrences.

Obviously $\mathrm{SKO}_E(\cdot)$ does not preserve logical equivalence. However it does preserve 1-satisfiability as the shown in the following.

Theorem 1. *Let $A_1, \ldots, A_m$ be prenex formulas. Then*

$$\Big(\bigwedge_{1\le i\le m} A_i\Big) \in 1\mathrm{SAT} \iff \Big(\exists x E(x) \wedge \bigwedge_{1\le i\le m} \mathrm{SKO}_E(A_i)\Big) \in 1\mathrm{SAT}$$

Proof. ($\Leftarrow$) We show that $\mathrm{SKO}_E(A_i)$ together with $\exists x E(x)$ implies A_i. Note that $\mathrm{SKO}_E(A_i)$ is of the form

$$\forall x_1 \ldots \forall x_n \triangle (E(x_1) \to \ldots (E(x_n) \to A_i^{sk}) \ldots)$$

where A_i^{sk} denotes the quantifier free part of A_i with all existentially bound variable replaced by Skolem terms, as specified in Definition 1. Since $\models_{\mathbf{G}_\infty^\triangle} \triangle \forall x B \leftrightarrow \forall x \triangle B$ and $\models_{\mathbf{G}_\infty^\triangle} \triangle B \to B$ for all formulas B, we can remove the indicated occurrence of $\triangle$ in $\mathrm{SKO}_E(A_i)$. Then we use $\models_{\mathbf{G}_\infty^\triangle} A(f(x,\overline{y}) \to \exists v A(v)$, where v is a new variable, to replace the Skolem terms by existentially quantified variables. Next we use $\models_{\mathbf{G}_\infty^\triangle} \forall x(E(x) \to \exists v A(v)) \to (\exists x E(x) \to \exists v A(v))$, where x does not occur in $\exists v A(v))$, to move existial quantifiers immediately in front of all occurrences of E. Finally we use the assumption that $\exists x E(x)$ to detach all these occurrences. The resulting formula is A_i, as required.

($\Rightarrow$) Suppose $\mathcal{I}$ satisfies A_i for $1 \leq i \leq m$. To obtain a model $\mathcal{J}$ of the formula at the left hand side from we may augment $\mathcal{I}$ by a freely chosen interpretation of E and of the Skolem function symbols. In particular, to achieve $\|\exists x E(x)\|_{\mathcal{J}} = 1$ we use Proposition 1 and assign $v_{\mathcal{J}}(E)(d_i) = v_i$ for $d_i \in D_{\mathcal{J}}$ in such a manner that $\sup_i v_i = 1$, but $v_i \neq 1$ for all $i \in \omega$.

In interpreting the Skolem functions we have to make sure that

$$\|E(x) \to \Psi_E(A_i(f(x, \overline{y})))\|_{\mathcal{J}[d/x, \overline{e}/\overline{y}]} = 1,$$

for all $d \in D_{\mathcal{J}}$ and all $\overline{e} \in D^n_{\mathcal{J}}$, where n is the number of free variables in $\exists x A_i(x)$. To this aim we use the assumption that $\|\exists x A_i(x)\|_{\mathcal{I}[\overline{e}/\overline{y}]} = 1$. This means that for any $d \in D_{\mathcal{I}} = D_{\mathcal{J}}$ there is a further domain element d' such that $\|E(x)\|_{\mathcal{J}[d/x]} \leq \|A_i(x)\|_{\mathcal{I}[d'/x, \overline{e}/\overline{y}]}$. We assign $v_{\mathcal{J}}(f)(d, \overline{e}) = d'$. If there is no more existential quantifier in A_i then we are done, since then $\Psi_E(A_i) = A_i$ and therefore $\|A_i(x)\|_{\mathcal{I}[d'/x, \overline{e}/\overline{y}]} = \|\Psi_E(A_i(f(x, \overline{y}))))\|_{\mathcal{J}[d/x, \overline{e}/\overline{y}]}$. Otherwise we proceed by induction on the number of existential quantifiers replaced by applying Ψ_E, with (essentially) the presented argument as inductive step. $\square$

Remark 1. We have seen that only our alternative form of Skolemization is needed to preserve 1-satisfiability for prenex formulas A of $\mathbf{G}^{\triangle}_{\infty}$, in general. However standard Skolemization (where no additional predicate symbol is introduced) suffices if the quantifier free part of A is preceded by $\triangle$. This observation can be exploited to achieve more efficient translations. Those conjuncts of the formula in question that are of the form $\mathsf{Q}\overline{x}\triangle B$ can be Skolemized in the traditional manner, i.e., existentially quantified variables x are replaced by Skolem terms $f(\overline{y})$, where $\overline{y}$ denotes the variables bound by universal quantifiers in the scope of which x occurs. Only the remaining conjuncts will be treated using the fresh monadic predicate symbol E, as described above.

Note that the price we had to pay for preserving 1-satisfiability consists not only in the additional occurrences of the new predicate symbol E in the antecedents of universally quantified implications, but also in adding the conjunct $\exists x E(x)$. To obtain a purely universal formula that is ready for translation to chain normal form, we have to replace also $\exists x E(x)$ by a conjunction of universal formulas. To this aim we first introduce notation that will be useful also in defining chain normal forms in Section 4, below.

Definition 2. $A \trianglelefteq B \stackrel{\text{def}}{=} \triangle(A \to B)$ *and* $A \triangleleft B \stackrel{\text{def}}{=} \neg\triangle(B \to A)$.

It is straightforward to check that the suggestive symbols are justified by

$$\|A \trianglelefteq B\|_{\mathcal{I}} = \begin{cases} 1 & \text{if } \|A\|_{\mathcal{I}} \leq \|B\|_{\mathcal{I}} \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \|A \triangleleft B\|_{\mathcal{I}} = \begin{cases} 1 & \text{if } \|A\|_{\mathcal{I}} < \|B\|_{\mathcal{I}} \\ 0 & \text{otherwise.} \end{cases}$$

Definition 3. *Let $P_1, \ldots, P_k$ be the predicate symbols occurring in* $\mathrm{SKO}_E(A)$, *where A is conjunction of prenex formulas. (Note that $E \in \{P_1, \ldots, P_k\}$.)*

$$\mathcal{H}_{\exists}(E, A) \stackrel{\text{def}}{=} \bigwedge_{1 \leq i \leq k} \forall \overline{y_i}(\top \trianglelefteq P_i(\overline{y_i}) \vee P_i(\overline{y_i}) \triangleleft E(f_{P_i}(\overline{y_i}))),$$

where $\overline{y_i}$ is a sequence of fresh variables, according to the arity of P_i.

Lemma 1. *Let $A = \bigwedge_{1\leq i\leq m} A_i$ where for $1 \leq i \leq m$ A_i is of the form $\forall \overline{x_i} \triangle A_i^-$ for some is quantifier free formula A_i^-. Then*

$$\big(\exists x E(x) \wedge A\big) \in \text{1SAT} \iff \big(\mathcal{H}_\exists(E, A) \wedge \bigwedge_{1\leq i\leq m} \text{SKO}_E(A_i)\big) \in \text{1SAT}$$

Proof. ($\Rightarrow$) Let $\mathcal{I}$ be a model of $\exists x E(x) \wedge A$. For every $\overline{d} \in D_{\mathcal{I}}^n$, where n is the arity of P_i the following holds. Either $\|P_i(\overline{y_i})\|_{\mathcal{I}[\overline{d}/\overline{y_i}]} = 1$, implying that the first disjunct $\top \trianglelefteq P_i(\overline{y_i})$ of the relevant conjunct in $\mathcal{H}_\exists(E, A)$ evaluates to 1. Otherwise, since we have $\|\exists x E(x)\|_{\mathcal{I}} = 1$, we can extend $\mathcal{I}$ by an interpretation of the new function symbols f_{P_i} in such a manner that $\|E(f_{P_i}(\overline{y_i})\|_{\mathcal{I}[\overline{d}/\overline{y_i}]} > \|P_i(\overline{y_i})\|_{\mathcal{I}[\overline{d}/\overline{y_i}]}$ holds. But this implies that the second disjunct in $\mathcal{H}_\exists(E, A)$ is evaluated to 1.

($\Leftarrow$) Let $\mathcal{I}$ be a model of $\mathcal{H}_\exists(E, A) \wedge \bigwedge_{1\leq i\leq m} \text{SKO}_E(A_i)$. If $\|E(x)\|_{\mathcal{I}[d/x]} = 1$ for some $d \in D_{\mathcal{I}}$ then nothing remains to prove. Otherwise remember that $E \in \{P_1, \ldots, P_k\}$, the set of predicate symbols occurring in $\text{SKO}_E(A)$. Therefore $\|\mathcal{H}_\exists(E, A)\|_{\mathcal{I}} = 1$ implies that for every $d \in D_{\mathcal{I}}$ we have $\|E(x)\|_{\mathcal{I}[x/d]} < \|E(f_E(x))\|_{\mathcal{I}[x/d]}$, since $\|\top \trianglelefteq E(x)\|_{\mathcal{I}[x/d]} < 1$. Consequently there must exist some $v < 1$ such that $\sup_{d\in D_{\mathcal{I}}} \|E(x)\|_{\mathcal{I}[x/d]} = v$, but $\|E(x)\|_{\mathcal{I}[x/d]} \neq v$ for all $d \in D_{\mathcal{I}}$. $\|\mathcal{H}_\exists(E, A)\|_{\mathcal{I}} = 1$ also implies that for every $\overline{d} \in D_{\mathcal{I}}^n$, where n is the arity of P_i, we have either $\|P_i(\overline{y_i})\|_{\mathcal{I}[\overline{d}/\overline{y_i}]} = 1$ or $\|P_i(\overline{y_i})\|_{\mathcal{I}[\overline{d}/\overline{y_i}]} < v$. In other words: no atomic formula is assigned a value in the interval $[v, 1)$ by $\mathcal{I}$. We may therefore define a new interpretation $\mathcal{J}$ over the same domain $D_{\mathcal{I}}$ by setting $\|P_i(\overline{y_i})\|_{\mathcal{J}[\overline{d}/\overline{y_i}]} = \|P_i(\overline{y_i})\|_{\mathcal{I}[\overline{d}/\overline{y_i}]} + (1 - v)$, whenever $\|P_i(\overline{y_i})\|_{\mathcal{I}[\overline{d}/\overline{y_i}]} \neq 1$. Otherwise the corresponding truth value remains 1.

It remains to show that $\mathcal{J}$ is a model of $\exists x E(x) \wedge A$. By definition of $\mathcal{J}$ we have $\sup_{d\in D_{\mathcal{I}}} \|E(x)\|_{\mathcal{J}[x/d]} = 1$. To complete the argument remember that A_i is the form $\forall \overline{x} \triangle A_i^-$ for $1 \leq i \leq m$. Therefore $\|A_i^-\|_{\mathcal{I}[\overline{d}/\overline{x}]} = 1$ for every appropriate tuple $\overline{d}$ of domain elements. This means that the evaluation essentially reduces to that of a propositional formula of $\mathbf{G}_\infty^\triangle$. But an inspection of the truth functions for the propositional connectives shows that whether a given interpretation $\mathcal{I}$ satisfies a formula only depends on the relative order of assigned truth values below 1, but not on their absolute values. Therefore, just like $\mathcal{I}$, also $\mathcal{J}$ is a model of A_i for $1 \leq i \leq m$. □

4 Chain normal form

The results of the last section imply that the satisfiability problem for prenex $\mathbf{G}_\infty^\triangle$ can be reduced to checking satisfiability of conjunctions of purely universal formulas. To define so-called chain normal forms [3, 4] let us use, in addition to $\triangleleft$, also $A \triangleq B$ as an abbreviation for $\triangle(A \leftrightarrow B)$. Clearly $\|A \triangleq B\|_{\mathcal{I}} = 1$ iff $\|A\|_{\mathcal{I}} = \|B\|_{\mathcal{I}}$.

Definition 4. *Let F be a quantifier-free formulas of* $\mathbf{G}_\infty^\triangle$ *and* $A_1, \ldots, A_n$ *the atoms occurring in* F. *A* $\triangle$-chain *over* F *is a formula of the form*

$$(\bot \bowtie_0 A_{\pi(1)}) \wedge (A_{\pi(1)} \bowtie_1 A_{\pi(2)}) \wedge \cdots \wedge (A_{\pi(n-1)} \bowtie_{n-1} A_{\pi(n)}) \wedge (A_{\pi(n)} \bowtie_n \top)$$

where π *is a permutation of* $\{1, \ldots, n\}$, $\bowtie_i$ *is either* $\lhd$ *or* $\triangleq$, *and at least one of the* $\bowtie_i$*'s stands for* $\lhd$.

By $\mathrm{Chains}(F)$ *we denote the set of all* $\triangle$*-chains over* F.

The following follows immediately from Theorem 17 of [5].

Theorem 2. *Let* F *be of the form* $\bigwedge_{1 \leq i \leq n} \forall \overline{x_i} \triangle F_i$, *where* F_i *is quantifier free. Then there exist* $\Gamma_i \subseteq \mathrm{Chains}(F_i)$ *for all* $1 \leq i \leq n$ *such that*

$$\models_{\mathbf{G}_\infty^\triangle} F \leftrightarrow \bigwedge_{1 \leq i \leq n} \bigvee_{C \in \Gamma_i} C.$$

While Theorem 2 can be used, in principle, to reduce Skolemized formulas to clausal form, the result might be exceedingly complex, since $\mathrm{Chains}(F)$ contains a super-exponential number of different elements (with respect to the length of F) in general. Therefore we present an alternative structural translation that results in clause forms of linear size.

Definition 5. *For any quantifier free formula* F *of form* $F_1 \circ F_2$, *where* $\circ \in \{\wedge, \vee, \rightarrow\}$, *let*

$$\mathrm{df}(F) \stackrel{\mathrm{def}}{=} [p_F(\overline{x}) \triangleq (p_{F_1}(\overline{x_1}) \circ p_{F_2}(\overline{x_2}))]$$

where p_F, p_{F_1}, p_{F_2} *are new predicate symbols and* $\overline{x}, \overline{x_1}, \overline{x_2}$ *are the tuples of variables occurring in* F, F_1, F_2, *respectively. If* F *is of form* $\triangle F_1$ *then*

$$\mathrm{df}(F) \stackrel{\mathrm{def}}{=} [p_F(\overline{x}) \triangleq \triangle p_{F_1}(\overline{x_1})].$$

If F *is atomic then* $p_F(\overline{x})$ *is simply an alternative denotation for* F.

For any quantifier free formula G *the* definitional normal form *is defined as*

$$\mathrm{DEF}(G) \stackrel{\mathrm{def}}{=} \triangle p_G(\overline{x}) \wedge \big(\bigwedge_{F \in \mathrm{nsf}(G)} \mathrm{df}(F) \big)$$

where $\mathrm{nsf}(G)$ *is the set of all non-atomic subformulas of* G, $\overline{x}$ *is the tuple of variables occurring in* G, *and* p_G *is a new predicate symbol.*

The following is straightforwardly obtained from simple equivalences.

Proposition 2. *A quantifier free formula* F *is 1-satisfiable iff its definitional normal form* $\mathrm{DEF}(F)$ *is 1-satisfiable.*

Definition 6. *A* literal *is a formula of the form* $A \lhd B$ *or* $A \unlhd B$, *where* A *and* B *are atomic formulas (including* $\top$ *and* $\bot$*). A* clause *is a finite set of literals, denoting a disjunction of its elements. A set of clauses is called satisfiable if the universal closure of the corresponding conjunction of disjunction of literals is 1-satisfiable in* $\mathbf{G}_\infty^\triangle$.

For the following it does not matter, whether we think of a clause as a disjunction of formulas built up using $\to$ and $\triangle$; or rather as 'logic free' syntax, where the semantics is fixed according to the indicated correspondence to $\mathbf{G}_\infty^\triangle$.

Definition 7. *Let A, B, and C be atomic formulas.*

$$
\begin{array}{ll}
\mathrm{cl}(C \triangleq (A \wedge B)) & \stackrel{\text{def}}{=} \{\{C \trianglelefteq A\},\ \{C \trianglelefteq B\},\ \{A \trianglelefteq C, B \trianglelefteq C\}\} \\
\mathrm{cl}(C \triangleq (A \vee B)) & \stackrel{\text{def}}{=} \{\{A \trianglelefteq C\},\ \{B \trianglelefteq C\},\ \{C \trianglelefteq A, C \trianglelefteq B\}\} \\
\mathrm{cl}(C \triangleq (A \to B)) & \stackrel{\text{def}}{=} \{\{A \trianglelefteq B, C \trianglelefteq B\},\ \{\top \trianglelefteq C, B \triangleleft A\}, \\
 & \qquad \{\top \trianglelefteq C, C \trianglelefteq B\},\ \{B \trianglelefteq C\}\} \\
\mathrm{cl}(C \triangleq \triangle A) & \stackrel{\text{def}}{=} \{\{C \triangleleft \top, \top \trianglelefteq A\},\ \{\top \trianglelefteq C, A \triangleleft \top\}\}
\end{array}
$$

For a quantifier free formula G the definitional clause form *is defined as*

$$\mathrm{CF}^{\mathrm{d}}(G) \stackrel{\text{def}}{=} \{\{\top \trianglelefteq p_G(\overline{x})\}\} \cup \bigcup_{F \in \mathrm{nsf}(G)} \mathrm{cl}(\mathrm{df}(F)))$$

where $\mathrm{nsf}(G)$ *denotes the set of all non-atomic subformulas of G, $\overline{x}$ is the tuple of variables occurring in G, and p_G is a new predicate symbol.*

Theorem 3. *Let $A = \bigwedge_{1 \leq i \leq m} A_i$ where, for $1 \leq i \leq m$, A_i is a $\mathbf{G}_\infty^\triangle$-formula of the form $\forall \overline{x_i} \triangle A_i^-$ for some is quantifier free formula A_i^-. Then A is 1-satisfiable iff $\mathrm{CF}^{\mathrm{d}}(A) = \bigcup_{1 \leq i \leq m} \mathrm{CF}^{\mathrm{d}}(A_i^-)$ is satisfiable.*

Proof. In light of Proposition 2 we only need to check that the clauses specified in Definition 7 are equivalent to the corresponding definitional forms. (Note that, by the way they involve $\triangle$, every interpretation $\mathcal{I}$ evaluates the clauses to either 0 or 1.)

– $C \triangleq (A \to B)$: We have

$$\|\triangle(C \leftrightarrow (A \to B))\|_{\mathcal{I}} = \|(A \trianglelefteq B \wedge \triangle C) \vee (B \triangleleft A \wedge C \trianglelefteq B \wedge B \trianglelefteq C)\|_{\mathcal{I}}.$$

By applying the law of distribution to the formula at the right hand side we obtain the conjunction of the following six formulas:

$$
\begin{array}{rcll}
A \trianglelefteq B & \vee & B \triangleleft A & \qquad (1) \\
A \trianglelefteq B & \vee & C \trianglelefteq B & \qquad (2) \\
A \trianglelefteq B & \vee & B \trianglelefteq C & \qquad (3) \\
\triangle C & \vee & B \triangleleft A & \qquad (4) \\
\triangle C & \vee & C \trianglelefteq B & \qquad (5) \\
\triangle C & \vee & B \trianglelefteq C & \qquad (6)
\end{array}
$$

Note that conjunct (1) is valid and that $B \trianglelefteq C$ is entailed by (6). $B \trianglelefteq C$ in turn entails conjuncts (3) and (6). Moreover we can express $\triangle C$ by the equivalent literal $\top \trianglelefteq C$. Thus we obtain the following four conjuncts that directly correspond to $\mathrm{cl}(C \triangleq (A \to B))$:

$$\begin{array}{l} A \trianglelefteq B \;\vee\; C \trianglelefteq B \\ \top \trianglelefteq C \;\vee\; B \triangleleft A \\ \top \trianglelefteq C \;\vee\; C \trianglelefteq B \\ B \trianglelefteq C \end{array}$$

- $C \triangleq (A \wedge B)$: $\triangle(C \leftrightarrow (A \wedge B))$ is easily seen to be equivalent to the conjunction of the three disjunctions

$$\begin{array}{l} C \trianglelefteq A \\ C \trianglelefteq B \\ A \trianglelefteq C \;\vee\; B \trianglelefteq C \end{array}$$

  that directly correspond to $\mathrm{cl}(C \triangleq (A \wedge B))$.
- $C \triangleq (A \vee B)$: $\triangle(C \leftrightarrow (A \vee B))$ is to be equivalent to the conjunction of

$$\begin{array}{l} A \trianglelefteq C \\ B \trianglelefteq C \\ C \trianglelefteq A \;\vee\; C \trianglelefteq B \end{array}$$

  that directly correspond to $\mathrm{cl}(C \triangleq (A \vee B))$.
- $C \triangleq \triangle A$: $\triangle(C \leftrightarrow \triangle A)$ is equivalent to the conjunctions of the following two disjunctions

$$\begin{array}{l} C \triangleleft \top \;\vee\; \top \trianglelefteq A \\ \top \trianglelefteq C \;\vee\; A \triangleleft \top \end{array}$$

  that directly correspond to $\mathrm{cl}(C \triangleq \triangle A)$. □

Remark 2. A somewhat different structural clause form has been described in [5]. The fact that the language there is directly referring to order claims relating terms instead of formulas is rather irrelevant. Also the context of testing validity is immaterial. But the definitional clauses in [5] contain redundancies that are eliminated here. Moreover the description of clauses is only indirect in [5], due to the fact that $\{A = B\} \cup E$ is used to represent the set containing the clauses $\{A \trianglelefteq B\} \cup E$ and $\{B \trianglelefteq A\} \cup E$. This has to be iterated for each occurrence of an identity to obtain, e.g., 6 clauses for conjunction as well as for disjunction where we need only 3 clauses in each case. Similarly only 4 instead of 5 clauses are needed here for implication.

5 Chaining resolution

There are different methods to test the unsatisfiabilty of the sets of clauses $\mathrm{CF}^{\mathrm{d}}(A)$ obtained for a conjunction A of prenex $\mathbf{G}_\infty^\triangle$-formulas as described in the previous sections. In the following *literals* are understood to be of the form $s < t$ or $s \leq t$, for arbitrary first order terms s and t. This means that atomic formulas are considered as terms now and the order relation is directly expressed in the syntax, not indirectly using logical connectives. Note that by this final move we have reduced the 1-satisfiabilty problem for prenex $\mathbf{G}_\infty^\triangle$ to the satisfiabilty problem for sets of clauses referring to a dense total order with endpoints.

One possibility to proceed with $\mathrm{CF}^{\mathrm{d}}(A)$ is to add clausal forms of axioms $\mathcal{D}_<$ that express that $\lhd$ and $\unlhd$ refer to a total denses order with endpoints $\bot$ and $\top$. The resulting set of clauses can be feeded to any first order resolution theorem prover to $\mathrm{CF}^{\mathrm{d}}(A)$ to check whether the empty clause, signalling unsatisfiabilty, can be derived.

Another, more efficient method, that has been presented in [5], is to employ a so-called chaining calculus [7, 6] for this purpose. We will briefly describe this mechanism also here.

For the resulting *order clauses* we consider the following inference rules.

Irreflexivity Resolution:

$$\frac{C \cup \{s < t\}}{C\sigma}$$

where σ is the mgu of s and t

Factorized Chaining:

$$\frac{C \cup \{u_1 \lhd_1 s_1, \dots, u_m \lhd_m s_m\} \quad D \cup \{t_1 \lhd'_1 v_1, \dots, t_n \lhd'_n v_n\}}{C\sigma \cup D\sigma \cup \{u_i\sigma \lhd_{i,j} v_j\sigma \mid 1 \le i \le m, 1 \le n\}}$$

where σ is the mgu of $s_1, \dots, s_m, t_1, \dots, t_n$ and $\lhd_{i,j}$ is $<$ if and only if either $\lhd_i$ is $<$ or $\lhd'_j$ is $<$. Moreover, $t_1\sigma$ occurs in $D\sigma$ only in inequalities $v \lhd t_1\sigma$.

These two rules constitute a refutationally complete inference system for the theory of all total orders with endpoints $\bot$ and $\top$ in presence of set $\mathcal{E}q^{\mathbf{F}}$ of clauses

$$\{x_i < y_i, y_i < x_i \mid 1 \le i \le n\} \cup \{f(x_1, \dots, x_n) \le f(y_1, \dots, y_n)\}$$

where f ranges the set $\mathbf{F}$ of function symbols of the signature.

To achieve more efficient proof search, we impose the following conditions on the rules rules. These conditions refer to a complete reduction order $\succ$ (see[6]) declared on the set of terms. We write $s \not\succeq t$ if $\neg(s \succ t)$ and $s \neq t$.

Maximality Condition for Irreflexivity Resolution:

- $s\sigma$ is a maximal term with respect to $\succ$ in $C\sigma$.

Maximality Condition for Chaining:

- $u_i\sigma \not\succeq s_1\sigma$ for all $1 \le i \le n$,
- $v_i\sigma \not\succeq t_1\sigma$ for all $1 \le i \le m$,
- $u\sigma \not\succeq s_1\sigma$ for all terms u such that $u \lhd s \in C$ or $s \lhd u \in C$, and
- $v\sigma \not\succeq t_1\sigma$ for all terms v such that $v \lhd s \in C$ or $s \lhd v \in C$.

It is convenient to view the resulting inference system $\mathrm{MC}_\succ$ as a set operator.

Definition 8. $\mathrm{MC}_{\succ}(\mathcal{C})$ *is the set of all conclusions of Irreflexivity Resolution and Maximal Chaining where the premises are (variants of) members of the set of clauses* $\mathcal{C}$. *Moreover,* $\mathrm{MC}^0_{\succ}(\mathcal{C}) = \mathcal{C}$, $\mathrm{MC}^{i+1}_{\succ}(\mathcal{C}) = \mathrm{MC}_{\succ}(\mathrm{MC}^i_{\succ}(\mathcal{C})) \cup \mathrm{MC}^i_{\succ}(\mathcal{C})$, *and* $\mathrm{MC}^*_{\succ}(\mathcal{C}) = \bigcup_{i \geq 0} \mathrm{MC}^i_{\succ}(\mathcal{C})$.

The set consisting of the three clauses $\{\bot \leq x\}$, $\{x \leq \top\}$, and $\{\bot < \top\}$, corresponding to the endpoint axioms, is called $\mathcal{E}p$. Let d be a frech binary function symbol. The set $\{\{y \leq x,\ d(x,y) < y\},\ \{y \leq x,\ x < d(x,y)\}\}$, corresponding to the density axiom, is called $\mathcal{D}o$.

The following completeness result follows directly from Theorem 2 of [6].

Theorem 4. *A set of clause* $\mathcal{C}$ *has a dense total order with endpoints* $\bot$ *and* $\top$ *as a model iff* $\mathrm{MC}^*_{\succ}(\mathcal{C} \cup \mathcal{E}q^{\mathbf{F}} \cup \mathcal{E}p \cup \mathcal{D}o)$ *does not contain the empty clause.*

Remark 3. Yet more refined versions of chaining calculi are investigated in [6, 7]. However, standard breadth-first proof search based on $\mathrm{MC}_{\succ}$ seems to be quite appropriate in our context; in particular, since the problem of "variable chaining" does not occur for the sets of clauses obtained using $\mathrm{CF}^{\mathrm{d}}(\cdot)$.

6 Conclusion

Remember that for testing whether a given specification expressed in fuzzy logic $\mathbf{G}^{\triangle}_{\infty}$ has a model we cannot rely on proof procedures for checking validity in $\mathbf{G}^{\triangle}_{\infty}$. As demonstrated in this paper, one can nevertheless apply a resolution style method for testing (1-)satisfiability for *prenex* $\mathbf{G}^{\triangle}_{\infty}$-formulas. While prenex $\mathbf{G}^{\triangle}_{\infty}$ admits Skolemization with respect to *validity* in a standard manner [5], this result does not transfer to *satisfiability*. However, we have shown that a new non-standard form of Skolemization that involves the introduction of a fresh monadic predicate symbol preserves satisfiability. This in turn can be used to translate prenex $\mathbf{G}^{\triangle}_{\infty}$-formulas into a clausal form combining a definitional (structural) clause form translation with a generalized notion of literals that refers to the underlying order on the set of truth values $[0, 1]$. Finally efficient resolution style proof search, based on a so-called chaining calculus for the theory of dense total orders with endpoints, can be employed.

A number of related problems arise naturally. Can a similar method be found that covers full $\mathbf{G}^{\triangle}_{\infty}$, not only its prenex fragment? What about other t-norm based fuzzy logics, including other forms of Gödel logic? Instead of generalizing, it might also be interesting to go the other direction and focus on subclasses of prenex $\mathbf{G}^{\triangle}_{\infty}$. Of particular interest is the monadic class, where formulas contain only monadic predicate symbols. In fact, given the above results, it is straightforward to adapt the undecidability proof in [5] for the *validity* problem of the prenex monadic fragment of $\mathbf{G}^{\triangle}_{\infty}$ to the *1-satisfiability* problem for this class. Nevertheless it is likely that more refined proof search methods will allow to demonstrate the *decidability* of subclasses of $\mathbf{G}^{\triangle}_{\infty}$ in the style of, e.g., [9, 10].

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