A graded inference approach based on infinite-valued Łukasiewicz semantics

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***Abstract*—We present a consequence relation for graded inference within the frame of infinite-valued Łukasiewicz semantics. We consider the premises to be true to at least a certain degree η and consider as consequences those sentences entailed to have a degree of truth at least some suitable threshold ζ. We focus on the study of some aspects and features of the consequence relation presented and, in particular, on the effect of variations in the thresholds η, ζ.**

I. Introduction

The motivation of the present paper and the issues it deals with originated from the study and analysis of the consequence relation ${}^{\eta}\triangleright_{\zeta}$. The consequence relation ${}^{\eta}\triangleright_{\zeta}$ was defined in a simplified version in [8] (for $\eta = \zeta$) and further studied and extended in [9], [10] and [11]. Such consequence relation was mainly motivated by the following argument: Suppose the set of premises, say Γ, consists of the beliefs of a single *rational* agent. As such, a premise can be assigned a *degree of belief* that corresponds to the degree to which our agent believes the sentence to be true (which, in the context of ${}^{\eta}\triangleright_{\zeta}$, we identify with subjective probability). We then fix a lower bound belief threshold for Γ (η in ${}^{\eta}\triangleright_{\zeta}$) and another threshold for the conclusions (ζ in ${}^{\eta}\triangleright_{\zeta}$) on the basis that it might be argued that we should be willing to accept as consequences any other sentences which as a result have, by probability logic, a degree of belief at least as high as some suitable threshold ζ (the most natural choice being, arguably, $\zeta = \eta$).

Much attention is devoted to the function $F_{\Gamma,\theta}$ in [9], [10] and [11]. For $\Gamma \cup \{\theta\}$ a set of sentences, the function $F_{\Gamma,\theta}(\eta)$ is defined as follows: $F_{\Gamma,\theta}(\eta) = \sup\{\zeta \mid \Gamma^{\eta} \triangleright_{\zeta} \theta\}$. $F_{\Gamma,\theta}(\eta)$ gives us the highest belief threshold that we can place on θ that is consistent with any probability measure that assigns a degree of belief greater than or equal to η to all sentences in Γ. A representation theorem that fully characterizes the functions of the form $F_{\Gamma,\theta}$ and thus the sets of pairs (η, ζ) for which $\Gamma^{\eta} \triangleright_{\zeta} \theta$ holds is given in the above mentioned references.

In this paper we present the consequence relation ${}^{\eta}\blacktriangleright_{\zeta}$, of the same nature as ${}^{\eta}\triangleright_{\zeta}$, defined within the frame of Lukasiewicz semantics (see [3] or [4]). From it we define the function $\mathcal{L}_{\Gamma,\theta}$, the counterpart of $F_{\Gamma,\theta}$, and analyse its behaviour. We will refer to the values η, ζ in the context of ${}^{\eta}\blacktriangleright_{\zeta}$ as *degrees of truth* rather than as degrees of belief, although an interpretation of them in terms of belief is not discarded.

There are some approaches in the literature to graded inference in the context of Łukasiewicz semantics, some of them very recent (see for example [1], [2]). However, all such approaches (at least those known to the author) differ greatly from the one we present here.

The paper is structured as follows: Section II contains some preliminary definitions and notational remarks. Section III introduces the notion of Ł$_{\eta}$-consistency. Section IV is devoted to the study of the functions of the form $\mathcal{L}_{\Gamma,\theta}$ and Section V deals with some characteristic graphs that $\mathcal{L}_{\Gamma,\theta}$ can yield, for suitable $\Gamma \cup \{\theta\}$. The ultimate goal of the results presented in all these sections is the representation theorem given in Section VI, which offers a full characterization of the functions of the form $\mathcal{L}_{\Gamma,\theta}$.

II. Preliminary definitions and notation

Throughout we will be working with a finite propositional language $L = \{p_1, ..., p_m\}$, for some $m \in \mathbb{N}$. We will denote by SL the closure of $L \cup \{\bot\}$ under the connective '$\rightarrow$'.

We will be using in this paper a large number of abbreviations which correspond to other common logical connectives within the context of many-valued logics. We consider the following abbreviations, for $\phi, \theta \in SL$:

- '$\bot \rightarrow \bot$' is abbreviated by '$\top$'
- '$\phi \rightarrow \bot$' by '$\neg\phi$'
- '$\neg(\phi \rightarrow \neg\theta)$' by '$\phi \& \theta$'
- '$\neg\phi \rightarrow \theta$' by '$\phi \underline{\vee} \theta$'
- '$\phi \& (\phi \rightarrow \theta)$' by '$\phi \wedge \theta$'
- '$((\phi \rightarrow \theta) \rightarrow \theta) \wedge ((\theta \rightarrow \phi) \rightarrow \phi)$' by '$\phi \vee \theta$'.

Next we define the notion of Ł-valuation.

Definition 1: Let $w : SL \longrightarrow [0, 1]$. We say that w is an Ł-valuation on L if, for ϕ, $\theta \in SL$, we have what follows:

1) $w(\phi \rightarrow \theta) = \min\{1, 1 - w(\phi) + w(\theta)\}$
2) $w(\bot) = 0$

From these two clauses we can define the behaviour of Ł-valuations for the other connectives introduced above. Let $\phi, \theta \in SL$. We have what follows:

- $w(\top) = 1$
- $w(\neg\phi) = 1 - w(\phi)$
- $w(\phi \& \theta) = \max\{0, w(\phi) + w(\theta) - 1\}$
- $w(\phi \underline{\vee} \theta) = \min\{1, w(\phi) + w(\theta)\}$
- $w(\phi \wedge \theta) = \min\{w(\phi), w(\theta)\}$

- $w(\phi \vee \theta) = \max\{w(\phi), w(\theta)\}$

Let $\Gamma = \{\phi_1, ..., \phi_k\} \subseteq SL$, for some $k \in \mathbb{N}$. We will denote by $\bigwedge \Gamma$ the sentence $\phi_1 \wedge ... \wedge \phi_k$. Similarly $\bigvee \Gamma$, $\underline{\bigvee}\Gamma$ and $\&\Gamma$ will denote the sentences $\phi_1 \vee ... \vee \phi_k$, $\phi_1 \underline{\vee} ... \underline{\vee} \phi_k$ and $\phi_1 \& ... \& \phi_k$ respectively.

Sentences of the form $\phi \wedge ... \wedge \phi$ where ϕ occurs k times, for some $k \in \mathbb{N}$, will be abbreviated by the expression $\bigwedge^k \theta$ (and similarly for the other connectives). It is customary to refer to $\&^k \phi$ (that is, $\phi \& ... \& \phi$, where ϕ occurs k times) by ϕ^k in the literature and we will follow this convention.

Let $\phi \in SL$. We will denote by $L_\phi = \{p_1, ..., p_k\} \subseteq L$ the set of propositional variables that occur in ϕ. We will sometimes use the notation $\phi(p_1, ..., p_k)$.

Let w be an Ł-valuation on L. We have that

$$w(\phi) = f(w(p_1), ..., w(p_k))$$

for some $f : [0,1]^k \to [0,1]$. We will denote this f by f_ϕ. We will write sometimes $f_\phi(x_1, ..., x_k)$.

Next we state a central theorem in Łukasiewicz logic that will play an important role in this paper.

Theorem 2: **McNaughton's Theorem** (see [7])

In order that a function $f : [0,1]^k \to [0,1]$ be of the form f_ϕ for some $\phi \in SL$ it is necessary and sufficient that f satisfy the following two conditions:

1) f is continuous on $[0,1]^k$.
2) There are a finite number of distinct polynomials with integer coefficients λ_i, $1 \leq i \leq \mu$,

$$\lambda_i = b_i + m_{1_i} x_1 + ... + m_{k_i} x_k,$$

such that for every $(x_1, ..., x_k)$, $0 \leq x_i \leq 1$ for all $i \in \{1, ..., k\}$, there is λ_j for some $j \in \{1, ..., \mu\}$ such that $f(x_1, ..., x_k) = \lambda_j(x_1, ..., x_k)$.

For a proof of this theorem see [7].

III. The notion of Ł$_\eta$-consistency

We now define Ł$_\eta$-consistency and maximal Ł$_\eta$-consistency.

Let $\Gamma \subseteq SL$ and $\eta \in [0,1]$.

Definition 3: We say that Γ is Ł$_\eta$-consistent if and only if there exists an Ł-valuation w on L such that $w(\bigwedge \Gamma) \geq \eta$.

Definition 4: We define the notion of maximal consistency of Γ – denoted $\text{mc}(\Gamma)$ – as follows:

$$\text{mc}(\Gamma) = \sup\{\eta \mid \Gamma \ \ is \ \ \text{Ł}_\eta\text{-consistent}\}$$

We say that Γ is maximally Ł$_\eta$-consistent to mean that $\text{mc}(\Gamma) = \eta$.

These definitions ressemble those of η-consistency and maximal η-consistency presented in [5].[1] Maximal η-consistency was defined as a probabilistic measure of the *degree of consistency* for classical sets of sentences (for more on these notions see [5] or [11]). Our definition of maximal Ł$_\eta$-consistency is not presented here as a measure for the degree of consistency of a set of sentences – for which it does not seem to be suited anyway – but simply as some sort of technical notion that will be needed in further sections (for more on maximal Ł$_\eta$-consistency as a measure of inconsistency see [11]).

[1] The essential difference being that such notions are defined based on probability functions in L instead of Ł-valuations.

Notice that Ł$_\eta$-consistency of a set of sentences Γ is the same as Ł$_\eta$-consistency of the sentence $\bigwedge \Gamma$. We will talk indistinctively about the consistency of sentences and sets of sentences.

Proposition 5: $\text{mc}(\Gamma)$ is attained by some Ł-valuation.

Proof: Let $\text{mc}(\Gamma) = \eta$. We can define an increasing sequence $\{\eta_n\}$ whose limit is η such that for all $n \in \mathbb{N}$ there exists an Ł-valuation w_n on L with $w_n(\bigwedge \Gamma) \geq \eta_n$. We can characterize every w_n by the values it assigns to the propositional variables in L. We will thus identify w_n with the vector $\vec{w}_n = (w_n(p_1), ..., w_n(p_m))$. We need to prove that there exists an Ł-valuation w on L such that $w(\bigwedge \Gamma) \geq \eta$.

We can take a convergent subsequence $\{\vec{w}^1_{n_k}\}$ in the first coordinates of $\{\vec{w}_n\}$. We know such a convergent subsequence needs to exist and converge in the interval $[0,1]$ by compactness. Next we can pick a convergent subsequence $\{\vec{w}^2_{n_k}\}$ in the second coordinates of $\{\vec{w}^1_{n_k}\}$. As before, such subsequence needs to exist and converge in the interval $[0,1]$ by compactness. We can proceed in the same way for the other coordinates. The final subsequence, $\{\vec{w}^{2^m}_{n_k}\}$, will have as limit an Ł-valuation $\vec{w}$ on L for which $w(\bigwedge \Gamma) \geq \eta$. ■

Proposition 6: For all $k \in \mathbb{N}$ we can construct a sentence $\phi \in SL$ (which we will denote by $\phi_{\frac{1}{k}}$) that is maximally Ł$_{\frac{1}{k}}$-consistent.

Proof: Let us define $\phi_{\frac{1}{k}}$ as follows:

$$\phi_{\frac{1}{k}} = \neg p \wedge p^{k-1}$$

It can be easily checked that $\phi_{\frac{1}{k}}$ is maximally Ł$_{\frac{1}{k}}$-consistent. Consider the Ł-valuation w on L that assigns to p the value $\frac{k-1}{k}$. We have that $w(\phi_{\frac{1}{k}}) = \frac{1}{k}$. It is also clear that any other Ł-valuation w^* on L for which $w^*(p) < \frac{k-1}{k}$ or $w^*(p) > \frac{k-1}{k}$ will be such that $w^*(\phi_{\frac{1}{k}}) < \frac{1}{k}$. ■

Proposition 7: Let $r \in \mathbb{Q} \cap [0,1]$. We can construct a sentence $\phi \in SL$ (which we will denote by ϕ_r) that is maximally Ł$_r$-consistent.

Proof: Let $r = \frac{u}{v}$ and $p \in L$. Let us define ϕ_r as follows:

$$\phi_r = \underline{\bigvee}^u \phi_{\frac{1}{v}}$$

By *Proposition 6* $\phi_{\frac{1}{v}}$ is maximally Ł$_{\frac{1}{v}}$-consistent and thus $\underline{\bigvee}^u \phi_{\frac{1}{v}}$ is maximally Ł$_{\frac{u}{v}}$-consistent. ■

Although obvious, it is worth mentioning that there exists an Ł-valuation w on L for which $w(\phi_r) = 0$. Thus, by continuity of f_{ϕ_r}, we will have an Ł-valuation w on L such that $w(\phi_r) = \lambda$ for each $\lambda \in [0, r]$.

IV. $^{\eta}\blacktriangleright_{\zeta}$ AND THE FUNCTION $\mathcal{L}_{\Gamma,\theta}$

Time now to define the consequence relation $^{\eta}\blacktriangleright_{\zeta}$ introduced in the first section and, from it, the function $\mathcal{L}_{\Gamma,\theta}$.

Throughout let $\Gamma \cup \{\theta\} \subseteq SL$ and $\eta, \zeta \in [0,1]$.

Definition 8: We say that Γ (η,ζ)-implies θ (denoted $\Gamma^{\eta}\blacktriangleright_{\zeta}\ \theta$) if and only if, for all Ł-valuations w on L, if $w(\bigwedge\Gamma) \geq \eta$ then $w(\theta) \geq \zeta$.

Definition 9: The function $\mathcal{L}_{\Gamma,\theta} : [0,1] \longrightarrow [0,1]$ is defined as follows, for all $\eta \in [0,1]$:

$$\mathcal{L}_{\Gamma,\theta}(\eta) = \sup\{\zeta \,|\, \Gamma^{\eta}\blacktriangleright_{\zeta}\ \theta\}.$$

As mentioned earlier, $\mathcal{L}_{\Gamma,\theta}$ characterizes the pairs (η,ζ) for which $\Gamma^{\eta}\blacktriangleright_{\zeta}\ \theta$.

A. Some properties of $\mathcal{L}_{\Gamma,\theta}$

Proposition 10: Let Γ be $Ł_{\eta}$-consistent. There exists an Ł-valuation w on L such that $w(\bigwedge\Gamma) \geq \eta$ and $w(\theta) = \mathcal{L}_{\Gamma,\theta}(\eta) = \zeta$.

Proof: We proceed in a way similar to that of the proof of *Proposition 5*. We can define a decreasing sequence $\{\zeta_n\}$ whose limit is ζ such that for all $n \in \mathbb{N}$ there exists an Ł-valuation w_n on L with $w_n(\theta) = \zeta_n$ and $w_n(\bigwedge\Gamma) \geq \eta$. As in *Proposition 5*, we identify w_n with the vector $\vec{w}_n = (w_n(p_1), ..., w_n(p_m))$. We have to prove that there exists an Ł-valuation w on L such that $w(\theta) = \zeta$ and $w(\bigwedge\Gamma) \geq \eta$.

As before, we take a convergent subsequence $\{\vec{w}^1_{n_k}\}$ in the first coordinates of $\{\vec{w}_n\}$. Next we pick a convergent subsequence $\{\vec{w}^2_{n_k}\}$ in the second coordinates of $\{\vec{w}^1_{n_k}\}$ and proceed in the same way for the other coordinates. That all these subsequences exist and converge in the interval $[0,1]$ follows from compactness. The final subsequence, $\{\vec{w}^{2^m}_{n_k}\}$, will have as limit an Ł-valuation $\vec{w}$ on L for which $w(\theta) = \zeta$ and $w(\bigwedge\Gamma) \geq \eta$. ∎

Proposition 11: $\mathcal{L}_{\Gamma,\theta}$ is increasing.

Proof: It follows directly from the definition of $^{\eta}\blacktriangleright_{\zeta}$. ∎

For the next proposition assume that $\mathrm{mc}(\Gamma) = \lambda > 0$.

Proposition 12: $\mathcal{L}_{\Gamma,\theta}$ is left continuous on $[0,\lambda]$.

Proof: Let us proceed by *reductio ad absurdum* by assuming that there exists $\eta \in (0,\lambda]$ and $\epsilon > 0$ such that

$$\mathcal{L}_{\Gamma,\theta}(\eta) - \mathcal{L}_{\Gamma,\theta}(x) > \epsilon$$

for all $x \in [0,\eta)$.

Let $\zeta = \sup\{\mathcal{L}_{\Gamma,\theta}(x) |\, x < \eta\}$. We can define an increasing sequence $\{\eta_n\}$ with limit η and a sequence $\{\zeta_n\}$ with limit ζ such that for all $n \in \mathbb{N}$ there exists an Ł-valuation w_n with $w_n(\bigwedge\Gamma) = \eta_n$ and $w_n(\theta) = \zeta_n$. As in previous proofs we identify w_n with the vector $\vec{w}_n = (w_n(p_1), ..., w_n(p_m))$.

We proceed as in previous proofs by taking suitable convergent subsequences of $\{\vec{w}_n\}$ at each step until we come to $\{\vec{w}^{2^m}_{n_k}\}$, which will have as limit an Ł-valuation $\vec{w}$ on L for which $w(\Gamma) = \eta$ and $w(\theta) = \zeta$ since $\mathcal{L}_{\Gamma,\theta}$ is increasing. Therefore $\mathcal{L}_{\Gamma,\theta}$ needs to be continuous from the left at η. ∎

Proposition 13: $\mathcal{L}_{\Gamma,\theta}$ is of the following form:

$$\mathcal{L}_{\Gamma,\theta}(\eta) = \begin{cases} a_1\eta + b_1 & \text{if } \eta \leq \lambda_1 \\ ... & \\ a_k\eta + b_k & \text{if } \lambda_{k-1} < \eta \leq \lambda_k \end{cases}$$

with $a_i, b_i, \lambda_i \in \mathbb{Q}$ and $k \in \mathbb{N}$, $i \in \{1, ..., k\}$.

Proof: Let $\mathcal{R} = \langle \mathbb{R}, +, -, <, =, 0, 1\rangle$.[2]

The set of pairs

$$\{(x,y) \in \mathbb{R}^2 |\ y = \mathcal{L}_{\Gamma,\theta}(x)\}$$

is $\mathcal{R}$-definable (notice that, since $\mathcal{R}$ is an elementary extension of the structure $\mathcal{Q} = \langle \mathbb{Q}, +, -, <, =, 0, 1\rangle$, it is $\mathcal{Q}$-definable too).

The theory of $\mathcal{R}$ has quantifier elimination (see for example [6]). Therefore the set of pairs

$$\{(x,y) \in \mathbb{R}^2 |\ y = \mathcal{L}_{\Gamma,\theta}(x)\}$$

is given by a finite boolean combination (which reduces to a finite union of intersections by the complement and distributive laws for sets) of sets of the form

$$\{(x,y) \in \mathbb{R}^2 |\ my < nx + k\}$$

and

$$\{(x,y) \in \mathbb{R}^2 |\ my = nx + k\}$$

for $n, m, k \in \mathbb{Z}$.

Notice that each non-empty intersection of sets of such form is convex so, since $\mathcal{L}_{\Gamma,\theta}$ is a function, such intersection has to be a line segment (with coefficients and bounds in $\mathbb{Q}$).

That $\mathcal{L}_{\Gamma,\theta}$ is left continuous was stated and proved in Proposition 12. ∎

V. GRAPHS FOR $\mathcal{L}_{\Gamma,\theta}$

In this section we present some characteristic graphs for $\mathcal{L}_{\Gamma,\theta}$ that can be constructed from a suitable set $\Gamma \cup \{\theta\} \subseteq SL$. We start with what we call *basic graphs* and then we go on to define the *compound graphs*.

A. Basic graphs

We define five basic types of graphs that $\mathcal{L}_{\Gamma,\theta}$ can yield for suitable sets $\Gamma \cup \{\theta\}$.

Proposition 14: **(Type 1)**

Let $r, s \in [0,1] \cap \mathbb{Q}$. We can find $\Gamma \cup \{\theta\} \subseteq SL$ for which $\mathcal{L}_{\Gamma,\theta}$ is as follows:

$$\mathcal{L}_{\Gamma,\theta}(\eta) = \begin{cases} s & \text{if } \eta \leq r \\ 1 & \text{otherwise} \end{cases}$$

Proof: Let $0 < r = \frac{u_1}{v_1}$ and $0 < s = 1 - \frac{u_2}{v_2} < 1$.

Let $\Gamma = \{\underline{\bigvee}^{u_1}\phi_{\frac{1}{v_1}}\}$, with $\phi_{\frac{1}{v_1}} = \neg p \wedge p^{v_1-1}$ and $p \in L$. As seen previously, Γ is maximally $Ł_r$-consistent.

On the other hand take $\phi_{\frac{1}{v_2}} = \neg q \wedge q^{v_2-1}$, for $q \in L$, $q \neq p$. The sentence $\underline{\bigvee}^{u_2}\phi_{\frac{1}{v_2}}$ is maximally $Ł_{\frac{u_2}{v_2}}$-consistent. Thus there is no Ł-valuation w on L such that

$$w(\neg(\underline{\bigvee}^{u_2}\phi_{\frac{1}{v_2}})) < 1 - \frac{u_2}{v_2} = s.$$

[2]Here by '$-$' we mean the map given by $x \longrightarrow -x$.

Set $\theta = \neg(\underline{\bigvee}^{u_2}\phi_{\frac{1}{v_2}})$. Clearly, for Γ and θ thus defined, $\mathcal{L}_{\Gamma,\theta}$ is as stated above.

For $r = 0$ we can take $\bigwedge\Gamma$ to be an Ł-contradiction. If $s = 0$ we can take θ to be an Ł-contradiction and, if $s = 1$, an Ł-tautology. ■

It is worth remarking the importance of a subclass of this type of graphs; namely, the graph given when $s = 0$.

Notice that in the above example Γ is not Ł$_1$-consistent. Later on, in order to prove the representation theorem for the functions $\mathcal{L}_{\Gamma,\theta}$, we will need to appeal to graphs of this form for Ł$_1$-consistent sets of premises. From McNaughton's Theorem we can claim that there exist sentences $\bigwedge\Gamma$ and θ involving only one propositional variable –say $p \in L$– with $\bigwedge\Gamma$ Ł$_1$-consistent such that $\mathcal{L}_{\Gamma,\theta}(\eta) = 0$ for $\eta \leq r$ and $\mathcal{L}_{\Gamma,\theta}(\eta) = 1$ for $\eta > r$, for any $r \in [0,1] \cap \mathbb{Q}$. To see this consider for example $f_{\bigwedge\Gamma}(x)$ and $f_\theta(x)$ to be of the following form:

$$f_{\bigwedge\Gamma}(x) = \begin{cases} a_1 x & \text{if } x \leq \frac{1+b_2}{a_1+a_2} \\ 1-(a_2x-b_2) & \text{if } \frac{1+b_2}{a_1+a_2} < x \leq \frac{1+b_2}{a_2} \\ a_3x - b_3 & \text{if } \frac{1+b_2}{a_2} < x \leq c \\ 1 & \text{otherwise} \end{cases}$$

Here a_1, a_2, a_3, b_2, b_3 are positive integers and c is a rational number. Other conditions on these values are that $a_1(\frac{1+b_2}{a_1+a_2}) = 1-(a_2(\frac{1+b_2}{a_1+a_2})-b_2) = r$, $1+b_2 < a_2$, $1-(a_2(\frac{1+b_2}{a_2})-b_2) = a_3(\frac{1+b_2}{a_2}) - b_3 = 0$ and $a_3c - b_3 = 1$.

$$f_\theta(x) = \begin{cases} 0 & \text{if } x \leq d_1 \\ a_4x - b_4 & \text{if } d_1 < x \leq d_2 \\ 1 & \text{otherwise} \end{cases}$$

Here a_4, b_4 are positive integers and d_1, d_2 are rational numbers. Other conditions on these values are $a_4d_1 - b_4 = 0$, $a_4d_2 - b_4 = 1$ and $\frac{1+b_2}{a_1+a_2} \leq d_1 < d_2 \leq \frac{1+b_2}{a_2}$.

For $\bigwedge\Gamma$ and θ of this form the function $\mathcal{L}_{\Gamma,\theta}$ will be as desired. Notice that $f_{\bigwedge\Gamma}(\frac{1+b_2}{a_1+a_2}) = r$, $f_\theta(\frac{1+b_2}{a_1+a_2}) = 0$ and, for all $x \in [0,1]$ for which $f_{\bigwedge\Gamma}(x) > r$ we have that $f_\theta(x) = 1$.

Proposition 15: **(Type 2)**

Let $r, s \in [0,1] \cap \mathbb{Q}$, with $r < s$. We can find $\Gamma \cup \{\theta\} \subseteq SL$ for which $\mathcal{L}_{\Gamma,\theta}$ is of the following form:

$$\mathcal{L}_{\Gamma,\theta}(\eta) = \begin{cases} 0 & \text{if } \eta \leq r \\ \frac{\eta-r}{s-r} & \text{if } r < \eta < s \\ 1 & \text{otherwise} \end{cases}$$

Proof: Let $0 < r = \frac{u_1}{v_1} < s = \frac{u_2}{v_2}$.

Take $s - r = \frac{u_2v_1 - u_1v_2}{v_1v_2}$ and define ψ_1 and θ as follows:

$$\psi_1 = \underline{\bigvee}^{u_2v_1-u_1v_2}\phi_{\frac{1}{v_1v_2}}$$

$$\theta = \underline{\bigvee}^{v_1v_2}\phi_{\frac{1}{v_1v_2}}.$$

Here $\phi_{\frac{1}{v_1v_2}} = \neg p \wedge p^{v_1v_2-1}$, for $p \in L$.

Define ψ_2 as follows:

$$\psi_2 = \underline{\bigvee}^{u_1}\phi_{\frac{1}{v_1}}.$$

We take $\phi_{\frac{1}{v_1}}$ to be $\neg q \wedge q^{v_1-1}$, for $q \in L$ with $q \neq p$, and set $\Gamma = \{\psi_1 \underline{\vee} \psi_2\}$.

$\mathcal{L}_{\Gamma,\theta}$ is as required. To see this notice that, since ψ_2 is maximally Ł$_r$-consistent, $\mathcal{L}_{\Gamma,\theta}(x) = 0$ for all $x \in [0,r]$ and that any Ł-valuation w on L for which $w(\psi_1) = \lambda(s-r)$, for $\lambda \in [0,1]$, is such that $w(\theta) = \lambda$. If $r = 0$ then we can dispense with ψ_2 and take $\Gamma = \{\psi_1\}$. ■

As with *Type 1* McNaughton's Theorem guarantees the existence of $\bigwedge\Gamma$ Ł$_1$-consistent and θ such that $\mathcal{L}_{\Gamma,\theta}$ is as above. To see this consider $\phi(p)$ and $\theta(p)$ (with $p \in L$) for which $f_\phi(x)$ and $f_\theta(x)$ are of the following form:

$$f_\phi(x) = \begin{cases} bx & \text{if } x \leq \frac{1}{b} \\ 1 & \text{otherwise} \end{cases}$$

$$f_\theta(x) = \begin{cases} ax & \text{if } x \leq \frac{1}{a} \\ 1 & \text{otherwise} \end{cases}$$

Here $a, b \in \mathbb{N}$ and $\frac{a}{b} = \frac{1}{s-r}$. Notice that $\mathcal{L}_{\{\phi\},\theta}(\eta) = \frac{a\eta}{b}$ for all $\eta \leq \frac{b}{a}$. We can then set $\Gamma = \{\phi \underline{\vee} \psi_2\}$, where ψ_2 is as defined above. The function $\mathcal{L}_{\Gamma,\theta}$ will be as stated, with Γ Ł$_1$-consistent.

Proposition 16: **(Type 3)**

Let $r, s \in [0,1] \cap \mathbb{Q}$. We can define $\Gamma \cup \{\theta\} \subseteq SL$ for which $\mathcal{L}_{\Gamma,\theta}$ has the following form:

$$\mathcal{L}_{\Gamma,\theta}(\eta) = \begin{cases} 0 & \text{if } \eta \leq r \\ \frac{s(\eta-r)}{1-r} & \text{otherwise} \end{cases}$$

Proof: Let $r = \frac{u_1}{v_1}$ and $s = \frac{u_2}{v_2}$. We have to distinguish two possible cases here:

Case 1. $\frac{s}{1-r} \leq 1$.

Consider $\frac{s}{1-r} = \frac{u_2v_1}{v_2(v_1-u_1)}$.

We first define ψ_1 and θ as follows:

$$\psi_1 = \underline{\bigvee}^{v_2(v_1-u_1)}\phi_{\frac{1}{v_2(v_1-u_1)}},$$

$$\theta = \underline{\bigvee}^{u_2v_1}\phi_{\frac{1}{v_2(v_1-u_1)}}.$$

Here $\phi_{\frac{1}{v_2(v_1-u_1)}} = \neg p \wedge p^{v_2(v_1-u_1)-1}$, for $p \in L$.

Let us now define ψ_2 for $r > 0$ as follows:

$$\psi_2 = \underline{\bigvee}^{u_1}\phi_{\frac{1}{v_1}}.$$

Here $\phi_{\frac{1}{v_1}} = \neg q \wedge q^{v_1-1}$, for $q \in L$ with $q \neq p$. We set $\Gamma = \{\psi_1 \underline{\vee} \psi_2\}$. We can clearly see that $\mathcal{L}_{\Gamma,\theta}$ is as stated.

Notice that if $r = 0$ then we can dispense with ψ_2 and set $\Gamma = \{\psi_1\}$.

Case 2. $\frac{s}{1-r} > 1$.

Consider $\frac{1-r}{s}=\frac{v_2(v_1-u_1)}{u_2v_1}$.

We now define ψ_1 and θ in the following way:

$$\psi_1=\underline{\bigvee}^{v_2(v_1-u_1)}\phi_{\frac{1}{u_2v_1}},$$

$$\theta=\underline{\bigvee}^{u_2v_1}\phi_{\frac{1}{u_2v_1}}$$

with $\phi_{\frac{1}{u_2v_1}}=\neg p\wedge p^{u_2v_1-1}$, for $p\in L$.

Define ψ_2 as in Case 1 and set $\Gamma=\{\psi_1\underline{\vee}\psi_2\}$. $\mathcal{L}_{\Gamma,\theta}$ will be as stated. ■

Proposition 17: (**Type 4**)

Let $r,s\in[0,1]\cap\mathbb{Q}$, with $r<s$. We can define $\Gamma\cup\{\theta\}\subseteq SL$ for which $\mathcal{L}_{\Gamma,\theta}(\eta)=(s-r)\eta+r$.

Proof: Let $r=\frac{u_1}{v_1}<s=\frac{u_2}{v_2}$. Take $s-r=\frac{u_2v_1-u_1v_2}{v_1v_2}$ and define ψ and θ_1 as follows:

$$\psi=\underline{\bigvee}^{v_1v_2}\phi_{\frac{1}{v_1v_2}},$$

$$\theta_1=\underline{\bigvee}^{u_2v_1-u_1v_2}\phi_{\frac{1}{v_1v_2}},$$

where $\phi_{\frac{1}{v_1v_2}}=\neg p\wedge p^{v_1v_2-1}$, for $p\in L$.

Let us define θ_2 as follows:

$$\theta_2=\neg(\underline{\bigvee}^{u_1}\phi_{\frac{1}{v_1}}).$$

Here $\phi_{\frac{1}{v_1}}=\neg q\wedge q^{v_1-1}$, with $q\in L$ and $q\neq p$.

By setting $\theta=\theta_1\underline{\vee}\theta_2$ and $\Gamma=\{\psi\}$ we get $\mathcal{L}_{\Gamma,\theta}$ of the form desired.

If $r=0$ then we set $\theta=\theta_1$. ■

Proposition 18: (**Type 5**)

Let $r,s\in[0,1]\cap\mathbb{Q}$. We can define $\Gamma\cup\{\theta\}\subseteq SL$ for which $\mathcal{L}_{\Gamma,\theta}$ has the following form:

$$\mathcal{L}_{\Gamma,\theta}(\eta)=\begin{cases}\eta(\frac{1-r}{s})+r & \text{if } \eta\leq s\\ 1 & \text{otherwise}\end{cases}$$

Proof: Let $0<r=\frac{u_1}{v_1}$ and $s=\frac{u_2}{v_2}$. We have to distinguish two possible cases:

Case 1. $\frac{1-r}{s}>1$.

Consider $\frac{s}{1-r}=\frac{u_2v_1}{v_2(v_1-u_1)}$ and define ψ and θ_1 as follows:

$$\psi=\underline{\bigvee}^{u_2v_1}\phi_{\frac{1}{v_2(v_1-u_1)}},$$

$$\theta_1=\underline{\bigvee}^{v_2(v_1-u_1)}\phi_{\frac{1}{v_2(v_1-u_1)}},$$

with $\phi_{\frac{1}{v_2(v_1-u_1)}}=\neg p\wedge p^{v_2(v_1-u_1)-1}$, for $p\in L$.

On the other hand define θ_2 as follows:

$$\theta_2=\neg(\underline{\bigvee}^{u_1}\phi_{\frac{1}{v_1}}),$$

with $\phi_{\frac{1}{v_1}}=\neg q\wedge q^{v_1-1}$, for $q\in L$ and $q\neq p$.

Set $\theta=\theta_1\underline{\vee}\theta_2$ and $\Gamma=\{\psi\}$. The function $\mathcal{L}_{\Gamma,\theta}$ will be as desired.

If $r=0$ then we can set $\theta=\theta_1$.

As with Type 1 and Type 2, McNaughton's Theorem guarantees the existence of sentences $\bigwedge\Gamma$ and θ in one variable (say $p\in L$), with $\bigwedge\Gamma$ Ł$_1$-consistent, such that $\mathcal{L}_{\Gamma,\theta}$ is of the required form. Consider for example ϕ and ψ for which $f_\phi(x)$ and $f_\psi(x)$ are defined as those seen previously for Type 2:

$$f_\phi(x)=\begin{cases}bx & \text{if } x\leq\frac{1}{b}\\ 1 & \text{otherwise}\end{cases}$$

$$f_\psi(x)=\begin{cases}ax & \text{if } x\leq\frac{1}{a}\\ 1 & \text{otherwise}\end{cases}$$

Here $a,b\in\mathbb{N}$ and $\frac{a}{b}=\frac{1-r}{s}$.

Set $\Gamma=\{\phi\}$ and $\theta=\{\psi\underline{\vee}\theta_2\}$, where

$$\theta_2=\neg(\underline{\bigvee}^{u_1}\phi_{\frac{1}{v_1}})$$

and $\phi_{\frac{1}{v_1}}=\neg q\wedge q^{v_1-1}$, with $q\in L$ and $q\neq p$.

Clearly $\mathcal{L}_{\Gamma,\theta}$ will be as stated, with Γ Ł$_1$-consistent.

Case 2. $\frac{1-r}{s}\leq 1$.

Consider $\frac{1-r}{s}=\frac{v_2(v_1-u_1)}{u_2v_1}$ and define ψ and θ_1 as follows:

$$\psi=\underline{\bigvee}^{u_2v_1}\phi_{\frac{1}{u_2v_1}},$$

$$\theta_1=\underline{\bigvee}^{v_2(v_1-u_1)}\phi_{\frac{1}{u_2v_1}},$$

where $\phi_{\frac{1}{u_2v_1}}=\neg p\wedge p^{u_2v_1-1}$, for $p\in L$.

Define θ_2 as in Case 1 and set $\theta=\theta_1\underline{\vee}\theta_2$ and $\Gamma=\{\psi\}$. The function $\mathcal{L}_{\Gamma,\theta}$ will be as desired.

For $r=0$ we dispense again with θ_2. ■

B. Compound graphs

Let L_1, L_2 be two disjoint languages and SL_1, SL_2 their respective sets of sentences. Take $\Gamma_1\subseteq SL_1$, $\Gamma_2\subseteq SL_2$ and $\theta_1\in SL_1$, $\theta_2\in SL_2$. Assume that $\Gamma=\Gamma_1\cup\Gamma_2$ is maximally Ł$_\lambda$-consistent.

Proposition 19: For all $\eta\in[0,1]$,

$$\max\{\mathcal{L}_{\Gamma_1,\theta_1}(\eta),\mathcal{L}_{\Gamma_2,\theta_2}(\eta)\}=\mathcal{L}_{\Gamma_1\cup\Gamma_2,\theta_1\vee\theta_2}(\eta).$$

Proof: It follows trivially from the interpretation of the connective '$\vee$'. ■

Proposition 20: For all $\eta\in[0,\lambda]$,

$$\min\{\mathcal{L}_{\Gamma_1,\theta_1}(\eta),\mathcal{L}_{\Gamma_2,\theta_2}(\eta)\}=\mathcal{L}_{\Gamma_1\cup\Gamma_2,\theta_1\wedge\theta_2}(\eta).$$

Proof: It follows trivially from the interpretation of '$\wedge$'. ■

We can extend these propositions to any finite collection of sets of sentences $\Gamma_1\subseteq SL_1,...,\Gamma_k\subseteq SL_k$ and $\theta_1\in SL_1,...,\theta_k\in SL_k$, for some $k\in\mathbb{N}$, with $L_1,...,L_k$ a collection of pairwise disjoint languages.

VI. Representation theorem

At this point we have all the intermediate results necessary for the representation theorem that we finally present in this section.

Theorem 21: **Representation Theorem**. The function $\mathcal{F} : [0,1] \longrightarrow [0,1]$ is of the form $\mathcal{L}_{\Gamma,\theta}$ for some $\Gamma \cup \{\theta\} \subseteq SL$ if and only if $\mathcal{F}$ is an increasing function of the following form:

$$\mathcal{F}(x) = \begin{cases} a_1x + b_1 & \text{if } x \leq \lambda_1 \\ ... & \\ a_kx + b_k & \text{if } \lambda_{k-1} < x \leq \lambda_k \end{cases}$$

with $a_i, b_i, \lambda_i \in \mathbb{Q}$ and $k \in \mathbb{N}$, $i \in \{1, ..., k\}$.

Proof: If the function $\mathcal{F} : [0,1] \longrightarrow [0,1]$ is of the form $\mathcal{L}_{\Gamma,\theta}$ for some $\Gamma \cup \{\theta\} \subseteq SL$ then we know, by Propositions 11 and 13, that $\mathcal{F}$ will be an increasing function of the form stated in the theorem.

Let us prove now the left implication.

Let $\mathcal{F} : [0,1] \rightarrow [0,1]$ be as stated.

We will denote the line segment given by $a_ix + b_i$ and $\lambda_{i-1} < x \leq \lambda_i$ by l_i, for $i \in \{2, ..., k\}$ (l_1 will be the line segment given by $a_1x + b_1$ and $x \leq \lambda_1$).

Let us define Γ and θ for which $\mathcal{L}_{\Gamma,\theta}(\eta) = \mathcal{F}(\eta)$ for all $\eta \in [0,1]$.

First, let l_i be a line segment of $\mathcal{F}$, $i \in \{1, ..., k\}$ (without loss of generality we can assume that $i \neq 1$). We can define $\Gamma_i \subseteq SL$ Ł$_1$-consistent and $\theta_i \in SL$ for which $\mathcal{L}_{\Gamma_i,\theta_i}$ is as follows:

$$\mathcal{L}_{\Gamma_i,\theta_i}(x) = \begin{cases} a_i\lambda_{i-1} + b_i & \text{if } x \leq \lambda_{i-1} \\ a_ix + b_i & \text{if } \lambda_{i-1} < x \leq \lambda_i \\ 1 & \text{otherwise} \end{cases}$$

To see this set

$$\mathcal{L}_{\Gamma_i,\theta_i}(\eta) = \max\{\mathcal{L}_{\Delta_1,\psi_1}(\eta), \max\{\mathcal{L}_{\Delta_2,\psi_2}(\eta), \mathcal{L}_{\Delta_3,\psi_3}(\eta)\}\}$$

for all $\eta \in [0,1]$, with $\Delta_j \subseteq SL_j$ Ł$_1$-consistent and $\psi_j \in SL_j$ for all $j \in \{1,2,3\}$, where L_1, L_2, L_3 are pairwise disjoint languages.

$\mathcal{L}_{\Delta_1,\psi_1}$ and $\mathcal{L}_{\Delta_2,\psi_2}$ are of *Type 1*:

$$\mathcal{L}_{\Delta_1,\psi_1}(x) = \begin{cases} 0 & \text{if } x \leq \lambda_i \\ 1 & \text{otherwise} \end{cases}$$

$$\mathcal{L}_{\Delta_2,\psi_2}(x) = \quad a_i\lambda_{i-1} + b_i \quad for\ all\ x \in [0,1]$$

The nature of the straight line $a_ix + b_i$ will determine the type of graph of $\mathcal{L}_{\Delta_3,\psi_3}$. We will choose Δ_3 and ψ_3 such that the graph of $\mathcal{L}_{\Delta_3,\psi_3}$ contains the straight segment $a_ix + b_i$, for $\lambda_{i-1} < x \leq \lambda_i$. That $\mathcal{L}_{\Delta_3,\psi_3}$ will be of one of the types described in the previous subsection is clear.

It can easily be seen that

$$\mathcal{F}(\eta) = \mathcal{L}_{\bigcup \Gamma_i, \bigwedge \theta_i}(\eta) = \min\{\mathcal{L}_{\Gamma_i,\theta_i}(\eta) \mid i \in \{1, ..., k\}\}$$

for all $\eta \in [0,1]$, with $\Gamma_1 \subseteq SL_1, ..., \Gamma_k \subseteq SL_k$, $\theta_1 \in SL_1, ..., \theta_k \in SL_k$ and $L_1, ..., L_k$ a pairwise disjoint collection of languages. ∎

VII. Conclusion

We have introduced the consequence relation ${}^{\eta} \blacktriangleright_{\zeta}$ for graded inference built upon Łukasiewicz semantics. We have studied the behaviour of ${}^{\eta} \blacktriangleright_{\zeta}$ in relation to variations of the thresholds η, ζ through the analysis of the functions of the form $\mathcal{L}_{\Gamma,\theta}$ defined in Section IV, for $\Gamma \cup \{\theta\}$ a set of sentences. Both in Section IV and Section V we have produced results necessary for the proof of the representation theorem given in Section VI, which fully characterizes the functions of the form just mentioned and thus the pairs of thresholds (η, ζ) for which $\Gamma^{\eta} \blacktriangleright_{\zeta} \theta$ holds.

Much is left to be analysed about the consequence relation ${}^{\eta} \blacktriangleright_{\zeta}$. A sound and complete proof system for such relation is yet to be found.

Acknowledgment

The author would like to thank Jeff B. Paris for his helpful remarks. This research has been supported by the UK Engineering and Physical Sciences Research Council (EPSRC) in the form of a Research Studentship and by the Vienna Science and Technology Fund ($WWTF$), Grant $MA07-016$.

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