Plausible reasoning and graded information: a unified approach

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Abstract

We propose a formal method for reasoning both under uncertainty and under vagueness in a coherent way. We deal with implicational relationships where an explicit numerical degree is used to express uncertainty. The approach relies on Dubois and Prade's Possibilistic Logic. Furthermore, we take the possible vagueness of the involved properties into account. Namely, we deal with properties of the form that some vague criterion is fulfilled to a specific degree. Thus vague properties are treated as parametrised sets of crisp properties. Finally, a rule is included to ensure smoothness of the uncertainty degree with regard to changes of the degrees to which the properties under consideration hold.

The calculus is applicable wherever graded properties are subject to uncertainty. Vagueness and uncertainty are treated independently, but can optionally be interconnected in a controlled way. A specific application suggests itself in the field of medical expert systems.

Keywords: Reasoning about uncertainty, Reasoning under vagueness, Possibilistic Logic, Gradedness, Fuzzy set theory.

1 Introduction

The difficult task to represent experience-based knowledge includes the necessity to account for two basic aspects: uncertainty and vagueness. For the formalisation of knowledge in an area of natural sciences these aspects are met inevitably. For instance, knowledge representation in medicine requires a framework which is both capable of representing uncertain information and flexible enough to cope with the vagueness of the involved notions.

In this paper we propose a formalism which treats both uncertainty and vagueness. Uncertainty is understood as plausibility and our general framework is Dubois and Prade's Possibilistic Logic [5]. Furthermore, we will not deal directly with vagueness, as done, e.g., in [7, 4]; we will rather formalise the gradedness of properties.

Modelling uncertainty

The development of formal systems dealing with uncertainty is the subject of a lively research field. Numerous formal systems for reasoning under uncertainty have been proposed in the past and several branches have emerged. For a recent overview we refer to the comprehensive paper [6] and the references given there; among the introductory monographs we may mention [16] and [12]. The picture is rather inhomogeneous; the approaches are motivated by different types of applications and by different ways to understand uncertainty.

Here, we are guided by the following considerations. We assume to be in a situation where we do not have the possibility to tell about the truth or falsity of certain facts. We treat this uncertainty as ignorance, leaving it open if the facts under consideration actually generally hold, sometimes hold, or generally do not hold. We furthermore quantify our ignorance numerically; this simply mirrors the fact that we can be uncertain about something to a smaller or larger extent. To this end, we use real values between 0 and 1.

Assume, for instance, that we want to express that some property α implies another property β. Assume furthermore that the truth of this claim is actually not established and the claim is just found plausible to a certain extent. We shall formally express such statements by

$$\alpha \overset{d}{\Rightarrow} \beta, \tag{1}$$

where d is an element of the real unit interval $[0, 1]$ and chosen the higher the more plausible the implication is. The role of d is as follows. To conclude β from α is the more plausible the less plausible the situation specified by $\alpha \wedge \neg\beta$ is. We will in this paper actually not work with degrees of plausibility, but dually with degrees of "implausibility" or "surprise"; cf. [10]. Thus, d quantifies the degree of implausibility of $\alpha \wedge \neg\beta$.

Our logic deals with what can be derived from available facts by means of implicational relationships; and both the available facts and the implications are possibly of limited plausibility. (1) is the example of an implication. Moreover, to express that it is plausible to the degree t that a fact α holds, we use the implication

$$\top \overset{d}{\Rightarrow} \alpha, \tag{2}$$

where $\top$ is the constant "true". (2) actually says that $\neg\alpha$ is implausible to the degree d; but this amounts to the statement that α is plausible to the degree d.

We note that we do not need anything more in our logic than implication of the form (1). In fact, only implications (1) will be allowed, together with the possibility to use multiple assumptions. In particular, there will be no implication connective. This procedure has a practical advantage: the meaning of nested implications is in general hard to explain and is avoided here since not needed. In addition, as we will see, the procedure has also advantages from the theoretical viewpoint: the formulae we deal with can be used without modification as the syntactical objects of our proof system.

The next point to be clarified is how the degree of implausibility should be related to the partial order. It is natural to assume antitonicity: if α is a property stronger than β, then β should be less implausible than α. We shall go further: We also let the implausibility of disjunctions be lower bounded by a function combining the implausibilities of the disjuncts. As a combining function we allow any fixed t-norm.

In practise, we guess that one of the three basic continuous t-norms, the Łukasiewicz, product, or Gödel t-norm, will be used; and among these, the Gödel t-norm will be most important. In the first two cases we are led to (the dual analog of) subadditive measures. In the third case we arrive at possibility measures; the implausibility of $\alpha \vee \beta$ is in this case uniquely determined as the smaller value among the implausibility of α and β.

In this latter case we are led to Possibilistic Logic, a logic which was introduced by Dubois and Prade and has been intensively developed since then. For an overview see, e.g., [5]. Moreover, an axiomatisation of first-order Possibilistic Logic is presented by J. Lang in [14].

Modelling vagueness

Having chosen a calculus to deal with uncertainty, the aim of the paper is to incorporate the vagueness of properties. Vague properties have been dealt with in the framework of Possibilistic Logic several times; see, for instance, [7, 4]. However, these approaches are not comparable to ours. Here, we do not directly deal with vague properties; we rather deal with sets of crisp properties which stepwise or continuously lead from the full truth to the full falsity of a vague property. Possibilistic reasoning about "properly" vague properties is the subject of [7, 4], where possibilistic measures are generalised from Boolean algebras to MV-algebras and Gödel algebras of fuzzy sets, respectively.

Vagueness can be characterised along the following lines [18]. Vague properties involve two levels of perception. A property expressed in natural language, like "large" relies on a coarse level of perception. To say that something is large means to say it is observably larger than average-sized. To model the situation it is sufficient to take, for instance, the three categories "small", "average-sized", and "large". We furthermore have a fine level of perception, which is the result of an iterative process reflecting the underlying intuition, which is "size" in our example. We are then led to the structures commonly used in mathematics, like, e.g., the positive reals. If both levels are dealt with, the elements of the coarse structure need to be represented within the fine structure. A common way to do so is use fuzzy sets.

In this paper, we follow this standard approach. Let a property α be modelled by a fuzzy set u with the domain S. Let furthermore a degree $t \in [0,1]$ be given, and let A be the set of $x \in S$ such that $u(x) = t$. We may then say that A models the property that α holds to the degree t. In other word, to each property α and each $t \in [0,1]$ we may associate the statement that α holds to the degree t. We will symbolise this statement by (α, t); note that (α, t) is crisp.

It is this kind of statements we shall deal with. Their collection forms a Boolean algebra and possibilistic reasoning can be applied to them in a straightforward way.

The paper is organised as follows. In Section 2, we introduce our framework, which is a slight generalisation of Possibilistic Logic. But in fact, only Possibilistic Logic itself plays a role in the remainder of the paper. In Section 3, we extend the calculus so as to include reasoning with graded vague properties, whose model will be refined in Section 4. A rule with the effect that the implausibility degree cannot arbitrarily "jump" under slight changes of the degrees to which the involved properties hold is introduced in Section 5. Finally, Section 6 contains, in addition to a summary, the example of an application of the formalism in a medical expert system.

2 Generalised possibilistic logic

In his Ph.D. Thesis [14], Lang presents an axiomatisation of Possibilistic Logic. In the present section, we will provide a logical calculus which is very similar to Lang's. We will however make use of the fact that a simple generalisation is possible.

As usual we distinguish the content level and the belief level. On the content level, we refer to the factual content of our reasoning. This is a set of situations about which we assume that exactly one of it always holds. We distinguish between these situations classically: we choose a set of yes-no properties; each property holds in a given situation or not; and each considered situation is uniquely determined by knowing which properties hold and which do not hold in it. So in particular, a property is identified with a subset of a fixed set of situations.

We do not assume that a property can be checked to hold or not to hold in any situation. This gives rise to a second level, called the belief level. Here the subjective sphere of the "agent", that is, the one who reasons about the given set of situations, comes into play. Namely, we allow statements about the mutual relationship of properties which rely on possibly non-conclusive experience. That is, we allow to take into account knowledge about relationships between properties even if this knowledge is speculative. If $\alpha_1, \ldots, \alpha_k, \beta$ denote properties, a typical statement will look like

$$\alpha_1, \ldots, \alpha_k \stackrel{d}{\Rightarrow} \beta; \tag{3}$$

here d is an element of the real unit interval expressing the agent's ignorance in a quantitative way. Starting from $d = 1$ expressing certainty, we may decrease d to express a reduced confidence in the correctness of the implication (3), which for $d = 0$ is true in any case.

Note that, at least in principle, the value d in (3) can always be provided; in the worst case, if the agent does not have a clue, $d = 0$ is chosen.

Let's now proceed to specify our formal framework. There is nothing special on the content level. As usual, situations are modelled by an unstructured set S and each property by a subset of S. We use the usual connectives, namely conjunction, disjunction, and negation; and we will include the always false and always true property as well. We note that the classical implication plays no role. The interpretation of the connectives is standard. In short, the collection of properties is modelled by a Boolean algebra $\mathcal{B}$ of subsets of S.

To see how we proceed on the belief level, consider two elements $A, B \in \mathcal{B}$ and let us identify them with the two properties which they model. The degree of uncertainty about the question if we can conclude B from A should depend in some way on numerical values associated to the elements of the Boolean subalgebra generated by A and B. In contrast to the probabilistic approach, where we have a measure on $\mathcal{B}$ and take the quotient of the values associated to $A \wedge B$ and A, we follow here a much simpler way, adopting the concepts of Possibilistic Logic. Namely, we will use only one element of the subalgebra, namely $A \wedge \neg B$, to which we associate a "degree of surprise" $d = \varrho(A \wedge \neg B)$. The larger d is, the less plausible is a situation in which A holds and B does not hold. This will be our interpretation of the statement that A implies B to the degree d.

We may understand this degree of implausibility also "positively", namely as a degree of certainty, in the straightforward way. To say that A implies B to the degree 1 means that A implies B; that is, 1 expresses full certainty. Similarly, to say that A implies B to the degree 0 means not to say anything; this relationship holds between an arbitrary pair of properties. The remaining values refer to a smaller or larger degree of certainty, and specific values of certainty strictly in between 0 and 1 refer to subjectively quantified amount of certainty.

So our basic model consists of the Boolean algebra $\mathcal{B}$ together with a mapping ϱ from $\mathcal{B}$ to the real unit interval $[0, 1]$. We call ϱ here a rejection function; with regard to the setting of [5], ϱ plays the role of a necessity measure (or alternatively of a possibility measure). If $A \in \mathcal{B}$, $\varrho(A)$ expresses the degree to which the property modelled by A would be found surprising if found to hold. We assume ϱ to be order-reversing and $\varrho(1) = 0$. Furthermore we assume that $\mathcal{B}$ does not model situations which are considered as definitely impossible; so we require that $\varrho(A) = 1$ holds exactly if $A = 0$. Furthermore, a property may consist of alternatives, say, we may have $A = B \vee C$. In this case, we allow the assumption that A is not more surprising than B or C and also not less than both; then $\varrho(A) = \min \{\varrho(B), \varrho(C)\}$. However, we also allow to use an alternative combining function as long as it is fixed; our actual assumption about ϱ is that $\varrho(A) \geq \varrho(B) \odot \varrho(C)$ for some fixed t-norm $\odot$.

Let us fix a t-norm $\odot \colon [0, 1] \times [0, 1] \to [0, 1]$. Furthermore, we will denote the operations minimum, maximum, and standard negation on $[0, 1]$ by $\wedge$, $\vee$, $\sim$, respectively.

Definition 2.1. Let $(\mathcal{B}; \wedge, \vee, \neg, 0, 1)$ be a Boolean algebra. A *rejection function* on $\mathcal{B}$ w.r.t. $\odot$ is a mapping $\varrho \colon \mathcal{B} \to [0, 1]$ such that, for any $A, B \in \mathcal{B}$, (i) $\varrho(1) = 0$, (ii) $\varrho(A) = 1$ if and only if $A = 0$, (iii) $A \leq B$ implies $\varrho(B) \leq \varrho(A)$, and (iv) we have

$$\varrho(A \vee B) \;\geq\; \varrho(A) \odot \varrho(B).$$

A pair $(\mathcal{B}, \varrho)$ of a Boolean algebra $\mathcal{B}$ and a rejection function ϱ on $\mathcal{B}$ will be called a *Boolean uncertainty algebra*.

Let us consider the case that $\odot = \wedge$; this choice for $\odot$ will actually be predominant in our paper. Then condition (iv) can be formulated as

$$\varrho(A \vee B) \;=\; \varrho(A) \wedge \varrho(B), \tag{4}$$

where we have made use of the antitonicity of ϱ. It further follows that

$$N\colon\, \mathcal{B} \to [0,1], \quad A \mapsto \varrho(\neg A)$$

is a necessity measure – see [5] –, and our logic will turn out to be equivalent with Possibilistic Logic.

We proceed with the model-theoretic definition of what we call Generalised Possibilistic Logic, denoted by $\mathtt{I}^{\odot}$, where the "$\mathtt{I}$" stands for "ignorance". The choice of this name is motivated by the fact that $\mathtt{I}^{\wedge}$ is essentially Possibilistic Logic and $\mathtt{I}^{\odot}$ is a straightforward generalisation of $\mathtt{I}^{\wedge}$. In this paper, $\mathtt{I}^{\wedge}$ is still most important, and we will write in the sequel $\mathtt{I}$ instead of $\mathtt{I}^{\wedge}$.

Our language will be finite; let's fix a number $N \geq 1$ of variable symbols. Several results in the sequel would remain the same if we allowed a countably infinite set of variables; however, we do not see an important reason to do so.

Definition 2.2. The *propositions* of $\mathtt{I}^{\odot}$ are built up from *variables* $\varphi_1, \ldots, \varphi_N$ and the *truth constants* $\bot, \top$ by means of the binary connectives $\wedge, \vee$ and the unary connective $\neg$. We will denote the set of propositions by $\mathcal{P}$.

An *implication* of $\mathtt{I}^{\odot}$ is a triple consisting of a finite non-empty set of propositions $\alpha_1, ..., \alpha_k$, a proposition β, and an element d of the real unit interval, denoted

$$\alpha_1, ..., \alpha_k \stackrel{d}{\Rightarrow} \beta.$$

Here $\alpha_1, ..., \alpha_k$ are called the *antecedents*, β is the *succedent*, and d is the *degree of certainty*.

An *evaluation* for $\mathtt{I}^{\odot}$ is a mapping v from $\mathcal{P}$ to a Boolean uncertainty algebra $(\mathcal{B}, \varrho)$ such that $v(\alpha \wedge \beta) = v(\alpha) \wedge v(\beta)$, $v(\alpha \vee \beta) = v(\alpha) \vee v(\beta)$, $v(\neg\alpha) = \neg v(\alpha)$ for $\alpha, \beta \in \mathcal{P}$ and $v(\bot) = 0$, $v(\top) = 1$. An implication $\alpha_1, ..., \alpha_k \stackrel{d}{\Rightarrow} \beta$ is then *satisfied* by v if

$$\varrho(v(\alpha_1 \wedge \ldots \wedge \alpha_k \wedge \neg\beta)) \;\geq\; d.$$

A *theory* is a set of implications. We say that a theory $\mathcal{T}$ *semantically entails* an implication $\alpha_1, ..., \alpha_k \stackrel{d}{\Rightarrow} \beta$ if, for all evaluations v, whenever all elements of $\mathcal{T}$ are satisfied by v, then also $\alpha_1, ..., \alpha_k \stackrel{d}{\Rightarrow} \beta$ is satisfied by v.

We axiomatise the logic $\mathtt{I}^{\odot}$ as follows. Here, rules are pairs of a possibly empty finite set of implications and one further implication. The Greek Γ denotes a finite set of antecedents, and as usual, expressions like Γ, α, β denote $\Gamma \cup \{\alpha, \beta\}$, where it is not assumed that these sets must not overlap or that α and β must be distinct. The set Γ can be empty; recall however that an implication has at least one antecedent; thus in an expression like $\Gamma \stackrel{d}{\Rightarrow} \alpha$, Γ must be non-empty.

For the case that $\odot = \wedge$ the following calculus is the propositional part of Lang's calculus in [14], just presented in a modified way. The main point concerns our use of a rejection function. Indeed, instead of a necessity measure, we use the complemented possibility measure.

Definition 2.3. The following are the rules of $\mathtt{I}^{\odot}$, where α, β, γ are propositions, Γ is a finite set of propositions, and $c, d \in [0,1]$:

$$\bot \overset{d}{\Rightarrow} \alpha \qquad \alpha \overset{d}{\Rightarrow} \alpha \qquad \alpha \overset{d}{\Rightarrow} \top \qquad \alpha, \neg\alpha \overset{d}{\Rightarrow} \bot$$

$$\alpha \overset{0}{\Rightarrow} \beta \qquad \frac{\Gamma \overset{d}{\Rightarrow} \alpha}{\Gamma \overset{c}{\Rightarrow} \alpha} \text{ where } c < d \qquad \frac{\Gamma \overset{c}{\Rightarrow} \alpha \quad \alpha \overset{d}{\Rightarrow} \beta}{\Gamma \overset{c \odot d}{\Rightarrow} \beta}$$

$$\frac{\Gamma \overset{d}{\Rightarrow} \alpha}{\Gamma, \beta \overset{d}{\Rightarrow} \alpha} \qquad \frac{\Gamma, \alpha, \beta \overset{d}{\Rightarrow} \gamma}{\Gamma, \alpha \wedge \beta \overset{d}{\Rightarrow} \gamma} \qquad \frac{\Gamma \overset{c}{\Rightarrow} \alpha \quad \Gamma \overset{d}{\Rightarrow} \beta}{\Gamma \overset{c \odot d}{\Rightarrow} \alpha \wedge \beta}$$

$$\frac{\Gamma, \alpha \overset{c}{\Rightarrow} \gamma \quad \Gamma, \beta \overset{d}{\Rightarrow} \gamma}{\Gamma, \alpha \vee \beta \overset{c \odot d}{\Rightarrow} \gamma} \qquad \frac{\Gamma \overset{d}{\Rightarrow} \alpha}{\Gamma \overset{d}{\Rightarrow} \alpha \vee \beta} \qquad \frac{\Gamma \overset{d}{\Rightarrow} \beta}{\Gamma \overset{d}{\Rightarrow} \alpha \vee \beta}$$

$$\frac{\alpha \overset{d}{\Rightarrow} \beta}{\neg\beta \overset{d}{\Rightarrow} \neg\alpha} \qquad \frac{\neg\alpha \overset{d}{\Rightarrow} \beta}{\neg\beta \overset{d}{\Rightarrow} \alpha} \qquad \frac{\alpha \overset{d}{\Rightarrow} \neg\beta}{\beta \overset{d}{\Rightarrow} \neg\alpha}$$

The notion of a proof of an implication $\alpha \overset{d}{\Rightarrow} \beta$ from a theory $\mathcal{T}$ is defined in the usual way. We write $\mathcal{T} \vdash \alpha \overset{d}{\Rightarrow} \beta$ if there exists one.

A theory $\mathcal{T}$ is called *consistent* if $\mathcal{T} \vdash \top \overset{d}{\Rightarrow} \bot$ implies $d = 0$.

The proof of the completeness Theorem 2.7 for $\mathtt{I}^{\odot}$ below is possible along routine lines; we calculate the Lindenbaum algebra associated to a given theory, and the maximal d such that the theory proves $\alpha \overset{d}{\Rightarrow} \bot$ is taken as the value to which ϱ maps the equivalence class of α. For $\mathtt{I}$, a proof is moreover contained in [14].

In spite of this, we will devote the remainder of this section to present a fully detailed proof which is even more involved than necessary. We do so because in the subsequent sections, we will present three logics which are successively more special than $\mathtt{I}^{\odot}$; we shall proceed then in full analogy to the easy case discussed here.

For propositions α and β, we write $\alpha \to \beta$ to abbreviate $\neg\alpha \vee \beta$.

Lemma 2.4. *Let α, β be propositions of $\mathtt{I}^{\odot}$. Then $\vdash \alpha \overset{1}{\Rightarrow} \beta$ if and only if $\alpha \to \beta$ is a tautology of classical propositional logic.*

Proof. Let $\mathtt{I}^{\odot}$ prove $\alpha \overset{1}{\Rightarrow} \beta$. Since the degree associated to the conclusion is in each rule smaller or equal to each degree in the assertions, a proof $\alpha \overset{1}{\Rightarrow} \beta$ can be assumed to involve the degree 1 only. It follows $\alpha \to \beta$ is a classical tautology.

Conversely, assume that $\alpha \to \beta$ is a classical tautology. Then the sequent $\alpha \vdash \beta$ is derivable in Gentzen's calculus for classical propositional logic [8]. By obvious modifications this proof can be transformed into a proof of $\alpha \stackrel{1}{\Rightarrow} \beta$ in $\mathtt{I}^{\odot}$, where all degrees equal 1. $\square$

For some set Ω, let us denote the Boolean algebra of subsets of Ω by $P\Omega$. For instance, $P\{0,1\}$ is the set of subsets of $\{0,1\}$ and can be identified with the two-element Boolean algebra.

Furthermore, $P\{0,1\}^N$ denotes the free Boolean algebra with N generators. We will identify the latter with the Boolean algebra of subsets of $\{0,1\}^N$, that is, with $P(\{0,1\}^N)$.

We will now consider the Lindenbaum algebra associated to $\mathtt{I}^{\odot}$, where equivalence of propositions will mean that one implies the other one to the degree 1. Accordingly, the somewhat loose statements "α implies β" and "α and β are equivalent" mean that $\mathtt{I}^{\odot}$ proves $\alpha \stackrel{1}{\Rightarrow} \beta$, or both $\alpha \stackrel{1}{\Rightarrow} \beta$ and $\beta \stackrel{1}{\Rightarrow} \alpha$, respectively. Analogous remarks apply to all logics considered in subsequent sections as well.

Lemma 2.5. *For propositions α and β of $\mathtt{I}^{\odot}$, we put $\alpha \approx \beta$ if $\mathtt{I}^{\odot}$ proves $\alpha \stackrel{1}{\Rightarrow} \beta$ and $\beta \stackrel{1}{\Rightarrow} \alpha$. Then the quotient $\langle \mathcal{P} \rangle$ of $\mathcal{P}$ w.r.t. $\approx$, endowed with the induced operations $\wedge, \vee, \neg$ and the constants $\langle \perp \rangle, \langle \top \rangle$, is a Boolean algebra isomorphic to $\bar{\mathcal{B}} = P\{0,1\}^N$. The isomorphism is given by*

$$w(\langle \varphi_i \rangle) \;=\; \{(r_1, \ldots, r_N) \in \{0,1\}^N \colon r_i = 1\}, \quad i = 1, \ldots, N$$

Furthermore, let $\varrho \colon \bar{\mathcal{B}} \to [0,1]$ be 0 on all non-zero elements; then $(\bar{\mathcal{B}}, \varrho)$ is a Boolean uncertainty algebra.

Define $\bar{v} : \mathcal{P} \to \{0,1\}^N, \;\; \alpha \mapsto w(\langle \alpha \rangle)$. Then $\bar{v}$ is an evaluation of $\mathtt{I}^{\odot}$ such that $\bar{v}(\alpha) = \emptyset$ if and only if $\vdash \alpha \stackrel{1}{\Rightarrow} \perp$.

Proof. By Lemma 2.4, $\langle \mathcal{P} \rangle$ is the free Boolean algebra with N generators.

Clearly, $(\bar{\mathcal{B}}, \varrho)$ is a Boolean uncertainty algebra and $\bar{v}$ is an evaluation for $\mathtt{I}^{\odot}$. Furthermore, $\mathtt{I}^{\odot}$ proves $\alpha \stackrel{1}{\Rightarrow} \perp$ iff $\langle \alpha \rangle = \langle \perp \rangle$ iff $w(\langle \alpha \rangle) = w(\langle \perp \rangle)$. Given that $\bar{v}(\alpha) = w(\langle \alpha \rangle)$ and $w(\langle \perp \rangle) = \emptyset$, the last part follows. $\square$

In the next lemma, two theories proving the same implications are called equivalent.

Lemma 2.6. *Let $\mathcal{T}$ be a theory of $\mathtt{I}^{\odot}$. Then there is a theory $\mathcal{T}'$ which is equivalent to $\mathcal{T}$ and consists of*

$$\chi_0 \stackrel{d_0}{\Rightarrow} \perp, \;\; \chi_1 \stackrel{d_1}{\Rightarrow} \perp, \;\; \ldots, \;\; \chi_m \stackrel{d_m}{\Rightarrow} \perp,$$

where $\vdash \chi_i \wedge \chi_j \stackrel{1}{\Rightarrow} \perp$ for $i \neq j$, $\vdash \top \stackrel{1}{\Rightarrow} \chi_0 \vee \ldots \vee \chi_m$, and $1 = d_0 > d_1 \geq \ldots \geq d_{m-1} > d_m = 0$.

In case $\odot = \wedge$ we may require $d_0 > d_1 > \ldots > d_m$.

Proof. We claim that $\mathcal{T}$ is equivalent to a finite theory. Indeed, any proposition appearing in $\mathcal{T}$ can be substituted by any equivalent one. Furthermore, by Lemma 2.5 there are, up to equivalence, only finite many propositions. Finally, if there are two implications in $\mathcal{T}$ differing only in the degree of certainty, the implication with the lower degree of certainty can be dropped.

Next, using Lemma 2.4, it is not difficult to see that $\alpha_1, ..., \alpha_k \overset{d}{\Rightarrow} \beta$ and $\alpha_1 \wedge \ldots \wedge \alpha_k \wedge \neg\beta \overset{d}{\Rightarrow} \bot$ are in $\mathtt{I}^{\odot}$ mutually derivable. So we assume that $\mathcal{T}$ contains only implications of the latter form.

If there are $\alpha_1 \overset{e_1}{\Rightarrow} \bot$ and $\alpha_2 \overset{e_2}{\Rightarrow} \bot$, where $e_1 \leq e_2$ and $\nvdash \alpha_1 \wedge \alpha_2 \overset{1}{\Rightarrow} \bot$, we may replace the first implication by $\alpha_1 \wedge \neg\alpha_2 \overset{e_1}{\Rightarrow} \bot$.

Finally, the implications with the degree 0 or 1 can be replaced by a single implication, combining the antecedents disjunctively. If $\odot = \wedge$, the same can be done with any implications whose degrees of certainty coincide. $\square$

On the basis of these preliminaries we prove the completeness of $\mathtt{I}^{\odot}$.

Theorem 2.7. *Let $\mathcal{T}$ be a consistent theory of $\mathtt{I}^{\odot}$ and $\Gamma \overset{e}{\Rightarrow} \delta$ an implication of $\mathtt{I}^{\odot}$. Then $\mathcal{T}$ semantically entails $\Gamma \overset{e}{\Rightarrow} \delta$ if and only if $\mathcal{T}$ proves $\Gamma \overset{e}{\Rightarrow} \delta$.*

Proof. It is easily checked that all rules are sound. The “if” part follows.

By Lemma 2.6, we can assume that $\mathcal{T} = \{\chi_0 \overset{d_0}{\Rightarrow} \bot, \chi_1 \overset{d_1}{\Rightarrow} \bot, \ldots, \chi_m \overset{d_m}{\Rightarrow} \bot\}$, where the χ_i are pairwise disjoint and jointly exhaustive, and $1 = d_1 > d_2 \geq \ldots \geq d_{m-1} > d_m = 0$.

Let $\bar{v}: \mathcal{P} \to \bar{\mathcal{B}}$ be the evaluation according to Lemma 2.5. Let $S = \bar{v}(\neg\chi_0)$, and let $\mathcal{B} = [0, S]$ be endowed with the Boolean structure induced by $\bar{\mathcal{B}}$. Let

$$v\colon \mathcal{P} \to \mathcal{B}, \quad \alpha \mapsto \bar{v}(\alpha) \cap S.$$

We furthermore define

$$\varrho\colon \mathcal{B} \to [0,1], \quad A \mapsto \bigodot \{d_i\colon 1 \leq i \leq m \text{ and } A \wedge v(\chi_i) \neq 0\},$$

where the result is 1 in case the set is empty. This is obviously a rejection function on $\mathcal{B}$. Then v is an evaluation of $\mathtt{I}^{\odot}$ in the Boolean uncertainty algebra $(\mathcal{B}, \varrho)$. Since $\varrho(v(\chi_i)) = d_i$ for all i, all elements of $\mathcal{T}$ are satisfied by v.

Assume now that $\mathcal{T}$ does not prove $\Gamma \overset{e}{\Rightarrow} \delta$. So $\mathcal{T}$ does not prove $\alpha \overset{e}{\Rightarrow} \bot$, where α is the conjunction of $\Gamma \cup \{\neg\delta\}$. Let $d = \varrho(v(\alpha))$. We easily check that $\mathcal{T} \vdash \alpha \overset{d}{\Rightarrow} \bot$. It follows $d < e$, so in particular $\Gamma \overset{e}{\Rightarrow} \delta$ is not satisfied by v and $\mathcal{T}$ does not semantically entail $\Gamma \overset{e}{\Rightarrow} \delta$. The proof of the “only if” part is complete. $\square$

We note that the model constructed in this proof is finite; so Theorem 2.7 could be reformulated to involve finite Boolean uncertainty algebras only.

3 Inclusion of graded properties: the finite case

We extend our framework to include properties which are not in every situation assumed to hold or not to hold. We still work with a Boolean algebra of subsets of some set S, and properties which, as it has been the case by now, are modelled by subsets of S will be called crisp. The variable symbols however will from now on denote vague properties and are interpreted by fuzzy sets over S. It will still be possible to reason about crisp properties because crispness is expressible in our framework.

It must be stressed that we will not allow statements of the form $\varphi \overset{d}{\Rightarrow} \psi$ where φ and ψ model vague properties. We will rather denote by (φ, s) the property that φ holds to the degree s, modelled by the subset of S consisting of those points which are mapped to s. The value s will be referred to as the degree of presence of φ in such situations. We allow then statements of the form $(\varphi, s) \overset{d}{\Rightarrow} (\psi, t)$ and, as relationships between crisp properties, we will interpret them just as before.

We will furthermore allow vague properties to be combined by means of a conjunction $\wedge$, a disjunction $\vee$, and a negation $\sim$. These connectives are interpreted by the minimum, maximum, and standard negation applied pointwise to the respective fuzzy sets. For example, for some evaluation v, $(\varphi \vee \sim\psi, t)$ is interpreted by $\{a \in S : v(\varphi)(a) \vee \sim v(\psi)(a) = t\}$.

Our t-norm will from now on always be the Gödel t-norm, that is, we put $\odot = \wedge$. So in all what follows we stay in the realm of the Possibilistic Logic $\mathtt{I}$. The more general case remains to be explored.

Furthermore, for the moment we will restrict to a finite set of degrees of presence: let fix a finite set $V \subset [0, 1]$ containing 0 and closed under $\sim$. As a matter of fact, the approach chosen in this section cannot be easily generalised to the infinite case; it has turned out that the use of an infinite set like the rational unit interval would lead to technical difficulties. In the subsequent section, we will modify our approach and the restriction will be dropped.

Definition 3.1. Let S be a nonempty set. A V-valued *fuzzy set* over S is a mapping from S to V.

For two fuzzy sets $u, v\colon\ S \to V$, we let $u \wedge v$ and $u \vee v$ be the pointwise minimum and maximum of u and v, respectively; we let $\sim u$ be the pointwise standard negation of u; and we let $\bar{0}$ and $\bar{1}$ be the constant 0 and 1 fuzzy set, respectively. Let M be a collection of V-valued fuzzy sets over S containing $\bar{0}, \bar{1}$ and closed under $\wedge, \vee, \sim$; then we call $(M; \wedge, \vee, \sim, \bar{0}, \bar{1})$ a *Kleene algebra of fuzzy sets*.

For any $u \in M$ and $t \in V$, we define

$$[u]_t \;=\; \{a \in S\colon\ u(a) = t\}.$$

The Boolean algebra of subsets of S generated by $[u]_t$, where $u \in M$ and $t \in V$, will be called the *Boolean algebra associated with* M, denoted by $\mathcal{B}_M$.

Finally, let ϱ be a rejection function on $\mathcal{B}_M$ w.r.t. $\wedge$. Then (M, ϱ) is called a *Kleene uncertainty algebra*.

We will now define the Possibilistic Logic with Sharp Gradation, denoted by $\mathtt{IG}^0$.

Definition 3.2. The *gradable propositions* of $\mathtt{IG}^0$ are built up from *variables* $\varphi_1, \ldots, \varphi_N$ and the constants $0, 1$ by means of the binary connectives $\wedge, \vee$ and the unary connective $\sim$. We denote the set of gradable propositions by $\mathcal{F}$. The *graded propositions* of $\mathtt{IG}^0$ are of the form (φ, t) where φ is a gradable proposition and $t \in V$. The *crisp propositions* of $\mathtt{IG}^0$, or *propositions* for short, are built up from graded propositions and the *truth constants* $\perp, \top$ by means of the binary connectives $\wedge, \vee$ and the unary connective $\neg$. We denote the set of propositions by $\mathcal{P}$.

An *implication* of $\mathtt{IG}^0$ is a triple consisting of a finite non-empty set of propositions $\alpha_1, ..., \alpha_k$, a proposition β, and an element $d \in V$, denoted $\alpha_1, ..., \alpha_k \stackrel{d}{\Rightarrow} \beta$.

An *evaluation* for $\mathtt{IG}^0$ is, for some Kleene uncertainty algebra (M, ϱ), a pair of mappings $v_f \colon \mathcal{F} \to M$ and $v_b \colon \mathcal{P} \to M_B$ such that the following holds:

(i) $v_f(\varphi \wedge \psi) = v_f(\varphi) \wedge v_f(\psi),\ v_f(\varphi \vee \psi) = v_f(\varphi) \vee v_f(\psi),\ v_f(\sim\varphi) = \sim v_f(\varphi)$ for gradable propositions φ, ψ, and $v_f(0) = \bar{0},\ v_f(1) = \bar{1}$;

(ii) for $\varphi \in \mathcal{F}$ and $t \in V$ we have

$$v_b((\varphi, t)) \ = \ [v_f(\varphi)]_t \tag{5}$$

and furthermore $v_b(\alpha \wedge \beta) = v_b(\alpha) \wedge v_b(\beta),\ \ v_b(\alpha \vee \beta) = v_b(\alpha) \vee v_b(\beta),$ $v_b(\alpha \wedge \beta) = \neg v_b(\alpha)$ for propositions α, β, and $v_b(\perp) = 0,\ v_b(\top) = 1$.

The notions of *satisfaction*, of a *theory*, and of *semantic entailment* is defined for $\mathtt{IG}^0$ similarly as for $\mathtt{I}$.

Note that the variables are now gradable propositions and in fact are interpreted by fuzzy sets. If this is not intended for some variable φ, we may make use of the fact that the implications $(\varphi, 0) \stackrel{1}{\Rightarrow} \neg(\varphi, 1)$ and $(\varphi, 1) \stackrel{1}{\Rightarrow} \neg(\varphi, 0)$ are satisfied only if φ is interpreted by a characteristic function; they can be asserted to ensure crispness. Moreover, if a variable φ is not going to be connected with further gradable variables, this is not even necessary; simply $(\varphi, 1)$ can be used to model a crisp property.

We axiomatise the logic $\mathtt{IG}^0$ as follows.

Definition 3.3. The rules of $\mathtt{IG}^0$ split into three groups:

The *basic rules* are those of $\mathtt{I}$ (see Def. 2.3) where propositions are understood as those of $\mathtt{IG}^0$.

The *degree-of-presence rules* are the following, where φ is a gradable proposition, and $s, s_0, ..., t \in V$:

$$(\varphi, s) \stackrel{d}{\Rightarrow} \neg(\varphi, t) \text{ where } s \neq t$$

$$\neg(\varphi, s_0), \ldots, \neg(\varphi, s_M) \stackrel{d}{\Rightarrow} \perp \text{ where } V = \{s_0, \ldots, s_M\}$$

The *fuzzy-set rules* are the following, where Γ is a finite set of propositions, φ, ψ are gradable propositions, α is a proposition, $c, d \in [0, 1]$, and $r, s, t \in V$:

$$\frac{\Gamma, (\varphi \wedge \psi, s{\wedge}t) \stackrel{d}{\Rightarrow} \alpha}{\Gamma, (\varphi, s), (\psi, t) \stackrel{d}{\Rightarrow} \alpha} \qquad \frac{\Gamma, \neg(\varphi \wedge \psi, t) \stackrel{d}{\Rightarrow} \alpha}{\Gamma, (\varphi, r), (\psi, s) \stackrel{d}{\Rightarrow} \alpha} \text{ where } r, s > t$$

$$\frac{\Gamma, \neg(\varphi \wedge \psi, t) \stackrel{d}{\Rightarrow} \alpha}{\Gamma, (\varphi, s) \stackrel{d}{\Rightarrow} \alpha} \text{ where } s < t \qquad \frac{\Gamma, \neg(\varphi \wedge \psi, t) \stackrel{d}{\Rightarrow} \alpha}{\Gamma, (\psi, s) \stackrel{d}{\Rightarrow} \alpha} \text{ where } s < t$$

$$\frac{\Gamma, (\varphi \vee \psi, s{\vee}t) \stackrel{d}{\Rightarrow} \alpha}{\Gamma, (\varphi, s), (\psi, t) \stackrel{d}{\Rightarrow} \alpha} \qquad \frac{\Gamma, \neg(\varphi \vee \psi, t) \stackrel{d}{\Rightarrow} \alpha}{\Gamma, (\varphi, r), (\psi, s) \stackrel{d}{\Rightarrow} \alpha} \text{ where } r, s < t$$

$$\frac{\Gamma, \neg(\varphi \vee \psi, t) \stackrel{d}{\Rightarrow} \alpha}{\Gamma, (\varphi, s) \stackrel{d}{\Rightarrow} \alpha} \text{ where } s > t \qquad \frac{\Gamma, \neg(\varphi \vee \psi, t) \stackrel{d}{\Rightarrow} \alpha}{\Gamma, (\psi, s) \stackrel{d}{\Rightarrow} \alpha} \text{ where } s > t$$

$$\frac{\Gamma, (\varphi, c) \stackrel{d}{\Rightarrow} \alpha}{\Gamma, ({\sim}\varphi, {\sim}c) \stackrel{d}{\Rightarrow} \alpha} \qquad \frac{\Gamma, ({\sim}\varphi, c) \stackrel{d}{\Rightarrow} \alpha}{\Gamma, (\varphi, {\sim}c) \stackrel{d}{\Rightarrow} \alpha}$$

The notion of a *proof* of some implication from a theory as well as the *consistency* of a theory is defined like for $\mathtt{I}$.

The soundness causes again no difficulties.

Theorem 3.4. *Let $\mathcal{T}$ be a theory of $\mathtt{IG}^0$ and $\Gamma \stackrel{e}{\Rightarrow} \delta$ an implication of $\mathtt{IG}^0$. Then $\mathcal{T}$ semantically entails $\Gamma \stackrel{e}{\Rightarrow} \delta$ if $\mathcal{T}$ proves $\Gamma \stackrel{e}{\Rightarrow} \delta$.*

For the completeness proof, several preparatory lemmas are necessary. Our procedure in case of the logic $\mathtt{I}^{\odot}$ will serve as a pattern.

In what follows, by a graded variable we will mean a graded proposition (φ, t) such that φ is a variable. In our first step we will show that compound gradual propositions are eliminable from the calculus; graded propositions are replaceable by Boolean combination of graded variables.

Lemma 3.5. *Let φ, ψ be gradable propositions of $\mathtt{IG}^0$, and let $t \in V$. Then $(\varphi \wedge \psi, t)$ is equivalent to*

$$\bigvee \{(\varphi, r) \wedge (\psi, s) \colon\ r = t \text{ and } s \geq t, \quad \text{or } r \geq t \text{ and } s = t\}; \tag{6}$$

$(\varphi \vee \psi, t)$ is equivalent to

$$\bigvee \{(\varphi, r) \wedge (\psi, s) \colon\ r = t \text{ and } s \leq t, \quad \text{or } r \leq t \text{ and } s = t\}, \tag{7}$$

and $({\sim}\varphi, t)$ is equivalent to

$$(\varphi, {\sim}t). \tag{8}$$

Furthermore, $\neg(\varphi, t)$ is equivalent to $\bigvee_{s \neq t} (\varphi, s)$. Finally, any proposition of $\mathtt{IG}^0$ is equivalent to the disjunction of conjunctions of graded variables.

Proof. Every disjunct in (6), so (6) itself, implies $(\varphi \wedge \psi, t)$. Furthermore, the negation of (6) is equivalent to the disjunction of $\bigvee_{r,s>t}((\varphi, r) \wedge (\psi, s))$ and $\bigvee_{r<t}(\varphi, r)$ and $\bigvee_{s<t}(\psi, s)$, each of which implies $\neg(\varphi \wedge \psi, t)$.

Similarly, we proceed for (7). The claims concerning $({\sim}\varphi, t)$ and $\neg(\varphi, t)$ are easy.

The last assertion follows, for a graded proposition, by induction over the complexity of the involved gradable proposition. For a proposition, the assertion follows by induction over its complexity. $\square$

We recall that PV is the Boolean algebra of subsets of V, where V is the set of degrees of presence. Furthermore, we denote by PV^N the N-fold free product of the Boolean algebras PV; for the notion of free products of algebras see, e.g., [11, Chapter VI]. We may, and we actually will, identify PV^N with $P(V^N)$, the Boolean algebra of subsets of V^N.

Lemma 3.6. *For propositions α and β of $\mathtt{IG}^0$, we put $\alpha \approx \beta$ if $\mathtt{IG}^0$ proves $\alpha \overset{1}{\Rightarrow} \beta$ and $\beta \overset{1}{\Rightarrow} \alpha$. Then the quotient $\langle \mathcal{P} \rangle$ of $\mathcal{P}$ w.r.t. $\approx$, endowed with the induced operations $\wedge, \vee, \neg$ and the constants $\langle \bot \rangle, \langle \top \rangle$, is a Boolean algebra isomorphic to PV^N. The isomorphism is given by*

$$w(\langle(\varphi_i, t)\rangle) \;=\; \{(r_1, ..., r_N) \in V^N \colon r_i = t\}, \quad i = 1, \ldots, N, \;\; t \in V. \tag{9}$$

Furthermore, let

$$u_i \colon V^N \to V, \;\; (r_1, ..., r_N) \mapsto r_i,$$

and let $\bar{M}$ be the Kleene algebra generated by $u_1, \ldots, u_N$. Then $\mathcal{B}_{\bar{M}} = PV^N$. Define $\varrho \colon \mathcal{B}_{\bar{M}} \to [0,1]$ to be 0 on all non-zero elements; then $(\bar{M}, \varrho)$ is a Kleene uncertainty algebra.

Define $\bar{v}_f(\varphi_i) = u_i$ for $i = 1, \ldots, N$, and extend $\bar{v}_f$ to $\mathcal{F}$ such that $\wedge, \vee, \sim$ and $\bar{0}, \bar{1}$ are preserved. Define $\bar{v}_b(\alpha) = w(\langle \alpha \rangle)$ for $\alpha \in \mathcal{P}$. Then $(\bar{v}_f, \bar{v}_b)$ is an evaluation of $\mathtt{IG}^0$ such that $\bar{v}_b(\alpha) = \emptyset$ if and only if $\vdash \alpha \overset{1}{\Rightarrow} \bot$.

Proof. Note first that in all the degree-of-presence and fuzzy-set rules, we may w.l.o.g. assume that $d = 1$. Let us modify $\mathtt{IG}^0$ as follows: We drop all fuzzy-set rules and add as new axioms the six implications expressing the equivalences of $(\varphi \wedge \psi, t)$, $(\varphi \vee \psi, t)$, $({\sim}\varphi, t)$ with the expressions (6), (7), and (8), respectively, where φ, ψ are gradable propositions and $t \in [-\zeta, 1+\zeta]$. By Lemma 3.5 all these implications are provable, and from the added axioms we may easily derive any of the dropped rules. So the change has no effect for the set of provable implications.

Note next that a proof of an implication of the form $\alpha \overset{1}{\Rightarrow} \beta$ in $\mathtt{IG}^0$ can be chosen such that all occurring degrees of certainty are equal to 1. Let $\mathtt{IG}^0_\mathtt{c}$ be the calculus differing from $\mathtt{IG}^0$ in that only degree of certainty 1 are allowed. In $\mathtt{IG}^0_\mathtt{c}$, the relation $\approx$ obviously does not change.

By Lemma 2.4, $\mathtt{IG}^0_\mathtt{c}$ can be viewed as an extension of classical propositional logic: the variables are identified with the graded propositions; and the extension consists of the axioms of $\mathtt{IG}^0_\mathtt{c}$ where each implication $\alpha \overset{1}{\Rightarrow} \beta$ is understood as $\alpha \to \beta$. We keep this viewpoint implicitly in the background. We get as an immediate consequence that $\langle\mathcal{P}\rangle$ is a Boolean algebra.

Each graded proposition in which a compound gradual proposition occurs, is by assumption equivalent to an expression in graded variables. So to determine the Boolean algebra $\langle\mathcal{P}\rangle$, we need to consider only $\langle(\varphi_i, t)\rangle$ where $i \in \{1, ..., N\}$ and $t \in V$.

Consider now the degree-of-presence rules. We can restrict them to the case of graded variables. Indeed, it is not difficult, based on an inductive argument, to derive these axioms for compound gradable propositions from those for graded variables.

So we are left with the degree-of-presence rules restricted to graded variables. These axioms split into N disjoint subsets, one for each i. Furthermore, for any $i \in \{1, ..., N\}$, the Boolean subalgebra $\langle\mathcal{P}\rangle_i$ of $\langle\mathcal{P}\rangle$ generated by $\langle(\varphi_i, t)\rangle$, $t \in V$, is clearly isomorphic to PV under the assignment

$$w_i \colon \langle\mathcal{P}\rangle_i \to PV, \quad \langle(\varphi_i, t)\rangle \mapsto \{t\}.$$

Consequently, $\langle\mathcal{P}\rangle$ itself is isomorphic to the free product of N copies of PV under that assignment (9). The proof of the first half of the theorem is complete.

Clearly, $(\bar{M}, \varrho)$ is a Kleene uncertainty algebra such that $\mathcal{B}_{\bar{M}} = PV^N$.

It is furthermore clear that $\bar{v}_b$ preserves the Boolean structure of $\mathcal{P}$, and like in the proof of Theorem 2.7 we see that $\bar{v}_b(\alpha) = \emptyset$ iff $\vdash \alpha \overset{1}{\Rightarrow} \bot$. Moreover, $\bar{v}_f$ preserves the Kleene structure of $\mathcal{F}$ by construction.

To establish that $(\bar{v}_f, \bar{v}_b)$ is an evaluation, it remains to check (5), that is, we have to show $w(\langle(\varphi, t)\rangle) = [\bar{v}_f(\varphi)]_t$ for all $\varphi \in \mathcal{F}$ and $t \in V$. If φ is a variable, this equation holds by construction. For the general case, we proceed by induction over the complexity of φ and use Lemma 3.5. □

Lemma 3.7. *Let $\mathcal{T}$ be a finite theory of* $\mathtt{IG}^0$. *Then there is a theory $\mathcal{T}'$ which is equivalent to $\mathcal{T}$ and consists of*

$$\chi_0 \overset{d_0}{\Rightarrow} \bot, \;\; \chi_1 \overset{d_1}{\Rightarrow} \bot, \;\; \ldots, \;\; \chi_m \overset{d_m}{\Rightarrow} \bot,$$

where $\vdash \chi_i \wedge \chi_j \overset{1}{\Rightarrow} \bot$ *for* $i \neq j$, $\vdash \top \overset{1}{\Rightarrow} \chi_0 \vee \ldots \vee \chi_m$, *and* $1 = d_0 > d_1 > \ldots > d_{m-1} > d_m = 0$.

Proof. As we assumed that $\mathcal{T}$ is finite, we may proceed like in the proof of Lemma 2.6. □

Theorem 3.8. *Let $\mathcal{T}$ be a consistent finite theory of* $\mathtt{IG}^0$ *and* $\Gamma \overset{e}{\Rightarrow} \delta$ *an implication of* $\mathtt{IG}^0$. *If $\mathcal{T}$ semantically entails $\Gamma \overset{e}{\Rightarrow} \delta$, then $\mathcal{T}$ proves $\Gamma \overset{e}{\Rightarrow} \delta$.*

Proof. It is easily checked that all rules are sound. The "if" part follows.

We can assume that $\mathcal{T} = \{\chi_0 \overset{1}{\Rightarrow} \bot, \; \chi_1 \overset{d_1}{\Rightarrow} \bot, \; \ldots, \; \chi_m \overset{d_m}{\Rightarrow} \bot\}$, where the χ_i are pairwise disjoint and jointly exhaustive, and $1 > d_1 > \ldots > d_m = 0$.

Let $(\bar{v}_f, \bar{v}_b)$ be the evaluation in $(\bar{M}, \varrho)$ according to Lemma 3.6. Let $S = \bar{v}_b(\neg\chi_0)$. By consistency, $\mathcal{T}$ does not prove $\neg\chi_0 \overset{1}{\Rightarrow} \bot$, hence $S \neq \emptyset$. Let M be the Kleene algebra generated by $u_i|_S$, $i = 1, \ldots, N$. Then $\mathcal{B}_M = PS$.

Let $v_f \colon \mathcal{F} \to M, \ \ \varphi \mapsto \bar{v}_f(\varphi)|_S$ and $v_b \colon \mathcal{P} \to \mathcal{B}_S, \ \ \alpha \mapsto \bar{v}_b(\alpha) \cap S$.

We define

$$\varrho \colon \mathcal{B}_M \to [0,1], \quad A \mapsto \min \{d_i \colon 1 \leq i \leq m \text{ and } A \cap v(\chi_i) \neq \emptyset\},$$

where the minimum of the empty set is 1. This is obviously a rejection function.

Then (v_f, v_b) is an evaluation in the Kleene uncertainty algebra (M, ϱ). Since $\varrho(\chi_i) = d_i$ for all i, all elements of $\mathcal{T}$ are satisfied by (v_f, v_b).

If $\mathcal{T}$ does not prove $\Gamma \overset{e}{\Rightarrow} \delta$, we conclude like in the proof of Theorem 2.7 that $\Gamma \overset{e}{\Rightarrow} \delta$ is not satisfied by v. This completes the proof of the "only if" part. $\square$

We again note that the completeness theorem could obviously be modified so as to involve finite Kleene uncertainty algebras only.

4 Inclusion of graded properties: the continuous case

A property φ is called vague if not under all circumstances it can be told if φ applies or not. We have proposed to model this generalised type of a property in the usual way: as a fuzzy set over the set of all considered situations. A vague property φ is furthermore characterised by a continuous transition from φ to non-φ. Hence it would actually make sense to allow φ to be assigned any degree of presence taken from the real unit interval $[0,1]$, rather than using a finite subset of $[0,1]$ as we did in the previous section.

The statements (φ, t), where t varies over $[0,1]$, let us then distinguish between an infinity of pairwise exclusive situations. This fact in turn is not well in line with the idea that (φ, t) reflects an agent's impression, given the fact that there is no infinity of situations observable as pairwise exclusive. Nevertheless our intention might be to work with a continuity of situations.

We have chosen the following solution. Roughly speaking, we will assume that situations which are close to each other, that is, situations characterised by similar degrees of presence of the involved properties, are not necessarily distinguishable. Let us wonder what it actually means that an agent is asked to evaluate φ and answers 0.3. In fact, such an answer might mean not more than that φ is neither true nor false but fits somewhat better to the latter possibility. Thus the agent could have chosen equally well, say, the value 0.28 or 0.32. Accordingly we postulate that graded propositions (φ, s) and (φ, t) are treated as mutually exclusive only if s and t differ at least by a fixed minimal value, denoted by ζ.

The choice of ζ is to a certain extent arbitrary. Note however that in practise, also the degrees themselves are to a certain extent arbitrary. The introduction of the value ζ

takes exactly this point into account: it does in general not make sense to distinguish between arbitrary close degrees of presence.

We will modify the interpretation of (φ, t), $t \in [0, 1]$, as follows. Let φ be interpreted by a fuzzy set u over a set S. Then (φ, t) will no longer be interpreted by $[u]_t$ but by the set $[u]_t^\zeta$ which is, roughly speaking, consists of those $x \in S$ such that $u(x)$ differs from t less than ζ.

We note this idea is unrelated to any of the formalisms based on interval-valued fuzzy sets. In fact, gradable propositions will still be modelled by ordinary fuzzy sets. What we intend to account for is rather the idea that statements involving truth degrees should have a more "tolerant" interpretation; close truth degrees are allowed to overlap in their interpretation.

Our solution forces us to overcome several technical difficulties. To replace $[u]_t$ by the larger set $[u]_t^\zeta$ makes perfect sense if t is an intermediate truth degree, in particular if $\zeta \leq t \leq 1 - \zeta$. For sharp truth degrees, the situation is different; we should still be able to express that a property is clearly false or clearly true. In our formalism, the sets $[u]_0$ and $[u]_1$ will no longer appear; however, it will be possible to express the property to be clearly false or clearly true in an approximate way. We will simply extend the set of available degrees of presence from $[0, 1]$ to $(-\zeta, 1 + \zeta)$. The negative degrees and the degree larger than 1 are so-to-say virtual ones. A degree $t \in (-\zeta, 0)$ represents falsity, like 0, but in contrast to 0 the tolerance around 0 is $t + \zeta$, a value which can be arbitrarily small. Similarly, we use the degrees of presence above 1.

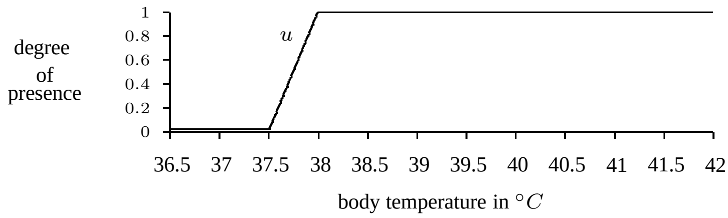
Remark 4.1. *Our formalism could be simplified in an easy way: we could interpret (φ, t) by the set of all $a \in S$ which are mapped to t or a larger value. This interpretation is indeed common in fuzzy logics. It would include, up to marginal points, the concept of intervals used in this section and the concept of points from the previous section. Even better, the fuzzy-set rules would simplify and in fact look more elegant. However, we do not adopt this approach here. We would see it as a lack of elegance to declare statements like "property φ holds to a degree of at least 0.4" as basic.*

In the area of application which we have in mind, an agent's utterance of this form, or a technical specification in this form, would come as a surprise. Statements providing a lower bound for the degree of presence of some property may still reasonably occur as the result of some inference step; but not typically as its assumption.

As a further consequence of our decision to work with the sets $[u]_t^\zeta$ which involve an "extended" set of degrees around t, we will no longer use the set-theoretical operations to interpret Boolean connectives. Our motivation is that single degrees of presence should no longer play a role. It should not matter if the marginal points $t - \zeta$ or $t + \zeta$ are included or not, and by use of Boolean connectives it should never be possible to arrive at sets of the form $[u]_t$.

Hence we need to endow our fuzzy set model with more structure than before. As a prototype consider the following simple fuzzy set u, modelling "having fever".

A fuzzy set modelling the property to have fever



We abstract from this example the following facts. The base set and the set of degrees of presence are endowed with a topology in a natural way, and w.r.t. these topologies the fuzzy set u is continuous. Furthermore, each intermediate truth degree is assigned to a single point only; each set $[u]_t$, where $0 < t < 1$, has an empty interior. In contrast, each set $\{s \in [36.5, 42]\colon\ t - \zeta < u(s) < t + \zeta\}$ is the interior of a closed set.

Remark 4.2. We compile for what follows the basics about the used topological notions. For more information see, e.g., [9].

Let S be a topological space. For $A \subseteq S$, we denote by A° the open interior of A, and by A^- the closure of A. A set $A \subseteq S$ is called regular open if it is the open interior of a closed set. So exactly all sets of the form $A^{-\circ}$ are regular open; we have

$$A^{-\circ} \;=\; \{x \in S\colon\ A \text{ is dense in some open neighbourhood of } x\}.$$

We denote by $\mathcal{R}(S)$ the set of all regular open subsets of S. Under set-theoretical inclusion, $\mathcal{R}(S)$ is a distributive $0,1$-lattice. For $A, B \in \mathcal{R}(S)$, the infimum is $A \cap B$; the supremum is $A \vee B \;=\; (A \cup B)^{-\circ}$; and $\emptyset, S$ are the bottom and top element, respectively. Furthermore,

$$^\perp\colon\ \mathcal{R}(S) \to \mathcal{R}(S), \quad A \mapsto (S \backslash A)^\circ$$

is a complementation function; in particular, $A^{\perp\perp} \;=\; A^{-\circ} \;=\; A$, $\;A \cap A^\perp \;=\; \emptyset$, and $A \cup A^\perp$ is dense in S. So $(\mathcal{R}(S); \cap, \vee, ^\perp, \emptyset, S)$ is a Boolean algebra.

For later use we remark the following. For open sets $A, B \subseteq S$ we have

$$(A \cup B)^{-\circ} \;=\; A^{-\circ} \vee B^{-\circ}, \tag{10}$$

$$(A \cap B)^{-\circ} \;=\; A^{-\circ} \cap B^{-\circ}, \tag{11}$$

where the supremum refers to the poset $\mathcal{R}(S)$.

Note first that for any $C, D \subseteq S$ we have $(C \cup D)^\perp = C^\perp \cap D^\perp$ and $C^{\perp\perp} = C^{-\circ}$. We conclude

$$\begin{aligned}(A \cup B)^{-\circ} =& (A \cup B)^{\perp\perp} = (A^\perp \cap B^\perp)^\perp = (A^{\perp\perp\perp} \cap B^{\perp\perp\perp})^\perp = \\ & (A^{\perp\perp} \cup B^{\perp\perp})^{\perp\perp} = (A^{-\circ} \cup B^{-\circ})^{-\circ} = A^{-\circ} \vee B^{-\circ}.\end{aligned}$$

This is (10); for (11) see [9, Lemma 4 of Chapter 10]. Finally, (11) implies that for closed sets $A, B \subseteq S$

$$(A \cup B)^\circ \;=\; (A^\circ \cup B^\circ)^{-\circ}. \tag{12}$$

Definition 4.3. Let S be a topological space. A fuzzy set $u: S \to [0,1]$ is called *regular* if the following conditions hold:

(R1) u is continuous w.r.t. the standard topology of $[0,1]$;

(R2) for any $t \in (0,1)$, $[u]_t$ has an empty interior.

A Kleene algebra M of regular fuzzy sets over S is called *regular*.

The notion of a regular Kleene algebra would not make sense if the conditions (R1) and (R2) were not preserved under the Kleene algebra operations.

Lemma 4.4. *Let M be a Kleene algebra of fuzzy sets over some topological space. Assume that M is generated by regular fuzzy sets. Then M is a regular Kleene algebra.*

Proof. Evidently, the constant fuzzy sets are regular. Let $u, v \in M$ be regular. Clearly, $u \wedge v$, $u \vee v$, and $\sim u$ are continuous.

Let $0 < t < 1$; we have to show that $[u \wedge v]_t$ has an empty interior. Let $a \in S$ be such that $(u \wedge v)(a) = t$, and let U be an open neighbourhood of a. W.l.o.g. assume $t = u(a) \leq v(a)$. If there is a $b \in U$ such that $u(b) < t$, we have $(u \wedge v)(b) = u(b) \wedge v(b) < t$. Otherwise there is a $b \in U$ such that $u(b) > t$ and consequently there is an open $V \subseteq U$ such that $u(b) > t$ for all $b \in V$. Choose some $c \in V$ such that $v(c) \neq t$; then $(u \wedge v)(b) \neq t$. It follows that $[u \wedge v]_t$ does not contain an open set.

Similarly we argue in case of $[u \vee v]_t$. Finally $[\sim u]_t = [u]_{\sim t}$ clearly has an empty interior as well. $\square$

We fix now a rational value $0 < \zeta < \frac{1}{2}$. ζ is supposed to quantify the distinguishability between different degrees to which a vague property holds; (φ, s) and (φ, t) will be modelled as disjoint only if $|s - t| \geq 2\zeta$. We will switch from $[u]_t$ to $[u]_t^\zeta$; here $[u]_t^\zeta$ does not simply denote the set of all points mapping to the interval $[t - \zeta, t + \zeta]$ or $(t - \zeta, t + \zeta)$; we will rather use a definition which ensures that $[u]_t^\zeta$ is regular open. Accordingly, rather than using the Boolean algebra generated by the sets $[u]_t$, we will work with the Boolean algebra of regular open sets generated by the sets of the form $[u]_t^\zeta$.

For a generalised degree of presence t, we will denote by t' the degree of presence which is actually meant by t, disregarding the amount of tolerance: for $t \in (-\zeta, 1+\zeta)$, we put $t' = (t \vee 0) \wedge 1$. For $I \subseteq (-\zeta, 1 + \zeta)$, we will write $I' = \{t' : t \in I\}$.

Finally, for $I \subseteq [0,1]$, we put $[u]_I = \{a \in S : u(a) \in I\}$.

Definition 4.5. Let M be a regular Kleene algebra of fuzzy sets over a topological space S. For $u \in M$ and $t \in (-\zeta, 1 + \zeta)$, we define

$$[u]_t^\zeta \; = \; [u]_{[t-\zeta, t+\zeta]'}{}^\circ.$$

Furthermore, the Boolean subalgebra of $\mathcal{R}(S)$ generated by $[u]_t^\zeta$, where $u \in M$ and $t \in (-\zeta, 1 + \zeta)$, will be called the *Boolean algebra associated with* M, denoted by $\mathcal{R}_M$.

Finally, let ϱ be a rejection function on $\mathcal{R}_M$ w.r.t. $\wedge$. Then (M, ϱ) is called a *regular Kleene uncertainty algebra*.

The following lemma provides an explicit description of the interpretation which we are going to apply.

Lemma 4.6. *Let* (M, ϱ) *be a regular Kleene uncertainty algebra. Let* $u \in M$. *For* $\zeta < t < 1 - \zeta$ *we have*

$$\begin{aligned}[u]_t^\zeta &= [u]_{[t-\zeta,t+\zeta]}{}^\circ \\ &= \{a \in S\colon\ u(a) \in (t-\zeta, t+\zeta) \\ &\qquad \text{or } u(a) \in \{t-\zeta, t+\zeta\} \text{ and } u \text{ maps a neighbourhood of } a \text{ to } [t-\zeta, t+\zeta]\};\end{aligned}$$

for $-\zeta < t \leq \zeta$*, we have*

$$\begin{aligned}[u]_t^\zeta &= [u]_{[0,t+\zeta]}{}^\circ \\ &= \{a \in S\colon\ u(a) \in [0, t+\zeta) \\ &\qquad \text{or } u(a) = t+\zeta \text{ and } u \text{ maps a neighbourhood of } a \text{ to } [0, t+\zeta]\};\end{aligned}$$

and similarly for $1 - \zeta \leq t < 1 + \zeta$.

So given a regular fuzzy set u, we see that $[u]_t^\zeta$ contains basically all points $a \in S$ such that $u(a) \in (t-\zeta, t+\zeta)$, but if, for instance, u has at the point $a \in S$ the strict local minimum $t - \zeta$ then a is joined to $[u]_t^\zeta$ as well.

In the above example, take $\zeta = 0.1$. Then we have, say, $[u]_{0.3}^\zeta = u^{-1}((0.2, 0.4)) = (37.6, 37.7)$, $[u]_{0.1}^\zeta = u^{-1}([0, 0.2)) = (36.5, 37.6)$. As the associated property to be clearly false we can take, e.g., $[u]_{-0.099}^\zeta = u^{-1}([0, 0.001)) = [36.5, 37.5005)$.

We define the Possibilistic Logic with Soft Gradation, denoted by $\mathtt{IG}^\zeta$, as follows.

Definition 4.7. The *propositions*, the set of which will still be denoted by $\mathcal{P}$, as well as the *implications* of $\mathtt{IG}^\zeta$ coincide with those of $\mathtt{IG}^0$, respectively (see Def. 3.2) except that we use the real interval $(-\zeta, 1+\zeta)$ as the set of degrees of presence.

An *evaluation* v of $\mathtt{IG}^\zeta$ is defined like for $\mathtt{IG}^0$ except that for $\varphi \in \mathcal{F}$ and $t \in (-\zeta, 1+\zeta)$ we define

$$v_b((\varphi, t)) \;=\; [v_f(\varphi)]_t^\zeta$$

and that v_b maps to $\mathcal{R}_M$.

A *theory* of $\mathtt{IG}^\zeta$ and *semantic entailment* for $\mathtt{IG}^\zeta$ is defined mutatis mutandis like for $\mathtt{I}$.

To axiomatise the logic $\mathtt{IG}^\zeta$ we have to modify all rules except the basic ones.

Definition 4.8. The rules of $\mathtt{IG}^\zeta$ split into three groups:

The *basic rules* are those of $\mathtt{I}$ (see Def. 2.3) where propositions are understood as those of $\mathtt{IG}^\zeta$.

The *degree-of-presence rules* are the following, where φ is a gradable proposition and $s, s_1, ..., t \in (-\zeta, 1+\zeta)$:

$$(\varphi, s) \overset{d}{\Rightarrow} \neg(\varphi, t) \text{ where } |s-t| \geq 2\zeta$$

$$(\varphi, s) \overset{d}{\Rightarrow} (\varphi, t) \text{ where } s \leq t \leq \zeta \text{ or } 1-\zeta \leq t \leq s$$

$$(\varphi, r) \overset{d}{\Rightarrow} (\varphi, s) \vee (\varphi, t) \text{ where } s \leq r \leq t \leq s+2\zeta$$

$$(\varphi, r), (\varphi, s) \overset{d}{\Rightarrow} (\varphi, t) \text{ where } r \leq t \leq s$$

$$\neg(\varphi, s_1), \ldots, \neg(\varphi, s_k) \overset{d}{\Rightarrow} \bot \text{ where } s_1 \leq \zeta;\ s_2 - s_1, \ldots, s_k - s_{k-1} \leq 2\zeta;\ s_k \geq 1-\zeta$$

The *fuzzy-set rules* are the following, where φ, ψ are gradable propositions, α is a proposition, Γ is a finite set of propositions, $c, d \in [0,1]$, and $s, t \in (-\zeta, 1+\zeta)$:

$$\frac{\Gamma, (\varphi \wedge \psi, s {\wedge} t) \overset{d}{\Rightarrow} \alpha}{\Gamma, (\varphi, s), (\psi, t) \overset{d}{\Rightarrow} \alpha} \qquad \frac{\Gamma, \neg(\varphi \wedge \psi, t) \overset{d}{\Rightarrow} \alpha}{\Gamma, (\varphi, r), (\psi, s) \overset{d}{\Rightarrow} \alpha} \text{ where } r, s \geq t+2\zeta$$

$$\frac{\Gamma, \neg(\varphi \wedge \psi, t) \overset{d}{\Rightarrow} \alpha}{\Gamma, (\varphi, s) \overset{d}{\Rightarrow} \alpha} \text{ where } s+2\zeta \leq t \qquad \frac{\Gamma, \neg(\varphi \wedge \psi, t) \overset{d}{\Rightarrow} \alpha}{\Gamma, (\psi, s) \overset{d}{\Rightarrow} \alpha} \text{ where } s+2\zeta \leq t$$

$$\frac{\Gamma, (\varphi \vee \psi, s {\vee} t) \overset{d}{\Rightarrow} \alpha}{\Gamma, (\varphi, s), (\psi, t) \overset{d}{\Rightarrow} \alpha} \qquad \frac{\Gamma, \neg(\varphi \vee \psi, t) \overset{d}{\Rightarrow} \alpha}{\Gamma, (\varphi, r), (\psi, s) \overset{d}{\Rightarrow} \alpha} \text{ where } r+2\zeta, s+2\zeta \leq t$$

$$\frac{\Gamma, \neg(\varphi \vee \psi, t) \overset{d}{\Rightarrow} \alpha}{\Gamma, (\varphi, s) \overset{d}{\Rightarrow} \alpha} \text{ where } s \geq t+2\zeta \qquad \frac{\Gamma, \neg(\varphi \vee \psi, t) \overset{d}{\Rightarrow} \alpha}{\Gamma, (\psi, s) \overset{d}{\Rightarrow} \alpha} \text{ where } s \geq t+2\zeta$$

$$\frac{\Gamma, (\varphi, c) \overset{d}{\Rightarrow} \alpha}{\Gamma, (\sim\varphi, {\sim}c) \overset{d}{\Rightarrow} \alpha} \qquad \frac{\Gamma, (\sim\varphi, c) \overset{d}{\Rightarrow} \alpha}{\Gamma, (\varphi, {\sim}c) \overset{d}{\Rightarrow} \alpha}$$

The notion of a *proof* as well as the *consistency* of a theory is defined like for `I` (see Def. 2.3).

We split up the soundness and completeness proof for $\mathtt{IG}^\zeta$ in a series of lemmas.

To establish the soundness of the rules, we have to examine the structure of our model in some more detail. In the next lemma we see how the Boolean operations act, for some fixed fuzzy set u, on the sets $[u]^\zeta_t$ where $t \in (-\zeta, 1+\zeta)$.

We will assume that the real unit interval $[0,1]$ is endowed with the subspace topology inherited from the reals endowed with the standard topology. $\mathcal{R}([0,1])$, the regular subsets of $[0,1]$ contains then the intervals of the form $[0,a)$ or (a,b) or $(b,1]$, where $0 < a < b < 1$, as well as the unions of such intervals if no two of them have a common endpoint. By $\mathcal{R}^{\mathrm{f}}([0,1])$, we will denote the Boolean subalgebra of $\mathcal{R}([0,1])$ consisting only of the finite unions of intervals of the form $[0,a)$ or (a,b) or $(b,1]$, where $0 < a < b < 1$.

Lemma 4.9. *Let (M, ϱ) be a regular Kleene uncertainty algebra. Let $u \in M$, and let $\mathcal{R}_u$ be the Boolean subalgebra generated by $[u]_t^\zeta$, $t \in (-\zeta, 1+\zeta)$ in $\mathcal{R}_M$.*

The mapping

$$\iota_u \colon \mathcal{R}^{\mathrm{f}}([0,1]) \to \mathcal{R}_u, \quad I \mapsto [u]_{I^-}{}^\circ$$

is an epimorphism of Boolean algebras.

Proof. For $0 \leq s_1 < t_1 \leq s_2 < t_2 \leq 1$, we have by the regularity of u

$$[u]_{[s_1,t_1]}{}^\circ \vee [u]_{[s_2,t_2]}{}^\circ \;=\; \begin{cases} [u]_{[s_1,t_2]}{}^\circ & \text{if } t_1 = s_2, \\ [u]_{[s_1,t_1] \cup [s_2,t_2]}{}^\circ & \text{if } t_1 < s_2 \end{cases} \;=\; [u]_{(s_1,t_1) \vee (s_2,t_2)}{}^\circ;$$

we conclude that ι_u preserves $\vee$. Similarly, we check that ι_u preserves $\sim$.

ι_u is surjective by construction. □

For the next lemma we introduce some technical notation. For $t \in (-\zeta, 1+\zeta)$, we define the finite subset $V_{\geq t}$ of $(-\zeta, 1+\zeta)$ to contain all values $t, t+2\zeta, t+4\zeta, \ldots$ which are strictly smaller than $1+\zeta$. Similarly, we define $V_{\leq t}$ to contain those values $t, t-2\zeta, \ldots$ which are strictly larger than $-\zeta$. Finally, we let $V_{\neg t}$ contain those values $t-2\zeta, t-4\zeta, \ldots$ as well as $t+2\zeta, t+4\zeta, \ldots$ which are contained in $(-\zeta, 1+\zeta)$. Finally, for an element u of a regular Kleene algebra, we put $[u]_{\geq t}^\zeta = [u]_{[t-\zeta,1+\zeta]'}{}^\circ$ and $[u]_{\leq t}^\zeta = [u]_{[-\zeta,t+\zeta]'}{}^\circ$.

Lemma 4.10. *Let (M, ϱ) be a regular Kleene uncertainty algebra. Let $u, v \in M$ and let $t \in (-\zeta, 1+\zeta)$. Then*

$$\begin{aligned}
[u \wedge v]_t^\zeta &= ([u]_t \cap [v]_{\geq t}^\zeta) \vee ([u]_{\geq t}^\zeta \cap [v]_t) \\
&= \bigvee \{ [u]_r^\zeta \cap [v]_s^\zeta \colon \; r = t \text{ and } s \in V_{\geq t}, \; \text{ or } r \in V_{\geq t} \text{ and } s = t \}, \\
[u \vee v]_t^\zeta &= ([u]_t \cap [v]_{\leq t}^\zeta) \vee ([u]_{\leq t}^\zeta \cap [v]_t) \\
&= \bigvee \{ [u]_r^\zeta \cap [v]_s^\zeta \colon \; r = t \text{ and } s \in V_{\leq t}, \; \text{ or } r \in V_{\leq t} \text{ and } s = t \}, \\
[\sim u]_t^\zeta &= [u]_{\sim t}^\zeta.
\end{aligned}$$

Proof. Using (10)–(12) we calculate

$$\begin{aligned}
[u \wedge v]_t^\zeta &= [u \wedge v]_{[t-\zeta,t+\zeta]'}{}^\circ \\
&= (([u]_{[t-\zeta,t+\zeta]'} \cap [v]_{[t-\zeta,1]}) \cup ([u]_{[t-\zeta,1]} \cap [v]_{[t-\zeta,t+\zeta]'}))^\circ \\
&= (([u]_{[t-\zeta,t+\zeta]'} \cup [v]_{[t-\zeta,t+\zeta]'}) \cap [u]_{[t-\zeta,1]} \cap [v]_{[t-\zeta,1]})^\circ \\
&= ([u]_{[t-\zeta,t+\zeta]'} \cup [v]_{[t-\zeta,t+\zeta]'})^\circ \cap [u]_{[t-\zeta,1]}{}^\circ \cap [v]_{[t-\zeta,1]}{}^\circ \\
&= ([u]_{[t-\zeta,t+\zeta]'}{}^\circ \cup [v]_{[t-\zeta,t+\zeta]'}{}^\circ)^{-\circ} \cap [u]_{[t-\zeta,1]}{}^{\circ-\circ} \cap [v]_{[t-\zeta,1]}{}^{\circ-\circ} \\
&= (([u]_{[t-\zeta,t+\zeta]'}{}^\circ \cup [v]_{[t-\zeta,t+\zeta]'}{}^\circ) \cap [u]_{[t-\zeta,1]}{}^\circ \cap [v]_{[t-\zeta,1]}{}^\circ)^{-\circ} \\
&= (([u]_{[t-\zeta,t+\zeta]'}{}^\circ \cap [v]_{[t-\zeta,1]}{}^\circ) \cup ([u]_{[t-\zeta,1]}{}^\circ \cap [v]_{[t-\zeta,t+\zeta]'}{}^\circ))^{-\circ} \\
&= ([u]_t^\zeta \cap [v]_{\geq t}^\zeta) \vee ([u]_{\geq t}^\zeta \cap [v]_t^\zeta);
\end{aligned}$$

furthermore, we clearly have $[u]^\zeta_{\geq t} = \bigvee_{s \in V_{\geq t}} [u]^\zeta_s$ and $[v]^\zeta_{\geq t} = \bigvee_{s \in V_{\geq t}} [v]^\zeta_s$, and the assertion follows by distributivity. Similarly we proceed for $[u \vee v]^\zeta_t$. The expression for $[\sim u]^\zeta_t$ is obvious. $\square$

Theorem 4.11. *Let $\mathcal{T}$ be a theory of $\mathtt{IG}^\zeta$ and $\Gamma \overset{e}{\Rightarrow} \delta$ an implication of $\mathtt{IG}^\zeta_\mathcal{T}$. Then $\mathcal{T}$ semantically entails $\Gamma \overset{e}{\Rightarrow} \delta$ if $\mathcal{T}$ proves $\Gamma \overset{e}{\Rightarrow} \delta$.*

Proof. The basic rules are sound by Theorem 2.7.

The soundness of the degree-of-presence and fuzzy-set rules follows from Lemmas 4.9 and 4.10. $\square$

We will now work towards the completeness part. We will proceed in analogy to the case of the logic $\mathtt{IG}^0$ whenever possible.

We first see how graded propositions decompose to Boolean expressions in graded variables, in the same way as described in Lemma 4.10.

Lemma 4.12. *Let φ, ψ be gradable propositions of $\mathtt{IG}^\zeta$, and let $t \in (-\zeta, 1+\zeta)$. Then $(\varphi \wedge \psi, t)$ is equivalent to*

$$\bigvee \{((\varphi, r) \wedge (\psi, s))\colon\ r = t \text{ and } s \in V_{\geq t}, \quad \text{or } r \in V_{\geq t} \text{ and } s = t\}; \tag{13}$$

$(\varphi \vee \psi, t)$ is equivalent to

$$\bigvee \{((\varphi, r) \wedge (\psi, s))\colon\ r = t \text{ and } s \in V_{\leq t}, \quad \text{or } r \in V_{\leq t} \text{ and } s = t\}, \tag{14}$$

and $(\sim \varphi, t)$ is equivalent to $(\varphi, \sim t)$.

Furthermore, $\neg(\varphi, t)$ is equivalent to

$$\bigvee_{s \in V_{\neg t}} (\varphi, s). \tag{15}$$

Finally, any proposition of $\mathtt{IG}^\zeta$ is equivalent to the disjunction of conjunctions of graded variables.

Proof. (13) implies $(\varphi \wedge \psi, t)$. Furthermore, the negation of this proposition is equivalent to a finite disjunction of propositions $(\varphi, r) \wedge (\psi, s)$ where either $r, s \geq t + 2\zeta$ or $r \leq t - 2\zeta$ or $s \leq t - 2\zeta$, each of which implies $\neg(\varphi \wedge \psi, t)$.

Similarly, we proceed to show that $(\varphi \vee \psi), t)$ is equivalent to (14). The assertion about $(\sim \varphi, t)$ is easy.

It is easily seen that $\neg(\varphi, t)$ is equivalent to (15).

By induction over its complexity we conclude that each graded proposition is the disjunction of conjunctions of graded variables. It follows that the same is the case for each proposition. $\square$

For a Boolean algebra $\mathcal{R}$ of open regular sets of a topological space S, we again denote by $\mathcal{R}^N$ the N-fold free product of $\mathcal{R}$. We can, and will, identify $\mathcal{R}^N$ with a subalgebra of $\mathcal{R}(S^N)$, the algebra of regular open sets in the product space S^N; $\mathcal{R}^N$ is generated by the sets of the form $A_1 \times \ldots \times A_N$, where $A_1, \ldots, A_N \in \mathcal{R}$. We will call the sets of the latter form cubic.

In the following lemma, we define for $t \in (-\zeta, 1+\zeta)$ the open set of $[0,1]$

$$U_\zeta(t) \; = \; [(t-\zeta)\vee 0, (t+\zeta)\wedge 1]^\circ.$$

Lemma 4.13. *For propositions α and β of $\mathtt{IG}^\zeta$, we put $\alpha \approx \beta$ if $\mathtt{IG}^\zeta$ proves $\alpha \overset{1}{\Rightarrow} \beta$ and $\beta \overset{1}{\Rightarrow} \alpha$. Then the quotient $\langle \mathcal{P} \rangle$ of $\mathcal{P}$ w.r.t. $\approx$, endowed with the induced operations $\wedge, \vee, \neg$ and the constants $\langle \bot \rangle, \langle \top \rangle$, is a Boolean algebra isomorphic to $\mathcal{R}^{\mathrm{f}}([0,1])^N$. The isomorphism w is given by*

$$\begin{aligned} w(\langle(\varphi_i, t)\rangle) \; &= \{(r_1, ..., r_N) \in [0,1]^N : \; r_i \in U_\zeta(t)\}, \\ & \qquad\qquad i = 1, \ldots, N, \;\; t \in (-\zeta, 1+\zeta). \end{aligned} \tag{16}$$

Furthermore, let

$$u_i \colon [0,1]^N \to [0,1], \;\; (r_1, ..., r_N) \mapsto r_i,$$

and let $\bar{M}$ be the Kleene algebra generated by $u_1, \ldots, u_N$. Then $\mathcal{R}_{\bar{M}} = \mathcal{R}^{\mathrm{f}}([0,1])^N$. Define $\varrho : \mathcal{R}_{\bar{M}} \to [0,1]$ to be 0 on all non-zero elements; then $(\bar{M}, \varrho)$ is a regular Kleene uncertainty algebra.

Define $\bar{v}_f(\varphi_i) = u_i$ for $i = 1, \ldots, N$, and extend $\bar{v}_f$ to $\mathcal{F}$ such that $\wedge, \vee, \sim$ and $\bar{0}, \bar{1}$ are preserved. Define $\bar{v}_b(\alpha) = w(\langle \alpha \rangle)$ for $\alpha \in \mathcal{P}$. Then $(\bar{v}_f, \bar{v}_b)$ is an evaluation of $\mathtt{IG}^\zeta$ such that $\bar{v}_b(\alpha) = \emptyset$ if and only if $\vdash \alpha \overset{1}{\Rightarrow} \bot$.

Proof. Again, for the degree-of-presence and fuzzy-set rules we may assume $d = 1$. We modify $\mathtt{IG}^\zeta$: We drop the fuzzy-set rules and add six axiom schemes expressing the equivalences of $(\varphi \wedge \psi, t)$, $(\varphi \vee \psi, t)$, $(\sim\varphi, t)$ with the expressions (13), (14), and (15), respectively. By Lemma 4.12 we see that this change has no effect.

Let $\mathtt{IG}^\zeta_{\mathtt{c}}$ be the restriction of $\mathtt{IG}^\zeta$ to degrees of certainty 1. In the same way as in the proof of Lemma 3.6, we may view $\mathtt{IG}^\zeta_{\mathtt{c}}$ as an extension of classical propositional logic.

We have to determine the Boolean algebra $\langle \mathcal{P} \rangle$. It is tedious but not difficult to check that the degree-of-presence rules can be restricted to the case of graded variables. Consequently, we again have N disjoint groups of axioms involving for each $i \in \{1, ..., N\}$ the graded variables (φ_i, t), $t \in (-\zeta, 1+\zeta)$.

Fix an $i \in \{1, ..., N\}$. We have to show that the subalgebra $\langle \mathcal{P} \rangle_i$ of $\langle \mathcal{P} \rangle$ generated by $\langle(\varphi_i, t)\rangle$, $t \in (-\zeta, 1+\zeta)$, is isomorphic to $\mathcal{R}^{\mathrm{f}}([0,1])$ under the assignment

$$w_i \colon \langle \mathcal{P} \rangle_i \to \mathcal{R}^{\mathrm{f}}([0,1]), \;\; \langle(\varphi_i, t)\rangle \mapsto U_\zeta(t).$$

It will then follow that $\langle \mathcal{P} \rangle$ is isomorphic to the free product of N copies of $\mathcal{R}^{\mathrm{f}}([0,1])$, the isomorphism being determined by (16).

It is easily checked that all inequalities holding in $\langle\mathcal{P}\rangle_i$ due to the degree-of-presence rules between $\langle(\varphi_i, t)\rangle$ are preserved by w_i. So w_i indeed extends to a Boolean homomorphism from the whole $\langle\mathcal{P}\rangle_i$ to $\mathcal{R}^{\mathrm{f}}([0,1])$. By construction w_i is surjective.

It remains to show that w_i is injective. Let $\{s_j : j \in J\}$ and $\{t_k : k \in K\}$ be two finite subsets of $(-\zeta, 1+\zeta)$. We have to prove that $\bigcap_j U_\zeta(s_j) \subseteq \bigvee_k U_\zeta(t_k)$ implies that $\mathtt{IG}^\zeta$ proves $\bigwedge_j(\varphi_i, s_j) \overset{1}{\Rightarrow} \bigvee_k(\varphi_i, t_k)$.

Case 1. J is empty. Then $\bigvee_k U_\zeta(t_k) = (-\zeta, 1+\zeta)$ and consequently $\{t_k\}$ contains elements $\leq \zeta$ and $\geq 1+\zeta$ and neighbouring values differ at most 2ζ. It follows that $\top \overset{1}{\Rightarrow} \bigvee_k(\varphi_i, t_k)$ is provable and the assertion follows.

Case 2. $J = \{j\}$ is one-element. Then either $s_j \leq t_k \leq \zeta$ for some k; or $1-\zeta \leq t_k \leq s_j$ for some k; otherwise either one value t_k coincides with s_j; or there are two values t_k with distance $\leq 2\zeta$ and such that s_j is in between. The assertion follows in each case.

Case 3. All ≥ 2 values in $\{s_j\}$ are $\leq \zeta$ or $\geq 1-\zeta$. This case reduces to Case 2.

Case 4. Two of the values in $\{s_j\}$ differ at least 2ζ, that is, the intersection $\bigcap_j U_\zeta(s_j)$ is empty. Then $\bigwedge_j(\varphi_i, s_j) \overset{1}{\Rightarrow} \perp$ is provable and the assertion follows.

Case 5. At least one value in $\{s_j\}$ is in $(\zeta, 1-\zeta)$ and all ≥ 2 values have a mutual distance of $\leq 2\zeta$. If there are more than two, we can delete all but the outermost ones. Let $s_j, s_{j'}$ be the two values and let $s_j < s_{j'}$. Then either there is a $s_j \leq t_k \leq s_{j'}$, or there are $t_k < s_j < s_{j'} < t_{k'}$ such that $t_{k'} - t_k < 2\zeta$. The assertion follows in both cases.

The proof of the first half of the theorem is complete. For the second we proceed like in the proof of Lemma 3.6. $\square$

We next note that Lemma 3.7 holds by the same proof also for $\mathtt{IG}^\zeta$.

Theorem 4.14. *Let $\mathcal{T}$ be a consistent finite theory of $\mathtt{IG}^\zeta$ and $\Gamma \overset{e}{\Rightarrow} \delta$ an implication of $\mathtt{IG}^\zeta_\tau$. If $\mathcal{T}$ semantically entails $\Gamma \overset{e}{\Rightarrow} \delta$, then $\mathcal{T}$ proves $\Gamma \overset{e}{\Rightarrow} \delta$.*

Proof. Assume to the contrary that $\mathcal{T}$ does not prove $\Gamma \overset{e}{\Rightarrow} \delta$. We can assume that $\mathcal{T} = \{\chi_0 \overset{1}{\Rightarrow} \perp,\ \chi_1 \overset{d_1}{\Rightarrow} \perp,\ \ldots,\ \chi_m \overset{d_m}{\Rightarrow} \perp\}$, where the χ_i are pairwise disjoint and jointly exhaustive, and $1 > d_1 > \ldots > d_m = 0$.

Let $(\bar{v}_f, \bar{v}_b)$ be the evaluation in $(\bar{M}, \varrho)$ according to Lemma 4.13. Let $S = \bar{v}_b(\neg\chi_0)$; then $S \neq \emptyset$ by consistency. Let M be the Kleene algebra generated by $u_i|_S$, $i = 1, \ldots, N$. Then M is a regular Kleene algebra. Furthermore $\mathcal{R}_M = \{A \cap S\colon\ A \in \mathcal{R}_{\bar{M}}\}$.

Let $v_f\colon\ \mathcal{F} \to M,\quad \varphi \mapsto \bar{v}_f(\varphi)|_S$ and $v_b\colon\ \mathcal{P} \to \mathcal{R}_S,\quad \alpha \mapsto \bar{v}_b(\alpha) \cap S$.

We define

$$\varrho\colon\ \mathcal{R}_M \to [0,1],\quad A \mapsto \min\{d_i\colon\ 0 \leq i \leq m \text{ and } A \cap \bar{v}_b(\chi_i) \neq \emptyset\},$$

where the minimum of the empty set is 1. This is obviously a rejection function on $\mathcal{R}_M$.

Then (v_f, v_b) is an evaluation in the regular Kleene uncertainty algebra (M, ϱ). As $\varrho(\bar{v}_b(\chi_i)) = d_i$ for all i, all elements of $\mathcal{T}$ are satisfied by (v_f, v_b).

If $\mathcal{T}$ does not prove $\Gamma \overset{e}{\Rightarrow} \delta$, we conclude like in the proof of Theorem 2.7 that $\Gamma \overset{e}{\Rightarrow} \delta$ is not satisfied by v. This completes the proof of the "only if" part. □

5 Smoothing the degree of certainty over gradation

Having started with the (slightly generalised) Possibilistic Logic as our general framework, we have included the possibility to express gradedness of information and we have subsequently modified the interpretation of graded properties. Doing this, we have not touched the underlying concept of uncertainty; the degree of certainty has remained unrelated to the degrees of presence.

Indeed, in $\mathtt{IG}^\zeta$ situations are specified by the propositions $(\varphi_1, t_1), \ldots, (\varphi_N, t_N)$, that is, by the N-tuple $(t_1, \ldots, t_N)$. To each such N-tuple, there is associated the degree of implausibility of the corresponding situation, namely $d = \varrho(v(\varphi_1, t_1) \wedge \ldots \wedge v(\varphi_N, t_N))$ for some interpretation v. We observe that the value d depends on $(t_1, \ldots, t_N)$ completely arbitrarily.

This arbitrariness might not be ideal for practical applications. Similar situations are presumably described by close N-tuples and so the implausibility should depend at least continuously on the N parameters. The problem is specific for the current approach; we refer to [2] for a broad overview of approaches to deal with different kinds of degrees in common frameworks.

In the present chapter, we add a simple rule to our logic with the sole effect of continuity. Disregarding inessential details, we propose the following approach. Situations are specified by N-tuples $(t_1, \ldots, t_N)$; assume that the associated degree of implausibility is d. We add a rule to ensure that a situation characterised by $(s_1, \ldots, s_N)$, where s_i differs from t_i less than λ, is assigned a degree of implausibility of at least $d - \tau\lambda$. In other words, we introduce Lipschitz continuity for ϱ if seen in dependence on the degrees of presence. So for instance, put $\tau = 2$, and assume that we know with certainty d that we can conclude that if property φ fully applies so does ψ, that is, $(\varphi, 1) \overset{d}{\Rightarrow} (\psi, 1)$. In the calculus introduced below we may conclude that $(\varphi, 0.9) \overset{d-0.2}{\Rightarrow} (\psi, 1)$ provided that $d > 0.2$.

Our new rule offers a simple way to prevent ϱ to "jump" when changing continuously from one situation to another, in that changes are bounded in dependence of the "distance" between situations. This procedure is certainly pragmatic. But it solves in a very direct and transparent way the problem which we have and its effect can be controlled by a deliberate choice of τ.

Let us fix a real parameter $\tau > 1$ throughout this section.

To realise our aim, we will not endow the underlying topological space with a metric. We will rather fix a subset U of M and restrict the way in which, for instance, $\varrho([u]_t)$, $u \in U$, can vary depending on t. The background is the following problem. Assume that $[u]_t$ is excluded, in which case $\varrho([u]_t) = 1$ and consequently $[u_i]_t = \emptyset$. Thus, if we define continuity by means of a metric of the underlying space, we could not infer anything about $\varrho([u_i]_s)$ for s close to t.

Definition 5.1. Let (M, ϱ) be a regular Kleene uncertainty algebra, and let $U \subseteq M$. Let $v_1 \ldots, v_k$ be (not necessarily different) elements of U; and let $s_1, t_1, \ldots, s_k, t_k \in [0,1]$ such that $|s_i - s_j|, |t_i - t_j| < 2\zeta$ whenever $v_i = v_j$. If under these conditions it is always the case that $|s_1 - t_1|, \ldots, |s_k - t_k| < \lambda$ implies

$$|\varrho([v_1]^\zeta_{s_1} \cap \ldots \cap [v_k]^\zeta_{s_k}) - \varrho([v_1]^\zeta_{t_1} \cap \ldots \cap [v_k]^\zeta_{t_k})| < \tau\lambda \tag{17}$$

we say that ϱ is *τ-smooth* w.r.t. U.

The arguments of ϱ in (17) have a rather specific form; a more general form is possible as well.

For the proof that follows, note that any $A_i, B_i \in \mathcal{R}_M$, $i \in I$, where M is a Kleene uncertainty algebra, we have

$$|\varrho(A_i) - \varrho(B_i)| < \lambda \quad \text{for each } i \text{ implies} \quad |\varrho(\textstyle\bigcup_i A_i) - \varrho(\bigcup_i B_i)| < \lambda. \tag{18}$$

Lemma 5.2. *Let (M, ϱ) be a regular Kleene uncertainty algebra such that ϱ is τ-smooth w.r.t. $U \subseteq M$. Let $v_1 \ldots, v_k \in M$ be expressible from elements of U in a way that each $u \in U$ occurs always positively or always negatively. Furthermore, let $s_1, t_1, \ldots, s_k, t_k \in [0,1]$ such that $|s_i - s_j|, |t_i - t_j| < 2\zeta$ whenever both in v_i and v_j some $u \in U$ occurs. Then $|s_1 - t_1|, \ldots, |s_k - t_k| < \lambda$ implies* (17).

Proof. The argument relies on a decomposition of the occurring terms by means of Lemma 4.10. Namely, we can write $[v_1]^\zeta_{s_1} \cap \ldots \cap [v_k]^\zeta_{s_k}$ as the disjunction of conjunctions of expressions of the form $[u]^\zeta_s$ or $[u]^\zeta_{\geq s}$ or $[u]^\zeta_{\leq s}$, where $u \in U$ and $s \in \{s_1, {\sim}s_1, \ldots, s_k, {\sim}s_k\}$. Consider one among these conjunctions, and let $u \in U$ occur in it. Then the conjunction of those conjuncts in which u appears equals $[u]^\zeta_s$ or $[u]^\zeta_{\geq s}$ or $[u]^\zeta_{\leq s}$ or $[u]^\zeta_s \cap [u]^\zeta_{s'}$, where $|s - s'| < 2\zeta$. As we can express the terms of the form $[u]^\zeta_{\leq s}$ or $[u]^\zeta_{\geq s}$ as a disjunction of expressions of the form $[u]^\zeta_t$, we conclude that $[v_1]^\zeta_{s_1} \cap \ldots \cap [v_k]^\zeta_{s_k}$ is the disjunction of terms of the form as specified in Definition 5.1.

Let us next decompose $[v_1]^\zeta_{t_1} \cap \ldots \cap [v_k]^\zeta_{t_k}$ in exact analogy. The disjuncts are then in a one-to-one correspondence, and corresponding values differ by less than λ.

From (18) we conclude the assertion. $\square$

Smoothness of a rejection function can be particularly easily expressed in terms of the natural parametrisation of the Boolean algebra $\mathcal{R}_M$ associated to a regular Kleene algebra (M, ϱ). Namely, we may associate with each cube $(r_1, s_1) \times \ldots \times (r_N, s_N) \subseteq [0,1]^N$ the (not necessarily non-zero) element $\{x \in S : \; r_1 < u_1(x) < s_1, \; \ldots, \; r_N < u_N(x) < s_N\}$ of $\mathcal{R}_M$.

Lemma 5.3. *Let (M, ϱ) be a regular Kleene uncertainty algebra, and let $U = \{u_1, \ldots, u_N\} \subseteq M$ generate M as a Kleene algebra. For $I_1 \times \ldots \times I_N \in \mathcal{R}^{\mathrm{f}}([0,1])^N$, where $I_1, \ldots, I_N \in \mathcal{R}^{\mathrm{f}}([0,1])$, put*

$$\iota(I_1 \times \ldots \times I_N) \;=\; \iota_{u_1} I_1 \cap \ldots \cap \iota_{u_N} I_N,$$

where ι_{u_i}, $i = 1, ..., N$ is given according to Lemma 4.9. Then ι extends to an epimorphism between Boolean algebras from $\mathcal{R}^{\mathrm{f}}([0,1])^N$ to $\mathcal{R}_M$.

Proof. For each i, ι_{u_i} is an homomorphism from $\mathcal{R}^{\mathrm{f}}([0,1])$ to $\mathcal{R}_M$ by Lemma 4.9. These homomorphisms are combined to the homomorphism ι from the N-fold free product of $\mathcal{R}^{\mathrm{f}}([0,1])$ to $\mathcal{R}_M$ as indicated [11, Chapter VI].

Because M is generated by U and because of Lemma 4.10, ι is surjective. $\square$

Before expressing smoothness with respect to a parametrisation of S, we have to fix some notation. We will use the supremum metric $\mathrm{d}_\infty(\cdot,\cdot)$ on $[0,1]^N$:

$$\mathrm{d}_\infty((x_1, ..., x_N), (y_1, ..., y_N)) \;=\; \max_i |x_i - y_i|, \qquad (x_1, ..., x_N), (y_1, ..., y_N) \in [0,1]^N.$$

Occasionally, we will use the ε-neighbourhoods w.r.t. this metric; for $p \in [0,1]^N$ and $\varepsilon > 0$, we put $U_\varepsilon(p) = \{q \in [0,1]^N \colon \mathrm{d}_\infty(p,q) < \varepsilon\}$. Moreover, the diameter of some $P \in \mathcal{R}^{\mathrm{f}}([0,1])^N$ will be meant to be the value $\sup\{\mathrm{d}_\infty(p_1,p_2) \colon p_1, p_2 \in P\}$.

We furthermore extend $\mathrm{d}_\infty(\cdot,\cdot)$ to pairs of subsets in the usual way, both asymmetrical and symmetrical. So $\mathrm{q_H}(\cdot,\cdot)$ is the Hausdorff quasimetric on $\mathcal{R}^{\mathrm{f}}([0,1])^N$:

$$\mathrm{q_H}(P,Q) \;=\; \sup_{p \in P} \inf_{q \in Q} \mathrm{d}_\infty(p,q), \qquad P, Q \in \mathcal{R}^{\mathrm{f}}([0,1])^N;$$

and $\mathrm{d_H}(\cdot,\cdot)$ is the Hausdorff metric on $\mathcal{R}^{\mathrm{f}}([0,1])^N$:

$$\mathrm{d_H}(P,Q) \;=\; \mathrm{q_H}(P,Q) \vee \mathrm{q_H}(Q,P), \qquad P, Q \in \mathcal{R}^{\mathrm{f}}([0,1])^N.$$

We are ready to state several alternative formulations of smoothness of the rejection function.

Definition 5.4. Let (M, ϱ) be a regular Kleene uncertainty algebra, and let $U = \{u_1, \ldots, u_N\} \subseteq M$ generate M. Let $\iota \colon \mathcal{R}^{\mathrm{f}}([0,1])^N \to \mathcal{R}_M$ be defined according to Lemma 5.3; then we will call ι the *natural parametrisation* of $\mathcal{R}_M$ w.r.t. U.

Furthermore, ϱ is said to be *induced* by a function $r \colon D \to [0,1]$, where D is a dense subset of $[0,1]^N$, if

$$\varrho(\iota P) \;=\; \inf_{p \in P \cap D} r(p), \qquad P \in \mathcal{R}^{\mathrm{f}}([0,1])^N.$$

Finally, a function $r \colon [0,1]^N \to [0,1]$ is called *τ-Lipschitz continuous* if, for any $p, q \in [0,1]^N$, we have $|r(p) - r(q)| < \tau\lambda$ whenever $\mathrm{d}_\infty(p,q) < \lambda$.

Lemma 5.5. *Let (M, ϱ) be a regular Kleene uncertainty algebra, and let U generate M. Let $\iota\colon\, \mathcal{R}([0,1])^N \to \mathcal{R}_M$ be the natural parametrisation of $\mathcal{R}_M$ w.r.t. U. Then the following statements are equivalent:*

(α) *ϱ is τ-smooth w.r.t. U.*

(β) *For any non-empty $P, Q \in \mathcal{R}^{\mathrm{f}}([0,1])^N$, $|\varrho(\iota P) - \varrho(\iota Q)| < \tau\lambda$ if $\mathrm{d}_{\mathrm{H}}(P,Q) < \lambda$.*

(γ) *ϱ is induced by a τ-Lipschitz continuous function $r\colon\, [0,1]^N \to [0,1]$.*

Proof. Assume (α). To show (β), let $P, Q \in \mathcal{R}^{\mathrm{f}}([0,1])^N$ such that $\mathrm{d}_{\mathrm{H}}(P,Q) < \lambda$. Assume first that P and Q are cubic and have a diameter $\leq 2\zeta$. Then $P = I_1 \times \ldots \times I_N$ and $Q = J_1 \times \ldots \times J_N$ for intervals $I_1, \ldots, J_N \in \mathcal{R}^{\mathrm{f}}([0,1])$ such that $\mathrm{d}_{\mathrm{H}}(I_1, J_1), \ldots, \mathrm{d}_{\mathrm{H}}(I_N, J_N) < \lambda$. Clearly, there are $s_1, s_2, t_1, t_2 \in (-\zeta, 1+\zeta)$ such that $I_1 = U_\zeta(s_1) \cap U_\zeta(s_2)$ and $J_1 = U_\zeta(t_1) \cap U_\zeta(t_2)$ and $|s_1 - t_1|, |s_2 - t_2| < \lambda$; and the same holds for the remaining indices $2, \ldots, N$. Because, for $t \in (-\zeta, 1+\zeta)$, $\iota(U_\zeta(t) \times [0,1] \times \ldots \times [0,1]) = [u_1]^\zeta_t$ and similarly for the indices $2, \ldots, N$, the assertion follows.

Let now $P, Q \in \mathcal{R}^{\mathrm{f}}([0,1])^N$ be arbitrary such that $\mathrm{d}_{\mathrm{H}}(P,Q) < \lambda$. Then there are $P_1, \ldots, P_k, Q_1, \ldots, Q_k$ with diameter $\leq 2\zeta$ such that $P = \bigcup_i P_i$ and $Q = \bigcup_i Q_i$ and furthermore $\mathrm{d}_{\mathrm{H}}(P_i, Q_i) < \lambda$ for all i. We conclude by (18) that (β) holds.

It is clear that (β) implies (α).

Assume (β); we proceed to show (γ). For $p \in [0,1]^N$, let

$$r(p) \;=\; \sup\,\{\varrho(\iota P)\colon\, P \in \mathcal{R}^{\mathrm{f}}([0,1])^N \text{ and } p \in P\}$$

We claim that r is τ-smooth. Indeed, let $p, q \in [0,1]^N$ such that $\mathrm{d}_\infty(p,q) < \lambda$. Let λ' be such that $\mathrm{d}_\infty(p,q) < \lambda' < \lambda$ and let $\varepsilon > 0$. Choose cubic neighbourhoods U_p of p and U_q of q with diameter $\leq 2\zeta$ such that $|r(p) - \varrho(\iota U_p)|, |r(q) - \varrho(\iota U_q)| \leq \varepsilon$ and $\mathrm{d}_{\mathrm{H}}(U_p, U_q) < \lambda'$. Then $|r(p) - r(q)| \leq |\varrho(\iota U_p) - \varrho(\iota U_q)| + 2\varepsilon < \tau\lambda' + 2\varepsilon$, and the claim follows.

We next claim that for $P \in \mathcal{R}^{\mathrm{f}}([0,1])^N$ we have $\varrho(\iota P) = \inf\,\{r(p)\colon\, p \in P\}$. Indeed, $\varrho(\iota P) \leq r(p)$ for any $p \in P$. Let $P \supset P_1 \supset P_2 \supset \ldots$ such that $\varrho(\iota P_i) = \varrho(\iota P)$ for all i and such that the diameter of P_i converges to 0. Let $\varepsilon > 0$; let i be large enough such that the diameter of P_i is below ε; then $\varrho(\iota Q) < \varrho(\iota P_i) + \tau\varepsilon = \varrho(\iota P) + \tau\varepsilon$ for any $Q \subseteq P_i$, and it follows $r(p) \leq \varrho(\iota P) + \tau\varepsilon$ for any $p \in P_i$. The claim follows, and (γ) is shown.

Assume (γ), and let ϱ be induced by the τ-smooth $r\colon\, [0,1]^N \to [0,1]$. Let $P, Q \in \mathcal{R}^{\mathrm{f}}([0,1])^N$ such that $\mathrm{d}_{\mathrm{H}}(P,Q) = \lambda' < \lambda$. W.l.o.g. we assume $\varrho(\iota P) \geq \varrho(\iota Q)$. Then $|\varrho(\iota P) - \varrho(\iota Q)| = \inf_{p \in P} r(p) - \inf_{q \in Q} r(q)$. Let $\varepsilon > 0$, and choose a $q \in Q$ such that $r(q) - \varrho(\iota Q) \leq \varepsilon$; and choose $p \in P$ such that $\mathrm{d}_\infty(p,q) < \lambda' + \varepsilon$. Then $|\varrho(\iota P) - \varrho(\iota Q)| \leq r(p) - r(q) + \varepsilon < \tau(\lambda' + \varepsilon) + \varepsilon$. ($\beta$) follows. $\square$

Lemma 5.6. *Let (M, ϱ) be a regular Kleene uncertainty algebra, and let $U \subseteq M$ generate M. Let $\iota\colon\ \mathcal{R}^{\mathrm{f}}([0,1])^N \to \mathcal{R}_M$ be the natural parametrisation of $\mathcal{R}_M$ w.r.t. U. Let $R_1, \ldots, R_m$ be a partition of $\mathcal{R}^{\mathrm{f}}([0,1])^N$; then $D = R_1 \cup \ldots \cup R_m$ is dense in $[0,1]^N$. Let furthermore $r_1, \ldots, r_m \in [0,1]$ such that $r_i = 0$ for at least one index i. Let*

$$r\colon\ D \to [0,1], \quad p \mapsto \begin{cases} r_1 & \text{if } p \in R_1, \\ \ldots & \\ r_m & \text{if } p \in R_m, \end{cases}$$

and let ϱ be the rejection function induced by r. Furthermore, let

$$r'\colon\ [0,1]^N \to [0,1], \quad p \mapsto \sup_{q \in D}\, (r(q) - \tau \mathrm{d}_\infty(p,q)) \vee 0,$$

and let ϱ' be the rejection function induced by r'. Then ϱ' is the smallest τ-smooth rejection function such that $\varrho' \geq \varrho$.

Moreover, let $P \in \mathcal{R}^{\mathrm{f}}([0,1])^N$. Then there are cubes $P_1, \ldots, P_n \in \mathcal{R}^{\mathrm{f}}([0,1])^N$ such that $P = P_1 \vee \ldots \vee P_n$, and for each $i = 1, \ldots, n$ there is a cubic Q_i contained in R_j for some $j \in \{1, ..., m\}$ such that either

$$\varrho'(\iota P_i) \ = \ \varrho(\iota Q_i) - \tau \mathrm{d}_{\mathrm{H}}(P_i, Q_i) = \varrho'(\iota P) \tag{19}$$

or

$$\varrho'(\iota P_i) \geq \varrho(\iota Q_i) - \tau \mathrm{d}_{\mathrm{H}}(P_i, Q_i) > \varrho'(\iota P), \tag{20}$$

where the first case applies for at least one i.

Proof. We first show that r' is τ-smooth. Let $p, q \in [0,1]^N$ such that $\mathrm{d}_\infty(p,q) < \lambda$. Let $\varepsilon > 0$, and choose $s_q \in D$ such that $r'(q) \leq r(s_q) - \tau \mathrm{d}_\infty(q, s_q) + \varepsilon$. Then $r'(p) \geq r(s_q) - \tau \mathrm{d}_\infty(p, s_q) \geq r(s_q) - \tau \mathrm{d}_\infty(p,q) - \tau \mathrm{d}_\infty(q, s_q) \geq r'(q) - \tau \mathrm{d}_\infty(p,q) - \varepsilon$. So $r'(q) - r'(p) < \tau\lambda$, and by symmetry we conclude $|r'(p) - r'(q)| < \tau\lambda$.

Clearly $r'|_D \geq r$, hence $\varrho' \geq \varrho$. Let now $\varrho'' \geq \varrho$ another τ-smooth rejection function. Let ϱ'' be induced by r''. Let $p \in D$, and let i be such that $p \in R_i$. Then $r''(p) \geq \varrho''(\iota R_i) \geq \varrho(\iota R_i) = r_i = r(p)$; hence $r'' \geq r$. For any $q \in D$, it follows $r''(p) \geq r''(q) - \tau \mathrm{d}_\infty(p,q) \geq r(q) - \tau \mathrm{d}_\infty(p,q)$; hence even $r'' \geq r'$.

It remains to show the last assertion. W.l.o.g. we may assume that $R_1, \ldots, R_m$ and P are all cubic. We have

$$\begin{aligned} \varrho'(\iota P) \ &= \ \inf_{p \in P} \sup_{q \in D}\, (r(q) - \tau \mathrm{d}_\infty(p,q)) \\ &= \ \inf_{p \in P} \max_i\, (r_i - \tau \mathrm{q}_{\mathrm{H}}(\{p\}, R_i)). \end{aligned} \tag{21}$$

Let us consider a point $p = (z_1, \ldots, z_N) \in P^-$. There are two cases:

(A) $\max_i\, (r_i - \tau \mathrm{q}_{\mathrm{H}}(\{p\}, R_i) = \varrho'(\iota P)$

(B) $\max_i\, (r_i - \tau \mathrm{q}_{\mathrm{H}}(\{p\}, R_i)) > \varrho'(\iota P)$.

If (A) applies, we will associate with p a cubic neighbourhood U_p and a partition of $U_p \cap P$ such that, for each element U of this partition, $\varrho(\iota U)$ can be calculated according to (19). Note that by the continuity of the mapping $p \mapsto \max_i \, (r_i - \tau \mathrm{q_H}(\{p\}, R_i)$ there is at least one $p \in P^-$ fulfilling (A). If (B) applies, we will associate with p a cubic neighbourhood U_p such that (20) holds for $U_p \cap P$. $(U_p)_{p \in P^-}$ will be a cover of P^- by open sets; as P^- is compact, we may choose a finite subcover, and we will be done.

Case (A)*:* Let $J = \{j \in \{1, ..., m\} \colon \varrho'(\iota P) = r_j - \tau \mathrm{q_H}(\{p\}, R_j)\}$. For $1 \leq i \leq N$, let $\pm_i \in \{+, -\}$ such that $E(\pm_1, ..., \pm_N) = \{(z_1 \pm_1 t_1, \ldots, z_N \pm_N t_N) \colon t_1, \ldots, t_N \geq 0\}$ intersects P non-emptily. Then there must be a $j \in J$ and an $\varepsilon > 0$ such that $U_\varepsilon(p) \cap E(\pm_1, ..., \pm_N) \subseteq P$ and $\mathrm{q_H}(\{(z_1 \pm_1 t, ..., z_N \pm_N t)\}, R_j) \leq \mathrm{q_H}(\{p\}, R_j)$ for $0 < t \leq \varepsilon$; indeed, otherwise the infimum (21) would not be attained at p.

It follows that $\mathrm{q_H}(\{q\}, R_j) \leq \mathrm{q_H}(\{p\}, R_j)$ for all $q \in U_\varepsilon(p) \cap E(\pm_1, ..., \pm_N)$, and we conclude $\mathrm{q_H}(U_\varepsilon(p) \cap E(\pm_1, ..., \pm_N), R_j) = \mathrm{q_H}(\{p\}, R_j)$. We select a cubic $Q \subseteq R_j$ such that $\mathrm{q_H}(U_\varepsilon(p) \cap E(\pm_1, ..., \pm_N), R_j) = \mathrm{d_H}(U_\varepsilon(p) \cap E(\pm_1, ..., \pm_N), Q)$. So we have $\varrho'(\iota(U_\varepsilon(p) \cap E(\pm_1, ..., \pm_N)) = \varrho'(\iota P) = r_j - \tau \mathrm{q_H}(\{p\}, R_j) = \varrho(\iota Q) - \tau \mathrm{d_H}(U_\varepsilon(p) \cap E(\pm_1, ..., \pm_N), Q)$. Decreasing ε if necessary, we put $U_p = U_\varepsilon(p)$.

Case (B)*:* Let j be such that $r_j - \tau \mathrm{q_H}(\{p\}, R_j) > \varrho'(\iota P)$. Let U_p be a cubic neighbourhood of p such that, for some r, we have $r_j - \tau \mathrm{q_H}(\{q\}, R_j) \geq r > \varrho'(\iota P)$ for all $q \in U_p$ and consequently $r_j - \tau \mathrm{q_H}(\{U_p \cap P\}, R_j) \geq r$. We select a cubic $Q \subseteq R_j$ such that $\mathrm{q_H}(\{U_p \cap P\}, R_j) = \mathrm{d_H}(\{U_p \cap P\}, Q)$. Then, by the smoothness of ϱ', we conclude $\varrho'(\iota(U_p \cap P)) \geq \varrho'(\iota Q) - \tau \mathrm{d_H}(\{U_p \cap P\}, Q) \geq \varrho(\iota Q) - \tau \mathrm{d_H}(\{U_p \cap P\}, Q) = r_j - \tau \mathrm{q_H}(\{U_p \cap P\}, R_j) \geq r > \varrho'(\iota P)$. ☐

We now modify $\mathtt{IG}^\zeta$ accordingly. The resulting logic will be called the Smooth Possibilistic Logic with Soft Gradation, denoted by $\mathtt{IG}^\zeta_\tau$.

Definition 5.7. The *propositions*, the set of which will still be denoted by $\mathcal{P}$, as well as the *implications* of $\mathtt{IG}^\zeta_\tau$ coincide with those of $\mathtt{IG}^\zeta$, respectively (see Def. 4.7).

An *evaluation* (v_f, v_b) of $\mathtt{IG}^\zeta_\tau$ in some regular Kleene uncertainty algebra (M, ϱ) is defined like for $\mathtt{IG}^\zeta$ except thatϱ is required to be τ-smooth w.r.t. $v_f(\varphi_1), \ldots, v_f(\varphi_N)$.

The notions of *satisfaction*, of a *theory*, and of *semantic entailment* for $\mathtt{IG}^\zeta$ is defined mutatis mutandis like for $\mathtt{I}$ (see Def. 2.3).

For an axiomatisation of $\mathtt{IG}^\zeta_\tau$ we have to add a rule reflecting the restriction to smooth rejection functions.

Definition 5.8. The rules of $\mathtt{IG}^\zeta_\tau$ are those of $\mathtt{IG}^\zeta$ (see Def. 4.8) and in addition the following *smoothing rule*. Here, $\psi_1, \ldots, \psi_k$ are gradable propositions such that each variable occurs in them at all places positively or at all places negatively; furthermore, $s_1, \ldots, s_k, t_1, \ldots, t_k \in (-\zeta, 1 + \zeta)$ such that $|s_1 - t_1|, \ldots, |s_k - t_k| < \lambda$ and if some variable occurs both in ψ_{i_1} and ψ_{i_2} then $s_{i_1} - s_{i_2}, t_{i_1} - t_{i_2} < 2\zeta$; and finally, α is a graded proposition which has no variable in common with $\psi_1, \dots, \psi_k$; and $d \in [0,1]$:

$$\frac{(\psi_1, t_1), \dots, (\psi_k, t_k) \stackrel{d}{\Rightarrow} \alpha}{(\psi_1, s_1), \dots, (\psi_k, s_k) \stackrel{(d - \tau\lambda) \vee 0}{\Rightarrow} \alpha}$$

The notion of a proof as well as consistency is defined like for $\mathtt{IG}^\zeta$.

Theorem 5.9. *Let $\mathcal{T}$ be a consistent theory of $\mathtt{IG}^\zeta_\tau$ and $\Gamma \stackrel{e}{\Rightarrow} \delta$ an implication of $\mathtt{IG}^\zeta_\tau$. Then $\mathcal{T}$ semantically entails $\Gamma \stackrel{e}{\Rightarrow} \delta$ if and only if $\mathcal{T}$ proves $\Gamma \stackrel{e}{\Rightarrow} \delta$.*

Proof. The soundness of the rules of $\mathtt{IG}^\zeta$ follows from Theorem 4.14; the soundness of the smoothing rule follows from Lemma 5.2.

To show completeness, assume that $\mathcal{T}$ does not prove $\Gamma \stackrel{e}{\Rightarrow} \delta$. Disregarding the smoothness rule, we proceed like in the proof of Theorem 4.14 to construct the evaluation (v_f, v_b) in the regular Kleene uncertainty algebra (M, ϱ) such that all elements of $\mathcal{T}$ are satisfied by (v_f, v_b). Let furthermore ϱ' be the smallest τ-smooth rejection function such that $\varrho' \geq \varrho$; then $\mathcal{T}$ is satisfied by (v_f, v_b) also in (M, ϱ'). Moreover, let α be any proposition and $d = \varrho'(v_b(\alpha))$. By Lemma 5.6 and the presence of the smoothness rule, $\mathcal{T}$ proves $\alpha \stackrel{d}{\Rightarrow} \perp$.

Let α be the conjunction of $\Gamma \cup \{\neg\delta\}$. Assume that $\Gamma \stackrel{e}{\Rightarrow} \delta$ is satisfied in (M, ϱ'). This means that $e' = \varrho'(v_b(\alpha)) \geq e$. It follows that $\mathcal{T}$ proves $\alpha \stackrel{e'}{\Rightarrow} \perp$, so that $\mathcal{T}$ proves $\Gamma \stackrel{e}{\Rightarrow} \delta$, in contradiction to the assumption. □

6 Conclusion

We have extended Dubois and Prade's Possibilistic Logic so as to allow the treatment of vague notions. Our guideline was to integrate, but not to mix, aspects of uncertainty and of vagueness in a uniform framework. Statements of the form that a property holds to a specific degree were integrated into the plausibility-based calculus. The degree of presence of a property has by default no influence on the degree of its plausibility; a smoothness rule, whose effect can be controlled by a real parameter, can however be added to ensure the continuity of the degree of uncertainty with regard to changes of the degrees of presence of the involved properties.

As regards the foundational problem of fuzzy logic, the method has, as we guess, the advantage that fuzzy sets are treated as parametrised sets of crisp properties, which in turn are treated classically. The question how to model vague properties by fuzzy sets is however assumed to be solved and the Kleene algebra structure has to be accepted as definitional. We may just underline that the choice of an appropriate fuzzy set for a given property works in practise very well and the decision about the shape of fuzzy sets can in fact be put on firm grounds as, for instance, the work [13] demonstrates. Even the adequacy of the Kleene algebra structure is supported by results of [13]. But we should certainly remain cautious – in general we should say that the operations between fuzzy sets chosen here are widely used but in most cases purely pragmatically.

In the remainder of the paper we will explain how our formalism can be of practical use. Namely, it is possible to mimic the inference mechanism of a medical expert system in our logic $\mathtt{IG}^{\zeta}_{\tau}$. We have in mind the expert system CADIAG-2 [1, 15], which has been developed at the Medical University of Vienna. It provides clinical decision support in several areas of internal medicine.

Assume that the following facts about a patient are known and that the following rule is contained in the knowledge base of CADIAG-2; we use the notation of [3]:

$$(\sigma_1,\ s),\quad (\sigma_2,\ t),\quad (\sigma_1 \wedge \sigma_2 \to \delta,\ d);$$

here, σ_1 and σ_2 denote symptoms, δ denotes a disease, and $s, t, d \in [0, 1]$. These statements code the following information: the symptom σ_1 holds to the degree s; the symptom σ_2 holds to the degree t; and if the conjunction of these two symptoms evaluate to 1, that is, if they both fully apply, we may conclude that δ is certain to the degree d. The following rules of the logic underlying CADIAG-2 – here we show the appropriate instances – are applied to draw a conclusion in case that $s, t, d > 0$ (see [3]):

$$\frac{(\sigma_1,\ s) \quad (\sigma_2,\ t)}{(\sigma_1 \wedge \sigma_2,\ s{\scriptstyle\wedge} t)} \qquad \frac{(\sigma_1 \wedge \sigma_2,\ s{\scriptstyle\wedge} t) \quad (\sigma_1 \wedge \sigma_2 \to \delta,\ d)}{(\delta,\ d \star (s{\scriptstyle\wedge} t))},$$

where $\star$ is a t-norm. We may for instance assume that $\star$ is the Łukasiewicz t-norm:

$$\star\colon\ [0, 1] \times [0, 1] \to [0, 1],\quad (t_1, t_2) \mapsto (t_1 + t_2 - 1){\scriptstyle\vee} 0.$$

Thus, in other words, CADIAG-2 concludes from $(\sigma_1,\ s)$ and $(\sigma_2,\ t)$ that δ is certain to the degree $d + s{\scriptstyle\wedge} t - 1$, provided that this value is strictly positive.

We switch now to the present framework. We choose $\zeta = 0.3$ and $\tau = 4$. The rule shown above translates to the following in $\mathtt{IG}^{0.3}_{4}$:

$$(\sigma_1 \wedge \sigma_2, 1) \overset{d}{\Rightarrow} (\delta, 1);$$

from this implication we derive in $\mathtt{IG}^{0.3}_{4}$

$$(\sigma_1, 1), (\sigma_2, 1) \overset{d}{\Rightarrow} (\delta, 1),$$

and using the smoothing rule furthermore

$$(\sigma_1, s), (\sigma_2, t) \overset{(d+4(s\wedge t)-4)\vee 0}{\Rightarrow} (\delta, 1).$$

Thus in this framework, the conclusion is that δ is present with the certainty degree $d + 4(s \wedge t) - 4 = d - 4(1 - s{\scriptstyle\wedge} t)$, provided that this value is positive. Comparing this with the value $d + s{\scriptstyle\wedge} t - 1 = d - (1 - s{\scriptstyle\wedge} t)$, we see that the conclusion is more cautious in $\mathtt{IG}^{0.3}_{4}$ than in CADIAG-2.

We conclude, first of all, that the shown inference of CADIAG-2, although in a modified form, is possible in $\mathtt{IG}^{0.3}_{4}$as well. The expert system can in fact be based on $\mathtt{IG}^{0.3}_{4}$.

However, we have not yet examined the question of performance. In the example, the certainty value provided by $\mathtt{IG}^{0.3}_{4}$ is smaller, so that the result is weaker than in the original version. There are two further types of rules in the knowledge base of CADIAG-2; in case of these rules the results will in general be stronger. Concerning the overall performance of CADIAG-2 on the one hand and of a system modified according to the ideas presented in this article on the other hand, this does not imply anything though. Practical tests with patient data are to follow.

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