A consequence relation for graded inference within the frame of infinite-valued Łukasiewicz logic

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Abstract

We present a family of consequence relations for graded inference among Łukasiewicz sentences. Given a set of premises and a threshold η, we consider consequences those sentences entailed to hold to at least some degree ζ whenever the premises hold to a degree at least η. We focus on the study of some aspects and features of the consequence relations presented and, in particular, on the effect of variations in the thresholds η, ζ.

1 Introduction

In this paper we present and study the family of consequence relations of the form ${}^{\eta}\blacktriangleright_{\zeta}$, for different thresholds η and ζ, aimed at formalizing graded inference among Łukasiewicz sentences in the following terms: given a set of premises and a threshold η, we consider as consequences those sentences entailed by Łukasiewicz logic (see, e.g., [7]) to hold to a degree at least ζ whenever the premises hold to a degree at least η.

The study and analysis of the family of consequence relations of the form ${}^{\eta}\blacktriangleright_{\zeta}$ carried out in this paper focuses mostly on the effect of variations of the thresholds η, ζ in relation to a given set of premises Γ and consequence θ. In particular, given η and a set of premises Γ, we are interested in determining to which degrees a particular sentence θ holds provided that the sentences in Γ hold at least to the degree η. This effect is formalized by means of the function $\mathcal{L}_{\Gamma,\theta}$ which, for a given degree η for our set of premises Γ, gives us the maximal degree ζ to which θ holds (i.e., the maximal value ζ such that $\Gamma^{\eta} \blacktriangleright_{\zeta} \theta$).

The motivation for the definition of the family of consequence relations ${}^{\eta} \blacktriangleright_{\zeta}$ and the issues we deal with in this paper partially originated from the previous study and analysis of the family of consequence relations of the form ${}^{\eta}\triangleright_{\zeta}$, first defined in a simplified version in [12] (for $\eta = \zeta$) and further studied and extended in [13], [14] and [15]. These consequence relations formalize graded inference in terms very similar to those of ${}^{\eta} \blacktriangleright_{\zeta}$, with the main difference being that the underlying semantics is probabilistic (see, e.g., [11]). In the context of ${}^{\eta}\triangleright_{\zeta}$ the thresholds η, ζ are considered to be probabilities (interpreted in the aforementioned references as degrees of belief) rather than, e.g., truth degrees—the standard interpretation for such values in the context of Łukasiewicz logic (see, e.g., [7] or [11]).

The present paper builds on [16] by first summarizing and extending results in it and, later, by introducing new ideas and methodology related to further analysis and computation of $\mathcal{L}_{\Gamma,\theta}$. It is structured as follows: in Section 2 we fix notation and give some preliminary definitions. In Section 3 we introduce the preliminary notion of Ł_{η}-consistency, for use in further sections. In Section 4 we formally introduce the family of consequence relations of the form ${}^{\eta} \blacktriangleright_{\zeta}$ and the functions of the form $\mathcal{L}_{\Gamma,\theta}$ described above. Some connections to related work in the literature are also given. Section 5 constitutes the core of the present paper. In it we provide a representation theorem that offers a full characterization of the functions of the form $\mathcal{L}_{\Gamma,\theta}$. Intermediate results needed for our representation theorem are provided in Sections 5.1, where some relevant properties of $\mathcal{L}_{\Gamma,\theta}$ are proved, and 5.2, where some characteristic functions of the form $\mathcal{L}_{\Gamma,\theta}$ are analyzed. In Section 6, some methodology and insights into the computation of functions of the form $\mathcal{L}_{\Gamma,\theta}$ is presented.

2 Preliminary definitions and notation

Throughout we will be working with a finite set of propositional variables $L = \{p_1, ..., p_m\}$, for some $m \in \mathbb{N}$. We will denote by SL the closure of $L \cup \{\perp\}$ under *implication*, i.e., the connective '$\rightarrow$'.

We define now the notion of Ł-valuation, based on the standard interpretation of sentences in propositional Łukasiewicz logic.

Definition 1 *Let $\omega : SL \longrightarrow [0, 1]$. We say that ω is an Ł-valuation on L if, for ϕ, $\theta \in SL$, we have what follows:*

1. $\omega(\phi \to \theta) = \min\{1, 1 - \omega(\phi) + \omega(\theta)\}$
2. $\omega(\bot) = 0$

We will be using in this paper a large number of abbreviations which correspond to other common logical connectives within the context of many-valued logics. We consider the following abbreviations, for $\phi, \theta \in SL$:

- $\bot \to \bot$ is abbreviated by $\top$
- $\phi \to \bot$ by $\neg\phi$
- $\neg(\phi \to \neg\theta)$ by $\phi \& \theta$
- $\neg\phi \to \theta$ by $\phi \underline{\vee} \theta$
- $\phi \& (\phi \to \theta)$ by $\phi \wedge \theta$
- $((\phi \to \theta) \to \theta) \wedge ((\theta \to \phi) \to \phi)$ by $\phi \vee \theta$.

From Definition 1 we can deduce the behaviour of Ł-valuations for the other connectives aforementioned.

Let $\phi, \theta \in SL$. We have what follows:

- $\omega(\top) = 1$
- $\omega(\neg\phi) = 1 - \omega(\phi)$
- $\omega(\phi \& \theta) = \max\{0, \omega(\phi) + \omega(\theta) - 1\}$
- $\omega(\phi \underline{\vee} \theta) = \min\{1, \omega(\phi) + \omega(\theta)\}$
- $\omega(\phi \wedge \theta) = \min\{\omega(\phi), \omega(\theta)\}$
- $\omega(\phi \vee \theta) = \max\{\omega(\phi), \omega(\theta)\}$

We will sometimes refer to Ł-valuations on L as m-tuples (based on some ordering of our set of propositional variables L). The set of all such tuples will be denoted by $\mathbb{D}_m$ (i.e., $\mathbb{D}_m = [0,1]^m$).

Let $\Gamma = \{\phi_1, ..., \phi_k\} \subset SL$, for some $k \in \mathbb{N}$. We will denote by $\bigwedge \Gamma$ the sentence $\phi_1 \wedge ... \wedge \phi_k$. Similarly $\bigvee \Gamma$, $\underline{\bigvee}\Gamma$ and $\&\Gamma$ will denote the sentences $\phi_1 \vee ... \vee \phi_k$, $\phi_1 \underline{\vee} ... \underline{\vee} \phi_k$ and $\phi_1 \& ... \& \phi_k$ respectively.

Sentences of the form $\phi \wedge ... \wedge \phi$ where ϕ occurs k times will be abbreviated by the expression $\bigwedge^k \theta$ (and similarly for the other connectives). It is customary to refer to $\&^k\phi$ (i.e., $\phi\&...\&\phi$, where ϕ occurs k times) by ϕ^k in the literature and we will follow this convention.

Let $\phi \in SL$. We will denote by $L_\phi = \{q_1, ..., q_k\} \subseteq L$ the set of propositional variables that occur in ϕ. We will sometimes use the notation $\phi(q_1, ..., q_k)$.

Let ω be an Ł-valuation on L. We have that

$$\omega(\phi) = f(\omega(q_1), ..., \omega(q_k))$$

for some $f : [0,1]^k \to [0,1]$. We will denote this f by f_ϕ. Sometimes, we will write $f_\phi(x_1, ..., x_k)$ or, in order to simplify notation slightly, $f_\phi(\vec{x})$, where $\vec{x}$ is an m-tuple in $\mathbb{D}_m$ (i.e., we consider as many paremeters in f as variables in L). Such functions we will call *NcNaughton functions.*

Throughout we will be working with finite subsets of sentences in SL. Unless otherwise stated we will consider only finite subsets of SL.

Next we state a central theorem in Łukasiewicz logic that will play an important role in this paper.

Theorem 2 ***McNaughton's Theorem*** *(see [10])*
In order that a function $f : [0,1]^k \to [0,1]$ be of the form f_ϕ for some $\phi \in SL$ it is necessary and sufficient that f satisfy the following two conditions:

1. *f is continuous on $[0,1]^k$.*

2. *There are a finite number of distinct polynomials with integer coefficients λ_i, $1 \leq i \leq \mu$,*

$$\lambda_i = b_i + m_{1_i}x_1 + ... + m_{k_i}x_k,$$

such that for every $(x_1, ..., x_k)$, $0 \leq x_i \leq 1$ for all $i \in \{1, ..., k\}$, there is λ_j for some $j \in \{1, ..., \mu\}$ such that $f(x_1, ..., x_k) = \lambda_j(x_1, ..., x_k)$.

For a proof of this theorem see [10].

3 The notion of Ł$_\eta$-consistency

We start this section by defining the notion of Ł$_\eta$-consistency. To do so, let us assume that $\Gamma \subset SL$ and $\eta \in [0,1]$.

Definition 3 *We say that Γ is Ł$_\eta$-consistent if there exists an Ł-valuation ω on L such that $\omega(\bigwedge \Gamma) \geq \eta$.*

From our notion of $Ł_\eta$-consistency we derive a degree of consistency for sets of sentences in SL.

Definition 4 *We define the consistency degree of* Γ*, denoted* $mc(\Gamma)$*, as follows:*

$$mc(\Gamma) = \sup\{\eta \mid \Gamma \textit{ is } Ł_\eta\textit{-consistent}\}$$

These two definitions resemble those of η-consistency and maximal η-consistency presented in [8].[1] Maximal η-consistency was defined as a probabilistic measure of the *degree of consistency* for sets of sentences in classical propositional calculus (for more on these notions see [8] or [15]). The notion of $Ł_\eta$-consistency and related degrees of consistency of sets of sentences are presented here simply as some sort of technical notion that will be needed in further sections (for more on the degree of consistency as a measure of inconsistency see [15]).

Notice that $Ł_\eta$-consistency of a set of sentences Γ is the same as $Ł_\eta$-consistency of the sentence $\bigwedge\Gamma$. We will talk indistinctively about the consistency of sentences and sets of sentences.

Proposition 5 $mc(\Gamma)$ *is attained by some Ł-valuation.*

Proof. Let $\mathrm{mc}(\Gamma) = \eta$. We can define an increasing sequence $\{\eta_n\}$ whose limit is η such that for all $n \in \mathbb{N}$ there exists an Ł-valuation $\vec{x}_n \in \mathbb{D}_m$ with $f_{\bigwedge\Gamma}(\vec{x}_n) \geq \eta_n$. We need to prove that there exists an Ł-valuation $\vec{x} \in \mathbb{D}_m$ such that $f_{\bigwedge\Gamma}(\vec{x}) \geq \eta$.

We can take a convergent subsequence $\{\vec{x}^1_{n_k}\}$ in the first coordinates of $\{\vec{x}_n\}$. We know such a convergent subsequence needs to exist and converge in the interval $[0,1]$ by compactness. Next we can pick a convergent subsequence $\{\vec{x}^2_{n_k}\}$ in the second coordinates of $\{\vec{x}^1_{n_k}\}$. As before, such subsequence needs to exist and converge in the interval $[0,1]$ by compactness. We can proceed in the same way for the other coordinates. The final subsequence, $\{\vec{x}^m_{n_k}\}$, will have as limit an Ł-valuation $\vec{x} \in \mathbb{D}_m$ for which, given the continuity of McNaughton functions, $f_{\bigwedge\Gamma}(\vec{x}) \geq \eta$. ■

The two types of sentences that we next define will prove useful in further sections. We start by defining $\phi_{\frac{1}{k}}$ as follows, with $p \in L$ and $k \in \mathbb{N}$:

$$\phi_{\frac{1}{k}} = \neg p \wedge p^{k-1}.$$

[1] The essential difference being that such notions are defined based on probability functions in L instead of Ł-valuations.

Proposition 6 *The sentence $\phi_{\frac{1}{k}}$ has the consistency degree $\frac{1}{k}$.*

Proof. It can be easily checked that $\phi_{\frac{1}{k}}$ has the consistency degree $\frac{1}{k}$. Consider the Ł-valuation ω on L that assigns to p the value $\frac{k-1}{k}$. We have that $\omega(\phi_{\frac{1}{k}}) = \frac{1}{k}$. It is also clear that any other Ł-valuation ω' on L for which $\omega'(p) < \frac{k-1}{k}$ or $\omega'(p) > \frac{k-1}{k}$ will be such that $\omega'(\phi_{\frac{1}{k}}) < \frac{1}{k}$. ■

We define now the sentence ϕ_r, with $r = \frac{u}{v}$ a rational number in $\mathbb{Q} \cap [0,1]$:

$$\phi_r = \underline{\bigvee}^u \phi_{\frac{1}{v}}.$$

Proposition 7 *The sentence ϕ_r has the consistency degree r.*

Proof. By Proposition 6 we have that $\phi_{\frac{1}{v}}$ has the consistency degree $\frac{1}{v}$ and thus $\underline{\bigvee}^u \phi_{\frac{1}{v}}$ will have the consistency degree $\frac{u}{v}$. ■

Although obvious, it is worth mentioning that there exists an Ł-valuation ω on L for which $\omega(\phi_r) = 0$. Thus, by continuity of f_{ϕ_r}, we will have an Ł-valuation ω on L such that $\omega(\phi_r) = \lambda$ for each $\lambda \in [0, r]$.

4 $^{\eta} \blacktriangleright_{\zeta}$ and the function $\mathcal{L}_{\Gamma,\theta}$

Time now to formally define the family of consequence relations of the form $^{\eta} \blacktriangleright_{\zeta}$ introduced in the first section and, from it, the function $\mathcal{L}_{\Gamma,\theta}$.

Throughout let $\Gamma \cup \{\theta\} \subset SL$ and $\eta, \zeta \in [0,1]$.

Definition 8 *We say that Γ (η,ζ)-implies θ if, for all Ł-valuations ω on L, if $\omega(\bigwedge \Gamma) \geq \eta$ then $\omega(\theta) \geq \zeta$.*

We write $\Gamma^{\eta} \blacktriangleright_{\zeta} \theta$ to denote that Γ (η,ζ)-implies θ.

Definition 9 *The function $\mathcal{L}_{\Gamma,\theta} : [0,1] \longrightarrow [0,1]$ is defined as follows, for all $\eta \in [0,1]$:*

$$\mathcal{L}_{\Gamma,\theta}(\eta) = \sup\{\zeta \mid \Gamma^{\eta} \blacktriangleright_{\zeta} \theta\}.$$

4.1 Related approaches

There exist some approaches in the literature that deal also with graded inference in the context of Łukasiewicz logic. Here we briefly provide some connections between ${}^{\eta}\blacktriangleright_{\zeta}$ and some of these approaches.

Probably the best-known logic that deals with graded truth in the context of Łukasiewicz semantics is Rational Pavelka Logic (RPL)—see, e.g., [7]. For the sake of comparing RPL and ${}^{\eta}\blacktriangleright_{\zeta}$ let us extend the set of sentences SL by introducing, for each $r \in \mathbb{Q} \cap [0,1]$, the symbol $\overline{r}$ and further restrict the behaviour of a valuation ω on L by adding the constraint $\omega(\overline{r}) = r$. We also restrict to consequence relations of the form ${}^{\eta}\blacktriangleright_{\zeta}$ with $\eta, \zeta \in \mathbb{Q} \cap [0,1]$. The following fact follows, for $\Gamma \cup \{\theta\} \subset SL$ and $\eta, \zeta \in \mathbb{Q} \cap [0,1]$:

$$\overline{\eta} \to \bigwedge \Gamma \ \vdash_{RPL} \ \overline{\lambda} \to \theta \quad \Longleftrightarrow \quad \Gamma^{\eta} \blacktriangleright_{\lambda} \theta.$$

A recent approach on graded inference within the context of Łukasiewicz logic is given by the consequence relation $\vdash_{\infty}^{\leqslant}$, defined and analyzed in [2] and [5]. For the sake of completeness we give the definition of $\vdash_{\infty}^{\leqslant}$. For $\Gamma \cup \{\theta\} \subset SL$, we define

$\Gamma \vdash_{\infty}^{\leqslant} \theta \iff$ for all $\eta \in [0,1]$ and all Ł-valulations ω on L, if $\omega(\bigwedge \Gamma) \geq \eta$ then $\omega(\theta) \geq \eta$.

The following equivalences follow trivially:

$\Gamma \vdash_{\infty}^{\leqslant} \theta \iff \mathcal{L}_{\Gamma,\theta}(\eta) \geq \eta$ for all $\eta \in [0,1] \iff \Gamma^{\eta} \blacktriangleright_{\eta} \theta$ for all $\eta \in [0,1]$.

Axiomatizations of sentences that are always above a certain threshold $\zeta \in [0,1]$ in the context of Łukasiewicz logic are provided in [3]. Thus, in our terms, [3] provides axiomatizations of those sets of sentences that are always inferred under ${}^{\eta}\blacktriangleright_{\zeta}$ regardless of the threshold η and the set of premises considered.

5 A representation theorem for $\mathcal{L}_{\Gamma,\theta}$

In this section we provide a representation theorem that fully characterizes functions of the form $\mathcal{L}_{\Gamma,\theta}$, for $\Gamma \cup \{\theta\} \subset SL$.

We start by showing some relevant features of functions of the form $\mathcal{L}_{\Gamma,\theta}$ that, as will be seen later, prove sufficient for a complete characterization of them. We continue, in Section 5.2, with some representative functions of the form $\mathcal{L}_{\Gamma,\theta}$. In particular, in Section 5.2.1 we introduce what we call in this paper *basic functions*: functions of the form $\mathcal{L}_{\Gamma,\theta}$ characterized by some specificic features that constitute in our approach the building blocks out of which *compound functions*, introduced in Section 5.2.2, can be built that can yield any function of the form $\mathcal{L}_{\Gamma,\theta}$, as stated by the representation theorem in Section 5.3.

5.1 Some properties of $\mathcal{L}_{\Gamma,\theta}$

The four propositions in this section show some general features of functions of the form $\mathcal{L}_{\Gamma,\theta}$, for $\Gamma \cup \{\theta\} \subset SL$.

Proposition 10 *Let Γ be Ł$_\eta$-consistent. There exists an Ł-valuation ω on L such that $\omega(\bigwedge\Gamma) \geq \eta$ and $\omega(\theta) = \mathcal{L}_{\Gamma,\theta}(\eta) = \zeta$.*

Proof. Let us assume that Γ is Ł$_\eta$-consistent. We proceed in a way similar to that in the proof of Proposition 5. We can define a decreasing sequence $\{\zeta_n\}$ whose limit is ζ such that for all $n \in \mathbb{N}$ there exists $\vec{x}_n \in \mathbb{D}_m$ with $f_\theta(\vec{x}_n) = \zeta_n$ and $f_{\bigwedge\Gamma}(\vec{x}_n) \geq \eta$. We have to prove that there exists $\vec{x} \in \mathbb{D}_m$ such that $f_\theta(\vec{x}) = \zeta$ and $f_{\bigwedge\Gamma}(\vec{x}) \geq \eta$.

As before, we take a convergent subsequence $\{\vec{x}^1_{n_k}\}$ in the first coordinates of $\{\vec{x}_n\}$. Next we pick a convergent subsequence $\{\vec{x}^2_{n_k}\}$ in the second coordinates of $\{\vec{x}^1_{n_k}\}$ and proceed in the same way for the other coordinates. That all these subsequences exist and converge in the interval $[0,1]$ follows from compactness. The final subsequence, $\{\vec{x}^m_{n_k}\}$, will have as limit $\vec{x} \in \mathbb{D}_m$ for which $f_\theta(\vec{x}) = \zeta$ and $f_{\bigwedge\Gamma}(\vec{x}) \geq \eta$. ■

In words, Proposition 10 simply states that the value $\mathcal{L}_{\Gamma,\theta}(\eta)$ is attained by some Ł-valuation ω on L, provided that Γ is Ł$_\eta$-consistent.

Proposition 11 *$\mathcal{L}_{\Gamma,\theta}$ is increasing.*

Proof. It follows directly from the definition of ${}^{\eta}\blacktriangleright_{\zeta}$. ■

For the next proposition assume that $\mathrm{mc}(\Gamma) = \lambda > 0$.

Proposition 12 *$\mathcal{L}_{\Gamma,\theta}$ is left continuous on $[0,\lambda]$.*

Proof. Let us proceed by *reductio ad absurdum* by assuming that there exists $\eta \in (0,\lambda]$ and $\epsilon > 0$ such that

$$\mathcal{L}_{\Gamma,\theta}(\eta) - \mathcal{L}_{\Gamma,\theta}(x) > \epsilon$$

for all $x \in [0, \eta)$.

Let $\zeta = \sup\{\mathcal{L}_{\Gamma,\theta}(x) |\, x < \eta\}$. We can define an increasing sequence $\{\eta_n\}$ with limit η and a sequence $\{\zeta_n\}$ with limit ζ such that for all $n \in \mathbb{N}$ there exists $\vec{x}_n \in \mathbb{D}_m$ with $f_{\bigwedge\Gamma}(\vec{x}_n) = \eta_n$ and $f_\theta(\vec{x}_n) = \zeta_n$.

We proceed as in previous proofs by taking suitable convergent subsequences of $\{\vec{x}_n\}$ at each step until we come to $\{\vec{x}^m_{n_k}\}$, which will have as limit a valuation $\vec{x} \in \mathbb{D}_m$ for which $f_{\bigwedge\Gamma}(\vec{x}) = \eta$ and $f_\theta(\vec{x}) = \zeta$ since $\mathcal{L}_{\Gamma,\theta}$ is increasing. Therefore $\mathcal{L}_{\Gamma,\theta}$ needs to be continuous from the left at η. ■

Proposition 13 *$\mathcal{L}_{\Gamma,\theta}$ is of the following form:*

$$\mathcal{L}_{\Gamma,\theta}(\eta) = \begin{cases} a_1\eta + b_1 & \text{if } \eta \le \lambda_1 \\ ... \\ a_k\eta + b_k & \text{if } \lambda_{k-1} < \eta \le \lambda_k \end{cases}$$

with $a_i, b_i, \lambda_i \in \mathbb{Q}$ and $k \in \mathbb{N}$, $i \in \{1, ..., k\}$.

Proof. Let $\mathcal{R} = \langle \mathbb{R}, +, -, <, =, 0, 1\rangle$.[2]

The set of pairs

$$\{(x, y) \in \mathbb{R}^2 |\, y = \mathcal{L}_{\Gamma,\theta}(x)\}$$

is $\mathcal{R}$-definable (notice that, since $\mathcal{R}$ is an elementary extension of the structure $\mathcal{Q} = \langle \mathbb{Q}, +, -, <, =, 0, 1\rangle$, it is $\mathcal{Q}$-definable too).

The theory of $\mathcal{R}$ has quantifier elimination (see for example [9]). Therefore the set of pairs

$$\{(x, y) \in \mathbb{R}^2 |\, y = \mathcal{L}_{\Gamma,\theta}(x)\}$$

is given by a finite boolean combination (which reduces to a finite union of intersections by the complement and distributive laws for sets) of sets of the form

$$\{(x, y) \in \mathbb{R}^2 |\, ay < bx + c\}$$

and

$$\{(x, y) \in \mathbb{R}^2 |\, ay = bx + c\}$$

for $a, b, c \in \mathbb{Z}$.

Notice that each non-empty intersection of sets of such form is convex so, since $\mathcal{L}_{\Gamma,\theta}$ is a function, such intersection has to be a line segment (with coefficients and bounds in $\mathbb{Q}$).

That $\mathcal{L}_{\Gamma,\theta}$ is left continuous was stated and proved in Proposition 12. ■

[2] Here by '$-$' we mean the map given by $x \longrightarrow -x$.

5.2 Building functions of the form $\mathcal{L}_{\Gamma,\theta}$

As mentioned earlier, the aim of this section is to provide basic functions of the form $\mathcal{L}_{\Gamma,\theta}$ for particular $\Gamma \cup \{\theta\} \subset SL$ out of which, by some suitable operations, any other function of that form can be obtained.

5.2.1 Basic functions

We define five types of basic functions.

Proposition 14 ***(Type 1)***
Let $r, s \in [0,1] \cap \mathbb{Q}$. We can find $\Gamma \cup \{\theta\} \subset SL$ for which $\mathcal{L}_{\Gamma,\theta}$ is as follows:

$$\mathcal{L}_{\Gamma,\theta}(\eta) = \begin{cases} s & \text{if } \eta \leq r \\ 1 & \text{otherwise} \end{cases}$$

Proof. Let $0 < r = \frac{u_1}{v_1}$ and $0 < s = 1 - \frac{u_2}{v_2} < 1$.

Let $\Gamma = \{\underline{\bigvee}^{u_1} \phi_{\frac{1}{v_1}}\}$, with $\phi_{\frac{1}{v_1}} = \neg p \wedge p^{v_1 - 1}$ and $p \in L$. As seen previously, Γ has the consistency degree r.

On the other hand take $\phi_{\frac{1}{v_2}} = \neg q \wedge q^{v_2 - 1}$, for $q \in L$, $q \neq p$. The sentence $\underline{\bigvee}^{u_2} \phi_{\frac{1}{v_2}}$ has the consistency degree $\frac{u_2}{v_2}$. Thus there is no Ł-valuation ω on L such that

$$\omega\big(\neg(\underline{\bigvee}^{u_2} \phi_{\frac{1}{v_2}})\big) < 1 - \frac{u_2}{v_2} = s.$$

Set $\theta = \neg\big(\underline{\bigvee}^{u_2} \phi_{\frac{1}{v_2}}\big)$. Clearly, for Γ and θ thus defined, $\mathcal{L}_{\Gamma,\theta}$ is as stated above.

For $r = 0$ we can take $\bigwedge \Gamma$ to be an Ł-contradiction. If $s = 0$ we can take θ to be an Ł-contradiction and, if $s = 1$, an Ł-tautology. ∎

It is worth remarking the importance of a subclass of this type of functions; namely, the function given for $s = 0$.

Notice that in the above example Γ is not $Ł_1$-consistent. Later on, in order to prove the representation theorem for the functions $\mathcal{L}_{\Gamma,\theta}$, we will need to appeal to functions of this form for $Ł_1$-consistent sets of premises. From McNaughton's Theorem we can claim that there exist sentences $\bigwedge \Gamma$ and θ involving only one propositional variable—say $p \in L$—with $\bigwedge \Gamma$ $Ł_1$-consistent such that $\mathcal{L}_{\Gamma,\theta}(\eta) = 0$ for $\eta \leq r$ and $\mathcal{L}_{\Gamma,\theta}(\eta) = 1$ for $\eta > r$, for any $r \in [0,1] \cap \mathbb{Q}$. To see this consider for example $f_{\bigwedge \Gamma}(x)$ to be of the following form:

$$f_{\bigwedge\Gamma}(x) = \begin{cases} a_1x & \text{if } x \le \frac{1+b_2}{a_1+a_2} \\ 1-(a_2x-b_2) & \text{if } \frac{1+b_2}{a_1+a_2} < x \le \frac{1+b_2}{a_2} \\ a_3x-b_3 & \text{if } \frac{1+b_2}{a_2} < x \le c \\ 1 & \text{otherwise} \end{cases}$$

Here a_1, a_2, a_3, b_2, b_3 are positive integers and c is a rational number. Other conditions on these values are that

$$a_1(\frac{1+b_2}{a_1+a_2}) = 1 - \big(a_2(\frac{1+b_2}{a_1+a_2}) - b_2\big) = r,\ 1+b_2 < a_2,$$

$$1 - \big(a_2(\frac{1+b_2}{a_2}) - b_2\big) = a_3(\frac{1+b_2}{a_2}) - b_3 = 0 \text{ and } a_3c - b_3 = 1.$$

Let us also consider $f_\theta(x)$ of the following form:

$$f_\theta(x) = \begin{cases} 0 & \text{if } x \le d_1 \\ a_4x-b_4 & \text{if } d_1 < x \le d_2 \\ 1 & \text{otherwise} \end{cases}$$

Here a_4, b_4 are positive integers and d_1, d_2 are rational numbers. Other conditions on these values are $a_4d_1 - b_4 = 0$, $a_4d_2 - b_4 = 1$ and

$$\frac{1+b_2}{a_1+a_2} \le d_1 < d_2 \le \frac{1+b_2}{a_2}.$$

For $\bigwedge\Gamma$ and θ of this form the function $\mathcal{L}_{\Gamma,\theta}$ will be as desired. Notice that

$$f_{\bigwedge\Gamma}(\frac{1+b_2}{a_1+a_2}) = r,\ f_\theta(\frac{1+b_2}{a_1+a_2}) = 0$$

and, for all $x \in [0,1]$ for which $f_{\bigwedge\Gamma}(x) > r$ we have that $f_\theta(x) = 1$.

Proposition 15 ***(Type 2)***

Let $r, s \in [0,1] \cap \mathbb{Q}$, with $r < s$. We can find $\Gamma \cup \{\theta\} \subset SL$ for which $\mathcal{L}_{\Gamma,\theta}$ is of the following form:

$$\mathcal{L}_{\Gamma,\theta}(\eta) = \begin{cases} 0 & \textit{if } \eta \le r \\ \frac{\eta-r}{s-r} & \textit{if } r < \eta < s \\ 1 & \textit{otherwise} \end{cases}$$

Proof. Let $0 < r = \frac{u_1}{v_1} < s = \frac{u_2}{v_2}$.

Take $s - r = \frac{u_2 v_1 - u_1 v_2}{v_1 v_2}$ and define ψ_1 and θ as follows:

$$\psi_1 = \underline{\bigvee}^{u_2 v_1 - u_1 v_2} \phi_{\frac{1}{v_1 v_2}}$$

$$\theta = \underline{\bigvee}^{v_1 v_2} \phi_{\frac{1}{v_1 v_2}}.$$

Here $\phi_{\frac{1}{v_1 v_2}} = \neg p \wedge p^{v_1 v_2 - 1}$, for $p \in L$.

Define ψ_2 as follows:

$$\psi_2 = \underline{\bigvee}^{u_1} \phi_{\frac{1}{v_1}}.$$

We take $\phi_{\frac{1}{v_1}}$ to be $\neg q \wedge q^{v_1 - 1}$, for $q \in L$ with $q \neq p$, and set $\Gamma = \{\psi_1 \underline{\vee} \psi_2\}$.

$\mathcal{L}_{\Gamma,\theta}$ is as required. To see this notice that, since ψ_2 has the consistency degree r, $\mathcal{L}_{\Gamma,\theta}(\eta) = 0$ for all $\eta \in [0, r]$ and that any Ł-valuation ω on L for which $\omega(\psi_1) = \lambda(s - r)$, for $\lambda \in [0, 1]$, is such that $\omega(\theta) = \lambda$. If $r = 0$ then we can dispense with ψ_2 and take $\Gamma = \{\psi_1\}$. ■

As with *Type 1* McNaughton's Theorem guarantees the existence of $\bigwedge \Gamma$ $Ł_1$-consistent and θ such that $\mathcal{L}_{\Gamma,\theta}$ is as above. To see this consider $\phi(p)$ and $\theta(p)$ (with $p \in L$) for which $f_\phi(x)$ and $f_\theta(x)$ are of the following form:

$$f_\phi(x) = \begin{cases} bx & \text{if } x \leq \frac{1}{b} \\ 1 & \text{otherwise} \end{cases}$$

$$f_\theta(x) = \begin{cases} ax & \text{if } x \leq \frac{1}{a} \\ 1 & \text{otherwise} \end{cases}$$

Here $a, b \in \mathbb{N}$ and $\frac{a}{b} = \frac{1}{s-r}$. Notice that $\mathcal{L}_{\{\phi\},\theta}(\eta) = \frac{a\eta}{b}$ for all $\eta \leq \frac{b}{a}$. We can then set $\Gamma = \{\phi \underline{\vee} \psi_2\}$, where ψ_2 is as defined above. The function $\mathcal{L}_{\Gamma,\theta}$ will be as stated, with Γ $Ł_1$-consistent.

Proposition 16 ***(Type 3)***

Let $r, s \in [0, 1] \cap \mathbb{Q}$. We can define $\Gamma \cup \{\theta\} \subset SL$ for which $\mathcal{L}_{\Gamma,\theta}$ has the following form:

$$\mathcal{L}_{\Gamma,\theta}(\eta) = \begin{cases} 0 & \textit{if } \eta \leq r \\ \frac{s(\eta - r)}{1 - r} & \textit{otherwise} \end{cases}$$

Proof. Let $r = \frac{u_1}{v_1}$ and $s = \frac{u_2}{v_2}$. We have to distinguish two possible cases here:

Case 1. $\frac{s}{1-r} \leq 1$.

Consider $\frac{s}{1-r} = \frac{u_2 v_1}{v_2(v_1-u_1)}$.
We first define ψ_1 and θ as follows:

$$\psi_1 = \underline{\bigvee}^{v_2(v_1-u_1)} \phi_{\frac{1}{v_2(v_1-u_1)}},$$

$$\theta = \underline{\bigvee}^{u_2 v_1} \phi_{\frac{1}{v_2(v_1-u_1)}}.$$

Here $\phi_{\frac{1}{v_2(v_1-u_1)}} = \neg p \wedge p^{v_2(v_1-u_1)-1}$, for $p \in L$.
Let us now define ψ_2 for $r > 0$ as follows:

$$\psi_2 = \underline{\bigvee}^{u_1} \phi_{\frac{1}{v_1}}.$$

Here $\phi_{\frac{1}{v_1}} = \neg q \wedge q^{v_1-1}$, for $q \in L$ with $q \neq p$. We set $\Gamma = \{\psi_1 \underline{\vee} \psi_2\}$. We can clearly see that $\mathcal{L}_{\Gamma,\theta}$ is as stated.

Notice that if $r = 0$ then we can dispense with ψ_2 and set $\Gamma = \{\psi_1\}$.

Case 2. $\frac{s}{1-r} > 1$.

Consider $\frac{1-r}{s} = \frac{v_2(v_1-u_1)}{u_2 v_1}$.
We now define ψ_1 and θ in the following way:

$$\psi_1 = \underline{\bigvee}^{v_2(v_1-u_1)} \phi_{\frac{1}{u_2 v_1}},$$

$$\theta = \underline{\bigvee}^{u_2 v_1} \phi_{\frac{1}{u_2 v_1}}$$

with $\phi_{\frac{1}{u_2 v_1}} = \neg p \wedge p^{u_2 v_1 - 1}$, for $p \in L$.

Define ψ_2 as in Case 1 and set $\Gamma = \{\psi_1 \underline{\vee} \psi_2\}$. $\mathcal{L}_{\Gamma,\theta}$ will be as stated. ■

Proposition 17 ***(Type 4)***
Let $r, s \in [0,1] \cap \mathbb{Q}$, with $r < s$. We can define $\Gamma \cup \{\theta\} \subset SL$ for which $\mathcal{L}_{\Gamma,\theta}(\eta) = (s-r)\eta + r$.

Proof. Let $r = \frac{u_1}{v_1} < s = \frac{u_2}{v_2}$. Take $s - r = \frac{u_2 v_1 - u_1 v_2}{v_1 v_2}$ and define ψ and θ_1 as follows:

$$\psi = \underline{\bigvee}^{v_1 v_2} \phi_{\frac{1}{v_1 v_2}},$$

$$\theta_1 = \underline{\bigvee}^{u_2 v_1 - u_1 v_2} \phi_{\frac{1}{v_1 v_2}},$$

where $\phi_{\frac{1}{v_1 v_2}} = \neg p \wedge p^{v_1 v_2 - 1}$, for $p \in L$.

Let us define θ_2 as follows:

$$\theta_2 = \neg(\underline{\bigvee}^{u_1} \phi_{\frac{1}{v_1}}).$$

Here $\phi_{\frac{1}{v_1}} = \neg q \wedge q^{v_1 - 1}$, with $q \in L$ and $q \neq p$.

By setting $\theta = \theta_1 \underline{\vee} \theta_2$ and $\Gamma = \{\psi\}$ we get $\mathcal{L}_{\Gamma,\theta}$ of the desired form.

If $r = 0$ then we set $\theta = \theta_1$. ■

Proposition 18 ***(Type 5)***

Let $r, s \in [0,1] \cap \mathbb{Q}$. We can define $\Gamma \cup \{\theta\} \subset SL$ for which $\mathcal{L}_{\Gamma,\theta}$ has the following form:

$$\mathcal{L}_{\Gamma,\theta}(\eta) = \begin{cases} \eta(\frac{1-r}{s}) + r & \text{if } \eta \leq s \\ 1 & \text{otherwise} \end{cases}$$

Proof. Let $0 < r = \frac{u_1}{v_1}$ and $s = \frac{u_2}{v_2}$. We have to distinguish two possible cases:

<u>Case 1</u>. $\frac{1-r}{s} > 1$.

Consider $\frac{s}{1-r} = \frac{u_2 v_1}{v_2(v_1 - u_1)}$ and define ψ and θ_1 as follows:

$$\psi = \underline{\bigvee}^{u_2 v_1} \phi_{\frac{1}{v_2(v_1 - u_1)}},$$

$$\theta_1 = \underline{\bigvee}^{v_2(v_1 - u_1)} \phi_{\frac{1}{v_2(v_1 - u_1)}},$$

with $\phi_{\frac{1}{v_2(v_1 - u_1)}} = \neg p \wedge p^{v_2(v_1 - u_1) - 1}$, for $p \in L$.

On the other hand define θ_2 as follows:

$$\theta_2 = \neg(\underline{\bigvee}^{u_1} \phi_{\frac{1}{v_1}}),$$

with $\phi_{\frac{1}{v_1}} = \neg q \wedge q^{v_1 - 1}$, for $q \in L$ and $q \neq p$.

Set $\theta = \theta_1 \underline{\vee} \theta_2$ and $\Gamma = \{\psi\}$. The function $\mathcal{L}_{\Gamma,\theta}$ will be as desired.

If $r = 0$ then we can set $\theta = \theta_1$.

As with Type 1 and Type 2, McNaughton's Theorem guarantees the existence of sentences $\bigwedge \Gamma$ and θ in one variable (say $p \in L$), with $\bigwedge \Gamma$ Ł$_1$-consistent, such that $\mathcal{L}_{\Gamma,\theta}$ is of the required form. Consider for example ϕ and ψ for which $f_\phi(x)$ and $f_\psi(x)$ are defined as those seen previously for Type 2:

$$f_\phi(x) = \begin{cases} bx & \text{if } x \leq \frac{1}{b} \\ 1 & \text{otherwise} \end{cases}$$

$$f_\psi(x) = \begin{cases} ax & \text{if } x \leq \frac{1}{a} \\ 1 & \text{otherwise} \end{cases}$$

Here $a, b \in \mathbb{N}$ and $\frac{a}{b} = \frac{1-r}{s}$.

Set $\Gamma = \{\phi\}$ and $\theta = \{\psi \underline{\vee} \theta_2\}$, where

$$\theta_2 = \neg\Big(\underline{\bigvee}^{u_1} \phi_{\frac{1}{v_1}}\Big)$$

and $\phi_{\frac{1}{v_1}} = \neg q \wedge q^{v_1 - 1}$, with $q \in L$ and $q \neq p$.

Clearly $\mathcal{L}_{\Gamma,\theta}$ will be as stated, with Γ Ł$_1$-consistent.

<u>Case 2</u>. $\frac{1-r}{s} \leq 1$.

Consider $\frac{1-r}{s} = \frac{v_2(v_1 - u_1)}{u_2 v_1}$ and define ψ and θ_1 as follows:

$$\psi = \underline{\bigvee}^{u_2 v_1} \phi_{\frac{1}{u_2 v_1}},$$

$$\theta_1 = \underline{\bigvee}^{v_2(v_1 - u_1)} \phi_{\frac{1}{u_2 v_1}},$$

where $\phi_{\frac{1}{u_2 v_1}} = \neg p \wedge p^{u_2 v_1 - 1}$, for $p \in L$.

Define θ_2 as in Case 1 and set $\theta = \theta_1 \underline{\vee} \theta_2$ and $\Gamma = \{\psi\}$. The function $\mathcal{L}_{\Gamma,\theta}$ will be as desired.

For $r = 0$ we dispense again with θ_2. ■

5.2.2 Compound functions

In order to introduce compound functions, let L_1, L_2 be two disjoint sets of propositional variables and SL_1, SL_2 their respective sets of sentences.

Take $\Gamma_1 \subset SL_1, \Gamma_2 \subset SL_2$ and $\theta_1 \in SL_1, \theta_2 \in SL_2$. Assume that $\Gamma = \Gamma_1 \cup \Gamma_2$ has the consistency degree λ.

Proposition 19 *For all $\eta \in [0, 1]$,*

$$\max\{\mathcal{L}_{\Gamma_1,\theta_1}(\eta), \mathcal{L}_{\Gamma_2,\theta_2}(\eta)\} = \mathcal{L}_{\Gamma_1\cup\Gamma_2,\theta_1\vee\theta_2}(\eta).$$

Proof. It follows trivially from the interpretation of the connective '$\vee$'. ■

Proposition 20 *For all $\eta \in [0, \lambda]$,*

$$\min\{\mathcal{L}_{\Gamma_1,\theta_1}(\eta), \mathcal{L}_{\Gamma_2,\theta_2}(\eta)\} = \mathcal{L}_{\Gamma_1\cup\Gamma_2,\theta_1\wedge\theta_2}(\eta).$$

Proof. It follows trivially from the interpretation of '$\wedge$'. ■

We can extend these propositions to any finite collection of sets of sentences $\Gamma_1 \subset SL_1, ..., \Gamma_k \subset SL_k$ and $\theta_1 \in SL_1, ..., \theta_k \in SL_k$, for some $k \in \mathbb{N}$, with $L_1, ..., L_k$ a collection of pairwise disjoint sets of propositional variables.

5.3 Representation theorem

At this point we have all the intermediate results necessary for the representation theorem that we finally present in this section.

Theorem 21 ***Representation Theorem.*** *The function $\mathcal{F} : [0, 1] \longrightarrow [0, 1]$ is of the form $\mathcal{L}_{\Gamma,\theta}$ for some $\Gamma \cup \{\theta\} \subset SL$ if and only if $\mathcal{F}$ is an increasing function of the following form:*

$$\mathcal{F}(x) = \begin{cases} a_1x + b_1 & \text{if } x \leq \lambda_1 \\ ... \\ a_kx + b_k & \text{if } \lambda_{k-1} < x \leq \lambda_k \end{cases}$$

with $a_i, b_i, \lambda_i \in \mathbb{Q}$ and $k \in \mathbb{N}$, $i \in \{1, ..., k\}$.

Proof. If the function $\mathcal{F} : [0, 1] \longrightarrow [0, 1]$ is of the form $\mathcal{L}_{\Gamma,\theta}$ for some $\Gamma \cup \{\theta\} \subset SL$ then we know, by Propositions 11 and 13, that $\mathcal{F}$ will be an increasing function of the form stated in the theorem.

Let us prove now the left implication.

Let $\mathcal{F} : [0, 1] \to [0, 1]$ be as stated.

We will denote the line segment given by $a_ix + b_i$ and $\lambda_{i-1} < x \leq \lambda_i$ by l_i, for $i \in \{2, ..., k\}$ (l_1 will be the line segment given by $a_1x + b_1$ and $x \leq \lambda_1$).

Let us define Γ and θ for which $\mathcal{L}_{\Gamma,\theta}(\eta) = \mathcal{F}(\eta)$ for all $\eta \in [0,1]$.

First, let l_i be a line segment of $\mathcal{F}$, $i \in \{1,...,k\}$ (without loss of generality we can assume that $i \neq 1$). We can define $\Gamma_i \subset SL$ Ł$_1$-consistent and $\theta_i \in SL$ for which $\mathcal{L}_{\Gamma_i,\theta_i}$ is as follows:

$$\mathcal{L}_{\Gamma_i,\theta_i}(x) = \begin{cases} a_i\lambda_{i-1} + b_i & \text{if } x \leq \lambda_{i-1} \\ a_i x + b_i & \text{if } \lambda_{i-1} < x \leq \lambda_i \\ 1 & \text{otherwise} \end{cases}$$

To see this set

$$\mathcal{L}_{\Gamma_i,\theta_i}(\eta) = \max\{\mathcal{L}_{\Delta_1,\psi_1}(\eta), \max\{\mathcal{L}_{\Delta_2,\psi_2}(\eta), \mathcal{L}_{\Delta_3,\psi_3}(\eta)\}\}$$

for all $\eta \in [0,1]$, with $\Delta_j \subseteq SL_j$ Ł$_1$-consistent and $\psi_j \in SL_j$ for all $j \in \{1,2,3\}$, where L_1, L_2, L_3 are pairwise disjoint.

$\mathcal{L}_{\Delta_1,\psi_1}$ and $\mathcal{L}_{\Delta_2,\psi_2}$ are of *Type 1*:

$$\mathcal{L}_{\Delta_1,\psi_1}(x) = \begin{cases} 0 & \text{if } x \leq \lambda_i \\ 1 & \text{otherwise} \end{cases}$$

$$\mathcal{L}_{\Delta_2,\psi_2}(x) = \quad a_i\lambda_{i-1} + b_i \quad for\ all\ x \in [0,1]$$

The nature of the straight line $a_i x + b_i$ will determine the type of function of $\mathcal{L}_{\Delta_3,\psi_3}$. We will choose Δ_3 and ψ_3 such that the function $\mathcal{L}_{\Delta_3,\psi_3}$ contains the straight segment $a_i x + b_i$, for $\lambda_{i-1} < x \leq \lambda_i$. That $\mathcal{L}_{\Delta_3,\psi_3}$ will be of one of the types described in the previous subsection is clear.

It can easily be seen that

$$\mathcal{F}(\eta) = \mathcal{L}_{\bigcup \Gamma_i, \bigwedge \theta_i}(\eta) = \min\{\mathcal{L}_{\Gamma_i,\theta_i}(\eta) \,|\, i \in \{1,...,k\}\}$$

for all $\eta \in [0,1]$, with $\Gamma_1 \subset SL_1, ..., \Gamma_k \subset SL_k$, $\theta_1 \in SL_1, ..., \theta_k \in SL_k$ and $L_1, ..., L_k$ a pairwise disjoint collection of sets of propositional variables. ■

6 The computation of $\mathcal{L}_{\Gamma,\theta}$

In this section we deal with the computation of $\mathcal{L}_{\Gamma,\theta}(\eta)$, for particular $\Gamma = \{\phi_1, ..., \phi_k\} \subset SL$ (for $k \in \mathbb{N}$), $\theta \in SL$ and $\eta \in [0,1]$. In particular, we show methodology based on finding the solution to certain constrained optimization problems in order to compute $\mathcal{L}_{\Gamma,\theta}(\eta)$. Similar methodology was presented in [17] in order to compute the amount of inconsistency (understood as distance to consistency) in knowledge bases with graded propositions, where grades represent truth values in the context of Łukasiewicz logic.

Throughout we will assume that $L = \bigcup_{i=1}^{k} L_{\phi_i}$ (recall that L_{ϕ_i} is the set of propositional variables that occur in ϕ_i).

Consider the following constrained optimization problem with *optimization variable* the m-tuple $\vec{x} \in \mathbb{R}^m$:

$$\text{minimize} \;\; f_\theta(\vec{x}) \tag{1}$$

subject to the following constraints:

- $f_{\phi_i}(\vec{x}) \geq \eta$ for each $i \in \{1, ..., k\}$,
- $\vec{x} \in \mathbb{D}_m$ (i.e., $0 \leq x_i \leq 1$ for each $i \in \{1, ..., m\}$),

Let us denote the constrained optimization problem (1) by $\mathcal{C}_{[\Gamma,\theta,\eta]}$. We define $\mathcal{SC}_{[\Gamma,\theta,\eta]}$, the solution to $\mathcal{C}_{[\Gamma,\theta,\eta]}$, as follows:

$$\mathcal{SC}_{[\Gamma,\theta,\eta]} = \inf_{\vec{x} \in \mathbb{D}_m} \{ f_\theta(\vec{x}) \,|\, \vec{x} \in \mathbb{R}^m \text{ is } \textit{feasible} \}.$$

By $\vec{x}$ being feasible we mean that $\vec{x}$ satisfies the constraints in $\mathcal{C}_{[\Gamma,\theta,\eta]}$. The collection of all such tuples is called the *feasible set* (of $\mathcal{C}_{[\Gamma,\theta,\eta]}$)—see, e.g., [4] for more on these concepts and, in general, on the terminology and basic definitions for constrained optimization problems.

Notice that $\mathcal{SC}_{[\Gamma,\theta,\eta]}$ does not exist if the feasible set of $\mathcal{C}_{[\Gamma,\theta,\eta]}$ is empty, in which case we will have that $\mathcal{L}_{\Gamma,\theta}(\eta) = 1$. Otherwise, as is clear from the definition of $\mathcal{L}_{\Gamma,\theta}$, we will have that

$$\mathcal{L}_{\Gamma,\theta}(\eta) = \mathcal{SC}_{[\Gamma,\theta,\eta]}.$$

In $\mathcal{C}_{[\Gamma,\theta,\eta]}$, neither the objective function f_θ nor functions f_{ϕ_i} in the constraints (for $i \in \{1, ..., k\}$) need to be convex, let alone linear. That places our problem $\mathcal{C}_{[\Gamma,\theta,\eta]}$ within *non-convex* optimization grounds and, unfortunately, there is no effective methodology for dealing with non-convex optimization problems for the general case—see, e.g., [1] or [4] for more on these issues. However, our case is peculiar in that McNaughton functions are *piecewise* linear, only the *min* operator breaks linearity.

Our aim is to express our problem $\mathcal{C}_{[\Gamma,\theta,\eta]}$ in an equivalent form involving only linear objective functions and constraints since, as it is well known, there exist effective and efficient algorithms for linear optimization—see [4]. We can do so in a pretty trivial way by observing that, for general functions g, h with domain in $\mathbb{R}^m$, the set

$$\{\vec{x} \mid \min\{g(\vec{x}), h(\vec{x})\} \in \mathcal{I}\},$$

for $\mathcal{I}$ a real interval in $\mathbb{R}$, is equal to the union of sets

$$\{\vec{x} \mid g(\vec{x}) \in \mathcal{I},\ g(\vec{x}) \leq h(\vec{x})\} \cup \{\vec{x} \mid h(\vec{x}) \in \mathcal{I},\ h(\vec{x}) \leq g(\vec{x})\}.$$

Notice that for g, h linear the constraints in the sets $\{g(\vec{x}) \in \mathcal{I},\ g(\vec{x}) \leq h(\vec{x})\}$ and $\{h(\vec{x}) \in \mathcal{I},\ h(\vec{x}) \leq g(\vec{x})\}$ are linear too.

We start by picking one of the innermost *min* operators occurring in the objective function f_θ in $\mathcal{C}_{[\Gamma,\theta,\eta]}$ (that is to say, that there is no other *min* operator within the scope of the chosen one—that if any such operator occurs at all in f_θ). Let us assume that the arguments of the *min* operator chosen are g, h. We eliminate such operator by generating two new optimization problems with distinct objective functions:

- An optimization problem with objective function f^1_θ, obtained from f_θ by replacing $\min\{g(\vec{x}), h(\vec{x})\}$ for $g(\vec{x})$, and constraint set that of $\mathcal{C}_{[\Gamma,\theta,\eta]}$ extended with the new linear constraint $g(\vec{x}) \leq h(\vec{x})$.
- An optimization problem with objective function $f^2_\theta(\vec{x})$, obtained from f_θ by replacing $\min\{g(\vec{x}), h(\vec{x})\}$ for $h(\vec{x})$, and constraint set that of $\mathcal{C}_{[\Gamma,\theta,\eta]}$ extended with the new linear constraint $h(\vec{x}) \leq g(\vec{x})$.

We proceed in the same way for the objective function of each new optimization problem that we obtain this way until all the *min* operators are eliminated. At the end of this process we will have a finite collection of new optimization problems with a linear objective function. For what follows, we will assume that the number of such problems is $t \in \mathbb{N}$ and will denote each one of them by $\mathcal{C}^i_{[\Gamma,\theta,\eta]}$, for $i \in \{1, ..., t\}$.

Now we need to eliminate the *min* operators occurring in the set of constraints of $\mathcal{C}^i_{[\Gamma,\theta,\eta]}$, for each $i \in \{1, ..., t\}$. To do so, we will proceed in a way similar to that above for eliminating the *min* operators in the objective function. As an example, let us consider a constraint of the form $f_\phi(\vec{x}) \geq \zeta$ in $\mathcal{C}^i_{[\Gamma,\theta,\eta]}$, for some $i \in \{1, ..., t\}$ and $\zeta \in [0, 1]$. As before, we will pick one of the innermost *min* operators in f_ϕ, with arguments g and h, and will eliminate it by producing two new optimization problems:

- An optimization problem obtained from $\mathcal{C}^i_{[\Gamma,\theta,\eta]}$ by replacing the constraint $f_\phi(\vec{x}) \geq \zeta$ for two new constraints: the first one, $f^1_\phi(\vec{x}) \geq \zeta$, obtained from $f_\phi(\vec{x}) \geq \zeta$ by replacing $\min\{g(\vec{x}), h(\vec{x})\}$ for $g(\vec{x})$, and the second one, given by $g(\vec{x}) \leq h(\vec{x})$.

- An optimization problem obtained from $\mathcal{C}^i_{[\Gamma,\theta,\eta]}$ as above, by replacing the constraint $f_\phi(\vec{x}) \geq \zeta$ for two new constraints: the first one, $f^2_\phi(\vec{x}) \geq \zeta$, obtained from $f_\phi(\vec{x}) \geq \zeta$ by replacing $\min\{g(\vec{x}), h(\vec{x})\}$ for $h(\vec{x})$, and the second one, given by $h(\vec{x}) \leq g(\vec{x})$.

We proceed in the same way for every newly generated optimization problem until all the *min* operators are eliminated. At the end of this process we will obtain a finite collection of linear optimization problems (i.e., linear programmes). We will denote the number of such linear programmes generated from $\mathcal{C}^i_{[\Gamma,\theta,\eta]}$ by s_i and the collection of linear programmes themselves by $\mathcal{C}^{i,j}_{[\Gamma,\theta,\eta]}$, for each $i \in \{1, ..., t\}$ and $j \in \{1, ..., s_i\}$.

We can view this whole process as the generation of a tree where the nodes are given by the distinct optimization problems generated, with *root* $\mathcal{C}_{[\Gamma,\theta,\eta]}$. An iteration in the process corresponds to the generation of exactly two new optimization problems that arise from the elimination of an innermost *min* operator in the objective function or in the constraints of the preceding optimization problem (in the tree, the *parent* of the two new problems obtained) in the way indicated above. The process terminates when the leaves of the generated tree have no *min* operators left.

For the next proposition let us consider the linear programme $\mathcal{C}^{i,j}_{[\Gamma,\theta,\eta]}$, for $i \in \{1, ..., t\}$ and $j \in \{1, ..., s_i\}$.

Proposition 22 *If the feasible set of $\mathcal{SC}^{i,j}_{[\Gamma,\theta,\eta]}$ is not empty then $\mathcal{SC}^{i,j}_{[\Gamma,\theta,\eta]}$ exists and is attained by some feasible point in $\mathbb{R}^m$.*

Proof. The result follows from compactness of the feasible set. ■

The next result summarizes the relation between our collection of linear programmes of the form $\mathcal{C}^{i,j}_{[\Gamma,\theta,\eta]}$, for $i \in \{1, ..., t\}$ and $j \in \{1, ..., s_i\}$, and our original problem $\mathcal{C}_{[\Gamma,\theta,\eta]}$.

Proposition 23 *Let us assume that $\mathcal{SC}_{[\Gamma,\theta,\eta]}$ exists. We then have the following identity:*

$$\mathcal{SC}_{[\Gamma,\theta,\eta]} = \min_{i,j}\{\mathcal{SC}^{i,j}_{[\Gamma,\theta,\eta]}\},$$

where i, j range over values in $\{1, ..., t\}$ and $\{1, ..., s_i\}$ respectively for which $\mathcal{SC}^{i,j}_{[\Gamma,\theta,\eta]}$ exists.

Proof. The result follows basically from the fact that the union of the feasible sets of the linear programmes of the form $\mathcal{C}^{i,j}_{[\Gamma,\theta,\eta]}$ is equal to the feasible set of $\mathcal{C}_{[\Gamma,\theta,\eta]}$. ■

Summarizing, the computation of $\mathcal{L}_{\Gamma,\theta}$ reduces to finding the solution of a collection of linear optimization problems (for which there exist efficient algorithms). However, the number of problems to be considered shows an exponential growth with respect to the *min* operators in the McNaughton functions that correspond to the sentences in $\Gamma \cup \{\theta\}$. Notice that, for $r \in \mathbb{N}$ the number of *min* operators occurring in $\mathcal{C}_{[\Gamma,\theta,\eta]}$, the number of linear programmes that we need to generate is, at least in principle, 2^r. However, it will not be necessarily so in most cases. At each step in the generation of our collection of optimization problems we can obtain a new problem whose feasible set is known to be empty and thus it will not need to be considered any further for the generation of new optimization problems (in graph terminology, a *branch* corresponding to a certain problem whose feasible set is known to be empty can be *closed*). This way we could reduce the number of linear programmes to be considered (the number of *leaves* in graph terminology) in order to calculate $\mathcal{L}_{\Gamma,\theta}(\eta)$.

7 Conclusion

The central notion of this paper is the family of consequence relations ${}^{\eta}\blacktriangleright_{\zeta}$, for distinct thresholds η and ζ, intended to formalize graded inference in the context of Łukasiewicz logic in the following terms: given set of premises and threshold η, we consider as consequences those sentences that, by Łukasiewicz logic, are entailed to hold to the degree at least ζ whenever the premises hold at least to the degree η.

Our analysis of consequence relations of the form ${}^{\eta}\blacktriangleright_{\zeta}$ in this paper mostly focused on the effect of variations of the thresholds η, ζ in relation to a given set of premises Γ and consequence θ. Such effect was formalized by means of the function $\mathcal{L}_{\Gamma,\theta}$ which, for a given threshold η for Γ, gives us the maximal threshold ζ such that $\Gamma^{\eta} \blacktriangleright_{\zeta} \theta$. The main result given in this paper, in relation to our analysis of functions of the form $\mathcal{L}_{\Gamma,\theta}$, was a full characterization of them (i.e., a representation theorem for these functions that provides necessary and sufficient conditions for a map to be of the form $\mathcal{L}_{\Gamma,\theta}$ for some set of sentences $\Gamma \cup \{\theta\}$).

Part of our analysis of functions of the form $\mathcal{L}_{\Gamma,\theta}$ was devoted to methodology for their computation. In particular, an equivalence between $\mathcal{L}_{\Gamma,\theta}(\eta)$ and the solution of a certain collection of linear optimization problems (i.e., linear programmes) was established. It is known that there exist efficient algorithms for solving linear programmes. However, the number of them to be considered for the computation of $\mathcal{L}_{\Gamma,\theta}$ shows an exponential growth with respect to the number of implications occurring in $\Gamma \cup \{\theta\}$, thus making our problem potentially infeasible for considerably large $\Gamma \cup \{\theta\}$.

Much is left to be analyzed about the family of consequence relations ${}^{\eta}\blacktriangleright_{\zeta}$. In particular, a logic based on these relations is yet to be found. In relation to the issues presented here, some further research could also be desirable (e.g., our methodology for computing $\mathcal{L}_{\Gamma,\theta}(\eta)$ could probably benefit from further research).

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