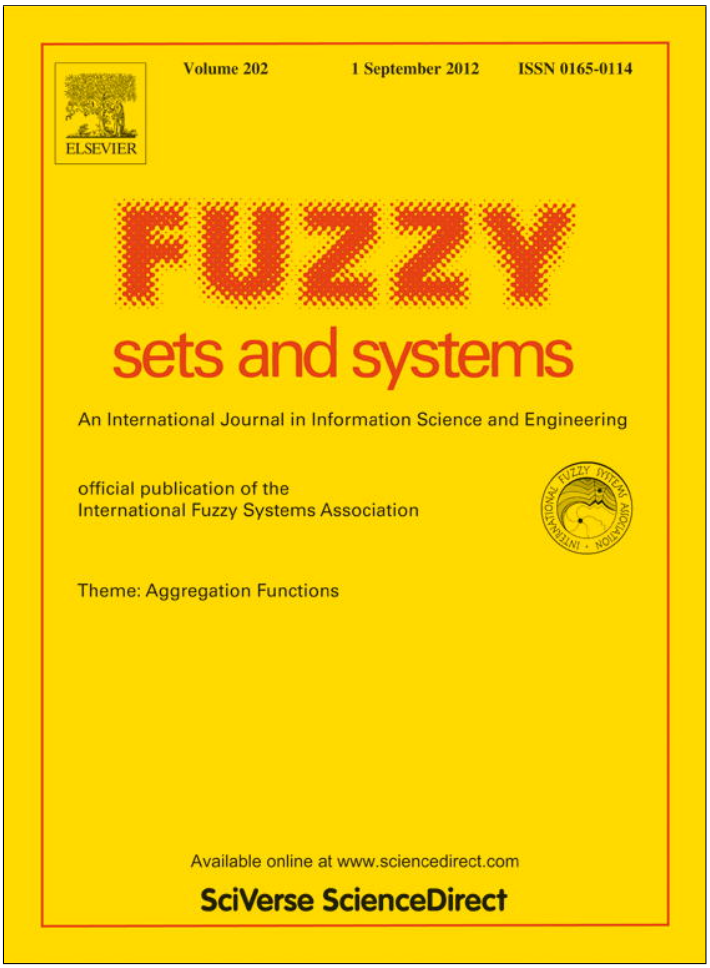

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Graded properties of unary and binary fuzzy connectives

Libor Běhounek*

Institute of Computer Science, Academy of Sciences of the Czech Republic, Pod Vodárenskou věží 2, 182 07 Prague, Czech Republic

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Abstract

The paper studies basic graded properties of unary and binary fuzzy connectives, i.e., unary and binary operations on the set of truth degrees of a background fuzzy logic extending the logic MTL of left-continuous t-norms. The properties studied in this paper are graded generalizations of monotony, Lipschitz continuity, null and unit elements, idempotence, commutativity, and associativity. The paper elaborates the initial study presented in previous papers and focuses mainly on parameterization of graded properties by conjunction-multiplicities of subformulae in the defining formulae, preservation of graded properties under compositions and slight variations of fuzzy connectives, the values of graded properties for basic connectives of the ground logic, and the dependence of the values on the ground logic. The results are proved in the formal framework of higher-order fuzzy logic MTL, also known as Fuzzy Class Theory (FCT). General theorems provable in FCT are illustrated on several semantic examples.

Keywords: Fuzzy connectives; Graded properties; Fuzzy Class Theory; Logic-based fuzzy mathematics; Defects of mathematical properties

1. Introduction

Fuzzy connectives are algebraic operations on the real unit interval [0, 1] or another suitable system L of degrees. Various classes of fuzzy connectives have been studied in the past decades, including t-norms and t-conorms, uninorms, copulas, semicopulas, fuzzy negations and implications, etc. Their commonly studied properties, such as commutativity, associativity, monotony, idempotence, etc., are, however, almost exclusively bivalent—i.e., defined by a crisp condition and thus either possessed by a fuzzy connective or not. It is, nevertheless, quite natural to consider also *graded* properties of fuzzy connectives, which they can possess to larger or smaller degrees. Such properties, being delimited by fuzzy (rather than crisp) conditions, can thus be identified with *fuzzy sets of fuzzy connectives.* Traditional non-graded properties are just their special cases delimited by bivalent membership functions.

A study of graded properties of fuzzy connectives has been initiated in the previous papers [12,13]. These papers focused on the graded generalizations of properties related mainly to t-norms [23] and the dominance relation [25,26] between aggregation operators [16,14]. Since fuzzy connectives $\mathrm{L}^n \to \mathrm{L}$ can equivalently be understood as n-ary fuzzy relations on the set L of truth degrees, a broader context of the enterprise is the study of graded properties of fuzzy relations, initiated by Gottwald in [19–21] and recently advanced in [11]. Since moreover the difference between the degree of a graded property and the full degree can be understood as a measure of the defect of the property, the study of graded properties also falls within the scope of the study of defects of mathematical properties [1]. Related to the topic

* Tel.: +420 266053921; fax: +420 286585789.
E-mail address: behounek@cs.cas.cz

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of the present paper (but based on criteria other than logical formulae), defects of properties of aggregation operators have been studied in [24]. The methods presented here can thus be viewed as a specific, logic-based approach to this area.

Like the previous papers [11–13], the present study is carried out in the formal framework of Fuzzy Class Theory (or higher-order fuzzy logic), introduced in [5] as an axiomatic theory of Zadeh's fuzzy sets [27] and fuzzy relations [28] of all finite arities and orders. The fuzzy logic MTL of [18] is used as the underlying logic for proving theorems, as it admits any left-continuous t-norm for the evaluation of graded properties in the [0, 1]-interval and is arguably [2, Section 4] one of the most general logics suitable for graded fuzzy mathematics. In semantic examples, on the other hand, standard Łukasiewicz logic is consistently used for the sake of concreteness and clarity.

The present paper extends the work of [12,13] in the following directions. First, it takes into account certain peculiar features that regularly occur in connection with graded properties in logic-based mathematics [9] and are caused by the general non-idempotence of t-norm conjunction in fuzzy logic. In particular, due to the non-idempotence of &, a multiple conjunction $\varphi \& \ldots \& \varphi$ of a formula φ is generally stronger than the formula φ itself. Consequently, a formula containing a multiple conjunction of some subformula defines a graded property which in general differs from the one defined without the multiplicity. The conjunction-multiplicities thus parameterize a family of mutually related graded properties, which generally differ in strength. The present paper introduces and investigates multiplicity-parameterized variants of the graded properties studied in [12,13]. Secondly, the paper studies the transmission of (multiplicity-parameterized) graded properties of unary and binary fuzzy connectives under functional composition, including its special case of slight variation of a fuzzy connective (i.e., its composition with a function close to the identity function). Thirdly, the values of graded properties are determined for the propositional connectives ($\neg$, &, $\vee$, $\rightarrow$, etc.) of the ground logic, i.e., the connectives that are themselves used for evaluation of graded properties. Finally, the dependence of the values of graded properties on the ground logic is discussed and some schematic results on this dependence are presented.

The importance and applicability of graded properties of fuzzy relations and fuzzy connectives has already been discussed in the previous papers [11,12]. Here let us additionally stress that a detailed investigation of graded properties of fuzzy connectives is indispensable for further development of logic-based fuzzy mathematics [8,4]. As shown in [3], the system L of truth degrees plays in logic-based fuzzy mathematics a role which is analogous to the role of real numbers in classical mathematics. Fuzzy connectives $\mathrm{L}^n \rightarrow \mathrm{L}$ thus correspond to algebraic operations on real numbers, and the study of their properties is therefore a counterpart, in formal fuzzy mathematics, of the classical algebra of real numbers. Moreover, since L is the system of fuzzy subclasses of crisp singletons [10], it is the basic structure of which the universe of fuzzy mathematics is composed. The role of a graded theory of fuzzy connectives in logic-based fuzzy mathematics is thus comparable to that of algebraic operations on Boolean algebras as well as on real numbers in classical mathematics.

A further important motivation for a part of the present investigation has been formulated in [12] as follows:

> Whenever, e.g., a t-norm is slightly distorted, for instance by noise or just by rounding, it is actually no longer a t-norm and theorems on t-norms say nothing about the function. For example, a function resulting from the product t-norm by adding a random noise with the maximal amplitude 0.001 need not be commutative nor associative, so the well-known theorems on t-norms are not applicable. Nevertheless, it is obvious that the function approximates the product t-norm very closely, and that many (though not all) of its properties will be very close to the properties of the product t-norm itself. However, unless we can (i) measure the degree of corruption of its commutativity, associativity, etc., and (ii) derive theorems on how these degrees propagate to other properties, we possess no information on which properties of t-norms almost apply to the distorted function (and to what degrees).

The study of graded properties in FCT provides us exactly with (i) a way of measuring the degrees of corruption of traditional properties, and theorems derived in this paper allow us to estimate (ii) how these degrees propagate to slightly distorted functions. [1]

The present elaboration, though fairly complete as regards provable theorems in the areas indicated above, is still not exhaustive. First, some less interesting possibilities of placing the multiplicity parameters are neglected (cf. Remark 3.5 below). Secondly, the optimality of the &-multiplicities in the premises of most theorems is not demonstrated, as this would clutter the paper with an enormous number of tedious semantic counterexamples (cf. Remark 3.17). In many cases, nevertheless, the optimality of the result is almost obvious from the proof, as counterexamples to the logical

[1] See, e.g., Example 6.13 below for a particular result of this type. The effects of noise and rounding are visualized in Figs. 3 and 4.

steps with lesser multiplicities would straightforwardly generate counterexamples for the whole theorem with lesser multiplicities.

2. Preliminaries

We shall work in higher-order fuzzy logic, also known as Fuzzy Class Theory (FCT). FCT is an axiomatic theory of Zadeh's notions of fuzzy set [27] and fuzzy relation [28] in formal fuzzy logic (in the sense of [22]). For reference, a (slightly simplified) definition of FCT is given below; for more details see the original paper [5] or the freely available primer [6].

We shall use the variant of FCT over the logic $\mathrm{MTL}_\triangle$ of all left-continuous t-norms [18], which in a certain specific sense (see [2, Section 4]) is the weakest fuzzy logic suitable for this type of graded fuzzy mathematics. We assume the reader's familiarity with the first-order logic $\mathrm{MTL}_{\triangle\forall}$; here we shall just briefly recall the standard semantics of its connectives and quantifiers over the real unit interval [0, 1]:

$\&$	...	any left-continuous t-norm $*$
$\to$	...	the residuum $\Rightarrow_*$ of $*$, defined as $x \Rightarrow_* y =_{\mathrm{df}} \sup\{z \mid z * x \le y\}$
$\wedge, \vee$	...	min, max
$\neg$	...	$\neg x =_{\mathrm{df}} x \Rightarrow_* 0$
$\leftrightarrow$	...	bi-residuum: $\min(x \Rightarrow_* y, y \Rightarrow_* x)$
$\triangle$	...	$\triangle x = 1$ if $x = 1$; $\triangle x = 0$ otherwise
$\forall, \exists$	...	inf, sup

By means of this 'dictionary', the formal results formulated and proved in $\mathrm{MTL}_{\triangle\forall}$ can be translated into the more common semantic notions: for instance, the formula (2) in Section 3,

$$\mathrm{MonCng}(\mathbf{u}) \equiv_{\mathrm{df}} (\forall\alpha\beta)((\alpha \to \beta) \to (\mathbf{u}\alpha \to \mathbf{u}\beta)),$$

expresses the following semantic definition of the degree of the fuzzy property MonCng for a fuzzy connective $\mathbf{u}$:

$$\mathrm{MonCng}(\mathbf{u}) = \bigwedge_{\alpha,\beta}((\alpha \Rightarrow_* \beta) \Rightarrow_* (\mathbf{u}(\alpha) \Rightarrow_* \mathbf{u}(\beta))),$$

for any given left-continuous t-norm $*$. The terms $\{x \mid \varphi(x)\}$ introduced later in this section are semantically interpreted as denoting the fuzzy set A such that $Ax = \varphi(x)$ for all x. Thus, e.g., $\{x \mid Bx \,\&\, Cx\}$ denotes the fuzzy set A such that $Ax = Bx * Cx$ for any x, i.e., the strong intersection of the fuzzy sets B and C (under any given left-continuous t-norm $*$). In this manner, all formulae encountered in this paper can be understood as denoting the corresponding semantic facts about standard fuzzy sets and fuzzy relations.

Recall further that since $x \Rightarrow_* y$ equals 1 iff $x \le y$, theorems with implication as the principal connective express the comparison of degrees. Thus, e.g., in Observation 3.3 (U1) below, the formula $\mathrm{Cng}_n(\mathbf{u}) \Rightarrow \mathrm{Cng}_m(\mathbf{u})$ expresses the semantic fact that $\mathrm{Cng}_n(\mathbf{u}) \le \mathrm{Cng}_m(\mathbf{u})$. (Notice that sometimes we use the sign $\Rightarrow$ for $\to$ and $\Leftrightarrow$ for $\leftrightarrow$, by Convention 2.1 below.) Similarly theorems with $\leftrightarrow$ as the principal connective express the identity of degrees; so for example Lemma 3.8 (L1) below, $(\mathrm{Hgt}\, A \to \mathrm{Plt}\, A) \Leftrightarrow (\forall xy)(Ax \to Ay)$, expresses the fact that $(\mathrm{Hgt}\, A \Rightarrow_* \mathrm{Plt}\, A) = \bigwedge_{x,y}(Ax \Rightarrow_* Ay)$, for any left-continuous t-norm $*$. For more details on the meaning of formulae in $\mathrm{MTL}_\triangle$ and FCT see [6,7].

By means of the above 'dictionary', the reader interested only in the results should be able to 'translate' the provable formulae into the more common semantic language of fuzzy set theory; such a reader can safely skip the rest of Preliminaries, using the section only as a reference for the defined notions, and skip all formal proofs in the paper. For the benefit of readers interested also in proofs and the formal aspects of the apparatus, the definition of Fuzzy Class Theory and some related technical notions are given below. For a more comprehensive account of FCT see [5–7].

Convention 2.1. For better readability of complex formulae, we shall alternatively use the comma (,) for &; the symbol $\Rightarrow$ for $\to$; and $\Leftrightarrow$ for $\leftrightarrow$. The symbols $\Rightarrow$ and $\Leftrightarrow$ can be chained, with $\varphi_1 \Rightarrow \varphi_2 \Rightarrow \cdots \Rightarrow \varphi_n$ representing the formula $(\varphi_1 \to \varphi_2) \,\&\, (\varphi_2 \to \varphi_3) \,\&\, \ldots \,\&\, (\varphi_{n-1} \to \varphi_n)$, and similarly for $\Leftrightarrow$. The sign $\equiv$ will indicate equivalence by definition. By convention, the symbols $\Rightarrow$, $\Leftrightarrow$ and $\equiv$ will have the lowest priority in formulae and the comma the second lowest priority. Of other symbols, $\to$ and $\leftrightarrow$ will have lower priority than other binary connectives, and quantifiers and unary connectives will have the highest priority.

We shall use the following notation:

$$\varphi^n \equiv_{\mathrm{df}} \varphi \,\&\, \ldots \,\&\, \varphi \quad (n \text{ times})$$

$$\varphi^0 \equiv_{\mathrm{df}} 1$$

$$\varphi^{\triangle} \equiv_{\mathrm{df}} \triangle\varphi$$

In formulae, the superscripts will have the highest priority; thus, e.g., $\neg\varphi^n$ will be understood as $\neg(\varphi^n)$.

Furthermore we shall use the following defined connectives that express the ordering and equality of truth degrees:

$$\varphi \leq \psi \equiv_{\mathrm{df}} \triangle(\varphi \to \psi)$$

$$\varphi = \psi \equiv_{\mathrm{df}} \triangle(\varphi \leftrightarrow \psi)$$

$$\varphi{\neq}\psi \equiv_{\mathrm{df}} \neg(\varphi = \psi)$$

$$\varphi < \psi \equiv_{\mathrm{df}} (\varphi \leq \psi) \,\&\, (\varphi{\neq}\psi)$$

and analogously for $\geq$, $>$. The priority of these connectives is the same as that of implication.

In proofs, we shall commonly use the following names of proof steps based on $\mathrm{MTL}_{\triangle\forall}$-valid theorems and rules (where x is not free in v in the quantifier shifts):

$\varphi \to (\psi \to \chi) \;\Leftrightarrow\; \varphi \,\&\, \psi \to \chi$	residuation
$\varphi \to (\psi \to \chi) \;\Leftrightarrow\; \psi \to (\varphi \to \chi)$	exchange
$\varphi \to \psi \;\Rightarrow\; (\chi \to \varphi) \to (\chi \to \psi)$	prefixing
$\varphi \to \psi \;\Rightarrow\; (\psi \to \chi) \to (\varphi \to \chi)$	suffixing
$\psi \;\Rightarrow\; \varphi \to \psi$	weakening
$\varphi_1 \to \psi_1,\ \varphi_2 \to \psi_2 \;\Rightarrow\; \varphi_1 \,\&\, \varphi_2 \to \psi_1 \,\&\, \psi_2$	combination of the antecedents and consequents
$\varphi_1 \vee \cdots \vee \varphi_n,\ \varphi_1 \to \psi,\ \ldots,\ \varphi_n \to \psi \;\Rightarrow\; \psi$	taking cases
$\triangle(\varphi \to \psi) \;\Rightarrow\; \triangle\varphi \to \triangle\psi$	$\triangle$-distribution
from φ infer $\triangle\varphi$	$\triangle$-necessitation
from φ infer $(\forall x)\varphi$	generalization (on x)
$(\forall x)\varphi(x) \;\Rightarrow\; \varphi(t)$	specification
$\varphi(t) \;\Rightarrow\; (\exists x)\varphi(x)$	dual specification
$(\forall x)(\varphi \to \psi) \;\Rightarrow\; (\forall x)\varphi \to (\forall x)\psi$	quantifier distribution (of $\forall$ as $\forall$)
$(\forall x)(\varphi \to \psi) \;\Rightarrow\; (\exists x)\varphi \to (\exists x)\psi$	quantifier distribution (of $\forall$ as $\exists$)
$(\exists x)(v \to \varphi) \;\Rightarrow\; v \to (\exists x)\varphi$	quantifier shift (of $\exists$ to the consequent)
$(\exists x)(v \to \varphi) \;\Rightarrow\; (\forall x)\varphi \to v$	quantifier shift (of $\exists$ to the antecedent)
$(\forall x)(\varphi \to v) \;\Leftrightarrow\; (\exists x)\varphi \to v$	quantifier shift (of $\forall$ to or $\exists$ from the antecedent)
$(\forall x)(v \to \varphi) \;\Leftrightarrow\; v \to (\forall x)\varphi$	quantifier shift (of $\forall$ to or from the consequent)

Fuzzy class theory FCT, or Henkin-style higher-order fuzzy logic $\mathrm{MTL}_{\triangle}$, is an axiomatic theory over multi-sorted first-order logic $\mathrm{MTL}_{\triangle}$, with sorts of variables for:

- *Atomic elements,* denoted by lowercase letters $x, y, \ldots$.
- *Fuzzy classes*[2] of atomic elements, denoted by uppercase letters $A, B, \ldots$.
- *Fuzzy classes of fuzzy classes* of atomic elements, denoted by calligraphic letters $\mathcal{A}, \mathcal{B}, \ldots$.
- Etc.; in general for *fuzzy classes of the n-th order*, written as $X^{(n)}, Y^{(n)}, \ldots$.

[2] In FCT, fuzzy sets are rendered as a primitive notion, rather than modeled by membership functions. To capture the distinction, we call the objects of the theory *fuzzy classes* and reserve the term *fuzzy sets* for their representation by membership functions in models of the theory. Nevertheless, since all theorems on fuzzy classes provable in FCT are true statements about $\mathrm{MTL}_{\triangle}$-valued fuzzy sets and relations, the reader can always safely substitute fuzzy sets for fuzzy classes in theorems of FCT (the distinction is only relevant in meta-theory).

The primitive symbols of FCT are:

- The *identity predicates* $=$ on each sort.
- The *membership predicates* $\in$ between successive sorts.
- The *class terms* $\{x \mid \varphi\}$ of order $n+1$, for each formula φ and each variable x of order $n \in \mathbb{N}$.
- The symbols for *tuples* $\langle x_1, \ldots, x_k\rangle$ of individuals $x_1, \ldots, x_k$ of any order and all arities $k \in \mathbb{N}$.

The formula $x \in A$ and the term $\langle x_1, \ldots, x_k\rangle$ may be abbreviated, respectively, as Ax and $x_1 \ldots x_k$. FCT has the following axioms, for all formulae φ and variables of any order:

- The logical axioms of multi-sorted first-order logic $\mathrm{MTL}_\triangle$.
- The *identity axioms:* $x = x$ and $x = y \to (\varphi(x) \leftrightarrow \varphi(y))$.
- The *tuple-identity axioms:* $\langle x_1, \ldots, x_k\rangle = \langle y_1, \ldots, y_k\rangle \to x_i = y_i$, for all $k \in \mathbb{N}$ and $1 \le i \le k$.
- The *comprehension axioms*: $y \in \{x \mid \varphi(x)\} \leftrightarrow \varphi(y)$.
- The *extensionality axioms*: $(\forall x)(Ax = Bx) \to A = B$.

The axioms for identity entail that the identity predicates $=$ on each sort are crisp (while the membership predicates $\in$ can in general be fuzzy). Notice that due to the logical axioms, theorems of FCT need to be proved by the rules of the logic $\mathrm{MTL}_{\triangle\forall}$ rather than classical logic.

The models of FCT are systems of fuzzy sets and fuzzy relations of all finite arities and orders over a fixed crisp set X (the universe of discourse) that are closed under all FCT-definable operations and whose membership degrees take values in any fixed $\mathrm{MTL}_\triangle$-chain (standardly, the real unit interval $[0, 1]$ equipped with a left-continuous t-norm). Models of FCT over standard $[0, 1]$-valued $\mathrm{MTL}_\triangle$-algebras are called *standard* models. Models that contain *all* fuzzy sets and fuzzy relations over the domain of discourse X (i.e., the intended models of FCT) are called *full* models. Standard full models, which represent Zadeh's original notion of fuzzy set, are called *Zadeh* models of FCT.

Definition 2.2. In FCT, we introduce the following defined notions[3]:

$A \subseteq B \equiv_{\mathrm{df}} (\forall x)(Ax \to Bx)$	inclusion
$A \sqsubseteq B \equiv_{\mathrm{df}} \triangle(A \subseteq B)$	crisp inclusion
$A \approx B \equiv_{\mathrm{df}} (\forall x)(Ax \leftrightarrow Bx)$	weak bi-inclusion
$A \cong B \equiv_{\mathrm{df}} (A \subseteq B)\ \&\ (B \subseteq A)$	strong bi-inclusion
$\mathrm{Hgt}\ A \equiv_{\mathrm{df}} (\exists x)Ax$	height
$\mathrm{Plt}\ A \equiv_{\mathrm{df}} (\forall x)Ax$	plinth
$\mathrm{Crisp}\ A \equiv_{\mathrm{df}} (\forall x)\triangle(Ax \vee \neg Ax)$	crispness
$A_1 \times \cdots \times A_k =_{\mathrm{df}} \{x_1 \ldots x_k \mid A_1x_1\ \&\ \ldots\ \&\ A_kx_k\}$	Cartesian product
$A^k =_{\mathrm{df}} \{x_1 \ldots x_k \mid Ax_1\ \&\ \ldots\ \&\ Ax_k\}$	Cartesian power
$R^{\mathrm{T}} =_{\mathrm{df}} \{xy \mid Ryx\}$	converse relation
$\mathrm{Ker}\ A =_{\mathrm{df}} \{x \mid \triangle Ax\}$	kernel
$\mathrm{Pow}\ A =_{\mathrm{df}} \{B \mid B \subseteq A\}$	power class

Convention 2.3. For a binary relation R and any $n \in \mathbb{N}$, we shall use the following notation[4]:

$$R^n =_{\mathrm{df}} \bigcap_{i=1}^{n} R, \quad R^{-n} =_{\mathrm{df}} (R^{\mathrm{T}})^n$$

[3] Since the axioms of FCT have the same form for each order, it is sufficient to formulate conventions, definitions, theorems, and proofs only for the lowest order, as they straightforwardly propagate to all higher orders.

[4] The distinction between the n-th Cartesian power A^n and the n-tuple intersection R^n will always be clear from the context. In this paper we shall only use Cartesian powers of the crisp class L defined below.

When using infix notation, we can occasionally write $(xRy)^{-n}$ instead of $xR^{-n}y$. Notice that while R^{-1} is the converse of R, the expressions R^n and R^{-n} denote the n-tuple *intersection* (rather than relational iteration) of R resp. R^{-1}. Accordingly, R^0 is the universal (rather than identity) relation.

Restricted quantification $(\forall x)(x \in A \to \varphi)$ and $(\exists x)(x \in A \,\&\, \varphi)$ will be written as $(\forall x \in A)\varphi$ and $(\exists x \in A)\varphi$, and similarly for quantification restricted by means of $=$, $\subseteq$, and other infix binary predicates.

In this paper we shall deal with fuzzy connectives, i.e., algebraic operations on truth degrees. Even though truth degrees are not part of the primitive language of FCT, they can be represented in the theory by subclasses of a crisp singleton (see [10, Section 3]).[5] The details of the representation are not important for our present purposes; we shall thus simply assume that variables $\alpha, \beta, \ldots$ for truth values are at our disposal in FCT, and that the ordering of truth values and the usual propositional connectives and the quantifiers $\forall, \exists$ are definable in FCT. The crisp class of the internal truth values will be denoted by L.

The study of fuzzy connectives in the framework of FCT was initiated in [12,13]. By *n-ary fuzzy connectives* we understand n-ary operations on truth degrees, i.e., crisp functions $\mathbf{c}\colon \mathrm{L}^n \to \mathrm{L}$. Being functions into L, they can equivalently be regarded as fuzzy relations on L^n; i.e., $\mathbf{c} \sqsubseteq \mathrm{L}^n$. Except in compositions (which are defined as compositions of crisp functions $\mathrm{L}^n \to \mathrm{L}$, see Definition 2.5 below), fuzzy connectives will in this paper be regarded in the latter sense, i.e., as fuzzy relations on L^n. Thus, e.g., fuzzy inclusion $\mathbf{c} \subseteq \mathbf{d}$ of binary fuzzy connectives $\mathbf{c}$, $\mathbf{d}$ will be understood as inclusion of fuzzy relations $\mathbf{c}, \mathbf{d}\colon \mathrm{L}^2 \to \mathrm{L}$, i.e., $\mathbf{c} \subseteq \mathbf{d} \equiv (\forall\alpha)(\mathbf{c}\alpha\beta \to \mathbf{d}\alpha\beta)$, rather than as inclusion of crisp functions. This dual nature applies as well to the propositional connectives $\&, \wedge, \vee, \neg, \ldots$ of the ground logic, which are particular instances of fuzzy connectives: they shall be therefore regarded as crisp functions in compositions (such as $\mathbf{a} \wedge \mathbf{b}$ or $\neg\mathbf{c}$) and as fuzzy relations in other cases (such as $\& \subseteq \wedge$). Nullary connectives $\mathbf{f}\colon \mathrm{L}^0 \to \mathrm{L}$, where $\mathrm{L}^0 = \{\mathrm{a}\}$ is an arbitrary fixed crisp singleton, can be identified with the truth values $\alpha = \mathbf{f}(\mathrm{a})$.

Convention 2.4. We shall always use Greek letters for truth values,[6] the letters $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ for unary connectives, and the letters $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, ... for binary connectives. The letters $\mathbf{f}$, $\mathbf{g}$, $\mathbf{h}$, ... will be used for connectives of arbitrary arity $0 \leq n \leq 2$ and $\vec{\alpha}, \vec{\beta}, \vec{\gamma}, \ldots$ for their arguments of the appropriate arity. Infix notation $\alpha\ \mathbf{c}\ \beta$ will usually be employed for binary connectives instead of prefix notation $\mathbf{c}\alpha\beta$. In formulae, infix binary connectives will by convention have the same priority as &: thus, e.g., $\neg\alpha\ \mathbf{c}\ \beta \to \gamma$ will mean $((\neg\alpha)\ \mathbf{c}\ \beta) \to \gamma$.

Fuzzy connectives, being crisp functions $\mathbf{c}\colon \mathrm{L}^2 \to \mathrm{L}$ (binary), $\mathbf{u}\colon \mathrm{L} \to \mathrm{L}$ (unary), and $\alpha\colon \mathrm{L}^0 \to \mathrm{L}$ (nullary), can be composed whenever their domains and codomains match. Recall the standard definitions for crisp functions:

Definition 2.5. For $f\colon X \to Y$, $g\colon Y \to Z$, the composition $gf\colon X \to Z$ is defined as $(gf)(x) =_{\mathrm{df}} g(f(x))$ for all $x \in X$.

Given the projections $p_1\colon X \times Y \to X$ and $p_2\colon X \times Y \to Y$, the product function of $f\colon Z \to X$ and $g\colon Z \to Y$ is defined as the unique function $(f, g)\colon Z \to X \times Y$ such that $(f, g)(z) = (f(z), g(z))$, i.e., $p_1((f, g)(z)) = f(z)$ and $p_2((f, g)(z)) = g(z)$.

The projections $\mathbf{p}_1, \mathbf{p}_2 : \mathrm{L}^2 \to \mathrm{L}$ are defined as $\mathbf{p}_1(\alpha, \beta) =_{\mathrm{df}} \alpha$ and $\mathbf{p}_2(\alpha, \beta) =_{\mathrm{df}} \beta$ for all $\alpha, \beta \in \mathrm{L}$. The identity on L will be denoted by $\mathbf{id}$, i.e., $\mathbf{id}\,\alpha =_{\mathrm{df}} \alpha$ for all $\alpha \in \mathrm{L}$. The constant function $X \to Y$ assigning a fixed $y \in Y$ to all $x \in X$ will be denoted by $\underline{y}_X$; the subscript X may be omitted if known from the context.

Various constructions are expressible by composition of fuzzy connectives, e.g.:

$$\mathbf{c}(\underline{\alpha}, \mathbf{id})\colon \mathrm{L} \to \mathrm{L} \quad \ldots \quad \mathbf{c}(\underline{\alpha}, \mathbf{id})\,(\beta) = \alpha\ \mathbf{c}\ \beta \qquad \text{(fixing the first argument)}$$
$$\mathbf{c}(\mathbf{id}, \mathbf{id})\colon \mathrm{L} \to \mathrm{L} \quad \ldots \quad \mathbf{c}(\mathbf{id}, \mathbf{id})\,(\alpha) = \alpha\ \mathbf{c}\ \alpha \qquad \text{(the diagonal)}$$

[5] The class L of internal truth values is defined as $\mathrm{L} \equiv_{\mathrm{df}} \mathrm{Ker}\,\mathrm{Pow}\{\mathrm{a}\}$, for a fixed atomic element a. The semantic truth value of a formula φ is represented by the element $\overline{\varphi} =_{\mathrm{df}} \{\mathrm{a} \mid \varphi\}$ of L. Conversely, a subclass α of $\{\mathrm{a}\}$ represents the semantic truth value of the formula $\mathrm{a} \in \alpha$. Basic class operations on L then represent the propositional connectives and quantifiers of the ground logic. For details of the construction and certain metamathematical qualifications regarding the representation see [10, Section 3].

[6] By a harmless abuse of notation, we shall not distinguish in formulae between the semantic truth values and their internal FCT representations, as doing so would complicate notation too much. Similarly we shall not distinguish between the semantic and internalized logical connectives and quantifiers.

$$\mathbf{c}(\mathbf{up}_1, \mathbf{vp}_2)\colon \mathrm{L}^2 \to \mathrm{L}\ldots \quad \mathbf{c}(\mathbf{up}_1, \mathbf{vp}_2)\,(\alpha, \beta) = (\mathbf{u}\alpha)\,\mathbf{c}\,(\mathbf{v}\beta) \qquad \text{(transformation of arguments)}$$
$$\mathbf{c}(\mathbf{p}_2, \mathbf{p}_1)\colon \mathrm{L}^2 \to \mathrm{L}\ldots \quad \mathbf{c}(\mathbf{p}_2, \mathbf{p}_1)\,(\alpha, \beta) = \beta\,\mathbf{c}\,\alpha \qquad \text{(the converse, } \mathbf{c}^{-1}\text{)}$$
$$\mathbf{c}(\mathbf{d}, \mathbf{e})\colon \mathrm{L}^2 \to \mathrm{L}\ldots \quad \mathbf{c}(\mathbf{d}, \mathbf{e})\,(\alpha, \beta) = (\alpha\,\mathbf{d}\,\beta)\,\mathbf{c}\,(\alpha\,\mathbf{e}\,\beta) \qquad \text{(composition of binary connectives)}$$

Notice that applying Convention 2.3 to fuzzy connectives we obtain $\mathbf{f}^n\vec{\alpha} \equiv_{\mathrm{df}} (\mathbf{f}\vec{\alpha})^n$, for $n \in \mathbb{N}$; thus $\mathbf{f}^n$ in this paper denotes the *conjunction* (i.e., intersection) rather than iteration of $\mathbf{f}$. Consequently, e.g., $\mathbf{id}^2\alpha = \alpha \,\&\, \alpha$; $\mathbf{id}^n\mathbf{f} = \mathbf{f}^n$; $\mathbf{f}^0 = \underline{1}$; and $\mathbf{f}^n \subseteq \mathbf{f}^m$ for $n \geq m \geq 0$.

3. Graded properties of unary connectives

In the framework of FCT, various graded properties of unary fuzzy connectives $\mathbf{u}\colon \mathrm{L} \to \mathrm{L}$ (or equivalently, $\mathbf{u} \sqsubseteq \mathrm{L}$) can be introduced. For instance, the following graded properties of unary connectives have been studied in [13]:

$$\mathrm{Mon}(\mathbf{u}) \equiv_{\mathrm{df}} (\forall\alpha\beta)((\alpha \leq \beta) \to (\mathbf{u}\alpha \to \mathbf{u}\beta)) \qquad \text{Graded monotony}$$
$$\mathrm{Ant}(\mathbf{u}) \equiv_{\mathrm{df}} (\forall\alpha\beta)((\alpha \leq \beta) \to (\mathbf{u}\beta \to \mathbf{u}\alpha)) \qquad \text{Graded antitony}$$
$$\mathrm{Cng}(\mathbf{u}) \equiv_{\mathrm{df}} (\forall\alpha\beta)((\alpha \leftrightarrow \beta) \to (\mathbf{u}\alpha \leftrightarrow \mathbf{u}\beta)) \qquad \text{Graded congruence w.r.t. } \leftrightarrow$$

The graded property Cng can be viewed as a generalized Lipschitz property, since $\mathrm{Cng}(\mathbf{u}) = 1$ in standard Łukasiewicz models iff $\mathbf{u}$ is a 1-Lipschitz function. Later on it will be useful to have the synonyms Mon_{+1} and Mon_{-1} for Mon and Ant, respectively; employing Convention 2.3, we can define them for $i = \pm 1$ as follows:

$$\mathrm{Mon}_i(\mathbf{u}) \equiv_{\mathrm{df}} (\forall\alpha\beta)((\alpha \leq^i \beta) \to (\mathbf{u}\alpha \to \mathbf{u}\beta)). \tag{1}$$

Furthermore, the following graded properties of unary fuzzy connectives have also been considered in [13]:

$$\mathrm{MonCng}(\mathbf{u}) \equiv_{\mathrm{df}} (\forall\alpha\beta)((\alpha \to \beta) \to (\mathbf{u}\alpha \to \mathbf{u}\beta)) \tag{2}$$
$$\mathrm{AntCng}(\mathbf{u}) \equiv_{\mathrm{df}} (\forall\alpha\beta)((\alpha \to \beta) \to (\mathbf{u}\beta \to \mathbf{u}\alpha)) \tag{3}$$

However, they have been shown to be superfluous, as they are equivalent to the min-conjunction of congruence and (positive resp. negative) monotony:

$$\mathrm{MonCng}(\mathbf{u}) \;\Leftrightarrow\; \mathrm{Mon}(\mathbf{u}) \wedge \mathrm{Cng}(\mathbf{u}) \tag{4}$$
$$\mathrm{AntCng}(\mathbf{u}) \;\Leftrightarrow\; \mathrm{Ant}(\mathbf{u}) \wedge \mathrm{Cng}(\mathbf{u}) \tag{5}$$

Moreover it can be observed that also the well-known characteristics of height $\mathrm{Hgt}(A) =_{\mathrm{df}} \sup_x Ax$ and plinth $\mathrm{Plt}(A) =_{\mathrm{df}} \inf_x Ax$ can be regarded as graded properties of fuzzy sets, as they, too, assign truth values to fuzzy sets. In this paper we shall therefore consider them alongside other graded properties of fuzzy connectives; their basic FCT-provable properties can be found in [5,10]. When applied to unary connectives $\mathbf{u} \sqsubseteq \mathrm{L}$, they are in FCT expressed by the following formulae:

$$\mathrm{Hgt}(\mathbf{u}) \equiv_{\mathrm{df}} (\exists\alpha)\mathbf{u}\alpha \qquad \text{Height}$$
$$\mathrm{Plt}(\mathbf{u}) \equiv_{\mathrm{df}} (\forall\alpha)\mathbf{u}\alpha \qquad \text{Plinth}$$

Example 3.1. Defined notions and theorems studied in this paper will be illustrated on several semantic examples over standard [0, 1]-models of FCT. For this purpose we shall define the following unary connectives on $\mathrm{L} = [0, 1]$[7]:

$$\mathbf{w}_{\mathrm{sq}}(\alpha) =_{\mathrm{df}} \alpha \cdot \alpha$$
$$\mathbf{w}_{\mathrm{N}}(\alpha) =_{\mathrm{df}} \begin{cases} \alpha + 0.2 & \text{for } \alpha \in [0, 0.4] \\ 1 - \alpha & \text{for } \alpha \in [0.4, 0.6] \\ \alpha - 0.2 & \text{for } \alpha \in [0.6, 1] \end{cases}$$
$$\mathbf{w}_{\mathrm{eps}}(\alpha) =_{\mathrm{df}} (\alpha + 0.01) \wedge 1 \tag{6}$$

[7] Since we are constructing semantic examples in standard models of FCT, these connectives can be defined by common mathematical operations on real numbers and need not even be definable by logical formulae in FCT over $\mathrm{MTL}_\triangle$ or another t-norm logic. As they are nevertheless present in a standard model of FCT, the graded properties defined and theorems derived in this paper still apply to them.

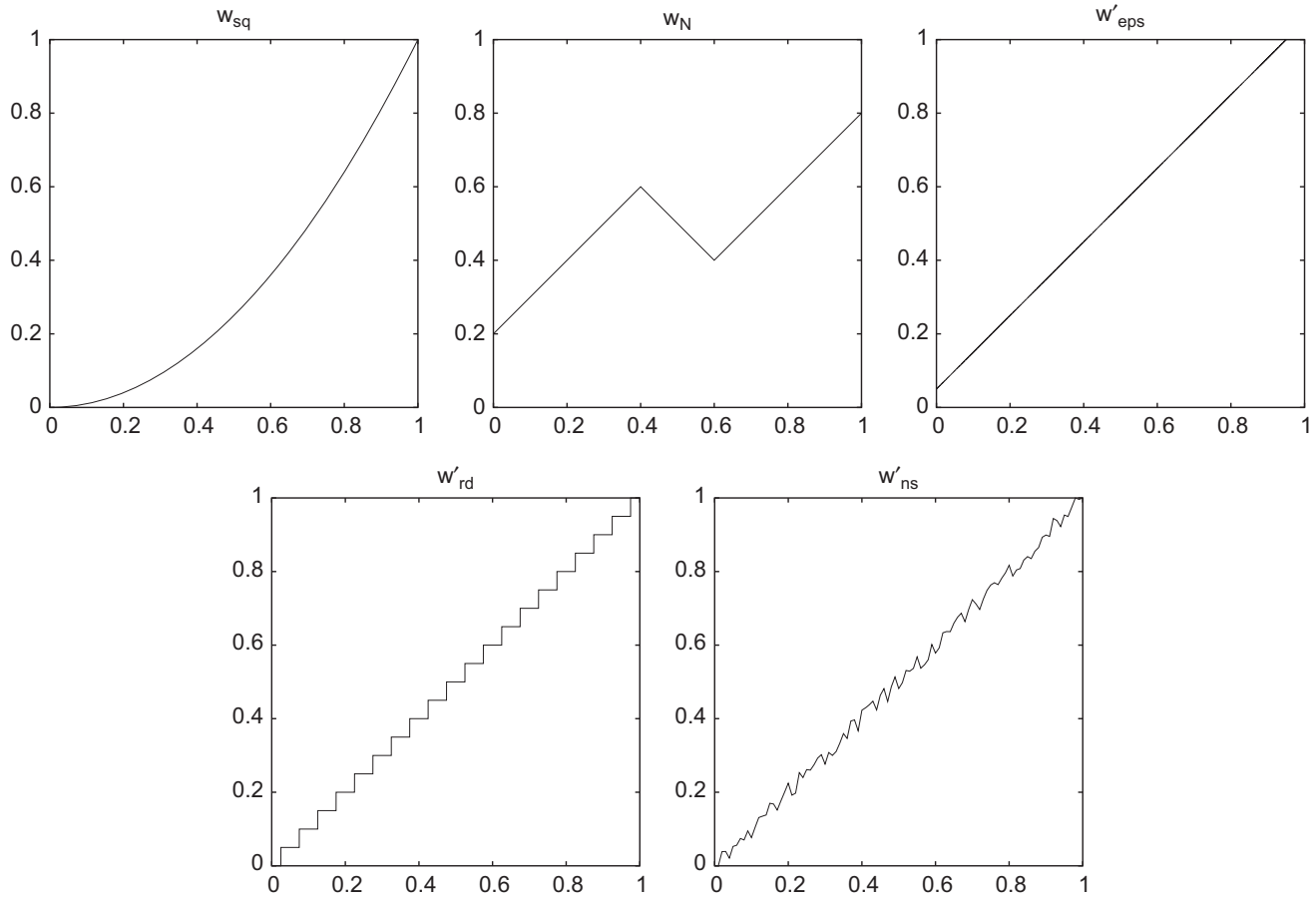

Fig. 1. Graphs of the fuzzy connectives $\mathbf{w}_{\mathrm{sq}}$, $\mathbf{w}_{\mathrm{N}}$, $\mathbf{w}_{\mathrm{eps}}$, $\mathbf{w}_{\mathrm{rd}}$, and $\mathbf{w}_{\mathrm{ns}}$ of Example 3.1. Since $\mathbf{w}_{\mathrm{eps}}$, $\mathbf{w}_{\mathrm{rd}}$, and $\mathbf{w}_{\mathrm{ns}}$ are visually almost indistinguishable from the identity function, their coarser variants $\mathbf{w}'_{\mathrm{eps}}$, $\mathbf{w}'_{\mathrm{rd}}$, and $\mathbf{w}'_{\mathrm{ns}}$ that use 0.05 and 20 instead of 0.01 and 100 in (6)–(8) are plotted in the figure; moreover, 0.01-steps in the argument have been used to approximate the graph of $\mathbf{w}'_{\mathrm{ns}}$. Note that the graph of $\mathbf{w}'_{\mathrm{ns}}$ is not graphically representable (its values forming with probability 1 a dense strip of points along the diagonal); consequently, the interpolating lines in the graphs in this paper only connect the grid values, and do not necessarily represent the actual interpolated values of the functions.

$$\mathbf{w}_{\mathrm{rd}}(\alpha) =_{\mathrm{df}} \frac{\lfloor 100\alpha + 0.5 \rfloor}{100} \tag{7}$$

$$\mathbf{w}_{\mathrm{ns}}(\alpha) =_{\mathrm{df}} \alpha + \frac{\mathrm{rand}(\alpha) - 0.5}{100} \tag{8}$$

where rand is a function assigning random numbers from $[0, 1]$ to its arguments from $[0, 1]$. The connective $\mathbf{w}_{\mathrm{sq}}$ returns the square of its argument; the connective $\mathbf{w}_{\mathrm{N}}$ is a slanted N-shaped piecewise linear function; $\mathbf{w}_{\mathrm{eps}}$ just adds a small constant (0.01) to the identity connective **id**; the connective $\mathbf{w}_{\mathrm{rd}}$ rounds its argument to two decimal places; and $\mathbf{w}_{\mathrm{ns}}$ adds a random noise with the amplitude 0.01 to the identity function **id**. The graphs of these functions are depicted in Fig. 1.

All examples in this paper will be evaluated in Zadeh models of FCT over standard Łukasiewicz logic. Via the function $1 - x$, the Łukasiewicz equivalence connective (which serves as fuzzified equality in the defining formulae of graded notions) corresponds to the Euclidean distance of values, as $1-(\alpha \leftrightarrow_{\mathrm{Ł}} \beta) = |(1-\alpha)-(1-\beta)|$. The Łukasiewicz conjunction similarly corresponds to the bounded addition of values, as $1-(\alpha \&_{\mathrm{Ł}} \beta) = ((1-\alpha)+(1-\beta)) \vee 0$. Evaluation in standard Łukasiewicz logic thus measures the 'Euclidean' defects of graded properties, while other left-continuous t-norms reflect other 'defect metrics' on the real unit interval: e.g., the product t-norm corresponds to the logarithmic Euclidean distance, the minimum t-norm to a 'maxitive' metric of defects, etc. Calculation of the values of graded properties in models over t-norms other than Łukasiewicz is left to the interested reader; the theorems of FCT apply to these logics as well.

In models over standard Łukasiewicz logic, the connectives $\mathbf{w}_{\mathrm{sq}}$, $\mathbf{w}_{\mathrm{N}}$, $\mathbf{w}_{\mathrm{eps}}$, $\mathbf{w}_{\mathrm{rd}}$, and $\mathbf{w}_{\mathrm{ns}}$ have the following values of graded properties[8]:

- Obviously $\mathrm{Plt}(\mathbf{w}_{\mathrm{N}}) = 0.2$ and $\mathrm{Hgt}(\mathbf{w}_{\mathrm{N}}) = 0.8$, while $\mathbf{w}_{\mathrm{sq}}$, $\mathbf{w}_{\mathrm{eps}}$, $\mathbf{w}_{\mathrm{rd}}$, and $\mathbf{w}_{\mathrm{ns}}$ all have plinth 0 and height 1.
- $\mathrm{Mon}(\mathbf{w}_{\mathrm{sq}}) = \mathrm{Mon}(\mathbf{w}_{\mathrm{eps}}) = \mathrm{Mon}(\mathbf{w}_{\mathrm{rd}}) = 1$, as these functions are fully monotone even according to the traditional non-graded definition of monotony. The function $\mathbf{w}_{\mathrm{N}}$ fails to be classically monotone, but the defect of its monotony is obviously not too large: indeed, according to our definition, $\mathrm{Mon}(\mathbf{w}_{\mathrm{N}}) = 0.8$, due to the decrease of $\mathbf{w}_{\mathrm{N}}$ by 0.2 in the interval $[0.4, 0.6]$. Thus $\mathbf{w}_{\mathrm{N}}$ is still fairly monotone, even if not fully so. Finally, $\mathrm{Mon}(\mathbf{w}_{\mathrm{ns}}) = 0.99$ (with probability 1), as the largest possible decrease of $\mathbf{w}_{\mathrm{ns}}$ equals the peak-to-peak amplitude of the noise added to **id**, i.e., 0.01.
- None of the five functions is classically antitone, and indeed $\mathrm{Ant}(\mathbf{w}_{\mathrm{sq}}) = \mathrm{Ant}(\mathbf{w}_{\mathrm{eps}}) = \mathrm{Ant}(\mathbf{w}_{\mathrm{rd}}) = 0$, as can be seen by specifying $\alpha = 0$ and $\beta = 1$ in the definition of graded antitony for these connectives, and also $\mathrm{Ant}(\mathbf{w}_{\mathrm{ns}}) = 0$, since the values $\mathbf{w}_{\mathrm{ns}}(\alpha) = 0$ and $\mathbf{w}_{\mathrm{ns}}(\beta) = 1$ are achieved for some $\alpha \in [0, 0.005]$ and $\beta \in [0.995, 1]$ with probability 1. The function $\mathbf{w}_{\mathrm{N}}$, on the other hand, is antitone to a non-zero degree: the definition yields $\mathrm{Ant}(\mathbf{w}_{\mathrm{N}}) = 0.4$, which corresponds to the fact that its largest possible increase (or, the lapse of decrease) between two values $\alpha \leq \beta$ is only 0.6.
- The connectives $\mathbf{w}_{\mathrm{N}}$ and $\mathbf{w}_{\mathrm{eps}}$, being 1-Lipschitz functions, have $\mathrm{Cng}(\mathbf{w}_{\mathrm{N}}) = \mathrm{Cng}(\mathbf{w}_{\mathrm{eps}}) = 1$. An elementary exercise in mathematical analysis yields $\mathrm{Cng}(\mathbf{w}_{\mathrm{sq}}) = 0.75$, the infimum in the definition of $\mathrm{Cng}(\mathbf{w}_{\mathrm{sq}})$ being achieved for $\alpha = 0.5$ and $\beta = 1$. The functions $\mathbf{w}_{\mathrm{rd}}$ and $\mathbf{w}_{\mathrm{ns}}$ are discontinuous, but since they differ from a 1-Lipschitz function only by at most 0.01, we still obtain $\mathrm{Cng}(\mathbf{w}_{\mathrm{rd}}) = \mathrm{Cng}(\mathbf{w}_{\mathrm{ns}}) = 0.99$, i.e., a very large degree of the 'graded 1-Lipschitz property' Cng.

Since the values of the graded properties of these connectives are going to be used in examples throughout this paper, we summarize them in the following table:

	Plt	Hgt	Mon	Ant	Cng	MonCng	AntCng
$\mathbf{w}_{\mathrm{sq}}$	0	1	1	0	0.75	0.75	0
$\mathbf{w}_{\mathrm{N}}$	0.2	0.8	0.8	0.4	1	0.8	0.4
$\mathbf{w}_{\mathrm{eps}}$	0	1	1	0	1	1	0
$\mathbf{w}_{\mathrm{rd}}$	0	1	1	0	0.99	0.99	0
$\mathbf{w}_{\mathrm{ns}}$	0	1	0.99	0	0.99	0.99	0

In this paper, we shall consider variants of graded properties that arise due to the general non-idempotence of conjunction in t-norm fuzzy logics and differ in the &-multiplicity of a subformula of the original definition. The occurrence of &-multiplicity parameters in definitions is actually a regular feature of formal theories over fuzzy logic, whenever dealing with graded notions [9]. Thus, for instance, we shall consider weaker variants of Cng that differ in the &-multiplicity of the antecedent condition $\alpha \leftrightarrow \beta$:

Definition 3.2. For a unary connective $\mathbf{u} \sqsubseteq \mathrm{L}$ and any $n \in \mathbb{N}$, we define in FCT:

$$\mathrm{Cng}_n(\mathbf{u}) \equiv_{\mathrm{df}} (\forall\alpha\beta)\big((\alpha \leftrightarrow \beta)^n \rightarrow (\mathbf{u}\alpha \leftrightarrow \mathbf{u}\beta)\big) \qquad \text{Graded } n\text{-congruence w.r.t. } \leftrightarrow$$

By convention, the index n in Cng_n can be omitted if equal to 1.

The graded assumption $(\alpha \leftrightarrow \beta)^n$ is in general stronger than $\alpha \leftrightarrow \beta$ if $n > 1$, as the truth degree of the former can in general be smaller than that of the latter. Consequently, the conditions Cng_n are in general *weaker* for larger n (as they have stronger antecedents):

Observation 3.3. *If* $m \geq n$, *then* FCT *proves*:

(U1) $\mathrm{Cng}_n(\mathbf{u}) \;\Rightarrow\; \mathrm{Cng}_m(\mathbf{u})$.

In semantics, the degree of $\mathrm{Cng}_m(\mathbf{u})$ is thus larger than or equal to that of $\mathrm{Cng}_n(\mathbf{u})$ for $m \geq n$. Since in standard Łukasiewicz logic the truth value of $(\alpha \leftrightarrow \beta)^n$ equals $(1 - n \cdot |\alpha - \beta|) \vee 0$, in standard Łukasiewicz models of FCT

[8] In the case of $\mathbf{w}_{\mathrm{ns}}$ we in this paper always refer to what happens with probability 1.

the condition $\mathrm{Cng}_n(\mathbf{u}) = 1$ expresses the *n-Lipschitz property* of $\mathbf{u}$ for $n \geq 1$. In standard product logic, the same correspondence holds on the logarithmic scale. In Gödel logic, which has idempotent conjunction, all properties Cng_n for $n \geq 1$ coincide with Cng.

Example 3.4. The connectives defined in Example 3.1 have the following values of the parameterized congruence property in models over standard Łukasiewicz logic:

- Since $\mathrm{Cng}(\mathbf{w}_\mathrm{N}) = \mathrm{Cng}(\mathbf{w}_\mathrm{eps}) = 1$, by (U1) we have $\mathrm{Cng}_n(\mathbf{w}_\mathrm{N}) = \mathrm{Cng}_n(\mathbf{w}_\mathrm{eps}) = 1$ for all $n \geq 1$.
- As $\mathbf{w}_\mathrm{sq}$ is a 2-Lipschitz function, we have $\mathrm{Cng}_2(\mathbf{w}_\mathrm{sq}) = 1$, and so $\mathrm{Cng}_n(\mathbf{w}_\mathrm{sq}) = 1$ for all $n \geq 2$ by (U1).
- The functions $\mathbf{w}_\mathrm{rd}$ and $\mathbf{w}_\mathrm{ns}$, being discontinuous, fail to be n-Lipschitz for any $n \in \mathbb{N}$. Consequently, the values of $\mathrm{Cng}_n(\mathbf{w}_\mathrm{rd})$ and $\mathrm{Cng}_n(\mathbf{w}_\mathrm{ns})$ are smaller than 1 for any $n \in \mathbb{N}$. In fact, since both have 0.01-jumps for arbitrarily close arguments, the &-multiplicity of the antecedent in the defining formula of Cng_n has no effect on these two connectives, and $\mathrm{Cng}_n(\mathbf{w}_\mathrm{rd}) = \mathrm{Cng}_n(\mathbf{w}_\mathrm{ns}) = 0.99$ for all $n \geq 1$. This behavior is only seemingly counter-intuitive: even though discontinuous functions are intuitively 'closer' to being n-Lipschitz for larger n, the logic-based measure of graded properties captures a notion of closeness different from this intuition. Rather, it is the pointwise 'distance' of functional values from the ideal ones that would be dictated by the fully true property that matters in graded logical inference, and is what therefore determines the values of logic-based graded properties. Since 0.01-jumps on arbitrarily close arguments are equally (namely, 0.01-) distant from what would be the values of n-Lipschitz functions on arbitrarily close arguments for any $n > 0$, it is just appropriate that the pointwise logic-measured defect of the n-Lipschitz continuity is 0.01 for all $n > 0$. (This remark refers to values in models over standard Łukasiewicz logic, but mutatis mutandis applies to other underlying logics as well, since other left-continuous t-norms just reflect different 'metrics' on [0, 1] by which the closeness is measured.)

The values of Cng_n for the five connectives defined in Example 3.1 are summarized in the following table. The values of Cng_0 in the first column will be explained later in Remark 3.9.

	Cng_0	Cng_1	Cng_2	Cng_3	Cng_4	...
$\mathbf{w}_\mathrm{sq}$	0	0.75	1	1	1	...
$\mathbf{w}_\mathrm{N}$	0.4	1	1	1	1	...
$\mathbf{w}_\mathrm{eps}$	0	1	1	1	1	...
$\mathbf{w}_\mathrm{rd}$	0	0.99	0.99	0.99	0.99	...
$\mathbf{w}_\mathrm{ns}$	0	0.99	0.99	0.99	0.99	...

For an example of non-trivial values of Cng_n, consider, e.g., the composition $\mathbf{w}_\mathrm{sq}\mathbf{w}_\mathrm{sq}$ (i.e., the fourth power of α; see Fig. 2). Since $\mathbf{w}_\mathrm{sq}\mathbf{w}_\mathrm{sq}$ is a 4-Lipschitz function on [0, 1], we have $\mathrm{Cng}_n(\mathbf{w}_\mathrm{sq}\mathbf{w}_\mathrm{sq}) = 1$ for all $n \geq 4$. For $n < 4$, an easy exercise in minimizing polynomials yields $\mathrm{Cng}_n(\mathbf{w}_\mathrm{sq}\mathbf{w}_\mathrm{sq}) = n - 3 \cdot (n/4)^{4/3}$, the infimum in the definition of Cng_n being achieved for $\alpha = (n/4)^{1/3}$ and $\beta = 1$. The numerical values (rounded to three decimal places) are tabulated below:

	Cng_0	Cng_1	Cng_2	Cng_3	Cng_4	Cng_5	...
$\mathbf{w}_\mathrm{sq}\mathbf{w}_\mathrm{sq}$	0	0.528	0.809	0.956	1	1	...

Remark 3.5. The &-multiplicity of the antecedent is not the only possible multiplicity-parameterization of the graded properties: there are several more places in the definitions where the multiplicity parameters can non-trivially be introduced. However, since such parameterizations are less interesting than the one introduced above, we shall not address them in this paper and will only briefly mention them here.

Besides the antecedent equivalence, a meaningful multiplicity parameter can also be introduced for the consequent equivalence in the definition of Cng. Thus we could define[9]:

$$\mathrm{Cng}_{n,m}(\mathbf{u}) \equiv_\mathrm{df} (\forall\alpha\beta)((\alpha \leftrightarrow \beta)^n \to (\mathbf{u}\alpha \leftrightarrow \mathbf{u}\beta)^m).$$

However, since FCT proves $\mathrm{Cng}^m_{\lceil n/m \rceil}(\mathbf{u}) \Rightarrow \mathrm{Cng}_{n,m}(\mathbf{u})$, we can always compensate the multiplicity m in $\mathrm{Cng}_{n,m}(\mathbf{u})$ by taking m copies of $\mathrm{Cng}_{\lceil n/m \rceil}(\mathbf{u})$, and prove (admittedly somewhat weaker) theorems of the form $\mathrm{Cng}^m_{\lceil n/m \rceil}(\mathbf{u}) \Rightarrow \varphi$ instead of $\mathrm{Cng}_{n,m}(\mathbf{u}) \Rightarrow \varphi$.

[9] Observe that in standard Łukasiewicz models of FCT, $\mathrm{Cng}_{n,m}(\mathbf{u}) = 1$ iff $\mathbf{u}$ is n/m-Lipschitz (for $n, m > 0$).

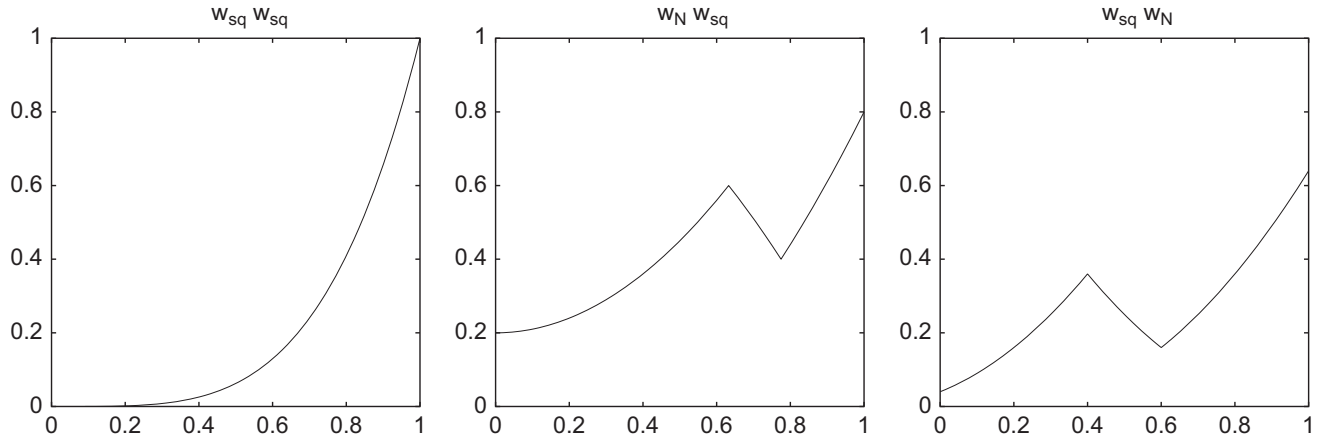

Fig. 2. Graphs of the compositions $\mathbf{w}_{\mathrm{sq}}\mathbf{w}_{\mathrm{sq}}$, $\mathbf{w}_{\mathrm{N}}\mathbf{w}_{\mathrm{sq}}$, and $\mathbf{w}_{\mathrm{sq}}\mathbf{w}_{\mathrm{N}}$ for the connectives $\mathbf{w}_{\mathrm{sq}}$ and $\mathbf{w}_{\mathrm{N}}$ defined in Example 3.1.

Similarly a distinction should be made between $\mathrm{Cng}^n(\mathbf{u})$, i.e., the n-tuple conjunction of $\mathrm{Cng}(\mathbf{u})$, and

$$\mathrm{Cng}^{(n)}(\mathbf{u}) \equiv_{\mathrm{df}} (\forall\alpha\beta)(((\alpha \leftrightarrow \beta) \to (\mathbf{u}\alpha \to \mathbf{u}\beta))^n),$$

where the n-tuple conjunction is within the scope the quantifier, as these are not equivalent in the logic $\mathrm{MTL}_\forall$, the latter being in general weaker than the former. An analogous remark applies to Mon, Ant, and other graded properties of the form $(\forall x)\varphi$. The distinction between $(\forall x)(\varphi^n)$ and $((\forall x)\varphi)^n$ in $\mathrm{MTL}_\forall$ is, however, rather subtle (e.g., in standard models, a difference only occurs in connection with right-discontinuity points of the left-continuous t-norm representing conjunction). Thus even though the weaker premises of the form $(\forall x)(\varphi^n)$ would be sufficient in most theorems proved in this paper (as a careful reader can check), we shall not introduce gradual properties the form $(\forall x)(\varphi^n)$ in order to avoid additional parameters in formulae.

The antecedent-multiplicity parameters can also be introduced in the definitions of MonCng and AntCng (for which see (2)–(3) in the beginning of this section):

$$\mathrm{MonCng}_n(\mathbf{u}) \equiv_{\mathrm{df}} (\forall\alpha\beta)((\alpha \to \beta)^n \to (\mathbf{u}\alpha \to \mathbf{u}\beta))$$
$$\mathrm{AntCng}_n(\mathbf{u}) \equiv_{\mathrm{df}} (\forall\alpha\beta)((\alpha \to \beta)^n \to (\mathbf{u}\beta \to \mathbf{u}\alpha))$$

It turns out that analogously to (4)–(5), these notions can be characterized in terms of Mon, Ant, and Cng_n:

Theorem 3.6. FCT *proves the following graded theorems, for all* $n \in \mathbb{N}$:

(U2) $\mathrm{MonCng}_n(\mathbf{u}) \Leftrightarrow \mathrm{Mon}(\mathbf{u}) \wedge \mathrm{Cng}_n(\mathbf{u})$.
(U3) $\mathrm{AntCng}_n(\mathbf{u}) \Leftrightarrow \mathrm{Ant}(\mathbf{u}) \wedge \mathrm{Cng}_n(\mathbf{u})$.

Thus, $\mathrm{MonCng}_n(\mathbf{u})$ has either the value of $\mathrm{Mon}(\mathbf{u})$ or $\mathrm{Cng}_n(\mathbf{u})$, whichever is smaller. (Analogously for AntCng_n.)

Proof. We shall prove just (U2), the proof of (U3) is analogous.

From left to right: First observe that trivially $\mathrm{MonCng}_n(\mathbf{u}) \Rightarrow \mathrm{Mon}(\mathbf{u})$, as $\triangle(\alpha \to \beta) \Rightarrow (\alpha \to \beta)^n$. Second, from $\mathrm{MonCng}_n(\mathbf{u})$ we get both $(\alpha \leftrightarrow \beta)^n \Rightarrow (\alpha \to \beta)^n \Rightarrow (\mathbf{u}\alpha \to \mathbf{u}\beta)$ and $(\alpha \leftrightarrow \beta)^n \Rightarrow (\beta \to \alpha)^n \Rightarrow (\mathbf{u}\beta \to \mathbf{u}\alpha)$. Thus $(\alpha \leftrightarrow \beta)^n \Rightarrow (\mathbf{u}\alpha \to \mathbf{u}\beta) \wedge (\mathbf{u}\beta \to \mathbf{u}\alpha) \Leftrightarrow (\mathbf{u}\alpha \leftrightarrow \mathbf{u}\beta)$ and the rest is simple.

For the converse direction we take the crisp cases $\alpha \le \beta$ and $\beta \le \alpha$, which are exhaustive due to the prelinearity axiom of MTL. For $\alpha \le \beta$ we obtain $(\alpha \to \beta)^n \Rightarrow \triangle(\alpha \to \beta) \Rightarrow (\mathbf{u}\alpha \to \mathbf{u}\beta)$ by $\mathrm{Mon}(\mathbf{u})$. For $\beta \le \alpha$ we have $(\alpha \to \beta)^n \to (\alpha \leftrightarrow \beta)^n$, and so by $\mathrm{Cng}_n(\mathbf{u})$ we obtain $(\alpha \to \beta)^n \to (\mathbf{u}\alpha \to \mathbf{u}\beta)$. Thus $\mathrm{Mon}(\mathbf{u}) \wedge \mathrm{Cng}_n(\mathbf{u}) \Rightarrow \mathrm{MonCng}_n(\mathbf{u})$. □

The following theorem shows that partly monotone and congruent fuzzy connectives are rather abundant: in particular, any connective $\mathbf{u}$ that has a non-zero degree of $\mathrm{Hgt}(\mathbf{u}) \to \mathrm{Plt}(\mathbf{u})$ has also a non-zero degree of monotony and congruence.

Thus, for instance, in models over standard Łukasiewicz logic, all connectives with a non-zero plinth as well as all connectives with a less than full height have a positive degree of monotony and congruence.

Theorem 3.7. FCT *proves, for all* $n \in \mathbb{N}$ *and* $i = \pm 1$:

(U4) $\mathrm{Hgt}(\mathbf{u}) \to \mathrm{Plt}(\mathbf{u}) \;\Rightarrow\; \mathrm{Cng}_n(\mathbf{u})$.
(U5) $\mathrm{Hgt}(\mathbf{u}) \to \mathrm{Plt}(\mathbf{u}) \;\Rightarrow\; \mathrm{Mon}_i(\mathbf{u})$.

The theorem is an instance of the following more general lemma that will also be useful in later sections:

Lemma 3.8. FCT *proves, for any formula* $\varphi(x, y, z_1, \ldots, z_n)$, $n \geq 0$:

(L1) $\mathrm{Hgt}\, A \to \mathrm{Plt}\, A \;\Leftrightarrow\; (\forall xy)(Ax \to Ay)$
(L2) $\mathrm{Hgt}\, A \to \mathrm{Plt}\, A \;\Leftrightarrow\; (\forall xy)(Ax \leftrightarrow Ay)$
(L3) $\mathrm{Hgt}\, A \to \mathrm{Plt}\, A \;\Rightarrow\; (\forall xyz_1 \ldots z_n)(\varphi(x, y, z_1, \ldots, z_n) \to (Ax \to Ay))$
(L4) $\mathrm{Hgt}\, A \to \mathrm{Plt}\, A \;\Rightarrow\; (\forall xyz_1 \ldots z_n)(\varphi(x, y, z_1, \ldots, z_n) \to (Ax \leftrightarrow Ay))$

Proof.

(L1) By quantifier shifts valid in $\mathrm{MTL}_\forall$, $(\exists x)Ax \to (\forall y)Ay \Leftrightarrow (\forall x)(\forall y)(Ax \to Ay)$.
(L2) The right-to-left direction follows from (L1), since $(\forall xy)(Ax \leftrightarrow Ay) \Rightarrow (\forall xy)(Ax \to Ay)$. Conversely, by specification in (L1) we obtain $\mathrm{Hgt}\, A \to \mathrm{Plt}\, A \Rightarrow Ax \to Ay$ and $\mathrm{Hgt}\, A \to \mathrm{Plt}\, A \Rightarrow Ay \to Ax$. Thus

$$\mathrm{Hgt}\, A \to \mathrm{Plt}\, A \;\Rightarrow\; (Ax \to Ay) \wedge (Ay \to Ax) \;\Leftrightarrow\; (Ax \leftrightarrow Ay).$$

Generalization on x, y and shifting the quantifier to the consequent then concludes the proof.
(L3) By (L1)and weakening we have

$$\mathrm{Hgt}\, A \to \mathrm{Plt}\, A \;\Rightarrow\; Ax \to Ay \;\Rightarrow\; \varphi(x, y, z_1, \ldots, z_n) \to (Ax \to Ay).$$

Generalization on $x, y, z_1, \ldots, z_n$ and shifting all quantifiers to the consequent then concludes the proof.
(L4) The proof is analogous to (L3), just using (L2) instead of (L1). □

Theorem 3.7 is now a direct corollary of the claims (L4) and (L3) for $A = \mathbf{u}$ and $\varphi(\alpha, \beta) \equiv (\alpha \leftrightarrow \beta)^n$ resp. $(\alpha \leq \beta)^i$.

Remark 3.9. Notice that Definition 3.2 did not exclude $n = 0$ in Cng_n, as most theorems proved in this section hold for $n = 0$ as well. In particular, FCT proves that $\mathrm{Cng}_0(\mathbf{u}) \Rightarrow \mathrm{Cng}_n(\mathbf{u})$ for any $n \in \mathbb{N}$, and as a special case of (U10) below we shall obtain $\mathrm{Cng}_0(\mathbf{u}) \Rightarrow \mathrm{Cng}_0(\mathbf{uv})$ and $\mathrm{Cng}(\mathbf{u}), \mathrm{Cng}_0(\mathbf{v}) \Rightarrow \mathrm{Cng}_0(\mathbf{uv})$. Furthermore it is easy to see that $\mathrm{Cng}_0(\underline{\alpha})$ and $\neg\, \mathrm{Cng}_0(\mathbf{id})$.

In fact, the property $\mathrm{Cng}_0(\mathbf{u}) \equiv (\forall \alpha\beta)(\mathbf{u}\alpha \leftrightarrow \mathbf{u}\beta)$ has a rather natural meaning, namely that of a 'degree of constantness': it expresses the (graded) fact that all functional values of $\mathbf{u}\colon \mathrm{L} \to \mathrm{L}$ are close to each other (in the sense of $\leftrightarrow$). Notice, however, that $\mathrm{Cng}_0(\mathbf{u})$ is not the only natural measure of constantness of $\mathbf{u}$, its competitors being, e.g., the properties $(\exists \alpha)(\mathbf{u} \approx \underline{\alpha})$ or $(\exists \alpha)(\mathbf{u} \approxeq \underline{\alpha})$. The graded property $\mathrm{Cng}_0(\mathbf{u})$ can be characterized as the 'difference' (in the sense of $\to$) between the height and plinth of the fuzzy set $\mathbf{u} \sqsubseteq \mathrm{L}$, as by (L1) and (L2) of Lemma 3.8,

$$\mathrm{Cng}_0(\mathbf{u}) \;\Leftrightarrow\; (\forall \alpha\beta)(\mathbf{u}\alpha \to \mathbf{u}\beta) \;\Leftrightarrow\; \mathrm{Hgt}(\mathbf{u}) \to \mathrm{Plt}(\mathbf{u}). \tag{9}$$

The first equivalence of (9) moreover shows that Cng_0 coincides with Mon_0, if the latter is defined by (1) with $i = 0$ (recall that $(\alpha \leq \beta)^0 \equiv 1$ by Convention 2.3). Cf. also the use of $\mathrm{Cng}_0(\mathbf{u})$ in Theorem 3.7, which by (9) is in fact an instance of (U1), and of an analogous property of binary connectives in (C8) and (14)–(15) in sections below.

Example 3.10. Of the connectives defined in Example 3.1, only $\mathbf{w}_\mathrm{N}$ has non-zero plinth and less than full height. In models over standard Łukasiewicz logic, the value $\mathrm{Cng}_0(\mathbf{w}_\mathrm{N}) = (\mathrm{Hgt}(\mathbf{w}_\mathrm{N}) \to \mathrm{Plt}(\mathbf{w}_\mathrm{N})) = (0.8 \to 0.2) = 0.4$ indeed lower-bounds the values $\mathrm{Mon}(\mathbf{w}_\mathrm{N}) = 0.8$, $\mathrm{Ant}(\mathbf{w}_\mathrm{N}) = 0.4$, $\mathrm{Cng}_n(\mathbf{w}_\mathrm{N}) = 1$, and consequently also $\mathrm{MonCng}_n(\mathbf{w}_\mathrm{N}) = 0.8$ and $\mathrm{AntCng}_n(\mathbf{w}_\mathrm{N}) = 0.4$ (for all n). The lower bounds given by Theorem 3.7 obviously need not be tight (though in the case of $\mathrm{Ant}(\mathbf{w}_\mathrm{N}) = 0.4$ they are).

The following theorem shows how the graded properties studied in this section are transmitted to connectives that are similar in the sense of $\approx$ or $\cong$ (see Definition 2.2). Even though the claims of Theorem 3.11 follow from a general metatheorem [17, Theorem 3.5], their direct proofs are given here as the proof of the metatheorem is omitted in [17]; the theorems (U6) and (U7) for $n = 1$ have also been proved in [13]. (Similar remarks apply as well to further theorems on preservation under $\approx$ or $\cong$ in this paper.)

Theorem 3.11. FCT *proves for any* $n \in \mathbb{N}$ *and* $i = \pm 1$:

(U6) $\mathrm{Cng}_n(\mathbf{u}), \mathbf{u} \approx^2 \mathbf{v} \Rightarrow \mathrm{Cng}_n(\mathbf{v})$.
(U7) $\mathrm{Mon}_i(\mathbf{u}), \mathbf{u} \cong \mathbf{v} \Rightarrow \mathrm{Mon}_i(\mathbf{v})$.
(U8) $\mathrm{Hgt}(\mathbf{u}), \mathbf{u} \subseteq \mathbf{v} \Rightarrow \mathrm{Hgt}(\mathbf{v})$.
(U9) $\mathrm{Plt}(\mathbf{u}), \mathbf{u} \subseteq \mathbf{v} \Rightarrow \mathrm{Plt}(\mathbf{v})$.

Proof.
(U6) $(\alpha \leftrightarrow \beta)^n$ implies $\mathbf{u}\alpha \leftrightarrow \mathbf{u}\beta$ by $\mathrm{Cng}_n(\mathbf{u})$, whence $\mathbf{v}\alpha \leftrightarrow \mathbf{v}\beta$ by $\mathbf{u} \approx^2 \mathbf{v}$ (as $\mathbf{u}\alpha \leftrightarrow \mathbf{v}\alpha$ by $\mathbf{u} \approx \mathbf{v}$ and $\mathbf{u}\beta \leftrightarrow \mathbf{v}\beta$ by $\mathbf{u} \approx \mathbf{v}$).
(U7) $\alpha \leq \beta$ (resp. $\beta \leq \alpha$) implies $\mathbf{u}\alpha \to \mathbf{u}\beta$ by $\mathrm{Mon}(\mathbf{u})$, whence $\mathbf{v}\alpha \to \mathbf{u}\beta$ by $\mathbf{v} \subseteq \mathbf{u}$, whence in turn $\mathbf{v}\alpha \to \mathbf{v}\beta$ by $\mathbf{u} \subseteq \mathbf{v}$.
(U8) and (U9) are trivial (cf., e.g., [5, Section 3]). □

Example 3.12. Since obviously $\mathrm{Cng}(\mathbf{id}) = \mathrm{Mon}(\mathbf{id}) = \mathrm{Hgt}(\mathbf{id}) = 1$, Theorem 3.11 guarantees large degrees of these three properties also for functions that are close (in the sense of $\approx$ or $\cong$) to the identity. For instance, it is easy to see that the connectives defined in Example 3.1 have the following values of inclusion or closeness to the identity connective **id**, in models over standard Łukasiewicz logic:

w	$\mathbf{w} \subseteq \mathbf{id}$	$\mathbf{id} \subseteq \mathbf{w}$	$\mathbf{w} \approx \mathbf{id}$	$\mathbf{w} \cong \mathbf{id}$
$\mathbf{w}_{\mathrm{sq}}$	1	0.75	0.75	0.75
$\mathbf{w}_{\mathrm{N}}$	0.8	0.8	0.8	0.6
$\mathbf{w}_{\mathrm{eps}}$	0.99	1	0.99	0.99
$\mathbf{w}_{\mathrm{rd}}$	0.995	0.995	0.995	0.99
$\mathbf{w}_{\mathrm{ns}}$	0.995	0.995	0.995	0.99

Recall from [11,9] that FCT proves:

$$A \approx^2 B \Rightarrow A \cong B \Rightarrow A \approx B. \tag{10}$$

As seen from the above table, over standard Łukasiewicz logic, $(\mathbf{w} \cong \mathbf{id}) = (\mathbf{w} \approx \mathbf{id})$ for $\mathbf{w} \in \{\mathbf{w}_{\mathrm{sq}}, \mathbf{w}_{\mathrm{eps}}\}$ and $(\mathbf{w} \cong \mathbf{id}) = (\mathbf{w} \approx^2 \mathbf{id})$ for $\mathbf{w} \in \{\mathbf{w}_{\mathrm{N}}, \mathbf{w}_{\mathrm{rd}}, \mathbf{w}_{\mathrm{ns}}\}$; for an example of the value of $\mathbf{w} \cong \mathbf{id}$ different from both $\mathbf{w} \approx^2 \mathbf{id}$ and $\mathbf{w} \approx \mathbf{id}$ consider, e.g., the function $(\mathbf{w}_{\mathrm{ns}} + 0.005) \wedge 1$ (i.e., a random noise added to **id** with the positive amplitude 0.015 and the negative amplitude 0.005).

By the above table, (U6) ensures that $\mathrm{Cng}(\mathbf{w}_{\mathrm{sq}}) \geq 0.75$ & $0.75 = 0.5$ in standard Łukasiewicz logic. This estimate is not tight, as actually $\mathrm{Cng}(\mathbf{w}_{\mathrm{sq}}) = 0.75$ (cf. the table at the end of Example 3.1). The exponent in (U6), nevertheless, cannot in general be discarded, as for instance $\mathrm{Cng}(\mathbf{w}_{\mathrm{rd}}) = 0.99 = (\mathbf{w}_{\mathrm{rd}} \approx \mathbf{id})^2$, which is the value guaranteed by (U6). Similarly (U7) guarantees $\mathrm{Mon}(\mathbf{w}_{\mathrm{rd}}) \geq 0.99$ (though actually $\mathrm{Mon}(\mathbf{w}_{\mathrm{rd}}) = 1$, as $\mathbf{w}_{\mathrm{rd}}$ is fully monotone), but the estimate is tight for $\mathrm{Mon}(\mathbf{w}_{\mathrm{ns}}) = 0.99 = (\mathbf{w}_{\mathrm{rd}} \cong \mathbf{id})$. Finally, since $(\mathbf{w}_{\mathrm{N}} \supseteq \mathbf{id}) = 0.8$, theorem (U8) ensures $\mathrm{Hgt}(\mathbf{w}_{\mathrm{N}}) \geq 0.8$; here again, the estimate is tight (though it is not, e.g., for $\mathbf{w}_{\mathrm{rd}}$, which has full height, but $\mathbf{w}_{\mathrm{rd}} \supseteq \mathbf{id}$ only equals 0.995).

Further we shall show how the graded properties of unary connectives are preserved under compositions.

Theorem 3.13. FCT *proves for any* $m, n \in \mathbb{N}$ *and* $i, j = \pm 1$:

(U10) $\mathrm{Cng}_m(\mathbf{u}), \mathrm{Cng}_n^m(\mathbf{v}) \Rightarrow \mathrm{Cng}_{m \cdot n}(\mathbf{uv})$.
(U11) $\mathrm{Mon}_i(\mathbf{u}), \triangle \mathrm{Mon}_j(\mathbf{v}) \Rightarrow \mathrm{Mon}_{i \cdot j}(\mathbf{uv})$.
(U12) $\mathbf{id} \subseteq \mathbf{u}, \mathrm{Hgt}(\mathbf{v}) \Rightarrow \mathrm{Hgt}(\mathbf{uv})$.
(U13) $\mathbf{id} \subseteq \mathbf{u}, \mathrm{Plt}(\mathbf{v}) \Rightarrow \mathrm{Plt}(\mathbf{uv})$.
(U14) $\mathrm{Hgt}(\mathbf{u}), \mathrm{Cng}_n(\mathbf{u}), \mathrm{Surj}_n(\mathbf{v}) \Rightarrow \mathrm{Hgt}(\mathbf{uv})$, *where* $\mathrm{Surj}_n(\mathbf{v}) \equiv_{\mathrm{df}} (\forall\beta)(\exists\alpha)(\beta \leftrightarrow^n \mathbf{v}\alpha)$.

(U15) $\mathrm{Plt}(\mathbf{u}) \;\Rightarrow\; \mathrm{Plt}(\mathbf{uv})$.
(U16) $\mathrm{Mon}_i(\mathbf{u}) \wedge \mathrm{Cng}_n(\mathbf{u}), \mathbf{v}_1 \subseteq^{i\cdot n} \mathbf{v}_2 \;\Rightarrow\; \mathbf{uv}_1 \subseteq \mathbf{uv}_2$.
(U17) $\mathbf{u}_1 \subseteq \mathbf{u}_2 \;\Rightarrow\; \mathbf{u}_1\mathbf{v} \subseteq \mathbf{u}_2\mathbf{v}$.
(U18) $\mathrm{Cng}_n(\mathbf{u}), \mathbf{v}_1 \approx^n \mathbf{v}_2 \;\Rightarrow\; \mathbf{uv}_1 \approx \mathbf{uv}_2$.
(U19) $\mathbf{u}_1 \approx \mathbf{u}_2 \;\Rightarrow\; \mathbf{u}_1\mathbf{v} \approx \mathbf{u}_2\mathbf{v}$.

Proof.

(U10) By $\mathrm{Cng}^m_n(\mathbf{v})$ we obtain $(\alpha \leftrightarrow \beta)^{m\cdot n} \to (\mathbf{v}\alpha \leftrightarrow \mathbf{v}\beta)^m$ and by $\mathrm{Cng}_m(\mathbf{u})$ we obtain $(\mathbf{v}\alpha \leftrightarrow \mathbf{v}\beta)^m \to (\mathbf{uv}\alpha \to \mathbf{uv}\beta)$. Transitivity of implication and generalization then conclude the proof.

(U11) We shall only prove the case for $i = j = -1$, i.e., $\mathrm{Ant}(\mathbf{u}), \triangle\,\mathrm{Ant}(\mathbf{v}) \;\Rightarrow\; \mathrm{Mon}(\mathbf{uv})$; the other cases are analogous. By $\triangle\,\mathrm{Ant}(\mathbf{v})$ we obtain $(\alpha \le \beta) \to (\mathbf{v}\beta \to \mathbf{v}\alpha)$ and by $\mathrm{Ant}(\mathbf{u})$ we obtain $(\mathbf{v}\beta \to \mathbf{v}\alpha) \to (\mathbf{uv}\alpha \to \mathbf{uv}\beta)$. Then proceed as in (U10).

The counterexamples showing that the $\triangle$ in (U11) cannot be avoided are easy to find (as the definitions of Mon and Ant say nothing about the relation between $\mathbf{u}\alpha$ and $\mathbf{u}\beta$ if $\alpha \not\le \beta$).

(U12) By $\mathbf{id} \subseteq \mathbf{u}$ we have $\mathbf{v}\alpha \to \mathbf{uv}\alpha$, whence by generalization $(\forall\alpha)(\mathbf{v}\alpha \to \mathbf{uv}\alpha)$, which in $\mathrm{MTL}_\forall$ implies $(\exists\alpha)\mathbf{v}\alpha \to (\exists\alpha)\mathbf{uv}\alpha$.

(U13) Proceed as in the proof of (U12), only use the quantifier distribution $(\forall\alpha)\mathbf{v}\alpha \to (\forall\alpha)\mathbf{uv}\alpha$ in the last step.

(U14) We shall first sketch an informal idea of the proof. By the graded assumption of n-surjectivity of $\mathbf{v}$ (cf. Remark 3.14 below), $\mathbf{v}$ has values close to the arguments that matter for $\mathrm{Hgt}(\mathbf{u})$, and by $\mathrm{Cng}_n(\mathbf{u})$ the values of $\mathbf{u}$ for arguments near (in the sense of $\leftrightarrow^n$) the spot cannot differ too much (in the sense of $\leftrightarrow$); thus also $\mathrm{Hgt}(\mathbf{uv})$ cannot be much lower.

We shall now give a formal proof, in two steps: first we shall show that the premises of (U14) imply $(\exists\beta)(\mathbf{u}\beta \,\&\, (\exists\alpha)(\mathbf{u}\beta \leftrightarrow \mathbf{uv}\alpha))$; then we shall show that the latter formula implies $\mathrm{Hgt}(\mathbf{uv})$.

The first step: By $\mathrm{Cng}_n(\mathbf{u})$ we obtain $(\beta \leftrightarrow^n \mathbf{v}\alpha) \to (\mathbf{u}\beta \leftrightarrow \mathbf{uv}\alpha)$. By generalization and distribution of $\forall$ (as $\exists$), we obtain

$$(\exists\alpha)(\beta \leftrightarrow^n \mathbf{v}\alpha) \to (\exists\alpha)(\mathbf{u}\beta \leftrightarrow \mathbf{uv}\alpha).$$

Another generalization and distribution of $\forall$ (now as $\forall$) yields

$$(\forall\beta)(\exists\alpha)(\beta \leftrightarrow^n \mathbf{v}\alpha) \to (\forall\beta)(\exists\alpha)(\mathbf{u}\beta \leftrightarrow \mathbf{uv}\alpha).$$

As the antecedent of the latter implication is $\mathrm{Surj}_n(\mathbf{v})$, by adding $\mathrm{Hgt}(\mathbf{u})$ to both its antecedent and consequent we obtain

$$\mathrm{Hgt}(\mathbf{u}), \mathrm{Cng}_n(\mathbf{u}), \mathrm{Surj}_n(\mathbf{v}) \;\Rightarrow\; (\exists\beta)\mathbf{u}\beta \,\&\, (\forall\beta)(\exists\alpha)(\mathbf{u}\beta \leftrightarrow \mathbf{uv}\alpha).$$

Now using the theorem $(\exists x)\varphi \,\&\, (\forall x)\psi \to (\exists x)(\varphi \,\&\, \psi)$ of $\mathrm{MTL}_\forall$ we obtain

$$\mathrm{Hgt}(\mathbf{u}), \mathrm{Cng}_n(\mathbf{u}), \mathrm{Surj}_n(\mathbf{v}) \;\Rightarrow\; (\exists\beta)(\mathbf{u}\beta \,\&\, (\exists\alpha)(\mathbf{u}\beta \leftrightarrow \mathbf{uv}\alpha)),$$

which concludes the first step of the proof.

In the second step we shall show that the latter consequent $(\exists\beta)(\mathbf{u}\beta \,\&\, (\exists\alpha)(\mathbf{u}\beta \leftrightarrow \mathbf{uv}\alpha))$ implies $\mathrm{Hgt}(\mathbf{uv})$:

From the trivial $(\mathbf{u}\beta \leftrightarrow \mathbf{uv}\alpha) \to (\mathbf{u}\beta \to \mathbf{uv}\alpha)$ we obtain $\mathbf{u}\beta \to ((\mathbf{u}\beta \leftrightarrow \mathbf{uv}\alpha) \to \mathbf{uv}\alpha)$ by exchange, whence

$$\mathbf{u}\beta \to ((\exists\alpha)(\mathbf{u}\beta \leftrightarrow \mathbf{uv}\alpha) \to (\exists\alpha)\mathbf{uv}\alpha)$$

by generalization and a quantifier shift and distribution (as $\exists$), whence

$$\mathbf{u}\beta \,\&\, (\exists\alpha)(\mathbf{u}\beta \leftrightarrow \mathbf{uv}\alpha) \to (\exists\alpha)\mathbf{uv}\alpha$$

by residuation, whence

$$(\exists\beta)(\mathbf{u}\beta \,\&\, (\exists\alpha)(\mathbf{u}\beta \leftrightarrow \mathbf{uv}\alpha)) \to (\exists\alpha)\mathbf{uv}\alpha$$

by generalization and shifting the quantifier to the antecedent (as $\exists$), which is the required implication from the consequent of the first step to $\mathrm{Hgt}(\mathbf{uv})$.

(U15) $(\forall\alpha)\mathbf{u}\alpha \to \mathbf{u}\mathbf{v}\alpha$ by specification; generalization and a quantifier shift complete the proof.
(U16) The proof follows from the fact that by (U2)–(U3),

$$\mathrm{Mon}_i(\mathbf{u}) \wedge \mathrm{Cng}_n(\mathbf{u}) \;\Rightarrow\; (\mathbf{v}_1\alpha \to^{i\cdot n} \mathbf{v}_2\alpha) \to (\mathbf{u}\mathbf{v}_1\alpha \to \mathbf{u}\mathbf{v}_2\alpha).$$

(U17) Specify $\mathbf{v}\alpha$ for γ in the definition $(\forall\gamma)(\mathbf{u}_1\gamma \to \mathbf{u}_2\gamma)$ of $\mathbf{u}_1 \subseteq \mathbf{u}_2$.
(U18) By $\mathrm{Cng}_n(\mathbf{u})$ we obtain that $(\mathbf{v}_1\alpha \leftrightarrow \mathbf{v}_2\alpha)^n$ implies $\mathbf{u}\mathbf{v}_1\alpha \leftrightarrow \mathbf{u}\mathbf{v}_2\alpha$.
(U19) The claim follows from (U17) and the fact that $A \approx B \Leftrightarrow (A \subseteq B) \wedge (B \subseteq A)$. □

Remark 3.14. The premise $\mathrm{Surj}_n(\mathbf{u}) \equiv (\forall\beta)(\exists\alpha)(\beta \leftrightarrow \mathbf{v}\alpha)$ in (U14) can be understood as a parameterized graded version of the surjectivity of $\mathbf{v}$, as it just replaces $=$ by $\leftrightarrow^n$ in the definition of crisp surjectivity $(\forall\beta)(\exists\alpha)(\beta = \mathbf{v}\alpha)$. Observe, however, that the crisp surjectivity of $\mathbf{v}$ is not equivalent to $\triangle\,\mathrm{Surj}_n(\mathbf{v})$, since $\triangle(\exists x)\varphi$ is not equivalent to $(\exists x)\triangle\varphi$ in $\mathrm{MTL}_{\triangle\forall}$: the difference is analogous to that between a fuzzy set having the height 1 and being normal (i.e., having non-empty kernel). For instance, in models over standard Łukasiewicz logic, $\mathrm{Surj}_n(\mathbf{u}) = 1$ even if the range of $\mathbf{u}$ is not equal to, but just dense in $[0, 1]$.

Example 3.15. Let us use (U10) to estimate $\mathrm{Cng}_n(\mathbf{w}_{\mathrm{sq}}\mathbf{w}_{\mathrm{sq}})$, whose exact values in models over standard Łukasiewicz logic we already know from Example 3.4:

- For $m = n = 1$, theorem (U10) yields $\mathrm{Cng}(\mathbf{w}_{\mathrm{sq}}), \mathrm{Cng}(\mathbf{w}_{\mathrm{sq}}) \Rightarrow \mathrm{Cng}(\mathbf{w}_{\mathrm{sq}}\mathbf{w}_{\mathrm{sq}})$, i.e., $\mathrm{Cng}(\mathbf{w}_{\mathrm{sq}}\mathbf{w}_{\mathrm{sq}}) \geq \mathrm{Cng}^2(\mathbf{w}_{\mathrm{sq}})$. Since $\mathrm{Cng}(\mathbf{w}_{\mathrm{sq}}) = 0.75$ (by Example 3.1), we obtain the estimate $\mathrm{Cng}(\mathbf{w}_{\mathrm{sq}}\mathbf{w}_{\mathrm{sq}}) \geq 0.5$. As in fact $\mathrm{Cng}(\mathbf{w}_{\mathrm{sq}}\mathbf{w}_{\mathrm{sq}}) \doteq 0.528$ (see Example 3.4), the theorem gives a rather tight estimate in this case, even if not the exact value.
- For $m = 1$ and $n = 2$, theorem (U10) yields $\mathrm{Cng}(\mathbf{w}_{\mathrm{sq}}), \mathrm{Cng}_2(\mathbf{w}_{\mathrm{sq}}) \Rightarrow \mathrm{Cng}_2(\mathbf{w}_{\mathrm{sq}}\mathbf{w}_{\mathrm{sq}})$. Since $\mathrm{Cng}(\mathbf{w}_{\mathrm{sq}}) = 0.75$ and $\mathrm{Cng}_2(\mathbf{w}_{\mathrm{sq}}) = 1$, we obtain $\mathrm{Cng}_2(\mathbf{w}_{\mathrm{sq}}\mathbf{w}_{\mathrm{sq}}) \geq 0.75$. In fact, as we know from Example 3.4, $\mathrm{Cng}_2(\mathbf{w}_{\mathrm{sq}}\mathbf{w}_{\mathrm{sq}}) \doteq 0.809$, thus the estimate given by the theorem is still fairly good. Notice that if we used (U10) with $m = 2$ and $n = 1$, we would only obtain a rather weak estimate $\mathrm{Cng}_2(\mathbf{w}_{\mathrm{sq}}\mathbf{w}_{\mathrm{sq}}) \geq 0.5$.
- Since $\mathrm{Cng}_2(\mathbf{w}_{\mathrm{sq}}) = 1$, the theorem for $m = n = 2$ yields the exact value $\mathrm{Cng}_4(\mathbf{w}_{\mathrm{sq}}\mathbf{w}_{\mathrm{sq}}) = 1$. In general, for full degrees the theorem (U10) reads $\triangle\,\mathrm{Cng}_m(\mathbf{u}), \triangle\,\mathrm{Cng}_n(\mathbf{v}) \Rightarrow \triangle\,\mathrm{Cng}_{m\cdot n}(\mathbf{u}\mathbf{v})$, which in models over standard Łukasiewicz logic expresses the well-known fact that the composition of an m-Lipschitz function with an n-Lipschitz one is $(m \cdot n)$-Lipschitz.

For the composition $\mathbf{w}_{\mathrm{N}}\mathbf{w}_{\mathrm{sq}}$ (see Fig. 2), Theorem 3.13 yields the following estimates:

- Since $\mathrm{Cng}(\mathbf{w}_{\mathrm{N}}) = 1$ (by Example 3.1), theorem (U10) yields $\mathrm{Cng}_n(\mathbf{w}_{\mathrm{sq}}) \Rightarrow \mathrm{Cng}_n(\mathbf{w}_{\mathrm{N}}\mathbf{w}_{\mathrm{sq}})$. Thus from the known values (see Example 3.4) of $\mathrm{Cng}_n(\mathbf{w}_{\mathrm{sq}})$ we obtain the estimates $\mathrm{Cng}(\mathbf{w}_{\mathrm{N}}\mathbf{w}_{\mathrm{sq}}) \geq 0.528$, $\mathrm{Cng}_2(\mathbf{w}_{\mathrm{N}}\mathbf{w}_{\mathrm{sq}}) \geq 0.809$, $\mathrm{Cng}_3(\mathbf{w}_{\mathrm{N}}\mathbf{w}_{\mathrm{sq}}) \geq 0.956$, and $\mathrm{Cng}_n(\mathbf{w}_{\mathrm{N}}\mathbf{w}_{\mathrm{sq}}) = 1$ for $n \geq 4$. Except for the latter, these estimates are not too tight, as it can be easily seen that the actual value of $\mathrm{Cng}(\mathbf{w}_{\mathrm{N}}\mathbf{w}_{\mathrm{sq}})$ is $1.6 - \sqrt{0.6} \doteq 0.825$ and $\mathrm{Cng}_2(\mathbf{w}_{\mathrm{N}}\mathbf{w}_{\mathrm{sq}}) = 1$.
- As $\mathbf{w}_{\mathrm{sq}}$ is fully monotone, (U11) yields $\mathrm{Mon}(\mathbf{w}_{\mathrm{N}}\mathbf{w}_{\mathrm{sq}}) \geq \mathrm{Mon}(\mathbf{w}_{\mathrm{N}}) = 0.8$ and $\mathrm{Ant}(\mathbf{w}_{\mathrm{N}}\mathbf{w}_{\mathrm{sq}}) \geq \mathrm{Ant}(\mathbf{w}_{\mathrm{N}}) = 0.4$, which happen to be the actual values.
- Since $\mathbf{w}_{\mathrm{sq}}$ has full height and $(\mathbf{id} \subseteq \mathbf{w}_{\mathrm{N}}) = 0.8$ (by Example 3.12), theorem (U12) yields $\mathrm{Hgt}(\mathbf{w}_{\mathrm{N}}\mathbf{w}_{\mathrm{sq}}) \geq 0.8$, which is its actual value. The same tight estimate can be obtained from (U14), as obviously $\mathrm{Surj}(\mathbf{w}_{\mathrm{sq}}) = 1$ in models over standard Łukasiewicz logic. Moreover, by (U15) we obtain $\mathrm{Plt}(\mathbf{w}_{\mathrm{N}}\mathbf{w}_{\mathrm{sq}}) \geq \mathrm{Plt}(\mathbf{w}_{\mathrm{N}}) = 0.2$, which is the actual value as well.

Finally, let us also compute estimates for the converse composition $\mathbf{w}_{\mathrm{sq}}\mathbf{w}_{\mathrm{N}}$ (see Fig. 2 for the graph):

- Since $\mathrm{Cng}(\mathbf{w}_{\mathrm{N}}) = 1$, theorem (U10) yields $\mathrm{Cng}_n(\mathbf{w}_{\mathrm{sq}}) \Rightarrow \mathrm{Cng}_n(\mathbf{w}_{\mathrm{sq}}\mathbf{w}_{\mathrm{N}})$, so we obtain the same estimates for $\mathrm{Cng}_n(\mathbf{w}_{\mathrm{sq}}\mathbf{w}_{\mathrm{N}})$ as we did for $\mathbf{w}_{\mathrm{N}}\mathbf{w}_{\mathrm{sq}}$. (The actual value of $\mathrm{Cng}(\mathbf{w}_{\mathrm{sq}}\mathbf{w}_{\mathrm{N}})$ is 0.91, so in this case the lower bound is even looser than in the former.)
- Since $(\mathbf{id} \subseteq \mathbf{w}_{\mathrm{sq}}) = 0.75$ and $\mathrm{Hgt}(\mathbf{w}_{\mathrm{N}}) = 0.8$, theorem (U12) yields $\mathrm{Hgt}(\mathbf{w}_{\mathrm{sq}}\mathbf{w}_{\mathrm{N}}) \geq 0.55$. The estimate is not tight (the actual value being 0.64), since the maximal difference between $\mathbf{w}_{\mathrm{sq}}$ and $\mathbf{id}$ occurs at a different value (namely, $\alpha = 0.5$) than where the maximum height is achieved (which is $\alpha = 1$). The same weak estimate is yielded by (U14) for $n = 1$, as $\mathrm{Hgt}(\mathbf{w}_{\mathrm{sq}}) = 1$, $\mathrm{Cng}(\mathbf{w}_{\mathrm{sq}}) = 0.75$, and $\mathrm{Surj}(\mathbf{w}_{\mathrm{N}}) = 0.8$ (the worst cases of β in the definition of $\mathrm{Surj}(\mathbf{w}_{\mathrm{N}})$ are obviously 0 and 1, for which the closest $\mathbf{w}_{\mathrm{N}}\alpha$ are 0.2-distant). A better estimate $\mathrm{Hgt}(\mathbf{w}_{\mathrm{sq}}\mathbf{w}_{\mathrm{N}}) \geq 0.6$ is obtained by (U14) for $n = 2$, as $\mathrm{Cng}_2(\mathbf{w}_{\mathrm{sq}}) = \mathrm{Hgt}(\mathbf{w}_{\mathrm{sq}}) = 1$ and $\mathrm{Surj}_2(\mathbf{w}_{\mathrm{N}}) = 0.8 \,\&\, 0.8 = 0.6$ in standard Łukasiewicz logic.

- (U11) only yields the useless estimate $\mathrm{Mon}(\mathbf{w}_{\mathrm{sq}}\mathbf{w}_{\mathrm{N}}) \geq 0$. Some estimate of the monotony degree can, nevertheless, be obtained by other theorems. For instance, by (U19) we obtain $(\mathbf{w}_{\mathrm{sq}}\mathbf{w}_{\mathrm{N}} \approx \mathbf{w}_{\mathrm{N}}) \geq 0.75$, whence by (10) we have $(\mathbf{w}_{\mathrm{sq}}\mathbf{w}_{\mathrm{N}} \mathbin{\tilde{\approx}} \mathbf{w}_{\mathrm{N}}) \geq (\mathbf{w}_{\mathrm{sq}}\mathbf{w}_{\mathrm{N}} \approx \mathbf{w}_{\mathrm{N}})^2 = 0.5$. Since $\mathrm{Mon}(\mathbf{w}_{\mathrm{N}}) = 0.8$, by (U7) we thus obtain $\mathrm{Mon}(\mathbf{w}_{\mathrm{sq}}\mathbf{w}_{\mathrm{N}}) \geq 0.3$. This is still a very loose bound, as the actual value of $\mathrm{Mon}(\mathbf{w}_{\mathrm{sq}}\mathbf{w}_{\mathrm{N}})$ is 0.8, caused by the drop of $\mathbf{w}_{\mathrm{sq}}\mathbf{w}_{\mathrm{N}}(\alpha)$ from 0.36 to 0.16 in the interval $[0.4, 0.6]$. The useless estimate $\mathrm{Plt}(\mathbf{w}_{\mathrm{sq}}\mathbf{w}_{\mathrm{N}}) \geq 0$ yielded by (U13), on the other hand, cannot be much improved, as the actual value is only 0.04.

Remark 3.16. It can be seen from Example 3.15 that theorems presented in this paper may sometimes give only very loose bounds for particular connectives. The role of the theorems, however, resides in their generality, i.e., the fact that they provide uniform estimates for *all* possible connectives (including the 'worst cases'). For particular connectives we are often able to compute exact values of graded properties by semantic calculations simply from their definitions. However, in cases where a direct calculation is too difficult (e.g., in the logic of some non-trivial left-continuous t-norm) or not at all possible (e.g., because the connective is insufficiently specified), general theorems may provide useful bounds. It could also be observed in Example 3.15 that different theorems may give us alternative ways of calculating the estimates of the same graded property, some of which may yield better results than others for particular fuzzy connectives.

Remark 3.17. The tight estimates yielded by some theorems for some connectives demonstrate that these theorems cannot be substantially strengthened (without restricting their range to only some connectives). A systematic search for such examples documenting the maximal strength of theorems is, nevertheless, omitted here, as it would clutter the paper with tedious semantic (counter)examples. The maximal strength of theorems is anyway important from the theoretical point of view only, as even maximally strong general theorems may still yield suboptimal bounds for *particular* connectives (cf. Remark 3.16). The optimality of a uniform theorem has therefore only a limited utility in practice: if we investigate a particular connective, then a suboptimally strong theorem may still give a better estimate for this connective (and perhaps worse for other connectives we are not interested in) than a maximally strong theorem that is less suited to this particular connective.

We shall now look at how basic unary connectives ($\neg$, $\triangle$, $\mathbf{id}^n$, $\underline{\gamma}$) satisfy the graded properties defined in this section. The following observations show the provability in FCT of known crisp properties of the connectives, plus a few additional graded ones:

Theorem 3.18. FCT *proves*, *for any* $n \geq 0$, $m \geq 1$, *and* $i = \pm 1$:

(U20) $\mathrm{Cng}(\neg)$, $\neg\,\mathrm{Mon}(\neg)$, $\mathrm{Ant}(\neg)$, $\mathrm{Hgt}(\neg)$, $\neg\,\mathrm{Plt}(\neg)$.
(U21) $\mathrm{Cng}_n(\mathbf{id}^n)$, $\mathrm{Mon}(\mathbf{id}^n)$, $\neg\,\mathrm{Ant}(\mathbf{id}^m)$, $\mathrm{Hgt}(\mathbf{id}^n)$, $\neg\,\mathrm{Plt}(\mathbf{id}^m)$.
(U22) $\mathrm{Cng}(\underline{\gamma})$, $\mathrm{Mon}(\underline{\gamma})$, $\mathrm{Ant}(\underline{\gamma})$.
(U23) $\mathrm{Hgt}(\underline{\gamma}) \Leftrightarrow \mathrm{Plt}(\underline{\gamma}) \Leftrightarrow \underline{\gamma}$.
(U24) $\mathrm{Mon}(\overline{\triangle})$, $\neg\,\mathrm{Ant}(\overline{\triangle})$, $\mathrm{Hgt}(\triangle)$, $\neg\,\mathrm{Plt}(\triangle)$.
(U25) $\mathbf{id} \subseteq^2 \triangle \Rightarrow \mathrm{Cng}_m(\triangle) \Rightarrow \mathbf{id}^m \subseteq \triangle$.
(U26) $(\triangle \subseteq \mathbf{id}^n)$, $\neg(\neg \subseteq^i \triangle)$, $\neg(\neg \subseteq^i \mathbf{id}^m)$, $(\underline{\gamma} \subseteq \underline{\delta}) \leftrightarrow (\gamma \to \delta)$.
(U27) $(\mathbf{id}^m \subseteq \underline{\gamma}) \leftrightarrow \gamma$, $(\underline{\gamma} \subseteq \mathbf{id}^m) \leftrightarrow \neg\gamma$; *consequently*, $(\mathbf{id}^m \approx \underline{\gamma}) \leftrightarrow \gamma \wedge \neg\gamma$ *and* $\neg(\mathbf{id}^m \mathbin{\tilde{\approx}} \underline{\gamma})$.

Proof. As the proofs are easy and similar to one another, we shall only give some typical examples:

- $\neg\,\mathrm{Ant}(\mathbf{id}^m)$ is obtained by specifying 0 for α and 1 for β in the definition of graded antitony.
- $\mathrm{Ant}(\neg)$ follows by generalization from the provability in $\mathrm{MTL}_\triangle$ of $\triangle(\beta \to \alpha) \to (\neg\alpha \to \neg\beta)$.
- For the proof of $\mathbf{id} \subseteq^2 \triangle \Rightarrow \mathrm{Cng}(\triangle)$ we employ the fact that $(\mathbf{id} \subseteq \triangle) \leftrightarrow (\mathbf{id} \mathbin{\tilde{\approx}} \triangle)$, since $\triangle \subseteq \mathbf{id}$ is a theorem. Thus $(\mathbf{id} \subseteq^2 \triangle) \to (\alpha \leftrightarrow \triangle\alpha)$ & $(\beta \leftrightarrow \triangle\beta)$, whence the required $(\mathbf{id} \subseteq^2 \triangle)$ & $(\alpha \leftrightarrow \beta) \to (\triangle\alpha \leftrightarrow \triangle\beta)$ is obtained by the transitivity of $\leftrightarrow$.
- $(\underline{\gamma} \subseteq \mathbf{id}^m) \leftrightarrow \neg\gamma$, i.e., $(\forall\alpha)(\gamma \to \alpha^m) \leftrightarrow (\gamma \to 0)$, is obtained from left to right by specifying $\alpha = 0$ and from right to left by generalization of the propositional theorem $(\gamma \to 0) \to (\gamma \to \alpha^m)$.

Proofs of the other claims are analogous or even simpler. □

Notice that the truth value of $\mathbf{id} \subseteq \triangle$ varies across particular models of FCT and is not fixed by the theory. The value of $\mathbf{id} \subseteq \triangle$ is in fact an important characteristic of the class L of inner truth values in a given model: roughly speaking, it indicates the degree to which 1 is isolated from the lesser values.[10] Thus, e.g., in crisp models (with $\mathrm{L} = \{0, 1\}$) it is true to degree 1, in Zadeh models (which have $\mathrm{L} = [0, 1]$) to degree 0, and in n-valued Łukasiewicz models to degree $1/(n-1)$. Similarly varied and depending on the structure of L in the model are the values of $\mathbf{id}^m \subseteq \triangle$ and $\mathrm{Cng}_m(\triangle)$.

The following corollaries of Theorems 3.13 and 3.18 show how graded properties are transmitted by basic unary connectives (only the interesting cases are listed):

Corollary 3.19. FCT *proves for all* $n, k \in \mathbb{N}$ *and* $m \geq n$:

(U28) $\mathrm{Cng}_n(\mathbf{v}) \;\Rightarrow\; \mathrm{Cng}_n(\neg\mathbf{v})$.
(U29) $\mathrm{Cng}^m_k(\mathbf{v}) \;\Rightarrow\; \mathrm{Cng}_{m\cdot k}(\mathbf{v}^n)$; *in particular*, $\mathrm{Cng}^m(\mathbf{v}) \Rightarrow \mathrm{Cng}_m(\mathbf{v}^n)$, *and* $\mathrm{Cng}^n(\mathbf{v}) \Rightarrow \mathrm{Cng}_n(\mathbf{v}^n)$.

Finally, we shall investigate the transmission of graded properties to slightly distorted unary connectives (cf. Section 1). To slightly distort a fuzzy connective $\mathbf{u}\colon \mathrm{L} \to \mathrm{L}$, e.g., by noise or rounding, is to compose it with a function $\mathbf{v}\colon \mathrm{L} \to \mathrm{L}$ that is very close (in the sense of $\approx$)[11] to the identity function **id**. In particular, if $\mathbf{v} \approx \mathbf{id}$ to a large degree, then the composition **uv** represents an application of a fuzzy connective **u** to values rounded or noised by **v**, and the composition **vu** represents rounding or noising (by **v**) the functional values of **u**. (Cf. Fig. 3.)

The following corollaries of Theorems 3.11, 3.13, and 3.18 show how graded properties of fuzzy connectives propagate through compositions with functions close (in the sense of $\approx$) to **id**.

Theorem 3.20. FCT *proves for all* $n \in \mathbb{N}$ *and* $i = \pm 1$:

(U30) $\mathbf{v} \approx \mathbf{id} \;\Rightarrow\; \mathbf{vu} \approx \mathbf{u}$.
(U31) $\mathbf{v} \approx^n \mathbf{id}, \mathrm{Cng}_n(\mathbf{u}) \;\Rightarrow\; \mathbf{uv} \approx \mathbf{u}$.
(U32) $\mathrm{Cng}_n(\mathbf{u}), \mathbf{v} \approx^2 \mathbf{id} \;\Rightarrow\; \mathrm{Cng}_n(\mathbf{vu})$.
(U33) $\mathrm{Cng}_n(\mathbf{u}), \mathbf{v} \approx^{2n} \mathbf{id} \;\Rightarrow\; \mathrm{Cng}_n(\mathbf{uv})$.
(U34) $\mathrm{Mon}_i(\mathbf{u}), \mathbf{v} \cong \mathbf{id} \;\Rightarrow\; \mathrm{Mon}_i(\mathbf{vu})$.
(U35) $\mathrm{Mon}_i(\mathbf{u}), \mathrm{Cng}^2_n(\mathbf{u}), \mathbf{v} \approx^{2n} \mathbf{id} \;\Rightarrow\; \mathrm{Mon}_i(\mathbf{uv})$.
(U36) $\mathrm{Hgt}(\mathbf{u}), \mathrm{Cng}_n(\mathbf{u}), \mathbf{v} \approx^n \mathbf{id} \;\Rightarrow\; \mathrm{Hgt}(\mathbf{uv})$.

Proof. (U30)–(U31) follow directly from (U18)–(U19) and the fact that $\mathbf{u}\,\mathbf{id} = \mathbf{id}\,\mathbf{u} = \mathbf{u}$.

(U32) We obtain $\mathbf{v} \approx^2 \mathbf{id} \Rightarrow \mathbf{u} \approx^2 \mathbf{vu}$ by (U30) and $\mathbf{u} \approx^2 \mathbf{vu}, \mathrm{Cng}_n(\mathbf{u}) \Rightarrow \mathrm{Cng}_n(\mathbf{vu})$ by (U6).
(U33) By (U21) we have $\mathrm{Cng}(\mathbf{id})$, thus $\mathbf{v} \approx^{2n} \mathbf{id} \Rightarrow \mathrm{Cng}^n(\mathbf{v})$ by (U6). The claim then follows from the fact that $\mathrm{Cng}_n(\mathbf{u}), \mathrm{Cng}^n(\mathbf{v}) \Rightarrow \mathrm{Cng}_n(\mathbf{uv})$, which is an instance of (U10).
(U34) We obtain $\mathbf{v} \cong \mathbf{id} \Rightarrow \mathbf{u} \cong \mathbf{vu}$ by (U17) and $\mathbf{u} \cong \mathbf{vu}, \mathrm{Mon}_i(\mathbf{u}) \Rightarrow \mathrm{Mon}_i(\mathbf{vu})$ by (U7).
(U35) We obtain $\mathbf{v} \approx^{2n} \mathbf{id}, \mathrm{Cng}^2_n(\mathbf{u}) \Rightarrow \mathbf{u} \approx^2 \mathbf{uv}$ by (U31); $\mathbf{u} \approx^2 \mathbf{uv} \Rightarrow \mathbf{u} \cong \mathbf{uv}$ by (10); and $\mathbf{u} \cong \mathbf{uv}, \mathrm{Mon}_i(\mathbf{u}) \Rightarrow \mathrm{Mon}_i(\mathbf{uv})$ by (U6).
(U36) The claim follows from (U14) by observing that $\mathbf{v} \approx^n \mathbf{id} \Rightarrow \mathrm{Surj}_n(\mathbf{v})$. The latter follows from a more general observation $\mathbf{u} \approx^n \mathbf{v}, \mathrm{Surj}_n(\mathbf{u}) \Rightarrow \mathrm{Surj}_n(\mathbf{v})$ by the fact that $\mathrm{Surj}_n(\mathbf{id})$ is a theorem of FCT, as is easily proved by taking β for α in the definition of $\mathrm{Surj}_n(\mathbf{id})$. The proof of the lemma $\mathbf{u} \approx^n \mathbf{v}, \mathrm{Surj}_n(\mathbf{u}) \Rightarrow \mathrm{Surj}_n(\mathbf{v})$ runs as follows:

Combining n copies of the MTL-theorem $(\mathbf{u}\alpha \leftrightarrow \mathbf{v}\alpha) \to ((\beta \leftrightarrow \mathbf{u}\alpha) \to (\beta \leftrightarrow \mathbf{v}\alpha))$, we obtain:

$$\begin{aligned}
&(\forall\alpha)((\mathbf{u}\alpha \leftrightarrow^n \mathbf{v}\alpha) \to ((\beta \leftrightarrow^n \mathbf{u}\alpha) \to (\beta \leftrightarrow^n \mathbf{v}\alpha)) && \text{by generalization on } \alpha\\
&\Rightarrow\; (\forall\alpha)(\mathbf{u}\alpha \leftrightarrow^n \mathbf{v}\alpha) \to (\forall\alpha)((\beta \leftrightarrow^n \mathbf{u}\alpha) \to (\beta \leftrightarrow^n \mathbf{v}\alpha)) && \text{by distribution of } \forall\\
&\Rightarrow\; (\forall\alpha)(\mathbf{u}\alpha \leftrightarrow^n \mathbf{v}\alpha) \to ((\exists\alpha)(\beta \leftrightarrow^n \mathbf{u}\alpha) \to (\exists\alpha)(\beta \leftrightarrow^n \mathbf{v}\alpha)) && \text{by distribution of } \forall \text{ as } \exists
\end{aligned}$$

Generalization on β and shifting and distributing the quantifier then concludes the proof. □

[10] By considering the crisp cases $\alpha = 1$ and $\alpha < 1$ it can be easily proved that $(\mathbf{id} \subseteq \triangle) \leftrightarrow (\forall\alpha < 1)\neg\alpha$, i.e., $\mathbf{id} \subseteq \triangle$ is equivalent to $\mathrm{Plt}(\neg \setminus \{1\})$.
[11] Since a natural measure of closeness between two values $\alpha, \beta \in \mathrm{L}$ in logic-based fuzzy mathematics is the degree of their logical equivalence $\alpha \leftrightarrow \beta$, a suitable logic-based notion of closeness between two fuzzy connectives is the graded bi-inclusion $\approx$ (see Definition 2.2).

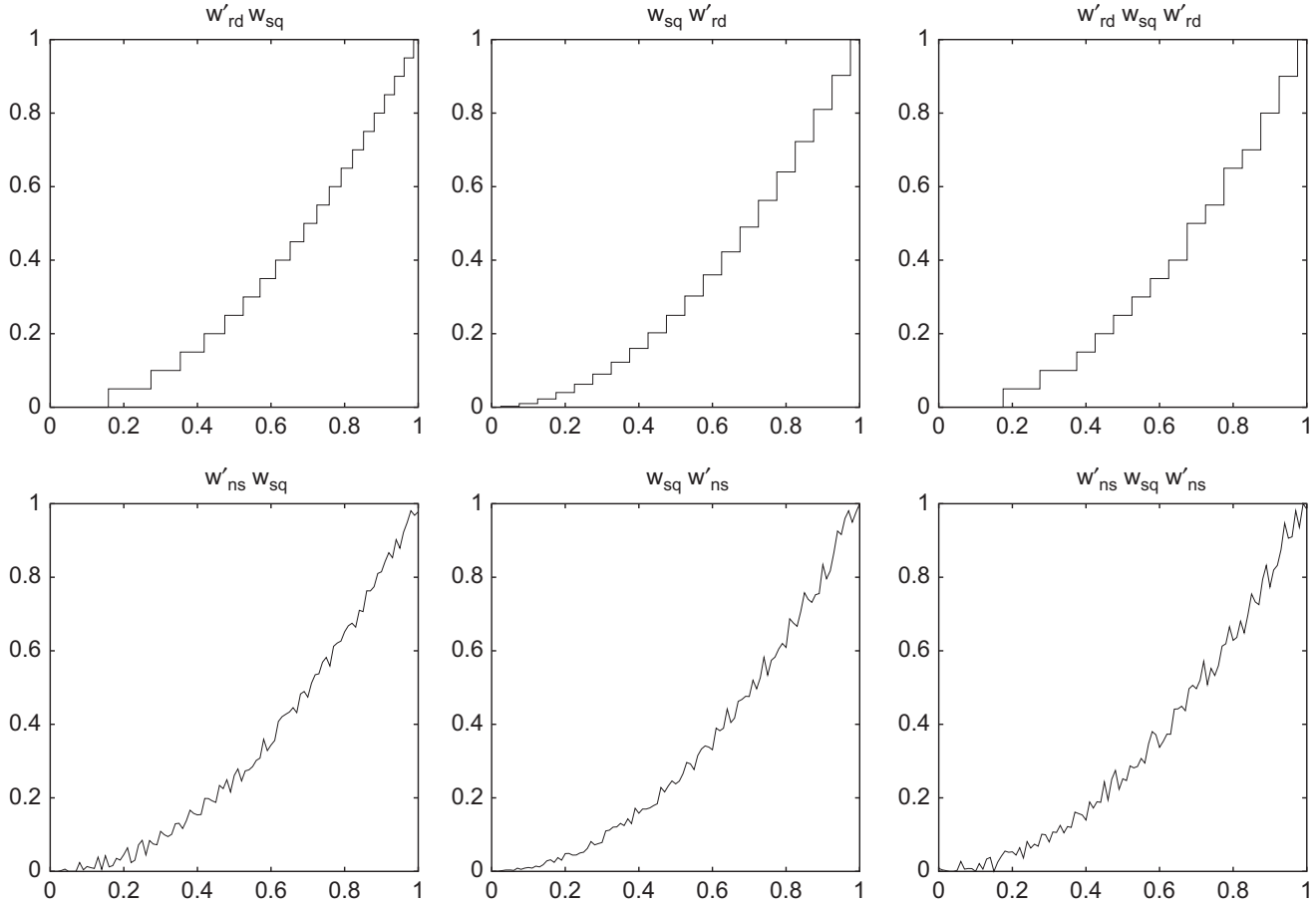

Fig. 3. The connective $\mathbf{w}_{\mathrm{sq}}$ (the square, see Example 3.1) distorted by rounding and noise. The upper row: graphs of the connectives $\mathbf{w}'_{\mathrm{rd}}\mathbf{w}_{\mathrm{sq}}$ (i.e., rounded squares), $\mathbf{w}_{\mathrm{sq}}\mathbf{w}'_{\mathrm{rd}}$ (i.e., squares of rounded values), and $\mathbf{w}'_{\mathrm{rd}}\mathbf{w}_{\mathrm{sq}}\mathbf{w}'_{\mathrm{rd}}$ (i.e., rounded squares of rounded values). The lower row: graphs of the connectives $\mathbf{w}'_{\mathrm{ns}}\mathbf{w}_{\mathrm{sq}}$ (i.e., noise-affected squares), $\mathbf{w}_{\mathrm{sq}}\mathbf{w}'_{\mathrm{ns}}$ (i.e., squares of noise-affected values), and $\mathbf{w}'_{\mathrm{ns}}\mathbf{w}_{\mathrm{sq}}\mathbf{w}'_{\mathrm{ns}}$ (i.e., noise-affected squares of noise-affected values). Visually exaggerated functions $\mathbf{w}'_{\mathrm{rd}}$ and $\mathbf{w}'_{\mathrm{ns}}$ have again been used instead of $\mathbf{w}_{\mathrm{rd}}$ and $\mathbf{w}_{\mathrm{ns}}$ in order to obtain optically distinguishable graphs (cf. Fig. 1).

Note that besides (U36), also the theorems (U12), (U13), and (U15) can be directly used to estimate the height and plinth of **uv** and **vu** if **v** is close to the identity (as $\mathbf{v} \approx \mathbf{id} \Rightarrow \mathbf{id} \subseteq \mathbf{v}$). Notice furthermore that an estimate for **uv** or **vu** can in some cases not covered by Theorem 3.20 be obtained directly from Theorem 3.13, even if the latter makes no assumption on the closeness of **v** to **id**: for instance, if **u** is insufficiently congruent for (U35) to be applied, $\mathrm{Mon}(\mathbf{uv})$ can still be obtained by (U11) if **v** is fully monotone (as is the case, e.g., if **v** represents rounding).

Example 3.21. Let us use Theorem 3.20 to estimate the standard Łukasiewicz values of graded properties for the composition $\mathbf{w}_{\mathrm{rd}}\mathbf{w}_{\mathrm{sq}}$, i.e., squares rounded to 2 decimal places (see Fig. 3 for an approximation of its graph):

- As $(\mathbf{w}_{\mathrm{rd}} \approx \mathbf{id}) = 0.995$ (see Example 3.12), by (U30) we obtain the (obviously tight) estimate $(\mathbf{w}_{\mathrm{rd}}\mathbf{w}_{\mathrm{sq}} \approx \mathbf{w}_{\mathrm{sq}}) \geq 0.995$.
- Since $\mathrm{Cng}(\mathbf{w}_{\mathrm{sq}}) = 0.75$ (see Example 3.4) and $(\mathbf{w}_{\mathrm{rd}} \approx^2 \mathbf{id}) = 0.99$, by (U32) we obtain $\mathrm{Cng}(\mathbf{w}_{\mathrm{rd}}\mathbf{w}_{\mathrm{sq}}) \geq 0.74$. (The calculation of the exact value, which lies in [0.74,0.75], is left to the interested reader.)
- As $\mathrm{Cng}_2(\mathbf{w}_{\mathrm{sq}}) = 1$ and $(\mathbf{w}_{\mathrm{rd}} \approx^2 \mathbf{id}) = 0.99$, by (U32) we obtain $\mathrm{Cng}_2(\mathbf{w}_{\mathrm{rd}}\mathbf{w}_{\mathrm{sq}}) \geq 0.99$, and the same estimate is obtained for any $n \geq 2$.
- Theorem (U34) only yields the estimate $\mathrm{Mon}(\mathbf{w}_{\mathrm{rd}}\mathbf{w}_{\mathrm{sq}}) \geq 0.99$. However, from (U11) we actually know that $\mathrm{Mon}(\mathbf{w}_{\mathrm{rd}}\mathbf{w}_{\mathrm{sq}}) = 1$, as both $\mathbf{w}_{\mathrm{sq}}$ and $\mathbf{w}_{\mathrm{rd}}$ are fully monotone.

Since $(\mathbf{w}_{\mathrm{ns}} \approx \mathbf{id}) = (\mathbf{w}_{\mathrm{rd}} \approx \mathbf{id})$, the same estimates are obtained for the connective $\mathbf{w}_{\mathrm{ns}}\mathbf{w}_{\mathrm{sq}}$ (i.e., the square with ± 0.005 random noise—see Fig. 3 for an approximation of its graph). However, as easily seen, $\mathrm{Mon}(\mathbf{w}_{\mathrm{ns}}\mathbf{w}_{\mathrm{sq}}) = 0.99$; thus here the estimate by (U34) is tight and $\mathbf{w}_{\mathrm{ns}}\mathbf{w}_{\mathrm{sq}}$, unlike $\mathbf{w}_{\mathrm{rd}}\mathbf{w}_{\mathrm{sq}}$, represents the worst case for (U34).

The converse compositions $\mathbf{w}_{\mathrm{sq}}\mathbf{w}_{\mathrm{rd}}$ and $\mathbf{w}_{\mathrm{sq}}\mathbf{w}_{\mathrm{ns}}$, i.e., the square of rounded or noise-affected arguments (see Fig. 3), obtain the following estimates by Theorem 3.20:

- Theorem (U31) for $n = 2$ yields $(\mathbf{w}_{\mathrm{sq}}\mathbf{w}_{\mathrm{rd}} \approx \mathbf{w}_{\mathrm{sq}}) \geq 0.99$, and the same estimate is obtained for $\mathbf{w}_{\mathrm{sq}}\mathbf{w}_{\mathrm{ns}}$. Notice that (U31) for $n = 1$ would only yield a worse lower bound 0.745.
- Theorem (U33) yields $\mathrm{Cng}(\mathbf{w}_{\mathrm{sq}}\mathbf{w}_{\mathrm{rd}}) \geq 0.74$ and $\mathrm{Cng}_2(\mathbf{w}_{\mathrm{sq}}\mathbf{w}_{\mathrm{rd}}) \geq 0.98$. For $n > 2$ theorem (U33) only yields $\mathrm{Cng}_n(\mathbf{w}_{\mathrm{sq}}\mathbf{w}_{\mathrm{rd}}) \geq 1 - 0.01n$, although we know by (U1) that the value of $\mathrm{Cng}_n(\mathbf{w}_{\mathrm{sq}}\mathbf{w}_{\mathrm{rd}})$ does not decrease with increasing n, and that in fact $\mathrm{Cng}_n(\mathbf{w}_{\mathrm{sq}}\mathbf{w}_{\mathrm{rd}}) \geq \mathrm{Cng}_2(\mathbf{w}_{\mathrm{sq}}\mathbf{w}_{\mathrm{rd}}) \geq 0.98$. The decrease in the estimate by (U33) is caused by the premise $\mathbf{w}_{\mathrm{rd}} \approx^{2n} \mathbf{id}$ getting stronger with increasing n, while the premise $\mathrm{Cng}_n(\mathbf{w}_{\mathrm{sq}})$ cannot get weaker after the value 1 is achieved for $n = 2$. A more useful theorem for connectives that are fully n-congruent for some n is thus the following corollary of (U1) and (U33):

$$\triangle\,\mathrm{Cng}_n(\mathbf{u}),\ \mathbf{v} \approx^{2\min\{m \mid \triangle\,\mathrm{Cng}_m(\mathbf{u})\}} \mathbf{id} \ \Rightarrow\ \mathrm{Cng}_n(\mathbf{uv}).$$

  The same estimates and remarks apply to $\mathbf{w}_{\mathrm{sq}}\mathbf{w}_{\mathrm{ns}}$ as well.
- By (U11) we know that $\mathbf{w}_{\mathrm{sq}}\mathbf{w}_{\mathrm{rd}}$ is fully monotone. Even though (U11) does not apply to $\mathbf{w}_{\mathrm{sq}}\mathbf{w}_{\mathrm{ns}}$, as $\mathrm{Mon}(\mathbf{w}_{\mathrm{ns}}) = 0.99 < 1$, by (U35) for $n = 2$ we obtain the estimate $\mathrm{Mon}(\mathbf{w}_{\mathrm{sq}}\mathbf{w}_{\mathrm{ns}}) \geq 0.98$. This estimate is rather tight, as the actual value is 0.9801. Notice that (U35) for $n = 1$ would only yield a much worse bound 0.49.

Iterated application of Theorem 3.20 allows us to estimate the values of iterated compositions with identity-close connectives, such as $\mathbf{w}_{\mathrm{rd}}\mathbf{w}_{\mathrm{sq}}\mathbf{w}_{\mathrm{rd}}$ (i.e., rounded squares of rounded arguments—see Fig. 3), which are encountered whenever some systematic disturbances occur (e.g., when working with a fixed number of decimal places). Thus, for instance, Theorem 3.20 yields the estimates $\mathrm{Cng}(\mathbf{w}_{\mathrm{rd}}\mathbf{w}_{\mathrm{sq}}\mathbf{w}_{\mathrm{rd}}) \geq 0.73$, $\mathrm{Mon}(\mathbf{w}_{\mathrm{ns}}\mathbf{w}_{\mathrm{sq}}\mathbf{w}_{\mathrm{ns}}) \geq 0.97$, etc., in standard Łukasiewicz logic.

4. Graded congruence and monotony of binary connectives

We now turn to binary fuzzy connectives. First we shall discuss the properties analogous to those we have studied for the unary case, namely monotony, antitony, congruence w.r.t. equivalence, height, and plinth. There are two possible ways how graded congruence and monotony for binary connectives can be defined: either coordinate-wise, or jointly in both coordinates. We shall start with the coordinate-wise variants.

The following left-argument graded properties of binary fuzzy connectives have been considered in [12,13]:

$$\begin{aligned}
\mathrm{LMon}(\mathbf{c}) &\equiv_{\mathrm{df}} (\forall\alpha\beta\gamma)((\alpha \leq \beta) \to (\alpha\,\mathbf{c}\,\gamma \to \beta\,\mathbf{c}\,\gamma))\\
\mathrm{LAnt}(\mathbf{c}) &\equiv_{\mathrm{df}} (\forall\alpha\beta\gamma)((\beta \leq \alpha) \to (\alpha\,\mathbf{c}\,\gamma \to \beta\,\mathbf{c}\,\gamma))\\
\mathrm{LCng}(\mathbf{c}) &\equiv_{\mathrm{df}} (\forall\alpha\beta\gamma)((\alpha \leftrightarrow \beta) \to (\alpha\,\mathbf{c}\,\gamma \leftrightarrow \beta\,\mathbf{c}\,\gamma))
\end{aligned}$$

The analogous right-argument properties have been defined as the left-argument properties for the converse connective, i.e., $\mathrm{RCng}(\mathbf{c}) \equiv_{\mathrm{df}} \mathrm{LCng}(\mathbf{c}^{-1})$ and similarly for RMon and RAnt. Again we shall use LMon_{+1} for LMon and LMon_{-1} for LAnt, and similarly for RMon_i with $i = \pm 1$.

Like in the case of unary Cng_n, the binary properties LCng and RCng can be parameterized by the &-multiplicity of the antecedent equivalence:

Definition 4.1. In FCT, we define for any $n \in \mathbb{N}$:

$$\begin{aligned}
\mathrm{LCng}_n(\mathbf{c}) &\equiv_{\mathrm{df}} (\forall\alpha\beta\gamma)((\alpha \leftrightarrow \beta)^n \to (\alpha\,\mathbf{c}\,\gamma \leftrightarrow \beta\,\mathbf{c}\,\gamma)) && \text{Left-argument binary } n\text{-congruence w.r.t. } \leftrightarrow\\
\mathrm{RCng}_n(\mathbf{c}) &\equiv_{\mathrm{df}} (\forall\alpha\beta\gamma)((\alpha \leftrightarrow \beta)^n \to (\gamma\,\mathbf{c}\,\alpha \leftrightarrow \gamma\,\mathbf{c}\,\beta)) && \text{Right-argument binary } n\text{-congruence w.r.t. } \leftrightarrow
\end{aligned}$$

The following observation shows that the coordinate-wise properties defined above can be reduced to the infima of the corresponding unary properties of "cross-sections" with one argument fixed:

Observation 4.2. FCT *obviously proves for each* $n \in \mathbb{N}$:

(C1) $\mathrm{LCng}_n(\mathbf{c}) \equiv (\forall\gamma)\,\mathrm{Cng}_n(\mathbf{c}(\mathbf{id}, \gamma))$.
(C2) $\mathrm{RCng}_n(\mathbf{c}) \equiv (\forall\gamma)\,\mathrm{Cng}_n(\mathbf{c}(\underline{\gamma}, \overline{\mathbf{id}}))$

and analogously for LMon_i *and* RMon_i, *where* $i = \pm 1$.

A similar theorem holds for the height, plinth, and inclusion of binary fuzzy connectives [12]:

Observation 4.3. FCT *obviously proves*:

(C3) $\mathrm{Hgt}(\mathbf{c}) \equiv (\exists\gamma)\,\mathrm{Hgt}(\mathbf{c}(\mathbf{id}, \underline{\gamma}))$.
(C4) $\mathrm{Plt}(\mathbf{c}) \equiv (\forall\gamma)\,\mathrm{Plt}(\mathbf{c}(\mathbf{id}, \underline{\gamma}))$.
(C5) $\mathbf{c} \subseteq \mathbf{d} \equiv (\forall\gamma)(\mathbf{c}(\mathbf{id}, \underline{\gamma}) \to \mathbf{d}(\mathbf{id}, \underline{\gamma}))$

and analogously also for $(\mathbf{id}, \underline{\gamma})$ *replaced by* $(\underline{\gamma}, \mathbf{id})$ *in the above formulae*.

Observations 4.2 and 4.3 make it possible to obtain many theorems on coordinate-wise properties of binary connectives from those on unary properties, as shall be seen below.

Similarly as in the unary case, we can introduce the analogue of LCng_n featuring implication in place of equivalence:

$$\mathrm{LMonCng}_n(\mathbf{c}) \equiv_{\mathrm{df}} (\forall\alpha\beta\gamma)((\alpha \to \beta)^n \to (\alpha\,\mathbf{c}\,\gamma \to \beta\,\mathbf{c}\,\gamma)),$$

and analogously for the right-sided and antitone variants. However, just like in the unary case, these graded properties are reducible to the min-conjunction of n-congruence and monotony:

Theorem 4.4. FCT *proves for all* $n \in \mathbb{N}$:

(C6) $\mathrm{LMonCng}_n(\mathbf{c}) \Leftrightarrow \mathrm{LMon}(\mathbf{c}) \wedge \mathrm{LCng}_n(\mathbf{c})$, *and similarly for the right-sided and antitone variants*.

Proof. The theorem is proved by the following chain of equivalences:

$$\begin{aligned}
\mathrm{LMonCng}_n(\mathbf{c}) &\Leftrightarrow (\forall\gamma)(\forall\alpha\beta)((\alpha \to \beta)^n \to (\alpha\,\mathbf{c}\,\gamma \to \beta\,\mathbf{c}\,\gamma)) && \text{by the definition of } \mathrm{LMonCng}_n\\
&\Leftrightarrow (\forall\gamma)\,\mathrm{MonCng}_n(\mathbf{c}(\mathbf{id}, \underline{\gamma})) && \text{by the definition of } \mathbf{c}(\mathbf{id}, \underline{\gamma})\\
&\Leftrightarrow (\forall\gamma)(\mathrm{Mon}(\mathbf{c}(\mathbf{id}, \underline{\gamma})) \wedge \mathrm{Cng}_n(\mathbf{c}(\mathbf{id}, \underline{\gamma}))) && \text{by (U2)}\\
&\Leftrightarrow (\forall\gamma)\,\mathrm{Mon}(\mathbf{c}(\mathbf{id}, \underline{\gamma})) \wedge (\forall\gamma)\,\mathrm{Cng}_n(\mathbf{c}(\mathbf{id}, \underline{\gamma})) && \text{by the distributivity of } \forall \text{ over } \wedge\\
&\Leftrightarrow \mathrm{LMon}(\mathbf{c}) \wedge \mathrm{LCng}_n(\mathbf{c}) && \text{by Observation 4.2}
\end{aligned}$$

and similarly for the variants. □

Also the following theorems are analogous to the unary case (we only give them for left-side properties):

Theorem 4.5. FCT *proves for all* $m, n \in \mathbb{N}$ *and* $i, j = \pm 1$:

(C7) $\mathrm{LCng}_n(\mathbf{c}) \Rightarrow \mathrm{LCng}_{n+m}(\mathbf{c})$.
(C8) $\mathrm{Hgt}(\mathbf{c}) \to \mathrm{Plt}(\mathbf{c}) \Rightarrow \mathrm{LMon}_i(\mathbf{c}) \wedge \mathrm{LCng}_n(\mathbf{c})$.
(C9) $\mathrm{LCng}_n(\mathbf{c}), \mathbf{c} \approx^2 \mathbf{d} \Rightarrow \mathrm{LCng}_n(\mathbf{d})$.
(C10) $\mathrm{LMon}_i(\mathbf{c}), \mathbf{c} \cong \mathbf{d} \Rightarrow \mathrm{LMon}_i(\mathbf{d})$.
(C11) $\mathrm{Cng}_m(\mathbf{u}), \mathrm{LCng}_n^m(\mathbf{c}) \Rightarrow \mathrm{LCng}_{m\cdot n}(\mathbf{uc})$.
(C12) $\mathrm{Mon}_i(\mathbf{u}), \triangle\,\mathrm{LMon}_j(\mathbf{c}) \Rightarrow \mathrm{Mon}_{i\cdot j}(\mathbf{uc})$.
(C13) $\mathbf{id} \subseteq \mathbf{u}, \mathrm{Hgt}(\mathbf{c}) \Rightarrow \mathrm{Hgt}(\mathbf{uc})$.
(C14) $\mathbf{id} \subseteq \mathbf{u}, \mathrm{Plt}(\mathbf{c}) \Rightarrow \mathrm{Plt}(\mathbf{uc})$.
(C15) $\mathrm{Hgt}(\mathbf{u}), \mathrm{Cng}_n(\mathbf{u}), \mathrm{Surj}_n(\mathbf{c}) \Rightarrow \mathrm{Hgt}(\mathbf{uc})$, *where* $\mathrm{Surj}_n(\mathbf{c}) \equiv_{\mathrm{df}} (\forall\beta)(\exists\gamma\delta)(\beta \leftrightarrow^n \gamma\,\mathbf{c}\,\delta)$.
(C16) $\mathrm{Plt}(\mathbf{u}) \Rightarrow \mathrm{Plt}(\mathbf{uc})$.
(C17) $\mathrm{Mon}_i(\mathbf{u}) \wedge \mathrm{Cng}_n(\mathbf{u}), \mathbf{c} \subseteq^{i\cdot n} \mathbf{d} \Rightarrow \mathbf{uc} \subseteq \mathbf{ud}$.
(C18) $\mathrm{Cng}_n(\mathbf{u}), \mathbf{c} \approx^n \mathbf{d} \Rightarrow \mathbf{uc} \approx \mathbf{ud}$.
(C19) $\mathbf{u} \subseteq \mathbf{v} \Rightarrow \mathbf{uc} \subseteq \mathbf{vc}$.
(C20) $\mathbf{u} \approx \mathbf{v} \Rightarrow \mathbf{uc} \approx \mathbf{vc}$.

Proof. We shall only prove (C12) and (C15); proofs of the other claims are similar to that of (C12).

[12] Their FCT definitions for binary connectives read: $\mathrm{Hgt}(\mathbf{c}) \equiv_{\mathrm{df}} (\exists\alpha\beta)(\alpha\,\mathbf{c}\,\beta)$, $\mathrm{Plt}(\mathbf{c}) \equiv_{\mathrm{df}} (\forall\alpha\beta)(\alpha\,\mathbf{c}\,\beta)$, and $\mathbf{c} \subseteq \mathbf{d} \equiv_{\mathrm{df}} (\forall\alpha\beta)(\alpha\,\mathbf{c}\,\beta \to \alpha\,\mathbf{d}\,\beta)$.

(C12) Applying (U11) to $\mathbf{v} = \mathbf{c}(\mathbf{id}, \underline{\gamma})$ we obtain: $\mathrm{Mon}(\mathbf{u})$, $\triangle\,\mathrm{Mon}(\mathbf{c}(\mathbf{id}, \underline{\gamma})) \Rightarrow \mathrm{Mon}(\mathbf{uc}(\mathbf{id}, \underline{\gamma}))$. By generalization on γ, a shift and distribution of the quantifier, and the theorem $(\forall x)\triangle\varphi \leftrightarrow \triangle(\forall x)\varphi$ of $\mathrm{MTL}_{\triangle\forall}$ we obtain

$$\mathrm{Mon}(\mathbf{u}),\ \triangle(\forall\gamma)\,\mathrm{Mon}(\mathbf{c}(\mathbf{id}, \underline{\gamma}))\ \Rightarrow\ (\forall\gamma)\,\mathrm{Mon}(\mathbf{uc}(\mathbf{id}, \underline{\gamma})),$$

which is the required formula by Observation 4.2.

(C15) The claim is proved by replacing $\mathbf{v}\alpha$ by $\gamma\ \mathbf{c}\ \delta$, and $(\exists\alpha)$ by $(\exists\gamma\delta)$, everywhere in the proof of (U14).

Observe that using Observation 4.2 for a proof would only yield a theorem with the premise $(\forall\gamma)\,\mathrm{Surj}_n(\mathbf{c}(\mathbf{id}, \underline{\gamma}))$, i.e., graded n-surjectivity of each γ-section $\mathbf{c}(\mathbf{id}, \underline{\gamma})$ of $\mathbf{c}$. Such a premise, however, would be much stronger than $\mathrm{Surj}_n(\mathbf{c})$ used in (C15): consider, e.g., $\mathbf{c}(\alpha, \gamma) = \gamma$, which is fully n-surjective, but each $\mathbf{c}(\mathbf{id}, \underline{\gamma})$ is constant (thus very little n-surjective). □

The following observation lists how basic binary connectives of the ground logic ($\wedge, \vee, \&, \rightarrow, \leftrightarrow$) satisfy the graded notions of coordinate-wise congruence and monotony, inclusion, height, and plinth. (Recall that $\mathbf{p}_1, \mathbf{p}_2$ denote the projections, see Definition 2.5.)

Theorem 4.6. FCT *proves, for* $k, l \in \{1, 2\}$, $k \neq l$:

(C21) $\& \subseteq \wedge \subseteq \leftrightarrow \subseteq \rightarrow,\ \wedge \subseteq \mathbf{p}_k \subseteq \vee,\ \mathbf{p}_2 \subseteq \rightarrow,\ (\neg\mathbf{p}_1) \subseteq \rightarrow$.
(C22) $\neg(\mathbf{c} \subseteq \mathbf{d})$, *for any inclusion* $\mathbf{d} \subseteq \mathbf{c}$ *of* (C21) *except* $\& \subseteq \wedge$.
Moreover, $\neg(\mathbf{p}_k \subseteq \mathbf{p}_l),\ \neg(\mathbf{p}_k \subseteq \leftrightarrow),\ \neg(\mathbf{p}_1 \subseteq \rightarrow),\ \neg(\vee \subseteq \rightarrow),\ \neg(\rightarrow \subseteq \vee)$.
The value of $\wedge \subseteq \&$, *on the other hand, depends on a particular model of* FCT.
(C23) $\mathrm{Hgt}(\mathbf{c}), \neg\,\mathrm{Plt}(\mathbf{c})$, *for* $\mathbf{c} \in \{\wedge, \vee, \&, \rightarrow, \leftrightarrow, \mathbf{p}_k\}$.
(C24) $\mathrm{LCng}(\mathbf{c})$, *for* $\mathbf{c} \in \{\wedge, \vee, \&, \rightarrow, \leftrightarrow, \mathbf{p}_k\}$, *and similarly for* RCng.
(C25) $\mathrm{LMon}(\mathbf{c})$, *for* $\mathbf{c} \in \{\wedge, \vee, \&, \mathbf{p}_k\}$, *and similarly for* RMon.
(C26) $\neg\,\mathrm{LMon}(\leftrightarrow), \neg\,\mathrm{LAnt}(\mathbf{c})$, *for* $\mathbf{c} \in \{\wedge, \vee, \&, \leftrightarrow, \mathbf{p}_k\}$, *and similarly for* RMon.
(C27) $\mathrm{LAnt}(\rightarrow), \mathrm{RMon}(\rightarrow), \neg\,\mathrm{LMon}(\rightarrow), \neg\,\mathrm{RAnt}(\rightarrow)$.

Proof. Proofs of most claims of Theorem 4.6 are trivial, following directly from the known (see [18]) properties of logical connectives in MTL: e.g., $\mathrm{LCng}(\leftrightarrow)$ follows directly from the transitivity and commutativity of $\leftrightarrow$. The negative claims are obtained by substituting the values 0 and 1, e.g.:

$$\mathrm{LMon}(\leftrightarrow)\ \Rightarrow\ ((0 \leq 1) \rightarrow ((0 \leftrightarrow 0) \rightarrow (1 \leftrightarrow 0)))\ \Leftrightarrow\ 0. \qquad \square$$

The following corollaries of Theorems 4.5 and 4.6 show how coordinate-wise properties are preserved by logical connectives (again, only interesting graded cases are listed):

Corollary 4.7. FCT *proves*:

(C28) $\mathrm{LCng}_n(\mathbf{c})\ \Rightarrow\ \mathrm{LCng}_n(\neg\mathbf{c})$.
(C29) $\mathrm{LCng}^m_k(\mathbf{c})\ \Rightarrow\ \mathrm{LCng}_{m\cdot k}(\mathbf{c}^n)$, *for* $m \geq n$; *in particular*, $\mathrm{LCng}^n(\mathbf{c}) \Rightarrow \mathrm{LCng}_n(\mathbf{c}^n)$

and analogously for the right-argument properties.

Now we turn to the both-coordinate properties Cng and Mon, which unlike their classical counterparts cannot in fuzzy logic be reduced to the component-wise ones (cf. Theorem 4.10 and Example 4.11 below). They are defined as follows:

Definition 4.8. In FCT, we define the following graded properties of binary connectives $\mathbf{c} \sqsubseteq \mathrm{L} \times \mathrm{L}$:

$$\mathrm{Cng}_{l,r}(\mathbf{c}) \equiv_{\mathrm{df}} (\forall\alpha\beta\alpha'\beta')((\alpha \leftrightarrow \alpha')^l\ \&\ (\beta \leftrightarrow \beta')^r \rightarrow (\alpha\ \mathbf{c}\ \beta \leftrightarrow \alpha'\ \mathbf{c}\ \beta')) \qquad \text{for } l, r \in \mathbb{N}$$

$$\mathrm{Mon}_{l,r}(\mathbf{c}) \equiv_{\mathrm{df}} (\forall\alpha\beta\alpha'\beta')((\alpha \leq \alpha')^l\ \&\ (\beta \leq \beta')^r \rightarrow (\alpha\ \mathbf{c}\ \beta \rightarrow \alpha'\ \mathbf{c}\ \beta')) \qquad \text{for } l, r = \pm 1$$

$$\mathrm{MonCng}_{l,r}(\mathbf{c}) \equiv_{\mathrm{df}} (\forall\alpha\beta\alpha'\beta')((\alpha \rightarrow \alpha')^l\ \&\ (\beta \rightarrow \beta')^r \rightarrow (\alpha\ \mathbf{c}\ \beta \rightarrow \alpha'\ \mathbf{c}\ \beta')) \qquad \text{for } l, r \in \mathbb{Z}$$

Recall Convention 2.3 for the meaning of negative exponents. By convention, the subscripts can be omitted if $l = r = 1$ (the type of the argument disambiguates between the unary and binary properties).

The meaning of the indices of these graded properties is analogous to the unary case. Like in the unary and component-wise cases, FCT proves $\mathrm{Cng}_{l,r}(\mathbf{c}) \Rightarrow \mathrm{Cng}_{l',r'}(\mathbf{c})$ for $l' \geq l$ and $r' \geq r$. Also analogously to unary and single-sided cases (cf. Theorem 3.7), non-zero degrees of binary congruence and monotony are ensured whenever $(\mathrm{Hgt}(\mathbf{c}) \to \mathrm{Plt}(\mathbf{c})) > 0$:

Corollary 4.9. *By Lemma* 3.8, FCT *proves for all* $l, r \in \mathbb{N}$ *and* $i, j = \pm 1$:

(C30) $\mathrm{Hgt}(\mathbf{c}) \to \mathrm{Plt}(\mathbf{c}) \;\Rightarrow\; \mathrm{Cng}_{l,r}(\mathbf{c})$.
(C31) $\mathrm{Hgt}(\mathbf{c}) \to \mathrm{Plt}(\mathbf{c}) \;\Rightarrow\; \mathrm{Mon}_{i,j}(\mathbf{c})$.
(C32) $\mathrm{Hgt}(\mathbf{c}) \to \mathrm{Plt}(\mathbf{c}) \;\Rightarrow\; \mathrm{MonCng}_{il,jr}(\mathbf{c})$.

The condition $\mathrm{Hgt}(\mathbf{c}) \to \mathrm{Plt}(\mathbf{c})$ coincides with $\mathrm{Cng}_{0,0}(\mathbf{c})$ and $\mathrm{MonCng}_{0,0}(\mathbf{c})$; cf. Remark 3.9.

The relationship between the coordinate-wise and both-coordinate properties of congruence and monotony is given by the following theorem. [13]

Theorem 4.10. FCT *proves*:

(C33) $\mathrm{LCng}_l(\mathbf{c})$ & $\mathrm{RCng}_r(\mathbf{c}) \;\Rightarrow\; \mathrm{Cng}_{l,r}(\mathbf{c}) \;\Rightarrow\; \mathrm{LCng}_l(\mathbf{c}) \wedge \mathrm{RCng}_r(\mathbf{c})$, *for any* $l, r \in \mathbb{N}$.
(C34) $\mathrm{LMon}_l(\mathbf{c})$ & $\mathrm{RMon}_r(\mathbf{c}) \;\Rightarrow\; \mathrm{Mon}_{l,r}(\mathbf{c}) \;\Rightarrow\; \mathrm{LMon}_l(\mathbf{c}) \wedge \mathrm{RMon}_r(\mathbf{c})$, *for* $l, r = \pm 1$.
(C35) $\mathrm{LMonCng}_l(\mathbf{c})$ & $\mathrm{RMonCng}_r(\mathbf{c}) \;\Rightarrow\; \mathrm{MonCng}_{l,r}(\mathbf{c}) \;\Rightarrow\; \mathrm{LMonCng}_l(\mathbf{c}) \wedge \mathrm{RMonCng}_r(\mathbf{c})$, *for* $l, r \in \mathbb{Z}$.

Proof. We shall only prove (C33), the proofs of (C34) and (C35) are analogous.

First we prove that $\mathrm{LCng}_l(\mathbf{c})$ & $\mathrm{RCng}_r(\mathbf{c}) \Rightarrow \mathrm{Cng}_{l,r}(\mathbf{c})$. By $\mathrm{LCng}_l(\mathbf{c})$ we obtain $(\alpha \leftrightarrow \alpha')^l \to (\alpha\, \mathbf{c}\, \beta \leftrightarrow \alpha'\, \mathbf{c}\, \beta)$. Similarly by $\mathrm{RCng}_r(\mathbf{c})$ we obtain $(\beta \leftrightarrow \beta')^r \to (\alpha'\, \mathbf{c}\, \beta \leftrightarrow \alpha'\, \mathbf{c}\, \beta')$. Combining the antecedents and consequents we therefore obtain $(\alpha \leftrightarrow \alpha')^l$ & $(\beta \leftrightarrow \beta')^r \to (\alpha\, \mathbf{c}\, \beta \leftrightarrow \alpha'\, \mathbf{c}\, \beta')$ by the transitivity of equivalence, and generalization completes the proof.

The second implication $\mathrm{Cng}_{l,r}(\mathbf{c}) \Rightarrow \mathrm{LCng}_l(\mathbf{c}) \wedge \mathrm{RCng}_r(\mathbf{c})$ holds trivially, as $\mathrm{Cng}_{l,r}(\mathbf{c}) \Rightarrow \mathrm{LCng}_l(\mathbf{c})$ is obtained by specifying $\beta' = \beta$ and $\mathrm{Cng}_{l,r}(\mathbf{c}) \Rightarrow \mathrm{RCng}_r(\mathbf{c})$ by specifying $\alpha' = \alpha$ in the definition of $\mathrm{Cng}_{l,r}(\mathbf{c})$. □

By (C33), the truth degree of $\mathrm{Cng}_{l,r}(\mathbf{c})$ is bounded by the truth degrees of strong resp. weak conjunction of the coordinate-wise congruences (and similarly for the other properties). Theorems of this form arise regularly in graded fuzzy mathematics [9]. The following counterexamples show that the implications in Theorem 4.10 cannot in general be converted.

Example 4.11. We shall only consider $l = r = 1$, as the counterexamples can easily be adapted for the other cases. All counterexamples are constructed in models over standard Łukasiewicz logic.

Let $\mathbf{c}_1(\alpha, \beta) = 2\alpha \wedge 2\beta \wedge 1$. Then it is easy to verify that $\mathrm{LCng}(\mathbf{c}_1) = \mathrm{RCng}(\mathbf{c}_1) = \mathrm{Cng}(\mathbf{c}_1) = 0.5$, which disproves the converse to the first implication of (C33).

Let $\mathbf{c}_2(\alpha, \beta) = 2(\alpha \wedge 0.1) + 2(\beta \wedge 0.1)$. Then it is easily seen that $\mathrm{LCng}(\mathbf{c}_2) = \mathrm{RCng}(\mathbf{c}_2) = 0.9$, while $\mathrm{Cng}(\mathbf{c}_2) = 0.8$, which disproves the converse to the second implication of (C33).

Let $\mathbf{c}_3(\alpha, \beta) = 1 - (\alpha \wedge \beta)/2$. Then obviously $\mathrm{LMon}(\mathbf{c}_3) = \mathrm{RMon}(\mathbf{c}_3) = \mathrm{Mon}(\mathbf{c}_3) = 0.5$, which disproves the converse to the first implication of (C34).

Let $\mathbf{c}_4(\alpha, \beta) = 1 - (\alpha + \beta)/4$. Then clearly $\mathrm{LMon}(\mathbf{c}_4) = \mathrm{RMon}(\mathbf{c}_4) = 0.75$ and $\mathrm{Mon}(\mathbf{c}_4) = 0.5$, which disproves the converse to the second implication of (C34).

As another example of the difference between the single-sided and both-sided properties we show that unlike $\mathrm{LMonCng}_n$, the both-sided property $\mathrm{MonCng}_{l,r}(\mathbf{c})$ cannot be reduced to the min-conjunction of Mon and Cng (with appropriate subscripts). Only the following implication holds:

Theorem 4.12. FCT *proves for* $i, j = \pm 1$ *and* $l, r \in \mathbb{N}$:

[13] It can be observed that admitting also the 'multiplicity-parameter' $\triangle$ in Definition 4.1 (cf. Convention 2.1 for $\varphi^{\triangle}$), $\mathrm{LCng}_n(\mathbf{c})$ could be written as $\mathrm{Cng}_{n,\triangle}(\mathbf{c})$ and $\mathrm{RCng}_n(\mathbf{c})$ as $\mathrm{Cng}_{\triangle,n}(\mathbf{c})$. We have not introduced this notation since the subscripts $\triangle$ would have to be treated separately in theorems anyway.

(C36) $\mathrm{MonCng}_{il,jr}(\mathbf{c}) \;\Rightarrow\; \mathrm{Mon}_{i,j}(\mathbf{c}) \wedge \mathrm{Cng}_{l,r}(\mathbf{c})$.

Proof. For simplicity, assume $i = j = +1$; the proofs for antitony are analogous.
First, $\mathrm{MonCng}_{l,r}(\mathbf{c}) \Rightarrow \mathrm{Mon}(\mathbf{c})$, since $(\alpha \le \alpha') \,\&\, (\beta \le \beta') \Rightarrow (\alpha \to \alpha')^l \,\&\, (\beta \to \beta')^r$.
Second, $\mathrm{MonCng}_{l,r}(\mathbf{c}) \Rightarrow \mathrm{Cng}_{l,r}(\mathbf{c})$, since from $\mathrm{MonCng}_{l,r}(\mathbf{c})$ we obtain by specification

$$(\alpha \to \alpha')^l \,\&\, (\beta \to \beta')^r \to (\alpha\,\mathbf{c}\,\beta \to \alpha'\,\mathbf{c}\,\beta'), \text{ as well as}$$
$$(\alpha' \to \alpha)^l \,\&\, (\beta' \to \beta)^r \to (\alpha'\,\mathbf{c}\,\beta' \to \alpha\,\mathbf{c}\,\beta).$$

Thus $\mathrm{MonCng}_{l,r}(\mathbf{c})$ implies

$$((\alpha \to \alpha')^l \,\&\, (\beta \to \beta')^r) \wedge ((\alpha' \to \alpha)^l \,\&\, (\beta' \to \beta)^r) \to (\alpha\,\mathbf{c}\,\beta \leftrightarrow \alpha'\,\mathbf{c}\,\beta'),$$

so a fortiori (using & instead of $\wedge$ in the antecedent),

$$(\alpha \leftrightarrow \alpha')^l \,\&\, (\beta \leftrightarrow \beta')^r \to (\alpha\,\mathbf{c}\,\beta \leftrightarrow \alpha'\,\mathbf{c}\,\beta'). \qquad \square$$

The following example shows that the converse to Theorem 4.12 is not generally valid. We only show the counterexample for $l = r = 1$ and $i = j = +1$; it can be easily adapted for other values $l, r \ge 1$ as well as for antitony.

Example 4.13. In models of FCT over standard Łukasiewicz logic define

$$\mathbf{c}(\alpha, \beta) = 0.5 - 2 \cdot (\alpha \wedge 0.1) + 2 \cdot (\beta \wedge 0.1)$$

for all $\alpha, \beta \in [0, 1]$. Then it is easy to check that $\mathrm{RMon}(\mathbf{c}) = 1$ and $\mathrm{LMon}(\mathbf{c}) = 0.8$; thus $\mathrm{Mon}(\mathbf{c}) = 0.8$. It can also be easily seen that $\mathrm{Cng}(\mathbf{c}) = 0.8$, while

$$\mathrm{MonCng}(\mathbf{c}) \le (0 \to 0.1) \,\&\, (0.1 \to 0) \to (0\,\mathbf{c}\,0.1 \to 0.1\,\mathbf{c}\,0) = 1 \,\&\, 0.9 \to (0.7 \to 0.3) = 0.7.$$

The following theorem shows how the both-sided properties are preserved under closeness (in the sense of $\approx$ or $\approxeq$) of connectives. Like in the unary and single-sided case, $\mathrm{Cng}_{l,r}$ is congruent w.r.t. $\approx^2$ and $\mathrm{Mon}_{i,j}$ w.r.t. $\approxeq$:

Theorem 4.14. FCT *proves for all* $l, r \in \mathbb{N}$ *and* $i, j = \pm 1$:

(C37) $\mathrm{Cng}_{l,r}(\mathbf{c}), \mathbf{c} \approx^2 \mathbf{d} \;\Rightarrow\; \mathrm{Cng}_{l,r}(\mathbf{d})$.
(C38) $\mathrm{Mon}_{i,j}(\mathbf{c}), \mathbf{c} \approxeq \mathbf{d} \;\Rightarrow\; \mathrm{Mon}_{i,j}(\mathbf{d})$.
(C39) $\mathrm{MonCng}_{il,jr}, \mathbf{c} \approxeq \mathbf{d} \;\Rightarrow\; \mathrm{MonCng}_{il,jr}(\mathbf{d})$.

Proof. The claim (C38) is proved by the following chain of implications:

$$(\alpha \le^i \alpha') \,\&\, (\beta \le^j \beta') \;\Rightarrow\; (\alpha\,\mathbf{c}\,\beta \to \alpha'\,\mathbf{c}\,\beta') \;\Rightarrow\; (\alpha\,\mathbf{d}\,\beta \to \alpha'\,\mathbf{d}\,\beta').$$

The first implication is obtained by $\mathrm{Mon}_{i,j}(\mathbf{c})$. The second implication follows from $\mathbf{c} \approxeq \mathbf{d}$ by the fact that $\mathbf{c} \subseteq \mathbf{d}$ implies $\alpha'\,\mathbf{c}\,\beta' \to \alpha'\,\mathbf{d}\,\beta'$ and $\mathbf{d} \subseteq \mathbf{c}$ implies $\alpha\,\mathbf{d}\,\beta \to \alpha\,\mathbf{c}\,\beta$. Generalization and quantifier shifts then conclude the proof. The proofs of (C37) and (C39) are analogous. $\square$

The following theorem shows how the both-argument properties are preserved under compositions and how inclusion, plinth, and height are preserved under compositions with binary connectives.

Theorem 4.15. FCT *proves for all* $m, n, l, r, l_{\mathbf{a}}, l_{\mathbf{b}}, l_{\mathbf{c}}, r_{\mathbf{a}}, r_{\mathbf{b}}, r_{\mathbf{c}} \in \mathbb{N}$ *and* $i, j, h, k = \pm 1$:

(C40) $\mathrm{Cng}_n(\mathbf{u}), \mathrm{Cng}^n_{l,r}(\mathbf{c}) \;\Rightarrow\; \mathrm{Cng}_{nl,nr}(\mathbf{uc})$.
(C41) $\mathrm{Cng}_{l,r}(\mathbf{c}), \mathrm{Cng}^l_m(\mathbf{u}), \mathrm{Cng}^r_n(\mathbf{v}) \;\Rightarrow\; \mathrm{Cng}_{ml+nr}(\mathbf{c}(\mathbf{u}, \mathbf{v}))$, *and analogously for* LCng *and* RCng *of* $\mathbf{c}(\mathbf{a}, \mathbf{b})$.
(C42) $\mathrm{Cng}_{l_{\mathbf{c}}, r_{\mathbf{c}}}(\mathbf{c}), \mathrm{Cng}^{l_{\mathbf{c}}}_{l_{\mathbf{a}}, r_{\mathbf{a}}}(\mathbf{a}), \mathrm{Cng}^{r_{\mathbf{c}}}_{l_{\mathbf{b}}, r_{\mathbf{b}}}(\mathbf{b}) \;\Rightarrow\; \mathrm{Cng}_{l_{\mathbf{a}} l_{\mathbf{c}} + l_{\mathbf{b}} r_{\mathbf{c}}, r_{\mathbf{a}} l_{\mathbf{c}} + r_{\mathbf{b}} r_{\mathbf{c}}}(\mathbf{c}(\mathbf{a}, \mathbf{b}))$.
(C43) $\mathrm{Mon}_i(\mathbf{u}), \triangle \mathrm{Mon}_{ih,ik}(\mathbf{c}) \;\Rightarrow\; \mathrm{Mon}_{h,k}(\mathbf{uc})$.

(C44) $\mathrm{Mon}_{ik,jk}(\mathbf{c}), \triangle\mathrm{Mon}_i(\mathbf{u}), \triangle\mathrm{Mon}_j(\mathbf{v}) \Rightarrow \mathrm{Mon}_k(\mathbf{c}(\mathbf{u},\mathbf{v}))$, *and analogously for* LMon *and* RMon *of* $\mathbf{c}(\mathbf{a},\mathbf{b})$.
(C45) $\mathrm{Mon}_{h,k}(\mathbf{c}), \triangle\mathrm{Mon}_{ih,jh}(\mathbf{a}), \triangle\mathrm{Mon}_{ik,jk}(\mathbf{b}) \Rightarrow \mathrm{Mon}_{i,j}(\mathbf{c}(\mathbf{a},\mathbf{b}))$.
(C46) $\mathrm{MonCng}_{in}(\mathbf{u}), \mathrm{MonCng}^n_{ihl,ikr}(\mathbf{c}) \Rightarrow \mathrm{MonCng}_{hl,kr}(\mathbf{uc})$.
(C47) $\mathrm{MonCng}_{ikl,jkr}(\mathbf{c}), \mathrm{MonCng}^l_{im}(\mathbf{u}), \mathrm{MonCng}^r_{jn}(\mathbf{v}) \Rightarrow \mathrm{MonCng}_{k(ml+nr)}(\mathbf{c}(\mathbf{u},\mathbf{v}))$,
and analogously for LMonCng *and* RMonCng of $\mathbf{c}(\mathbf{a},\mathbf{b})$.
(C48) $\mathrm{MonCng}_{hl_\mathbf{c},kr_\mathbf{c}}(\mathbf{c}), \mathrm{MonCng}^{l_\mathbf{c}}_{ihl_\mathbf{a},jhr_\mathbf{a}}(\mathbf{a}), \mathrm{MonCng}^{r_\mathbf{c}}_{ikl_\mathbf{b},jkr_\mathbf{b}}(\mathbf{b}) \Rightarrow \mathrm{MonCng}_{i(l_\mathbf{a}l_\mathbf{c}+l_\mathbf{b}r_\mathbf{c}),j(r_\mathbf{a}l_\mathbf{c}+r_\mathbf{b}r_\mathbf{c})}(\mathbf{c}(\mathbf{a},\mathbf{b}))$.
(C49) $\mathbf{c} \approx \mathbf{d} \Rightarrow \mathbf{c}(\mathbf{f},\mathbf{g}) \approx \mathbf{d}(\mathbf{f},\mathbf{g})$.
(C50) $\mathbf{c} \subseteq \mathbf{d} \Rightarrow \mathbf{c}(\mathbf{f},\mathbf{g}) \subseteq \mathbf{d}(\mathbf{f},\mathbf{g})$.
(C51) $\mathrm{Cng}_{l,r}(\mathbf{c}), \mathbf{f} \approx^l \mathbf{f}', \mathbf{g} \approx^r \mathbf{g}' \Rightarrow \mathbf{c}(\mathbf{f},\mathbf{g}) \approx \mathbf{c}(\mathbf{f}',\mathbf{g}')$.
(C52) $\mathrm{Mon}_{i,j}(\mathbf{c}), \mathbf{f} \sqsubseteq^i \mathbf{f}', \mathbf{g} \sqsubseteq^j \mathbf{g}' \Rightarrow \mathbf{c}(\mathbf{f},\mathbf{g}) \subseteq \mathbf{c}(\mathbf{f}',\mathbf{g}')$.
(C53) $\mathrm{MonCng}_{il,jr}(\mathbf{c}), \mathbf{f} \subseteq^{il} \mathbf{f}', \mathbf{g} \subseteq^{jr} \mathbf{g}' \Rightarrow \mathbf{c}(\mathbf{f},\mathbf{g}) \subseteq \mathbf{c}(\mathbf{f}',\mathbf{g}')$.
(C54) $\mathrm{Plt}(\mathbf{c}) \Rightarrow \mathrm{Plt}(\mathbf{c}(\mathbf{f},\mathbf{g}))$.
(C55) $\mathrm{Plt}(\mathbf{f}) \wedge \mathrm{Plt}(\mathbf{g}), \wedge \subseteq \mathbf{c} \Rightarrow \mathrm{Plt}(\mathbf{c}(\mathbf{f},\mathbf{g}))$.
(C56) $\mathrm{Hgt}(\mathbf{c}), \mathrm{Cng}_{l,r}(\mathbf{c}), \mathrm{Surj}_{l,r}(\mathbf{f},\mathbf{g}) \Rightarrow \mathrm{Hgt}(\mathbf{c}(\mathbf{f},\mathbf{g}))$, *where* $\mathrm{Surj}_{l,r}(\mathbf{f},\mathbf{g}) \equiv_{\mathrm{df}} (\forall\gamma\delta)(\exists\vec{\alpha})((\gamma \leftrightarrow \mathbf{f}\vec{\alpha})^l \,\&\, (\delta \leftrightarrow \mathbf{g}\vec{\alpha})^r)$.
(C57) $\mathrm{Hgt}(\mathbf{f}) \vee \mathrm{Hgt}(\mathbf{g}), \vee \subseteq \mathbf{c} \Rightarrow \mathrm{Hgt}(\mathbf{c}(\mathbf{f},\mathbf{g}))$.

Proof. The claims (C49)–(C50) and (C54) are trivial (by specification) and the proofs of (C51)–(C53) are straightforward. The proofs of (C46)–(C48) are similar to the proofs of (C40)–(C42) and (C43)–(C45). The remaining proofs run as follows:

(C40) By $\mathrm{Cng}_{l,r}(\mathbf{c})$ we have $(\alpha\,\mathbf{c}\,\alpha')^l, (\beta\,\mathbf{c}\,\beta')^r \Rightarrow (\alpha\,\mathbf{c}\,\beta \leftrightarrow \alpha'\,\mathbf{c}\,\beta')$. Combining n copies of this implication we obtain $(\alpha\,\mathbf{c}\,\alpha')^{nl}, (\beta\,\mathbf{c}\,\beta')^{nr} \Rightarrow (\alpha\,\mathbf{c}\,\beta \leftrightarrow \alpha'\,\mathbf{c}\,\beta')^n$, whence the claim follows by $\mathrm{Cng}_n(\mathbf{u})$.

(C41) By definition, we have $(\alpha \leftrightarrow \beta)^m, \mathrm{Cng}_m(\mathbf{u}) \Rightarrow \mathbf{u}\alpha \leftrightarrow \mathbf{u}\beta$ and $(\alpha \leftrightarrow \beta)^n, \mathrm{Cng}_n(\mathbf{v}) \Rightarrow \mathbf{v}\alpha \leftrightarrow \mathbf{v}\beta$ By combining l resp. r copies of these implications we obtain

$$(\alpha \leftrightarrow \beta)^{m\cdot l}, \mathrm{Cng}^l_m(\mathbf{u}) \Rightarrow (\mathbf{u}\alpha \leftrightarrow \mathbf{u}\beta)^l$$
$$(\alpha \leftrightarrow \beta)^{n\cdot r}, \mathrm{Cng}^r_n(\mathbf{v}) \Rightarrow (\mathbf{v}\alpha \leftrightarrow \mathbf{v}\beta)^r,$$

whence by $\mathrm{Cng}_{l,r}(\mathbf{c})$ we obtain the required consequent $\mathbf{u}\alpha\,\mathbf{c}\,\mathbf{v}\alpha \leftrightarrow \mathbf{u}\beta\,\mathbf{c}\,\mathbf{v}\beta$.

To prove the claim for LCng, generalize (on β) the unary case of (C41) for $\mathbf{u} = \mathbf{a}(\mathbf{id}, \underline{\beta})$ and $\mathbf{v} = \mathbf{b}(\mathbf{id}, \underline{\beta})$, and distribute the quantifier $(\forall\beta)$ to apply Observation 4.2 The claim for RCng is proved analogously.

(C42) Similarly as in the proof of (C41), we combine $l_\mathbf{c}$ resp. $r_\mathbf{c}$ copies of

$$(\alpha \leftrightarrow \alpha')^{l_\mathbf{a}}, (\beta \leftrightarrow \beta')^{r_\mathbf{a}}, \mathrm{Cng}_{l_\mathbf{a},r_\mathbf{a}}(\mathbf{a}) \Rightarrow \alpha\,\mathbf{a}\,\beta \leftrightarrow \alpha'\,\mathbf{a}\,\beta'$$
$$(\alpha \leftrightarrow \alpha')^{l_\mathbf{b}}, (\beta \leftrightarrow \beta')^{r_\mathbf{b}}, \mathrm{Cng}_{l_\mathbf{b},r_\mathbf{b}}(\mathbf{a}) \Rightarrow \alpha\,\mathbf{a}\,\beta \leftrightarrow \alpha'\,\mathbf{a}\,\beta'$$

to obtain

$$(\alpha \leftrightarrow \alpha')^{l_\mathbf{a}\cdot l_\mathbf{c}}, (\beta \leftrightarrow \beta')^{r_\mathbf{a}\cdot l_\mathbf{c}}, \mathrm{Cng}^{l_\mathbf{c}}_{l_\mathbf{a},r_\mathbf{a}}(\mathbf{a}) \Rightarrow (\alpha\,\mathbf{a}\,\beta \leftrightarrow \alpha'\,\mathbf{a}\,\beta')^{l_\mathbf{c}}$$
$$(\alpha \leftrightarrow \alpha')^{l_\mathbf{b}\cdot r_\mathbf{c}}, (\beta \leftrightarrow \beta')^{r_\mathbf{b}\cdot r_\mathbf{c}}, \mathrm{Cng}^{r_\mathbf{c}}_{l_\mathbf{b},r_\mathbf{b}}(\mathbf{a}) \Rightarrow (\alpha\,\mathbf{a}\,\beta \leftrightarrow \alpha'\,\mathbf{a}\,\beta')^{r_\mathbf{c}},$$

whence by $\mathrm{Cng}_{l_\mathbf{c},r_\mathbf{c}}(\mathbf{c})$ we obtain the desired $(\alpha\,\mathbf{a}\,\beta)\,\mathbf{c}\,(\alpha\,\mathbf{b}\,\beta) \leftrightarrow (\alpha'\,\mathbf{a}\,\beta')\,\mathbf{c}\,(\alpha'\,\mathbf{b}\,\beta')$.

(C43) From $\alpha \leq^h \alpha'$ and $\beta \leq^k \beta'$ we obtain $\alpha\,\mathbf{c}\,\beta \leq^i \alpha'\,\mathbf{c}\,\beta'$ by $\triangle\mathrm{Mon}_{ih,ik}(\mathbf{c})$, whence $\mathbf{u}(\alpha\,\mathbf{c}\,\beta) \to \mathbf{u}(\alpha'\,\mathbf{c}\,\beta')$ by $\mathrm{Mon}_i(\mathbf{u})$.

(C44) By definition we have $\alpha \leq^k \beta, \triangle\mathrm{Mon}_i(\mathbf{u}) \Rightarrow \mathbf{u}\alpha \leq^{i\cdot k} \mathbf{u}\beta$ and $\alpha \leq^k \beta, \triangle\mathrm{Mon}_j(\mathbf{v}) \Rightarrow \mathbf{v}\alpha \leq^{j\cdot k} \mathbf{v}\beta$, whence by $\mathrm{Mon}_{i\cdot k,j\cdot k}(\mathbf{c})$ we obtain $\mathrm{Mon}_k(\mathbf{c}(\mathbf{u},\mathbf{v}))$. The claims for LMon and RMon follow from the unary case in a similar manner as in the proof of (C41).

(C45) By definition we have

$$\alpha \leq^i \alpha', \beta \leq^j \beta', \triangle\mathrm{Mon}_{ih,jh}(\mathbf{a}) \Rightarrow \alpha\,\mathbf{a}\,\beta \leq^h \alpha'\,\mathbf{a}\,\beta'$$
$$\alpha \leq^i \alpha', \beta \leq^j \beta', \triangle\mathrm{Mon}_{ik,jk}(\mathbf{b}) \Rightarrow \alpha\,\mathbf{b}\,\beta \leq^k \alpha'\,\mathbf{b}\,\beta'$$

whence by $\mathrm{Mon}_{h,k}(\mathbf{c})$ we obtain $\mathrm{Mon}_{i,j}(\mathbf{c}(\mathbf{a},\mathbf{b}))$.

(C55) Let $\vec{\gamma}$ be a tuple of the appropriate arity for $\mathbf{f}$ and $\mathbf{g}$. By $\wedge \subseteq \mathbf{c}$ we have $\mathbf{f}\vec{\gamma} \wedge \mathbf{g}\vec{\gamma} \to \mathbf{c}(\mathbf{f}\vec{\gamma}, \mathbf{g}\vec{\gamma})$. By generalizing on $\vec{\gamma}$ and distributing the quantifier we obtain $(\forall\vec{\gamma})\mathbf{f}\vec{\gamma} \wedge (\forall\vec{\gamma})\mathbf{g}\vec{\gamma} \to (\forall\vec{\gamma})\mathbf{c}(\mathbf{f}\vec{\gamma}, \mathbf{g}\vec{\gamma})$, which is the required $\mathrm{Plt}(\mathbf{f}) \wedge \mathrm{Plt}(\mathbf{g}) \to \mathrm{Plt}(\mathbf{c}(\mathbf{f},\mathbf{g}))$.

(C56) The proof is analogous to that of (C15).
(C57) Similarly as in the proof of (C55), by $\vee \subseteq \mathbf{c}$ we have $\mathbf{f}\vec{\gamma} \vee \mathbf{g}\vec{\gamma} \to \mathbf{c}(\mathbf{f}\vec{\gamma}, \mathbf{g}\vec{\gamma})$. By generalizing on $\vec{\gamma}$ and distributing the quantifier (this time as $\exists$), we obtain $(\exists\vec{\gamma})\mathbf{f}\vec{\gamma} \vee (\exists\vec{\gamma})\mathbf{g}\vec{\gamma} \to (\exists\vec{\gamma})\mathbf{c}(\mathbf{f}\vec{\gamma}, \mathbf{g}\vec{\gamma})$, i.e., the required $\mathrm{Hgt}(\mathbf{f}) \vee \mathrm{Hgt}(\mathbf{g}) \to \mathrm{Hgt}(\mathbf{c}(\mathbf{f}, \mathbf{g}))$. □

The following observation lists how usual binary logical connectives ($\wedge$, $\vee$, &, $\to$, $\leftrightarrow$) satisfy the both-argument monotony and congruence properties.

Theorem 4.16. FCT *proves*:

(C58) $\mathrm{Cng}(\mathbf{c})$ *for* $\mathbf{c} \in \{\wedge, \vee, \&, \to, \leftrightarrow\}$.
(C59) $\mathrm{Mon}(\mathbf{c})$, $\mathrm{MonCng}(\mathbf{c})$ *for* $\mathbf{c} \in \{\wedge, \vee, \&\}$.
(C60) $\mathrm{Mon}_{-1,+1}(\to)$, $\mathrm{MonCng}_{-1,+1}(\to)$, $\neg\,\mathrm{Mon}_{i,j}(\to)$, $\neg\,\mathrm{MonCng}_{i,j}(\to)$ *for* $(i, j) \neq (-1, +1)$.
(C61) $\neg\,\mathrm{Mon}_{i,j}(\leftrightarrow)$, $\neg\,\mathrm{MonCng}_{i,j}(\leftrightarrow)$ *for* $i, j = \pm 1$.

Proof. The positive claims are corollaries of Theorems 4.6 and 4.10. The negative claims of (C60) and (C61) are obtained by specifying the values 0 and 1, e.g., $\mathrm{Mon}_{1,1}(\leftrightarrow) \Rightarrow ((0 \le 0)\,\&\,(0 \le 1) \to ((0 \leftrightarrow 0) \to (0 \leftrightarrow 1))) \Leftrightarrow 0$. □

The following corollary of Theorems 4.15 and 4.16 shows how the properties of congruence and monotony are preserved by basic logical connectives:

Corollary 4.17. FCT *proves for* $i, j = \pm 1$, *and any* $m, n, l_{\mathbf{a}}, l_{\mathbf{b}}, r_{\mathbf{a}}, r_{\mathbf{b}}, \in \mathbb{N}$:

(C62) $\mathrm{Cng}_m(\mathbf{u}), \mathrm{Cng}_n(\mathbf{v}) \Rightarrow \mathrm{Cng}_{m+n}(\mathbf{c}(\mathbf{u}, \mathbf{v}))$, *for* $\mathbf{c} \in \{\&, \wedge, \vee, \to, \leftrightarrow\}$.
In particular, $\mathrm{Cng}(\mathbf{u}), \mathrm{Cng}(\mathbf{v}) \Rightarrow \mathrm{Cng}_2(\mathbf{u}\,\&\,\mathbf{v})$, *and similarly for* $\wedge$, $\vee$, $\to$, *and* $\leftrightarrow$.
(C63) $\mathrm{LCng}_m(\mathbf{u}), \mathrm{LCng}_n(\mathbf{v}) \Rightarrow \mathrm{LCng}_{m+n}(\mathbf{c}(\mathbf{u}, \mathbf{v}))$, *for* $\mathbf{c} \in \{\&, \wedge, \vee, \to, \leftrightarrow\}$, *and analogously for* RCng.
(C64) $\mathrm{Cng}_{l_{\mathbf{a}}, r_{\mathbf{a}}}(\mathbf{a}), \mathrm{Cng}_{l_{\mathbf{b}}, r_{\mathbf{b}}}(\mathbf{b}) \Rightarrow \mathrm{Cng}_{l_{\mathbf{a}}+l_{\mathbf{b}}, r_{\mathbf{a}}+r_{\mathbf{b}}}(\mathbf{c}(\mathbf{a}, \mathbf{b}))$, *for* $\mathbf{c} \in \{\&, \wedge, \vee, \to, \leftrightarrow\}$.
In particular, $\mathrm{Cng}(\mathbf{a}), \mathrm{Cng}(\mathbf{b}) \Rightarrow \mathrm{Cng}_{2,2}(\mathbf{a} \to \mathbf{b})$, *and similarly for* $\wedge$, $\vee$, &, *and* $\leftrightarrow$.
(C65) $\mathrm{MonCng}_{im}(\mathbf{u}), \mathrm{MonCng}_{in}(\mathbf{v}) \Rightarrow \mathrm{MonCng}_{i(m+n)}(\mathbf{c}(\mathbf{u}, \mathbf{v}))$, *for* $\mathbf{c} \in \{\&, \wedge, \vee\}$.
(C66) $\mathrm{MonCng}_{-im}(\mathbf{u}), \mathrm{MonCng}_{in}(\mathbf{v}) \Rightarrow \mathrm{MonCng}_{i(m+n)}(\mathbf{u} \to \mathbf{v})$.
(C67) $\mathrm{LMonCng}_{im}(\mathbf{a}), \mathrm{LMonCng}_{in}(\mathbf{b}) \Rightarrow \mathrm{LMonCng}_{i(m+n)}(\mathbf{c}(\mathbf{a}, \mathbf{b}))$, *for* $\mathbf{c} \in \{\&, \wedge, \vee\}$, *and analogously for* RMonCng.
(C68) $\mathrm{LMonCng}_{-im}(\mathbf{a}), \mathrm{LMonCng}_{in}(\mathbf{b}) \Rightarrow \mathrm{LMonCng}_{i(m+n)}(\mathbf{a} \to \mathbf{b})$, *and analogously for* RMonCng.
(C69) $\mathrm{MonCng}_{il_{\mathbf{a}}, jr_{\mathbf{a}}}(\mathbf{a}), \mathrm{MonCng}_{il_{\mathbf{b}}, jr_{\mathbf{b}}}(\mathbf{b}) \Rightarrow \mathrm{MonCng}_{i(l_{\mathbf{a}}+l_{\mathbf{b}}), j(r_{\mathbf{a}}+r_{\mathbf{b}})}(\mathbf{c}(\mathbf{a}, \mathbf{b}))$, *for* $\mathbf{c} \in \{\&, \wedge, \vee\}$.
(C70) $\mathrm{MonCng}_{-il_{\mathbf{a}}, -jr_{\mathbf{a}}}(\mathbf{a}), \mathrm{MonCng}_{il_{\mathbf{b}}, jr_{\mathbf{b}}}(\mathbf{b}) \Rightarrow \mathrm{MonCng}_{i(l_{\mathbf{a}}+l_{\mathbf{b}}), j(r_{\mathbf{a}}+r_{\mathbf{b}})}(\mathbf{a} \to \mathbf{b})$.

Finally, we shall investigate the transmission of graded congruence and monotony by composition with connectives close (in the sense of $\approx$) to the identity. (Cf. Section 1 for motivation and the end of Section 3 for analogous results on unary connectives.)

In the binary case, a slight distortion (e.g., by rounding or noise) of the resulting value of $\mathbf{c}$ corresponds to the composition $\mathbf{vc}$, where $\mathbf{v}$ is close (in the sense of $\approx$) to $\mathbf{id}$, while a slight distortion of the arguments of $\mathbf{c}$ corresponds to the composition $\mathbf{c}(\mathbf{v}_1\mathbf{p}_1, \mathbf{v}_2\mathbf{p}_2)$, where $\mathbf{p}_1, \mathbf{p}_2$ are the projections [14] and $\mathbf{v}_1, \mathbf{v}_2$ are close (in the sense of $\approx$) to $\mathbf{id}$. The following theorem gives lower bounds for the values of graded congruence and monotony propagated to such compositions. The Theorems (C73)–(C76) are formulated for left-argument properties only; they hold analogously for the right-argument properties. Besides (C77), also theorems (C13), (C14), and (C54) give lower bounds for the height and plinth of $\mathbf{vc}$ or $\mathbf{c}(\mathbf{v}_1\mathbf{p}_1, \mathbf{v}_2\mathbf{p}_2)$.

Theorem 4.18. FCT *proves*, *for all* $n, l, r \in \mathbb{N}$ *and* $i, j = \pm 1$:

(C71) $\mathbf{v} \approx \mathbf{id} \Rightarrow \mathbf{vc} \approx \mathbf{c}$.
(C72) $\mathbf{v}_1 \approx^l \mathbf{id}$, $\mathbf{v}_2 \approx^r \mathbf{id}$, $\mathrm{Cng}_{l,r}(\mathbf{c}) \Rightarrow \mathbf{c}(\mathbf{v}_1\mathbf{p}_1, \mathbf{v}_2\mathbf{p}_2) \approx \mathbf{c}$.

[14] Recall that by Definition 2.5, $\mathbf{c}(\mathbf{v}_1\mathbf{p}_1, \mathbf{v}_2\mathbf{p}_2)(\alpha, \beta) = \mathbf{c}(\mathbf{v}_1\alpha, \mathbf{v}_2\beta)$.

(C73) $\mathrm{LCng}_n(\mathbf{c}), \mathbf{v} \approx^2 \mathbf{id} \;\Rightarrow\; \mathrm{LCng}_n(\mathbf{vc})$.
(C74) $\mathrm{LCng}_n(\mathbf{c}), \mathrm{Cng}^2_{l,r}(\mathbf{c}), \mathbf{v}_1 \approx^{2l} \mathbf{id}, \mathbf{v}_2 \approx^{2r} \mathbf{id} \;\Rightarrow\; \mathrm{LCng}_n(\mathbf{c}(\mathbf{v}_1\mathbf{p}_1, \mathbf{v}_2\mathbf{p}_2))$.
(C75) $\mathrm{LMon}_i(\mathbf{c}), \mathbf{v}{\cong}\mathbf{id} \;\Rightarrow\; \mathrm{LMon}_i(\mathbf{vc})$.
(C76) $\mathrm{LMon}_i(\mathbf{c}), \mathrm{Cng}^2_{l,r}(\mathbf{c}), \mathbf{v}_1 \approx^{2l} \mathbf{id}, \mathbf{v}_2 \approx^{2r} \mathbf{id} \;\Rightarrow\; \mathrm{LMon}_i(\mathbf{c}(\mathbf{v}_1\mathbf{p}_1, \mathbf{v}_2\mathbf{p}_2))$.
(C77) $\mathrm{Hgt}(\mathbf{c}), \mathrm{Cng}_n(\mathbf{c}), \mathbf{v} \approx^n \mathbf{id} \;\Rightarrow\; \mathrm{Hgt}(\mathbf{vc})$.
(C78) $\mathrm{Cng}_{l,r}(\mathbf{c}), \mathbf{v} \approx^2 \mathbf{id} \;\Rightarrow\; \mathrm{Cng}_{l,r}(\mathbf{vc})$.
(C79) $\mathrm{Cng}^3_{l,r}(\mathbf{c}), \mathbf{v}_1 \approx^{2l} \mathbf{id}, \mathbf{v}_2 \approx^{2r} \mathbf{id} \;\Rightarrow\; \mathrm{Cng}_{l,r}(\mathbf{c}(\mathbf{v}_1\mathbf{p}_1, \mathbf{v}_2\mathbf{p}_2))$.
(C80) $\mathrm{Mon}_{i,j}(\mathbf{c}), \mathbf{v}{\cong}\mathbf{id} \;\Rightarrow\; \mathrm{Mon}_{i,j}(\mathbf{vc})$.
(C81) $\mathrm{Mon}_{i,j}(\mathbf{c}), \mathrm{Cng}^2_{l,r}(\mathbf{c}), \mathbf{v}_1 \approx^{2l} \mathbf{id}, \mathbf{v}_2 \approx^{2r} \mathbf{id} \;\Rightarrow\; \mathrm{Mon}_{i,j}(\mathbf{c}(\mathbf{v}_1\mathbf{p}_1, \mathbf{v}_2\mathbf{p}_2))$.
(C82) $\mathrm{MonCng}_{il,jr}(\mathbf{c}), \mathbf{v}{\cong}\mathbf{id} \;\Rightarrow\; \mathrm{MonCng}_{il,jr}(\mathbf{vc})$.
(C83) $\mathrm{MonCng}_{il,jr}(\mathbf{c}), \mathrm{Cng}^2_{l,r}(\mathbf{c}), \mathbf{v}_1 \approx^{2l} \mathbf{id}, \mathbf{v}_2 \approx^{2r} \mathbf{id} \;\Rightarrow\; \mathrm{MonCng}_{il,jr}(\mathbf{c}(\mathbf{v}_1\mathbf{p}_1, \mathbf{v}_2\mathbf{p}_2))$.

Proof. The claim (C71) follows directly from (C20).

(C72) The claim follows straightforwardly from (C51), using the facts that $\mathbf{v}_i \approx \mathbf{id} \Rightarrow \mathbf{v}_i\mathbf{p}_i \approx \mathbf{p}_i$ for $i \in \{1, 2\}$ by (U30) and that $\mathbf{c}(\mathbf{id}\,\mathbf{p}_1, \mathbf{id}\,\mathbf{p}_2) = \mathbf{c}(\mathbf{p}_1, \mathbf{p}_2) = \mathbf{c}$.
(C73) The proof is analogous to that of (U32), from which it can also be obtained analogously to the proof of (C12).
(C74) The claim follows directly from (C72) and (C9).
(C75) The proof is analogous to that of (U34), from which it can also be obtained analogously to the proof of (C12).
(C76) The claim follows straightforwardly from (C72), (C10), and (10).
(C77) The proof is analogous to that of (U36); cf. the proof of (C15).
(C78) The claim follows from (C71) and (C37), or alternatively from (U21), (U6), and (C40) analogously to the proof of (U33).
(C79) The claim follows directly from (C37) and (C72).
(C80) By (C19) we obtain $\mathbf{v}{\cong}\mathbf{id} \Rightarrow \mathbf{vc}{\cong}\mathbf{c}$ and apply (C38).
(C81) The claim follows straightforwardly from (C72), (C38), and (10).
The proofs of (C82)–(C83) are analogous to those of (C80)–(C81), just using (C39)instead of (C38). □

Example 4.19. We shall use Theorem 4.18 to estimate the standard Łukasiewicz values of graded congruence and monotony of the product t-norm $\mathbf{t}_\Pi(\alpha, \beta) = \alpha \cdot \beta$ distorted by rounding (see Fig. 4) and noise. Recall from Example 3.21 that $(\mathbf{w}_{\mathrm{rd}} \approx \mathbf{id}) = 0.995$. Obviously $\mathrm{Mon}(\mathbf{t}_\Pi) = 1$ and $\mathrm{LCng}(\mathbf{t}_\Pi) = \mathrm{RCng}(\mathbf{t}_\Pi) = 1$; thus also $\mathrm{Cng}(\mathbf{t}_\Pi) = 1$ by (C33). Consequently, the claims of Theorem 4.18 yield the following estimates:

- By (C71)–(C72) we obtain $(\mathbf{w}_{\mathrm{rd}}\mathbf{t}_\Pi \approx \mathbf{t}_\Pi) \geq 0.995$, which is the exact value, and $(\mathbf{t}_\Pi(\mathbf{w}_{\mathrm{rd}}\mathbf{p}_1, \mathbf{w}_{\mathrm{rd}}\mathbf{p}_2) \approx \mathbf{t}_\Pi) \geq 0.99$, which is a very tight estimate, the actual value being just $0.995 \cdot 0.995 = 0.990025$.
- By (C73)–(C74) we have $\mathrm{LCng}(\mathbf{w}_{\mathrm{rd}}\mathbf{t}_\Pi) \geq 0.99$, which is the actual value, and $\mathrm{LCng}(\mathbf{t}_\Pi(\mathbf{w}_{\mathrm{rd}}\mathbf{p}_1, \mathbf{w}_{\mathrm{rd}}\mathbf{p}_2)) \geq 0.98$.
- By (C78)–(C79) we obtain $\mathrm{Cng}(\mathbf{w}_{\mathrm{rd}}\mathbf{t}_\Pi) \geq 0.99$ and $\mathrm{Cng}(\mathbf{t}_\Pi(\mathbf{w}_{\mathrm{rd}}\mathbf{p}_1, \mathbf{w}_{\mathrm{rd}}\mathbf{p}_2)) \geq 0.98$.

As $(\mathbf{w}_{\mathrm{rd}} \approx \mathbf{id}) = (\mathbf{w}_{\mathrm{ns}} \approx \mathbf{id})$, the same estimates are obtained for $\mathbf{w}_{\mathrm{ns}}\mathbf{t}_\Pi$ and $\mathbf{t}_\Pi(\mathbf{w}_{\mathrm{ns}}\mathbf{p}_1, \mathbf{w}_{\mathrm{ns}}\mathbf{p}_2)$. Since $\mathbf{w}_{\mathrm{rd}}$ as well as $\mathbf{t}_\Pi$ are fully monotone, so are $\mathbf{w}_{\mathrm{rd}}\mathbf{t}_\Pi$ and $\mathbf{t}_\Pi(\mathbf{w}_{\mathrm{rd}}\mathbf{p}_1, \mathbf{w}_{\mathrm{rd}}\mathbf{p}_2)$ by (C43)–(C44). Both (C43) and (C80) give us the same estimate $\mathrm{Mon}(\mathbf{w}_{\mathrm{rd}}\mathbf{t}_\Pi) \geq 0.99$; however, as $\mathbf{w}_{\mathrm{ns}}$ is less than fully monotone, no useful bound for $\mathrm{Mon}(\mathbf{t}_\Pi(\mathbf{w}_{\mathrm{ns}}\mathbf{p}_1, \mathbf{w}_{\mathrm{ns}}\mathbf{p}_2))$ is yielded by (C43); nevertheless, we can obtain $\mathrm{Mon}(\mathbf{t}_\Pi(\mathbf{w}_{\mathrm{ns}}\mathbf{p}_1, \mathbf{w}_{\mathrm{ns}}\mathbf{p}_2)) \geq 0.98$ by (C80).

5. Graded null and unit elements

In this section we shall investigate graded generalizations of unit and null elements, introduced in [12,13] by the following definitions (for any binary connective $\mathbf{c} \sqsubseteq \mathrm{L} \times \mathrm{L}$):

$$\mathrm{LUnit}(\mathbf{c}, \eta) \equiv_{\mathrm{df}} (\forall\alpha)(\eta\,\mathbf{c}\,\alpha \leftrightarrow \alpha) \qquad \text{Left-unit element}$$

$$\mathrm{RUnit}(\mathbf{c}, \eta) \equiv_{\mathrm{df}} (\forall\alpha)(\alpha\,\mathbf{c}\,\eta \leftrightarrow \alpha) \qquad \text{Right-unit element}$$

$$\mathrm{LNull}(\mathbf{c}, \eta) \equiv_{\mathrm{df}} (\forall\alpha)(\eta\,\mathbf{c}\,\alpha \leftrightarrow \eta) \qquad \text{Left-null element}$$

$$\mathrm{RNull}(\mathbf{c}, \eta) \equiv_{\mathrm{df}} (\forall\alpha)(\alpha\,\mathbf{c}\,\eta \leftrightarrow \eta) \qquad \text{Right-null element}$$

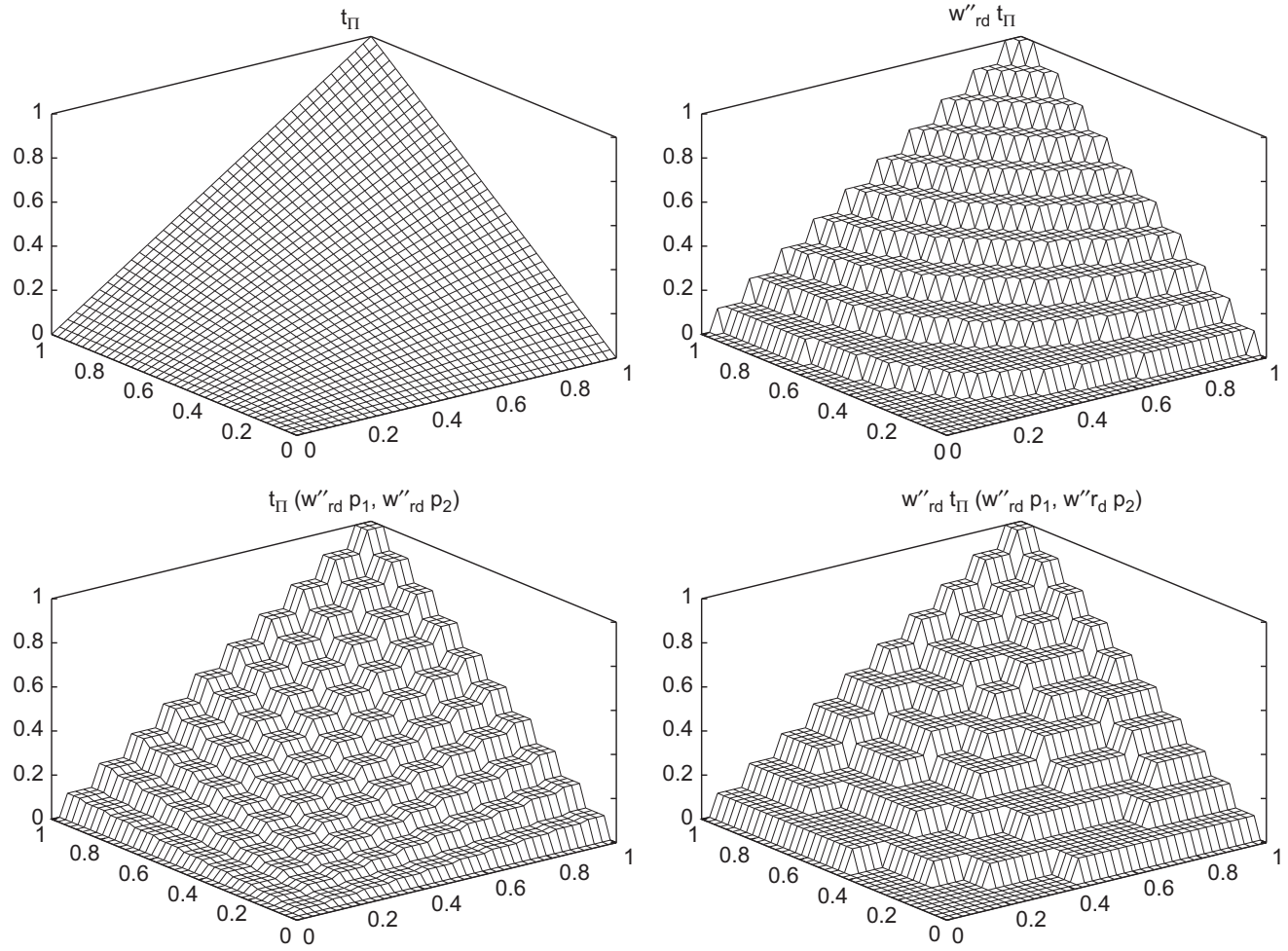

Fig. 4. The connective $\mathbf{t}_{\Pi}$ (the product t-norm) distorted by rounding. The upper row: graphs of the connectives $\mathbf{t}_{\Pi}$ (i.e., the product undistorted) and $\mathbf{w}''_{\mathrm{rd}}\mathbf{t}_{\Pi}$ (i.e., the rounded product). The lower row: graphs of the connectives $\mathbf{t}_{\Pi}(\mathbf{w}''_{\mathrm{rd}}\mathbf{p}_1, \mathbf{w}''_{\mathrm{rd}}\mathbf{p}_2)$, i.e., the product of rounded values, and $\mathbf{w}''_{\mathrm{rd}}\mathbf{t}_{\Pi}(\mathbf{w}''_{\mathrm{rd}}\mathbf{p}_1, \mathbf{w}''_{\mathrm{rd}}\mathbf{p}_2)$, i.e., the rounded product of rounded values. For better visual readability of the graphs (cf. Fig. 1), the rounding $\mathbf{w}''_{\mathrm{rd}}$ to just one decimal place has been used in the plot instead of the rounding $\mathbf{w}_{\mathrm{rd}}$ to two decimal places used for calculations in Example 4.19. The latter three graphs are discontinuous along the obvious lines or curves: as in the previous graphs, the interpolating lines do not represent the actual interpolated values, but only connect the grid values of the functions (cf. the note at Fig. 1).

Non-zero degrees of these properties are ensured for all connectives with a non-zero plinth and a less than full height:

Theorem 5.1. FCT *proves*:

(N1) $\mathrm{Plt}(\mathbf{c}) \wedge \neg\,\mathrm{Hgt}(\mathbf{c}) \;\Rightarrow\; \mathrm{LUnit}(\mathbf{c}, \eta)$
(N2) $\mathrm{Hgt}(\mathbf{c}) \to \mathrm{Plt}(\mathbf{c}) \;\Rightarrow\; \mathrm{LNull}(\mathbf{c}, \eta)$

and analogously for the right-sided properties.

Proof. The claim (N1) follows easily from the following chains of implications:

$$\begin{aligned} \neg\,\mathrm{Hgt}(\mathbf{c}) \;&\Leftrightarrow\; \mathrm{Hgt}(\mathbf{c}) \to 0 \;\Rightarrow\; \mathrm{Hgt}(\mathbf{c}) \to \alpha \;\Rightarrow\; \eta\,\mathbf{c}\,\alpha \to \alpha \\ \mathrm{Plt}(\mathbf{c}) \;&\Leftrightarrow\; 1 \to \mathrm{Plt}(\mathbf{c}) \;\Rightarrow\; \alpha \to \mathrm{Plt}(\mathbf{c}) \;\Rightarrow\; \alpha \to \eta\,\mathbf{c}\,\alpha \end{aligned}$$

The claim (N2) is proved in a similar manner. □

The following properties of units and nulls have been proved in [12,13]:

$$\begin{aligned}
&\mathrm{LNull}(\mathbf{c},\eta),\mathrm{RNull}(\mathbf{c},\zeta) \;\Rightarrow\; \eta\leftrightarrow\zeta\\
&\mathrm{LUnit}(\mathbf{c},\eta),\mathrm{RUnit}(\mathbf{c},\zeta) \;\Rightarrow\; \eta\leftrightarrow\zeta\\
&\mathrm{LNull}(\mathbf{c},\eta),\mathrm{LUnit}(\mathbf{c},\eta) \;\Rightarrow\; \eta\wedge\neg\eta \qquad (11)\\
&\mathrm{LUnit}(\mathbf{c}),\mathbf{c}\approx\mathbf{d} \;\Rightarrow\; \mathrm{LUnit}(\mathbf{d}) \qquad (12)\\
&\mathrm{LNull}(\mathbf{c}),\mathbf{c}\approx\mathbf{d} \;\Rightarrow\; \mathrm{LNull}(\mathbf{d}) \qquad (13)\\
&\mathrm{LUnit}(\mathbf{c},\eta),\mathrm{LCng}(\mathbf{c}),\eta\leftrightarrow\zeta \;\Rightarrow\; \mathrm{LUnit}(\mathbf{c},\zeta)\\
&\mathrm{LNull}(\mathbf{c},\eta),\mathrm{LCng}(\mathbf{c}),\eta\leftrightarrow^2\zeta \;\Rightarrow\; \mathrm{LNull}(\mathbf{c},\eta)
\end{aligned}$$

and analogously for RNull and RUnit. The latter two claims can be generalized for multiplicity parameters in LCng:

Theorem 5.2. FCT *proves for all* $n\in\mathbb{N}$:

(N3) $\mathrm{LUnit}(\mathbf{c},\eta),\mathrm{LCng}_n(\mathbf{c}),(\eta\leftrightarrow\zeta)^n \;\Rightarrow\; \mathrm{LUnit}(\mathbf{c},\zeta)$.
(N4) $\mathrm{LNull}(\mathbf{c},\eta),\mathrm{LCng}_n(\mathbf{c}),(\eta\leftrightarrow\zeta)^{n+1} \;\Rightarrow\; \mathrm{LNull}(\mathbf{c},\zeta)$.

and analogously for RUnit *and* RNull.

Proof.

(N3) $\zeta\,\mathbf{c}\,\alpha\Leftrightarrow\eta\,\mathbf{c}\,\alpha\Leftrightarrow\alpha$, where the first implication obtains by $(\eta\leftrightarrow\zeta)^n$ and $\mathrm{LCng}_n(\mathbf{c})$, and the second by $\mathrm{LUnit}(\mathbf{c},\eta)$.
(N4) The claim is proved by the following chain of implications:

$$\begin{aligned}
\zeta\,\mathbf{c}\,\alpha &\;\Leftrightarrow\; \eta\,\mathbf{c}\,\alpha \quad \text{by } \mathrm{LCng}_n(\mathbf{c}),(\eta\leftrightarrow\zeta)^n\\
&\;\Leftrightarrow\; \eta \quad \text{by } \mathrm{LNull}(\mathbf{c},\eta)\\
&\;\Leftrightarrow\; \zeta \quad \text{by } \eta\leftrightarrow\zeta \qquad \square
\end{aligned}$$

Example 5.3. The product t-norm $\mathbf{t}_\Pi$ has the unit 1 and the null 0 (both to degree 1). Since $\mathrm{LCng}(\mathbf{t}_\Pi)=1$ in models over standard Łukasiewicz logic (see Example 4.19), by (N3) we also have, for instance, $\mathrm{LUnit}(\mathbf{t}_\Pi,0.99)\geq 0.99$: indeed, the product $0.99\cdot\alpha$ can differ from α by at most 0.01, so it is natural to claim that 0.99 is a left unit of $\mathbf{t}_\Pi$ at least to degree 0.99. Since moreover $(0.99\cdot 1\leftrightarrow 1)=0.99$, the estimate is actually tight and $\mathrm{LUnit}(\mathbf{t}_\Pi,0.99)=0.99$.

Similarly, (N4) yields $\mathrm{LNull}(\mathbf{t}_\Pi,0.01)\geq 0.98$. This estimate is not tight, as $0.01\cdot\alpha$ can actually differ from 0.01 only by at most 0.01, so in fact $\mathrm{LNull}(\mathbf{t}_\Pi,0.01)=0.99$ as well. Nevertheless, the estimate given by (N4) cannot in general be improved, and the multiplicity of $(\eta\leftrightarrow\zeta)^{n+1}$ in (N4) is the best possible. To show this, e.g., for $n=1$, consider the connective $\mathbf{c}(\alpha,\beta)\equiv\mathbf{w}_{\mathrm{N}}(\alpha)$, where $\mathbf{w}_{\mathrm{N}}$ is the unary connective defined in Example 3.1. Since $0.5\;\mathbf{c}\;\beta=\mathbf{w}_{\mathrm{N}}(0.5)=0.5$ for any β, we have $\mathrm{LNull}(\mathbf{c},0.5)=1$. Furthermore, $0.6\;\mathbf{c}\;\beta=\mathbf{w}_{\mathrm{N}}(0.6)=0.4$ for all β, so $\mathrm{LNull}(\mathbf{c},0.6)=(0.4\leftrightarrow 0.6)=0.8$, which is the value of $(0.5\leftrightarrow 0.6)^2$ in standard Łukasiewicz logic. Since $\mathrm{Cng}(\mathbf{w}_{\mathrm{N}})=1$, the bound given by (N4) for $\mathbf{c}$ is therefore tight.

The following theorems address the preservation of nulls and units under composition. Notice that the property $\mathbf{c}(\mathbf{id},\mathbf{id})\approx\mathbf{id}$, needed in the theorem on preservation of units, is in fact the graded idempotence of $\mathbf{c}$ studied below in Section 6.

Theorem 5.4. FCT *proves*:

(N5) $\mathrm{LUnit}(\mathbf{a},\eta),\mathrm{LUnit}(\mathbf{b},\eta),\mathrm{Cng}(\mathbf{c}),\mathbf{c}(\mathbf{id},\mathbf{id})\approx\mathbf{id} \;\Rightarrow\; \mathrm{LUnit}(\mathbf{c}(\mathbf{a},\mathbf{b}),\eta)$, and analogously for RUnit.
(N6) $\mathrm{LNull}(\mathbf{a},\eta),\mathrm{LNull}(\mathbf{c},\eta),\mathrm{LCng}(\mathbf{c}) \;\Rightarrow\; \mathrm{LNull}(\mathbf{c}(\mathbf{a},\mathbf{b}),\eta)$.
(N7) $\mathrm{LNull}(\mathbf{b},\eta),\mathrm{RNull}(\mathbf{c},\eta),\mathrm{RCng}(\mathbf{c}) \;\Rightarrow\; \mathrm{LNull}(\mathbf{c}(\mathbf{a},\mathbf{b}),\eta)$.
(N8) $\mathrm{RNull}(\mathbf{a},\eta),\mathrm{LNull}(\mathbf{c},\eta),\mathrm{LCng}(\mathbf{c}) \;\Rightarrow\; \mathrm{RNull}(\mathbf{c}(\mathbf{a},\mathbf{b}),\eta)$.
(N9) $\mathrm{RNull}(\mathbf{b},\eta),\mathrm{RNull}(\mathbf{c},\eta),\mathrm{RCng}(\mathbf{c}) \;\Rightarrow\; \mathrm{RNull}(\mathbf{c}(\mathbf{a},\mathbf{b}),\eta)$.

Proof.

(N5) By $\mathrm{LUnit}(\mathbf{a},\eta)$ and $\mathrm{LUnit}(\mathbf{b},\eta)$ we have $\eta\,\mathbf{a}\,\alpha\leftrightarrow\alpha$ and $\eta\,\mathbf{b}\,\alpha\leftrightarrow\alpha$, thus

$$\begin{aligned}
(\eta\,\mathbf{a}\,\alpha)\,\mathbf{c}\,(\eta\,\mathbf{b}\,\alpha) &\;\Leftrightarrow\; \alpha\,\mathbf{c}\,\alpha \quad \text{by } \eta\,\mathbf{a}\,\alpha\leftrightarrow\alpha,\eta\,\mathbf{b}\,\alpha\leftrightarrow\alpha,\mathrm{Cng}_{1,1}(\mathbf{c})\\
&\;\Leftrightarrow\; \alpha \quad \text{by } \mathrm{Idem}(\mathbf{c})
\end{aligned}$$

(N6) By LNull($\mathbf{a}, \eta$) we have $\eta\,\mathbf{a}\,\alpha \leftrightarrow \eta$, thus

$$\begin{aligned}(\eta\,\mathbf{a}\,\alpha)\,\mathbf{c}\,(\eta\,\mathbf{b}\,\alpha) &\Leftrightarrow \eta\,\mathbf{c}\,(\eta\,\mathbf{b}\,\alpha) && \text{by LCng}(\mathbf{c})\\ &\Leftrightarrow \eta && \text{by LNull}(\mathbf{c}, \eta)\end{aligned}$$

The proofs of (N7)–(N9) are analogous. □

The following theorem shows the degrees of null and unit elements of logical connectives [15]:

Theorem 5.5. FCT *proves*:

(N10) $\mathrm{LUnit}(\wedge, \eta) \Leftrightarrow \eta$.
(N11) $\mathrm{LNull}(\wedge, \eta) \Leftrightarrow \neg\eta$.
(N12) $\mathrm{LUnit}(\vee, \eta) \Leftrightarrow \neg\eta$.
(N13) $\mathrm{LNull}(\vee, \eta) \Leftrightarrow \eta$.
(N14) $\eta \Rightarrow \mathrm{LUnit}(\&, \eta) \Rightarrow (\neg\eta \to \eta)$.
(N15) $\mathrm{LNull}(\&, \eta) \Leftrightarrow \neg\eta$.
(N16) $\eta \Rightarrow \mathrm{LUnit}(\to, \eta) \Rightarrow \neg\neg\eta$.
(N17) $\mathrm{LNull}(\to, \eta) \Rightarrow \eta \wedge (\eta \to \neg\eta)$.
(N18) $\neg\,\mathrm{RUnit}(\to, \eta)$.
(N19) $\mathrm{RNull}(\to, \eta) \Leftrightarrow \eta$.
(N20) $\eta \Rightarrow \mathrm{LUnit}(\leftrightarrow, \eta) \Rightarrow \neg\neg\eta$.
(N21) $\mathrm{LNull}(\leftrightarrow, \eta) \Rightarrow \eta \leftrightarrow \neg\eta$.

Proof. The proofs of (N11)–(N13), (N15), and (N19) are analogous to that of (N10):

(N10) The claim is proved by the following chain of equivalences:

$$\begin{aligned}\mathrm{LUnit}(\wedge, \eta) &\Leftrightarrow (\forall\alpha)(\alpha \to \eta \wedge \alpha) && \text{as } \eta \wedge \alpha \to \alpha \text{ is a theorem of MTL}\\ &\Leftrightarrow (\forall\alpha)(\alpha \to \eta) && \text{as } \alpha \to \eta \wedge \alpha \text{ is equivalent to } (\alpha \to \eta) \wedge (\alpha \to \alpha)\\ &\Leftrightarrow \eta && \text{as } \alpha = 1 \text{ yields the lowest value due to } \mathrm{LAnt}(\to)\end{aligned}$$

(N14) The first implication follows from (N3), (C24), and the trivial $\mathrm{LUnit}(\&, 1)$. (A direct proof of the first implication is also easy: it proceeds as in (N10), with $\Rightarrow$ instead of the second $\Leftrightarrow$.) The second implication: by (11) and (N15) we have

$$\mathrm{LUnit}(\&, \eta), \neg\eta \Rightarrow \eta \wedge \neg\eta,$$

i.e., $\mathrm{LUnit}(\&, \eta) \Rightarrow (\neg\eta \to (\eta \wedge \neg\eta))$, and observe that $\neg\eta \to (\eta \wedge \neg\eta) \Leftrightarrow (\neg\eta \to \neg\eta) \wedge (\neg\eta \to \eta)$.
(N16) Since $\alpha \to (\eta \to \alpha)$ is a theorem, $\mathrm{LUnit}(\to, \eta)$ is equivalent to $(\forall\alpha)((\eta \to \alpha) \to \alpha)$. Now the first implication follows from the fact that η implies $(\eta \to \alpha) \to \alpha$ and the second by the specification $\alpha = 0$.
(N17) Specify $\alpha = 0$ and $\alpha = 1$ in $(\forall\alpha)((\eta \to \alpha) \leftrightarrow \eta)$.
(N18) $(\forall\alpha)((\alpha \to \eta) \leftrightarrow \alpha) \Rightarrow ((0 \to \eta) \leftrightarrow 0) \Leftrightarrow 0$.
(N20) The first implication follows from (N3), (C24), and the trivial $\mathrm{LUnit}(\leftrightarrow, 1)$. For the second implication, specify $\alpha = 0$ in $(\forall\alpha)((\eta \leftrightarrow \alpha) \leftrightarrow \alpha)$.
(N21) Specify $\alpha = 0$ in $(\forall\alpha)((\eta \leftrightarrow \alpha) \leftrightarrow \eta)$. □

Corollary 5.6. *By Theorems* 5.4, 5.5, *and* (*C*58), FCT *proves for any* $i \in \{1, 2\}$:

(N22) $\mathrm{LUnit}(\mathbf{a}, \eta), \mathrm{LUnit}(\mathbf{b}, \eta) \Rightarrow \mathrm{LUnit}(\mathbf{c}(\mathbf{a}, \mathbf{b}), \eta)$, *for* $\mathbf{c} \in \{\wedge, \vee\}$.
(N23) $\mathrm{LNull}(\mathbf{a}_i, \eta), \eta \Rightarrow \mathrm{LNull}(\mathbf{a}_1 \vee \mathbf{a}_2, \eta)$.
(N24) $\mathrm{LNull}(\mathbf{a}_i, \eta), \neg\eta \Rightarrow \mathrm{LNull}(\mathbf{c}(\mathbf{a}_1, \mathbf{a}_2), \eta)$, *for* $\mathbf{c} \in \{\wedge, \&\}$.
(N25) $\mathrm{LNull}(\mathbf{b}, \eta), \eta \Rightarrow \mathrm{LNull}(\mathbf{a} \to \mathbf{b}, \eta)$
and analogously for RUnit *in* (N22) *and* RNull *in* (N23)–(N25).

[15] Theorems (N10)–(N19) have already been stated, with proofs omitted, in [12].

Finally, we shall investigate the transmission of unit and null elements to slightly distorted functions, i.e., to compositions $\mathbf{vc}$ and $\mathbf{c}(\mathbf{v}_1\mathbf{p}_1, \mathbf{v}_2\mathbf{p}_2)$, where $\mathbf{p}_1, \mathbf{p}_2$ are projections and $\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2$ are close (in the sense of $\approx$) to the identity (cf. the end of Sections 3 and 4).

Theorem 5.7. FCT *proves for all* $n, l, r \in \mathbb{N}$:

(N26) $\mathrm{LUnit}(\mathbf{c}, \eta), \mathbf{v} \approx \mathbf{id} \Rightarrow \mathrm{LUnit}(\mathbf{vc}, \eta)$.
(N27) $\mathrm{LNull}(\mathbf{c}, \eta), \mathbf{v} \approx \mathbf{id} \Rightarrow \mathrm{LNull}(\mathbf{vc}, \eta)$.
(N28) $\mathrm{LUnit}(\mathbf{c}, \eta), \mathrm{Cng}_{l,r}(\mathbf{c}), \mathbf{v}_1 \approx^l \mathbf{id}, \mathbf{v}_2 \approx^r \mathbf{id} \Rightarrow \mathrm{LUnit}(\mathbf{c}(\mathbf{v}_1\mathbf{p}_1, \mathbf{v}_2\mathbf{p}_2), \eta)$.
(N29) $\mathrm{LNull}(\mathbf{c}, \eta), \mathrm{Cng}_{l,r}(\mathbf{c}), \mathbf{v}_1 \approx^l \mathbf{id}, \mathbf{v}_2 \approx^r \mathbf{id} \Rightarrow \mathrm{LNull}(\mathbf{c}(\mathbf{v}_1\mathbf{p}_1, \mathbf{v}_2\mathbf{p}_2), \eta)$.
(N30) $\mathrm{LUnit}(\mathbf{c}, \eta), \mathrm{LCng}_n(\mathbf{c}), \mathbf{v} \approx^n \mathbf{id} \Rightarrow \mathrm{LUnit}(\mathbf{c}, \mathbf{v}\eta)$.
(N31) $\mathrm{LNull}(\mathbf{c}, \eta), \mathrm{LCng}_n(\mathbf{c}), \mathbf{v} \approx^{n+1} \mathbf{id} \Rightarrow \mathrm{LNull}(\mathbf{c}, \mathbf{v}\eta)$

and analogously for RUnit *and* RNull.

Proof. The claims (N26)–(N27) follow directly from (C71) and (12)–(13); the claims (N28)–(N29) from (C72) and (12)–(13); and the claims (N30)–(N31) from (N3)–(N4) and the fact that $\mathbf{v} \approx^n \mathbf{id} \Rightarrow (\mathbf{v}\eta \leftrightarrow \eta)^n$. □

Example 5.8. In models over standard Łukasiewicz logic, Theorem 5.7 yields the following estimates for the product t-norm $\mathbf{t}_\Pi$ distorted by rounding $\mathbf{w}_{\mathrm{rd}}$ or noise $\mathbf{w}_{\mathrm{ns}}$ (cf. Fig. 4 and Examples 3.1, 3.21, and 4.19):

- $\mathrm{LUnit}(\mathbf{w}_{\mathrm{ns}}\mathbf{t}_\Pi, 1) \geq 0.995$ (tight) and $\mathrm{LUnit}(\mathbf{t}_\Pi(\mathbf{w}_{\mathrm{ns}}\mathbf{p}_1, \mathbf{w}_{\mathrm{ns}}\mathbf{p}_2), 1) \geq 0.99$ (almost tight, actually 0.990025 for the worst case $\mathbf{w}_{\mathrm{ns}}(1) = 0.995$). Since $(\mathbf{w}_{\mathrm{ns}} \approx \mathbf{id}) = (\mathbf{w}_{\mathrm{rd}} \approx \mathbf{id})$, the same bounds are obtained for $\mathbf{w}_{\mathrm{rd}}\mathbf{t}_\Pi$ and $\mathbf{t}_\Pi(\mathbf{w}_{\mathrm{rd}}\mathbf{p}_1, \mathbf{w}_{\mathrm{rd}}\mathbf{p}_2)$; the actual value of $\mathrm{LUnit}(\mathbf{t}_\Pi(\mathbf{w}_{\mathrm{rd}}\mathbf{p}_1, \mathbf{w}_{\mathrm{rd}}\mathbf{p}_2), 1)$ is 0.995, though, due to the fact that $\mathbf{w}_{\mathrm{rd}}(1) = 1$.
- Since $\mathbf{w}_{\mathrm{rd}}(0) = 0$ and $0 \cdot \alpha = 0$ for all α, the value 0 is in fact a left null of both $\mathbf{w}_{\mathrm{rd}}\mathbf{t}_\Pi$ and $\mathbf{t}_\Pi(\mathbf{w}_{\mathrm{rd}}\mathbf{p}_1, \mathbf{w}_{\mathrm{rd}}\mathbf{p}_2)$ to degree 1. For product with noise we obtain the bounds $\mathrm{LNull}(\mathbf{w}_{\mathrm{ns}}\mathbf{t}_\Pi, 0) \geq 0.995$ and $\mathrm{LNull}(\mathbf{t}_\Pi(\mathbf{w}_{\mathrm{ns}}\mathbf{p}_1, \mathbf{w}_{\mathrm{ns}}\mathbf{p}_2), 0) \geq 0.99$.
- Considering the possibility that the value of the unit or null element itself may be affected by noise as well, also the following estimates by (N30)–(N31) can be useful:

$$\mathrm{LUnit}(\mathbf{w}_{\mathrm{ns}}\mathbf{t}_\Pi, \mathbf{w}_{\mathrm{ns}}(1)) \geq \mathrm{LUnit}(\mathbf{w}_{\mathrm{ns}}\mathbf{t}_\Pi, 1) \,\&\, \mathrm{LCng}(\mathbf{w}_{\mathrm{ns}}\mathbf{t}_\Pi) \,\&\, (\mathbf{w}_{\mathrm{ns}} \approx \mathbf{id}) \geq 0.995 \,\&\, 0.99 \,\&\, 0.995 = 0.98$$

$$\mathrm{LNull}(\mathbf{w}_{\mathrm{ns}}\mathbf{t}_\Pi, \mathbf{w}_{\mathrm{ns}}(0)) \geq \mathrm{LNull}(\mathbf{w}_{\mathrm{ns}}\mathbf{t}_\Pi, 1) \,\&\, \mathrm{LCng}(\mathbf{w}_{\mathrm{ns}}\mathbf{t}_\Pi) \,\&\, (\mathbf{w}_{\mathrm{ns}} \approx^2 \mathbf{id}) \geq 0.995 \,\&\, 0.99 \,\&\, 0.99 = 0.975$$

6. Graded idempotence, commutativity, and associativity

In [12,13], the following graded versions of the properties of idempotence, commutativity, and associativity of binary fuzzy connectives have been defined:

$\mathrm{Idem}(\mathbf{c}) \equiv_{\mathrm{df}} (\forall\alpha)(\alpha\,\mathbf{c}\,\alpha \leftrightarrow \alpha)$	Idempotence
$\mathrm{Com}(\mathbf{c}) \equiv_{\mathrm{df}} (\forall\alpha\beta)(\alpha\,\mathbf{c}\,\beta \leftrightarrow \beta\,\mathbf{c}\,\alpha)$	Commutativity
$\mathrm{Ass}(\mathbf{c}) \equiv_{\mathrm{df}} (\forall\alpha\beta\gamma)((\alpha\,\mathbf{c}\,\beta)\,\mathbf{c}\,\gamma) \leftrightarrow (\alpha\,\mathbf{c}\,(\beta\,\mathbf{c}\,\gamma))$	Associativity

Here we shall moreover distinguish two components of idempotence:

Definition 6.1. We define in FCT for any $\mathbf{c} \sqsubseteq \mathrm{L} \times \mathrm{L}$:

$\mathrm{SupIdem}(\mathbf{c}) \equiv_{\mathrm{df}} (\forall\alpha)(\alpha \to \alpha\,\mathbf{c}\,\alpha)$ Super-idempotence

$\mathrm{SubIdem}(\mathbf{c}) \equiv_{\mathrm{df}} (\forall\alpha)(\alpha\,\mathbf{c}\,\alpha \to \alpha)$ Sub-idempotence

For a more compact form of some theorems we shall also write SupIdem as SIdem_{+1} and SubIdem as SIdem_{-1}; i.e., using Convention 2.3 we define for $i = \pm 1$:

$\mathrm{SIdem}_i(\mathbf{c}) \equiv_{\mathrm{df}} (\forall\alpha)(\alpha \to^i \alpha\,\mathbf{c}\,\alpha)$

Notice that a similar distinction would be meaningless for graded commutativity, as it has been proved in [13] that

$$\mathrm{Com}(\mathbf{c}) \;\Leftrightarrow\; (\forall\alpha\beta)(\alpha\,\mathbf{c}\,\beta \to \beta\,\mathbf{c}\,\alpha) \;\Leftrightarrow\; \mathbf{c} \approx \mathbf{c}^{-1} \;\Leftrightarrow\; \mathbf{c} \subseteq \mathbf{c}^{-1}.$$

Observation 6.2. FCT *proves for* $i = \pm 1$:

(A1) $\mathrm{SIdem}_i(\mathbf{c}) \;\Leftrightarrow\; \mathbf{id} \subseteq^i \mathbf{c}(\mathbf{id}, \mathbf{id})$.
(A2) $\mathrm{Idem}(\mathbf{c}) \;\Leftrightarrow\; \mathrm{SubIdem}(\mathbf{c}) \wedge \mathrm{SupIdem}(\mathbf{c}) \;\Leftrightarrow\; \mathbf{c}(\mathbf{id}, \mathbf{id}) \approx \mathbf{id}$.
(A3) $\mathrm{Ass}(\mathbf{c}) \;\Leftrightarrow\; \mathbf{c}(\mathbf{c}(\mathbf{p}_1, \mathbf{p}_2), \mathbf{p}_3) \approx \mathbf{c}(\mathbf{p}_1, \mathbf{c}(\mathbf{p}_2, \mathbf{p}_3))$.

The following theorems proved in [13] show the abundance of partly commutative and associative connectives:

$$\mathrm{Hgt}(\mathbf{c}) \to \mathrm{Plt}(\mathbf{c}) \;\Rightarrow\; \mathrm{Com}(\mathbf{c}) \tag{14}$$

$$\mathrm{Hgt}(\mathbf{c}) \to \mathrm{Plt}(\mathbf{c}) \;\Rightarrow\; \mathrm{Ass}(\mathbf{c}) \tag{15}$$

A similar theorem holds also for idempotence and its components:

Theorem 6.3. FCT *proves*:

(A4) $\mathrm{Plt}(\mathbf{c}(\mathbf{id}, \mathbf{id})) \;\Rightarrow\; \mathrm{SupIdem}(\mathbf{c})$.
(A5) $\neg\,\mathrm{Hgt}(\mathbf{c}(\mathbf{id}, \mathbf{id})) \;\Rightarrow\; \mathrm{SubIdem}(\mathbf{c})$.
(A6) $\mathrm{Plt}(\mathbf{c}(\mathbf{id}, \mathbf{id})) \wedge \neg\,\mathrm{Hgt}(\mathbf{c}(\mathbf{id}, \mathbf{id})) \;\Rightarrow\; \mathrm{Idem}(\mathbf{c})$.

Proof. The first two claims are proved by the following chains of provable implications:

(A4) $\mathrm{Plt}(\mathbf{c}(\mathbf{id}, \mathbf{id})) \Leftrightarrow (\forall\alpha)(1 \to \alpha\,\mathbf{c}\,\alpha) \Rightarrow (\forall\alpha)(\alpha \to \alpha\,\mathbf{c}\,\alpha)$.
(A5) $\neg\,\mathrm{Hgt}(\mathbf{c}(\mathbf{id}, \mathbf{id})) \Leftrightarrow ((\exists\alpha)(\alpha\,\mathbf{c}\,\alpha) \to 0) \Leftrightarrow (\forall\alpha)(\alpha\,\mathbf{c}\,\alpha \to 0) \Rightarrow (\forall\alpha)(\alpha\,\mathbf{c}\,\alpha \to \alpha)$.

The claim (A6) follows from (A4) and (A5) by (A2). □

The following theorems proved in [13] show that graded commutativity converts graded left-argument properties to right-argument ones and vice versa:

$$\begin{aligned}
&\mathrm{Com}^2(\mathbf{c}) \;\Rightarrow\; \mathrm{LCng}(\mathbf{c}) \leftrightarrow \mathrm{RCng}(\mathbf{c}) \\
&\mathrm{Com}^2(\mathbf{c}) \;\Rightarrow\; \mathrm{LMon}_i(\mathbf{c}) \leftrightarrow \mathrm{RMon}_i(\mathbf{c}), \quad \text{for } i = \pm 1 \\
&\mathrm{Com}(\mathbf{c}) \;\Rightarrow\; \mathrm{LUnit}(\mathbf{c}) \leftrightarrow \mathrm{RUnit}(\mathbf{c}) \\
&\mathrm{Com}(\mathbf{c}) \;\Rightarrow\; \mathrm{LNull}(\mathbf{c}) \leftrightarrow \mathrm{RNull}(\mathbf{c})
\end{aligned} \tag{16}$$

It can be easily observed that (16) holds as well for multiplicity-parameterized congruence and that double commutativity swaps the left- and right-multiplicities in both-argument properties as expected:

Observation 6.4. FCT *proves for any* $n, l, r \in \mathbb{N}$ *and* $i, j = \pm 1$:

(A7) $\mathrm{Com}^2(\mathbf{c}) \;\Rightarrow\; \mathrm{LCng}_n(\mathbf{c}) \leftrightarrow \mathrm{RCng}_n(\mathbf{c})$.
(A8) $\mathrm{Com}^2(\mathbf{c}) \;\Rightarrow\; \mathrm{Cng}_{l,r}(\mathbf{c}) \leftrightarrow \mathrm{Cng}_{r,l}(\mathbf{c})$.
(A9) $\mathrm{Com}^2(\mathbf{c}) \;\Rightarrow\; \mathrm{Mon}_{i,j}(\mathbf{c}) \leftrightarrow \mathrm{Mon}_{j,i}(\mathbf{c})$.
(A10) $\mathrm{Com}^2(\mathbf{c}) \;\Rightarrow\; \mathrm{MonCng}_{il,jr}(\mathbf{c}) \leftrightarrow \mathrm{MonCng}_{jr,il}(\mathbf{c})$.

The transmission of commutativity, associativity, and idempotence to connectives that are close in the sense of $\approx$ or $\cong$ has been shown in [13]:

$$\mathrm{Com}(\mathbf{c}), \mathbf{c} \cong \mathbf{d} \;\Rightarrow\; \mathrm{Com}(\mathbf{d}) \tag{17}$$

$$\mathrm{Ass}(\mathbf{c}_1), \mathrm{LCng}(\mathbf{c}_i), \mathrm{RCng}(\mathbf{c}_j), \mathbf{c}_1 \approx^4 \mathbf{c}_2 \;\Rightarrow\; \mathrm{Ass}(\mathbf{c}_2) \quad \text{for } i, j \in \{1, 2\} \tag{18}$$

$$\mathrm{Idem}(\mathbf{c}), \mathbf{c}(\mathbf{id}, \mathbf{id}) \approx \mathbf{d}(\mathbf{id}, \mathbf{id}) \;\Rightarrow\; \mathrm{Idem}(\mathbf{d}) \tag{19}$$

The theorem (18) can be generalized for parameterized (i.e., possibly weaker) left and right congruence, compensated by tighter closeness of $\mathbf{c}_1, \mathbf{c}_2$:

Theorem 6.5. FCT *proves for all* $m, n \in \mathbb{N}$ *and* $i, j \in \{1, 2\}$:

(A11) $\mathrm{Ass}(\mathbf{c}_1), \mathrm{LCng}_m(\mathbf{c}_i), \mathrm{RCng}_n(\mathbf{c}_j), \mathbf{c}_1 \approx^{m+n+2} \mathbf{c}_2 \;\Rightarrow\; \mathrm{Ass}(\mathbf{c}_2)$.

Proof. We shall only prove the case $i = j = 1$, by the following chain of implications:

$$\begin{aligned}
(\alpha\, \mathbf{c}_2\, \beta)\, \mathbf{c}_2\, \gamma &\Rightarrow (\alpha\, \mathbf{c}_2\, \beta)\, \mathbf{c}_1\, \gamma && \text{by } \mathbf{c}_1 \approx \mathbf{c}_2\\
&\Rightarrow (\alpha\, \mathbf{c}_1\, \beta)\, \mathbf{c}_1\, \gamma && \text{by } \mathbf{c}_1 \approx^m \mathbf{c}_2 \text{ and } \mathrm{LCng}_m(\mathbf{c}_1)\\
&\Rightarrow \alpha\, \mathbf{c}_1\, (\beta\, \mathbf{c}_1\, \gamma) && \text{by } \mathrm{Ass}(\mathbf{c}_1)\\
&\Rightarrow \alpha\, \mathbf{c}_1\, (\beta\, \mathbf{c}_2\, \gamma) && \text{by } \mathbf{c}_1 \approx^n \mathbf{c}_2 \text{ and } \mathrm{RCng}_n(\mathbf{c}_1)\\
&\Rightarrow \alpha\, \mathbf{c}_2\, (\beta\, \mathbf{c}_2\, \gamma) && \text{by } \mathbf{c}_1 \approx \mathbf{c}_2
\end{aligned}$$

The other cases only differ in the order of replacing $\mathbf{c}_1$ and $\mathbf{c}_2$, which determines whether the left and right congruence of $\mathbf{c}_1$ or $\mathbf{c}_2$ is used. □

The components of idempotence are obviously transmitted by inclusion of the diagonal:

Observation 6.6. FCT *proves for* $i = \pm 1$:

(A12) $\mathrm{SIdem}_i(\mathbf{c}), \mathbf{c}(\mathbf{id}, \mathbf{id}) \subseteq^i \mathbf{d}(\mathbf{id}, \mathbf{id}) \;\Rightarrow\; \mathrm{SIdem}_i(\mathbf{d})$.

The following theorem shows how commutativity and idempotence are preserved by compositions. (Recall that $\mathbf{p}_1$ and $\mathbf{p}_2$ denote the projections, see Definition 2.5.)

Theorem 6.7. FCT *proves, for* $i, j, h = \pm 1$, *any* $l, r \in \mathbb{N}$, *and* $k \in \{1, 2\}$:

(A13) $\mathrm{Com}(\mathbf{a}), \mathrm{Com}(\mathbf{b}), \mathrm{Cng}(\mathbf{c}) \;\Rightarrow\; \mathrm{Com}(\mathbf{c}(\mathbf{a}, \mathbf{b}))$.
(A14) $\mathrm{SIdem}_i(\mathbf{c}), \mathbf{id} \subseteq^i \mathbf{u} \;\Rightarrow\; \mathrm{SIdem}_i(\mathbf{uc})$.
(A15) $\mathrm{Idem}(\mathbf{c}), \mathbf{u} \approx \mathbf{id} \;\Rightarrow\; \mathrm{Idem}(\mathbf{uc})$.
(A16) $\mathrm{SIdem}_i(\mathbf{c}), \mathrm{MonCng}_{jl,hr}(\mathbf{c}), \mathbf{id} \subseteq^{ijl} \mathbf{u}, \mathbf{id} \subseteq^{ihr} \mathbf{v} \;\Rightarrow\; \mathrm{SIdem}_i(\mathbf{c}(\mathbf{u}, \mathbf{v}))$.
(A17) $\mathrm{SIdem}_i(\mathbf{c}), \mathrm{Mon}_{j,h}(\mathbf{c}), \mathbf{id} \sqsubseteq^{ij} \mathbf{u}, \mathbf{id} \sqsubseteq^{ih} \mathbf{v} \;\Rightarrow\; \mathrm{SIdem}_i(\mathbf{c}(\mathbf{u}, \mathbf{v}))$.
(A18) $\mathrm{Idem}(\mathbf{c}), \mathrm{Cng}_{l,r}(\mathbf{c}), \mathbf{u} \approx^l \mathbf{id}, \mathbf{v} \approx^r \mathbf{id} \;\Rightarrow\; \mathrm{Idem}(\mathbf{c}(\mathbf{u}, \mathbf{v}))$.
(A19) $\mathbf{p}_k \subseteq^i \mathbf{c}, \mathrm{SIdem}_i(\mathbf{a}_k) \;\Rightarrow\; \mathrm{SIdem}_i(\mathbf{c}(\mathbf{a}_1, \mathbf{a}_2))$.
(A20) $\mathbf{c} \approx \mathbf{p}_k, \mathrm{Idem}(\mathbf{a}_k) \;\Rightarrow\; \mathrm{Idem}(\mathbf{c}(\mathbf{a}_1, \mathbf{a}_2))$.
(A21) $\mathrm{SIdem}_i(\mathbf{c}), \mathrm{SIdem}_i(\mathbf{a}), \mathrm{SIdem}_i(\mathbf{b}), \mathrm{MonCng}(\mathbf{c}) \;\Rightarrow\; \mathrm{SIdem}_i(\mathbf{c}(\mathbf{a}, \mathbf{b}))$.
(A22) $\mathrm{SIdem}_i(\mathbf{c}), \triangle\,\mathrm{SIdem}_i(\mathbf{a}), \triangle\,\mathrm{SIdem}_i(\mathbf{b}), \mathrm{Mon}(\mathbf{c}) \;\Rightarrow\; \mathrm{SIdem}_i(\mathbf{c}(\mathbf{a}, \mathbf{b}))$.
(A23) $\mathrm{Idem}(\mathbf{c}), \mathrm{Idem}(\mathbf{a}), \mathrm{Idem}(\mathbf{b}), \mathrm{Cng}(\mathbf{c}) \;\Rightarrow\; \mathrm{Idem}(\mathbf{c}(\mathbf{a}, \mathbf{b}))$.

Proof. We shall assume $i = -1$ and $k = 1$. The proofs for $i = +1$ and $k = 2$ are analogous.

(A13) By $\mathrm{Com}(\mathbf{a})$ and $\mathrm{Com}(\mathbf{b})$ we have, respectively, $\alpha\, \mathbf{a}\, \beta \leftrightarrow \beta\, \mathbf{a}\, \alpha$ and $\alpha\, \mathbf{b}\, \beta \leftrightarrow \beta\, \mathbf{b}\, \alpha$, thus by $\mathrm{Cng}_{1,1}(\mathbf{c})$ we obtain

$$(\alpha\, \mathbf{a}\, \beta)\, \mathbf{c}\, (\alpha\, \mathbf{b}\, \beta) \leftrightarrow (\beta\, \mathbf{a}\, \alpha)\, \mathbf{c}\, (\beta\, \mathbf{b}\, \alpha).$$

(A14) $\mathbf{u}(\alpha\, \mathbf{c}\, \alpha) \Rightarrow \alpha\, \mathbf{c}\, \alpha \Rightarrow \alpha$, by $\mathbf{u} \subseteq \mathbf{id}$ for the first implication and $\mathrm{SubIdem}(\mathbf{c})$ for the second. The claim is also a direct corollary of (A12) and (U17).

(A16) By $\mathbf{u} \subseteq^{jl} \mathbf{id}$ and $\mathbf{v} \subseteq^{hr} \mathbf{id}$ we have, respectively, $\mathbf{u}\alpha \to^{jl} \alpha$ and $\mathbf{v}\alpha \to^{hr} \alpha$. Thus we obtain $\mathbf{u}\alpha\mathbf{c}\,\mathbf{v}\alpha \Rightarrow \alpha\, \mathbf{c}\, \alpha \Rightarrow \alpha$, by $\mathrm{MonCng}_{jl,hr}(\mathbf{c})$ for the first implication and $\mathrm{SubIdem}(\mathbf{c})$ for the second.

(A19) By $\mathbf{c} \subseteq \mathbf{p}_1$ we have $(\forall\alpha\beta)(\alpha\, \mathbf{c}\, \beta \to \alpha)$, whence we obtain $(\alpha\, \mathbf{a}_1\, \alpha)\, \mathbf{c}\, (\alpha\, \mathbf{a}_2\, \alpha) \Rightarrow \alpha\, \mathbf{a}_1\, \alpha \Rightarrow \alpha$, using $\mathrm{SubIdem}(\mathbf{a}_1)$ for the latter implication.

The proofs of (A15), (A17)–(A18), and (A20) are analogous to those of (A14), (A16), and (A19), respectively. The claims (A21)–(A23) follow from (A16)–(A18) for $\mathbf{u} = \mathbf{a}(\mathbf{id}, \mathbf{id})$ and $\mathbf{v} = \mathbf{b}(\mathbf{id}, \mathbf{id})$. □

The following observation lists how the basic binary logical connectives ($\wedge$, $\vee$, $\&$, $\rightarrow$, $\leftrightarrow$) satisfy the properties of idempotence, commutativity, and associativity:

Theorem 6.8. FCT *proves*:

(A24) $\mathrm{SubIdem}(\mathbf{c})$ *for* $\mathbf{c} \in \{\&, \wedge, \vee\}$.
(A25) $\neg\,\mathrm{SubIdem}(\mathbf{c})$ *for* $\mathbf{c} \in \{\rightarrow, \leftrightarrow\}$.
(A26) $\mathrm{SupIdem}(\mathbf{c})$ *for* $\mathbf{c} \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$.
(A27) $\mathrm{Idem}(\mathbf{c})$ *for* $\mathbf{c} \in \{\wedge, \vee\}$.
(A28) $\neg\,\mathrm{Idem}(\mathbf{c})$ *for* $\mathbf{c} \in \{\rightarrow, \leftrightarrow\}$.
(A29) $\mathrm{Com}(\mathbf{c})$ *for* $\mathbf{c} \in \{\wedge, \vee, \&, \leftrightarrow\}$.
(A30) $\neg\,\mathrm{Com}(\rightarrow)$.
(A31) $\mathrm{Ass}(\mathbf{c})$ *for* $\mathbf{c} \in \{\wedge, \vee, \&\}$.
(A32) $\neg\,\mathrm{Ass}(\rightarrow)$.

Proof. Proofs of most claims of Theorem 6.8 are trivial, following directly from the known properties of logical connectives in MTL (see [18]). The negative claims are obtained by suitable substitutions of 0 and 1: e.g., $((0 \rightarrow 0) \rightarrow 0) = 0$ and $(0 \rightarrow (0 \rightarrow 0)) = 1$ proves $\neg\,\mathrm{Ass}(\rightarrow)$. □

Remark 6.9. The degree of Idem($\&$), which equals that of SupIdem($\&$), is model-dependent: e.g., it is obviously fully true in Gödel logic, half-true in standard Łukasiewicz logic, and arbitrarily small in standard models over nilpotent Schweizer–Sklar t-norms (see [23]).

Similarly, Ass($\leftrightarrow$) is fully true in crisp models, but fully false in standard or $(2n+1)$-valued Łukasiewicz models (as $((0 \leftrightarrow 0.5) \leftrightarrow 0.5) = 1$, while $(0 \leftrightarrow (0.5 \leftrightarrow 0.5)) = 0$ in Ł).

Furthermore we shall investigate how logical connectives transmit commutativity and idempotence. By Theorems 6.7, 6.8, and 4.16 we obtain the following corollaries:

Corollary 6.10. FCT *proves*, *for* $i \in \{1, 2\}$:

(A33) $\mathrm{Com}(\mathbf{a}), \mathrm{Com}(\mathbf{b}) \;\Rightarrow\; \mathrm{Com}(\mathbf{c}(\mathbf{a}, \mathbf{b}))$ *for* $\mathbf{c} \in \{\&, \wedge, \vee, \rightarrow, \leftrightarrow\}$
(A34) $\mathrm{SubIdem}(\mathbf{a}_i) \;\Rightarrow\; \mathrm{SubIdem}(\mathbf{c}(\mathbf{a}_1, \mathbf{a}_2))$ *for* $\mathbf{c} \in \{\&, \wedge\}$.
(A35) $\mathrm{SupIdem}(\mathbf{a}_i) \;\Rightarrow\; \mathrm{SupIdem}(\mathbf{a}_1 \vee \mathbf{a}_2)$.
(A36) $\mathrm{SupIdem}(\mathbf{b}) \;\Rightarrow\; \mathrm{SupIdem}(\mathbf{a} \rightarrow \mathbf{b})$.

Moreover, by special properties of lattice connectives in MTL we are able to prove the following theorems on transmission of idempotence, which are stronger than what would follow from Theorems 6.7, 6.8, and 4.16:

Theorem 6.11. FCT *proves*:

(A37) $\mathrm{SubIdem}(\mathbf{a}) \wedge \mathrm{SubIdem}(\mathbf{b}) \;\Rightarrow\; \mathrm{SubIdem}(\mathbf{a} \vee \mathbf{b})$.
(A38) $\mathrm{SupIdem}(\mathbf{a}) \wedge \mathrm{SupIdem}(\mathbf{b}) \;\Rightarrow\; \mathrm{SupIdem}(\mathbf{a} \wedge \mathbf{b})$.
(A39) $\mathrm{Idem}(\mathbf{a}) \wedge \mathrm{Idem}(\mathbf{b}) \;\Rightarrow\; \mathrm{Idem}(\mathbf{c}(\mathbf{a}, \mathbf{b}))$. *for* $\mathbf{c} \in \{\wedge, \vee\}$.

Proof.

(A37) The claim follows from the fact that $(\alpha\,\mathbf{a}\,\alpha \rightarrow \alpha) \wedge (\alpha\,\mathbf{b}\,\alpha \rightarrow \alpha) \rightarrow ((\alpha\,\mathbf{a}\,\alpha) \vee (\alpha\,\mathbf{b}\,\alpha) \rightarrow \alpha)$.
(A38) The claim follows similarly from $(\alpha \rightarrow \alpha\,\mathbf{a}\,\alpha) \wedge (\alpha \rightarrow \alpha\,\mathbf{b}\,\alpha) \rightarrow (\alpha \rightarrow (\alpha\,\mathbf{a}\,\alpha) \wedge (\alpha\,\mathbf{b}\,\alpha))$.
(A39) The claim is a corollary of (A34), (A35), (A37), and (A38). □

Finally, we shall investigate the transmission of associativity, commutativity, and idempotence to slightly distorted functions. The transmission of idempotence and its components to $\mathbf{vc}$ and $\mathbf{c}(\mathbf{v}_1\mathbf{p}_1, \mathbf{v}_2\mathbf{p}_2)$ for $\mathbf{v}$ close (in the sense of $\approx$) to $\mathbf{id}$ has already been given in (A14), (A15), (A16), and (A18).[16] The transmission of commutativity and associativity to slightly distorted connectives is given by the following theorem. Notice, however, that due to the nested application

[16] In (A14) and (A16), the fact that $\mathbf{v} \approx^n \mathbf{id} \Rightarrow \mathbf{id} \subseteq^{in} \mathbf{v}$ for any $n \in \mathbb{N}$ and $i = \pm 1$ can be used.

of $\mathbf{c}$ in the definition of $\mathrm{Ass}(\mathbf{c})$, the associativity of $\mathbf{c}$ on distorted data is not captured by $\mathrm{Ass}(\mathbf{c}(\mathbf{v}_1\mathbf{p}_1, \mathbf{v}_2\mathbf{p}_2))$, as $\mathbf{v}_1$ and $\mathbf{v}_2$ would also be applied to results of $\mathbf{c}$ in the definition of $\mathrm{Ass}(\mathbf{c}(\mathbf{v}_1\mathbf{p}_1, \mathbf{v}_2\mathbf{p}_2))$. A meaningful correlate of associativity for functions on distorted data is rather the closeness of the values $\mathbf{c}(\mathbf{c}(\mathbf{v}_1\alpha, \mathbf{v}_2\beta), \mathbf{v}_3\gamma)$ and $\mathbf{c}(\mathbf{v}_1\alpha, \mathbf{c}(\mathbf{v}_2\beta, \mathbf{v}_3\gamma))$, i.e., $\mathbf{c}(\mathbf{c}(\mathbf{v}_1\mathbf{p}_1, \mathbf{v}_2\mathbf{p}_2), \mathbf{v}_3\mathbf{p}_3) \approx \mathbf{c}(\mathbf{v}_1\mathbf{p}_1, \mathbf{c}(\mathbf{v}_2\mathbf{p}_2, \mathbf{v}_3\mathbf{p}_3))$; compare the equivalent definition (A3) of associativity for undistorted data. To avoid too many parameters, we shall only consider the case of uniformly distorted arguments (i.e., $\mathbf{v}_1 = \mathbf{v}_2 = \mathbf{v}_3$) in the theorem.

Theorem 6.12. FCT *proves for all* $m, n, l, r \in \mathbb{N}$:

(A40) $\mathrm{Com}(\mathbf{c}), \mathbf{v} \cong \mathbf{id} \;\Rightarrow\; \mathrm{Com}(\mathbf{vc})$.
(A41) $\mathrm{Com}(\mathbf{c}), \mathrm{Cng}^2_{l,r}(\mathbf{c}), \mathbf{v}_1 \approx^{2l} \mathbf{id}, \mathbf{v}_2 \approx^{2r} \mathbf{id} \;\Rightarrow\; \mathrm{Com}(\mathbf{c}(\mathbf{v}_1\mathbf{p}_1, \mathbf{v}_2\mathbf{p}_2))$.
(A42) $\mathrm{Ass}(\mathbf{c}), \mathrm{LCng}_m(\mathbf{c}), \mathrm{RCng}_n(\mathbf{c}), \mathbf{v} \approx^{m+n+2} \mathbf{id} \;\Rightarrow\; \mathrm{Ass}(\mathbf{vc})$.
(A43) $\mathrm{Ass}(\mathbf{c}), \mathrm{Cng}^{l+r+2}_{l,r}(\mathbf{c}), \mathbf{v} \approx^{(l+r)(l+r+1)} \mathbf{id} \;\Rightarrow\; \mathbf{c}(\mathbf{c}(\mathbf{vp}_1, \mathbf{vp}_2), \mathbf{vp}_3) \approx \mathbf{c}(\mathbf{vp}_1, \mathbf{c}(\mathbf{vp}_2, \mathbf{vp}_3))$.

Proof.

(A40) By (C19) we obtain $\mathbf{v} \cong \mathbf{id} \Rightarrow \mathbf{vc} \cong \mathbf{c}$ and apply (17).
(A41) The claim follows straightforwardly from (U72), (17), and (10).
(A42) The claim follows directly from (C71) and (A11).
(A43) By $\mathbf{v} \approx^{l(l+r)} \mathbf{id}$ and $\mathrm{Cng}^l_{l,r}(\mathbf{c})$ we obtain $\mathbf{v}\alpha\,\mathbf{c}\,\mathbf{v}\beta \leftrightarrow^l \alpha\,\mathbf{c}\,\beta$, whence

$$(\mathbf{v}\alpha\,\mathbf{c}\,\mathbf{v}\beta)\,\mathbf{c}\,\mathbf{v}\gamma \leftrightarrow (\alpha\,\mathbf{c}\,\beta)\,\mathbf{c}\,\gamma \quad \text{by } \mathbf{v} \approx^{l(l+r)} \mathbf{id}, \mathrm{Cng}^l_{l,r}(\mathbf{c}), \mathbf{v} \approx^r \mathbf{id}, \mathrm{Cng}_{l,r}(\mathbf{c}); \text{ similarly,}$$
$$\mathbf{v}\alpha\,\mathbf{c}\,(\mathbf{v}\beta\,\mathbf{c}\,\mathbf{v}\gamma) \leftrightarrow \alpha\,\mathbf{c}\,(\beta\,\mathbf{c}\,\gamma) \quad \text{by } \mathbf{v} \approx^{r(l+r)} \mathbf{id}, \mathrm{Cng}^r_{l,r}(\mathbf{c}), \mathbf{v} \approx^l \mathbf{id}, \mathrm{Cng}_{l,r}(\mathbf{c}); \text{ and}$$
$$(\alpha\,\mathbf{c}\,\beta)\,\mathbf{c}\,\gamma \leftrightarrow \alpha\,\mathbf{c}\,(\beta\,\mathbf{c}\,\gamma) \quad \text{by } \mathrm{Ass}(\mathbf{c}).$$

Put together this yields $(\mathbf{v}\alpha\,\mathbf{c}\,\mathbf{v}\beta)\,\mathbf{c}\,\mathbf{v}\gamma \leftrightarrow \mathbf{v}\alpha\,\mathbf{c}\,(\mathbf{v}\beta\,\mathbf{c}\,\mathbf{v}\gamma)$ by $\mathrm{Ass}(\mathbf{c}), \mathrm{Cng}^{l+r+2}_{l,r}(\mathbf{c}), \mathbf{v} \approx^{(l+r)(l+r+1)} \mathbf{id}$. □

Example 6.13. By Theorem 6.12, product with ± 0.005-noise (i.e., $\mathbf{w}_{\mathrm{ns}}\mathbf{t}_{\Pi}$) is still at least 0.99-commutative and 0.98-associative, in models over standard Łukasiewicz logic. Similarly, the product of ± 0.005-noise affected arguments is still at least 0.98-commutative and 0.97-'associative', in the sense of (A43).

7. Variable ground logic

In the previous sections, the logical connectives of the ground logic had a fixed meaning: their realizations in any model of FCT were based on a given left-continuous t-norm that interpreted conjunction in the algebra of truth degrees of the model. A natural question arises as to what relationships hold between the values of graded properties such as $\mathrm{Mon}(\mathbf{u})$ when they are evaluated under *different* ground t-norms.[17] For instance, the Łukasiewicz t-norm is pointwise smaller than the product t-norm: Does this fact entail that the value of $\mathrm{Mon}(\mathbf{u})$ is larger if evaluated under the product t-norm than under the Łukasiewicz one?

This section shall deal with this type of questions. Even though we will only be able to derive non-graded results, which could as easily be derived directly in the semantics of first-order $\mathrm{MTL}_\triangle$, we shall nonetheless do our calculations in FCT, even if that necessitates some preliminary work on the internalization of known results on t-norms, as the formal framework will allow us to formulate several schematic theorems applicable to a broad class of graded properties of not only fuzzy connectives, but also general fuzzy relations. The translation of known facts about t-norms into FCT can also be useful for the number-free approach to mathematical notions (see [3]). The results moreover demonstrate that FCT over $\mathrm{MTL}_\triangle$ is capable of internalizing the connectives based on particular left-continuous t-norms directly as operations on the system L of its internal truth degrees, without a detour through the real unit interval [0, 1]. Note that while FCT was originally [5] introduced over the logic ŁΠ to make connectives based on various particular t-norms available, the present construction shows that FCT over $\mathrm{MTL}_\triangle$ already possesses them as internal connectives.[18]

Later in this section we shall need several (meta)lemmata:

[17] The question was asked by B. De Baets at a discussion related to [13] at FSTA 2010.

[18] For instance, the Łukasiewicz connectives can be introduced in FCT over $\mathrm{MTL}_\triangle$ by Definitions 7.3–7.4 below, assuming additionally that the connective $\mathbf{t}$ satisfies the distinctive axioms of Łukasiewicz logic over MTL, i.e., $(\forall\alpha\beta)(\alpha\,\mathbf{t}\,(\alpha \rightarrow_{\mathbf{t}} \beta) = \alpha \wedge \beta)$ and $(\forall\alpha)(\neg_{\mathbf{t}}\neg_{\mathbf{t}}\alpha = \alpha)$; this makes $\mathbf{t}$ isomorphic to the Łukasiewicz t-norm in all Zadeh models of FCT.

Lemma 7.1. *Let* $\varphi(x, y)$ *and* $\psi(x)$ *be formulae of* FCT *and* t *a term substitutable for both* x *and* y *in* φ *and for* x *in* ψ. *Then* FCT *proves*:

(L5) $\varphi(t,t) \Rightarrow (\exists x)\,\varphi(x,t)$.
(L6) $\psi(t) \Leftrightarrow (\forall x = t)\,\psi(x)$.
(L7) $\psi(t) \Leftrightarrow (\exists x = t)\,\psi(x)$.
(L8) $(\exists \gamma \leq \beta)\gamma \Leftrightarrow \beta$.

Proof. Even though formulated for FCT here, the claims (L5)–(L7) are actually provable already in first-order $\mathrm{MTL}_\triangle$ (with crisp identity in the latter two cases):

(L5) Immediate by dual specification of x for t in φ.
(L6) Left to right: from the identity axiom $\psi(t) \to (x = t \to \psi(x))$ by generalization on x and shifting the quantifier. Right to left: by specifying t for x.
(L7) Left to right: $\psi(t)$ implies $t = t \;\&\; \psi(t)$, which by (L5) implies $(\exists x)(x = t \;\&\; \psi(x))$. Right to left: the claim follows from the identity axiom $x = t \;\&\; \psi(x) \to \psi(t)$ by generalization on x and shifting the quantifier to the antecedent.
(L8) The left-to-right direction follows from $(\gamma \leq \beta) \;\&\; \gamma \to \beta$ and the converse direction from

$$\beta \Rightarrow \beta \vee (\exists \gamma < \beta)\gamma \Leftrightarrow (\exists \gamma = \beta)\gamma \vee (\exists \gamma < \beta)\gamma \Leftrightarrow (\exists \gamma \leq \beta)\gamma. \qquad \square$$

Lemma 7.2. *Any formula of* FCT *is equivalent to a formula in which logical functions are applied only to variables and occur only in atomic subformulae of the form* $y = F(x_1, \ldots, x_k)$.

Proof. The claim is actually provable already in first-order $\mathrm{MTL}_\triangle$ with crisp identity: using (L7), recursively decompose nested terms $s(t)$ by $\varphi(s(t)) \Leftrightarrow (\exists x)(x = t \;\&\; \varphi(s(x)))$ and finally by $\varphi(F(x_1, \ldots, x_k)) \Leftrightarrow (\exists y)(y = F(x_1, \ldots, x_k) \;\&\; \varphi(y))$ for all F. $\square$

To handle the dependence of the values of graded properties on the ground t-norm in FCT, we first need to internalize the apparatus of the ground logic in the FCT framework. Following the semantics of the logic MTL, we shall, therefore, first define the notions of left-continuous t-norm and its residuum as crisp classes of internal connectives and reprove some basic facts about these notions in FCT. Then we shall redefine our graded properties relative to these internalized t-norms and residua, and derive theorems on the relationships between the values of graded properties in dependence on the t-norms used.

First it can be observed that the crisp (non-graded) notion of t-norm can be internalized easily by means of the full truth of the properties Mon, Com, Ass, and LUnit introduced earlier. The property of left-continuity, which ensures the existence of the residuum, can be internalized in FCT by an axiom expressing that the t-norm commutes with the suprema (i.e., existential quantification) of crisp classes of truth degrees (one inequality is sufficient in the axiom, as the converse is a theorem of first-order $\mathrm{MTL}_\triangle$). To stress that the defined properties are crisp, we add the exponent $\triangle$, even though we do not define their graded variants here.

Definition 7.3. In FCT we define the following crisp properties of a binary connective $\mathbf{t} \sqsubseteq \mathrm{L} \times \mathrm{L}$:

$$\begin{aligned}
\mathrm{TNorm}^\triangle(\mathbf{t}) &\equiv_{\mathrm{df}} \triangle\,\mathrm{Mon}(\mathbf{t}) \;\&\; \triangle\,\mathrm{Com}(\mathbf{t}) \;\&\; \triangle\,\mathrm{Ass}(\mathbf{t}) \;\&\; \triangle\,\mathrm{LUnit}(\mathbf{t}, 1)\\
\mathrm{LC}^\triangle(\mathbf{t}) &\equiv_{\mathrm{df}} (\forall A \sqsubseteq \mathrm{L}, \mathrm{Crisp}\,A)(\alpha\,\mathbf{t}\,(\exists \beta \in A)\beta \leq (\exists \beta \in A)(\alpha\,\mathbf{t}\,\beta)))\\
\mathrm{LCTNorm}^\triangle(\mathbf{t}) &\equiv_{\mathrm{df}} \mathrm{TNorm}^\triangle(\mathbf{t}) \;\&\; \mathrm{LC}^\triangle(\mathbf{t})
\end{aligned}$$

By the semantics of FCT it is obvious that if $\mathrm{L} = [0, 1]$, then $(\mathrm{LC})\mathrm{TNorm}^\triangle(\mathbf{t})$ has the value 1 iff $\mathbf{t}$ is realized as a (left-continuous) t-norm. Following the semantics of MTL, we are now able to define the logical connectives based on any $\mathbf{t} \in \mathrm{LCTNorm}^\triangle$. The internalized definition of the residuum $\to_{\mathbf{t}}$ directly translates its explicit definition $\alpha \to_{\mathbf{t}} \beta = \bigvee\{\gamma \mid \gamma\,\mathbf{t}\,\alpha \leq \beta\}$. Other $\mathbf{t}$-based connectives are defined in terms of $\mathbf{t}$ and $\to_{\mathbf{t}}$, following their definitions in MTL.

Definition 7.4. If LCTNorm$^{\triangle}(\mathbf{t})$, we define the $\mathbf{t}$-based logical connectives by setting for all $\alpha, \beta \in \mathrm{L}$:

$$\begin{aligned}
\alpha \&_{\mathbf{t}} \beta &\equiv_{\mathrm{df}} \alpha\, \mathbf{t}\, \beta \\
\alpha \rightarrow_{\mathbf{t}} \beta &\equiv_{\mathrm{df}} (\exists\gamma)((\gamma\, \mathbf{t}\, \alpha \leq \beta)\ \&\ \gamma) \\
\alpha \leftrightarrow_{\mathbf{t}} \beta &\equiv_{\mathrm{df}} (\alpha \rightarrow_{\mathbf{t}} \beta) \wedge (\beta \rightarrow_{\mathbf{t}} \alpha) \\
\neg_{\mathbf{t}} \alpha &\equiv_{\mathrm{df}} \alpha \rightarrow_{\mathbf{t}} 0
\end{aligned}$$

The following well-known facts can be internalized and proved in FCT:

Theorem 7.5. FCT *proves*, *for* $\mathbf{t}, \mathbf{t}_1, \mathbf{t}_2 \in$ LCTNorm$^{\triangle}$:

(V1) $\alpha\, \mathbf{t}\, 0 \Leftrightarrow 0$.
(V2) $1 \rightarrow_{\mathbf{t}} \beta \Leftrightarrow \beta$.
(V3) $\alpha\, \mathbf{t}\, (\alpha \rightarrow_{\mathbf{t}} \beta) \Rightarrow \beta$.
(V4) $\triangle(\alpha \rightarrow_{\mathbf{t}} \beta) \Leftrightarrow (\alpha \leq \beta)$.
(V5) $\triangle(\alpha \leftrightarrow_{\mathbf{t}} \beta) \Leftrightarrow (\alpha = \beta)$.
(V6) $\triangle\varphi \&_{\mathbf{t}} \psi \Leftrightarrow \triangle\varphi \wedge \psi$.
(V7) $\triangle\varphi \rightarrow_{\mathbf{t}} \psi \Leftrightarrow \neg\triangle\varphi \vee \psi$.
(V8) $\mathbf{t}_1 \sqsubseteq \mathbf{t}_2 \Rightarrow ((\alpha \rightarrow_{\mathbf{t}_2} \beta) \leq (\alpha \rightarrow_{\mathbf{t}_1} \beta))$.
(V9) $\mathbf{t}_1 \sqsubseteq \mathbf{t}_2 \Rightarrow ((\alpha \leftrightarrow_{\mathbf{t}_2} \beta) \leq (\alpha \leftrightarrow_{\mathbf{t}_1} \beta))$.
(V10) $\mathbf{t}_1 \sqsubseteq \mathbf{t}_2 \Rightarrow (\neg_{\mathbf{t}_2} \alpha \leq \neg_{\mathbf{t}_1} \alpha)$.
(V11) LAnt($\rightarrow_{\mathbf{t}}$), RMon($\rightarrow_{\mathbf{t}}$), Ant($\neg_{\mathbf{t}}$), Mon($\&_{\mathbf{t}}$).
(V12) LCTNorm$^{\triangle}(\&)$, $\rightarrow_{\&} = \rightarrow$.
(V13) LCTNorm$^{\triangle}(\wedge)$, $\mathbf{t} \sqsubseteq \wedge$.

Proof.

(V1) $\alpha\, \mathbf{t}\, 0 \Rightarrow 1\, \mathbf{t}\, 0 \Leftrightarrow 0$ by Mon($\mathbf{t}$) and LUnit($\mathbf{t}$, 1).

(V2) By definition, $(1 \rightarrow_{\mathbf{t}} \beta) \Leftrightarrow (\exists\gamma)((\gamma\, \mathbf{t}\, 1 \leq \beta)\ \&\ \gamma) \Leftrightarrow (\exists\gamma \leq \beta)\gamma \Leftrightarrow \beta$, where the last equivalence follows from (L8).

(V3) The claim is proved as follows:

$$\begin{aligned}
\alpha\, \mathbf{t}\, (\alpha \rightarrow_{\mathbf{t}} \beta) &\Leftrightarrow \alpha\, \mathbf{t}\, (\exists\gamma)((\gamma\, \mathbf{t}\, \alpha \leq \beta)\ \&\ \gamma) && \text{by definition} \\
&\Leftrightarrow (\exists\gamma)((\gamma\, \mathbf{t}\, \alpha \leq \beta)\ \&\ (\gamma\, \mathbf{t}\, \alpha)) && \text{by LC}^{\triangle}(\mathbf{t}) \text{ and } \triangle\,\mathrm{Com}(\mathbf{t}) \\
&\Rightarrow \beta
\end{aligned}$$

The last implication is proved by generalizing the theorem $\triangle(\gamma\, \mathbf{t}\, \alpha \rightarrow \beta)\ \&\ (\gamma\, \mathbf{t}\, \alpha) \rightarrow \beta$ of MTL$_{\triangle}$ on γ and shifting the quantifier as $(\exists\gamma)$ to the antecedent.

(V4) First we prove that $\triangle(\alpha \rightarrow_{\mathbf{t}} \beta) \Leftrightarrow (\exists\gamma)\triangle((\gamma\, \mathbf{t}\, \alpha \leq \beta)\ \&\ \gamma)$. The left-to-right direction is proved by the following chain of implications:

$$\begin{aligned}
\triangle(\alpha \rightarrow_{\mathbf{t}} \beta) &\Leftrightarrow (\alpha\, \mathbf{t}\, (\alpha \rightarrow_{\mathbf{t}} \beta) \leq \beta)\ \&\ \triangle(\alpha \rightarrow_{\mathbf{t}} \beta) && \text{as } \alpha\, \mathbf{t}\, (\alpha \rightarrow_{\mathbf{t}} \beta) \leq \beta \text{ by (V3)} \\
&\Leftrightarrow \triangle(\alpha\, \mathbf{t}\, (\alpha \rightarrow_{\mathbf{t}} \beta) \leq \beta)\ \&\ (\alpha \rightarrow_{\mathbf{t}} \beta)) && \text{as MTL}_{\triangle} \text{ proves } \triangle\varphi\ \&\ \triangle\psi \leftrightarrow \triangle(\varphi\ \&\ \psi) \\
&\Rightarrow (\exists\gamma)\triangle((\gamma\, \mathbf{t}\, \alpha \leq \beta)\ \&\ \gamma) && \text{by dual specification}
\end{aligned}$$

The converse direction holds a fortiori, as $(\exists x)\triangle\varphi \rightarrow \triangle(\exists x)\varphi$ is a theorem of first-order MTL$_{\triangle}$. The proof of the main claim is then concluded as follows:

$$\begin{aligned}
\triangle(\alpha \rightarrow_{\mathbf{t}} \beta) &\Leftrightarrow (\exists\gamma)\triangle((\gamma\, \mathbf{t}\, \alpha \leq \beta)\ \&\ \gamma) && \text{by the first step above} \\
&\Leftrightarrow (\exists\gamma = 1)\triangle((\gamma\, \mathbf{t}\, \alpha \leq \beta)\ \&\ \gamma) && \text{as } (\exists\gamma < 1)\triangle((\gamma\, \mathbf{t}\, \alpha \leq \beta)\ \&\ \gamma) = 0 \\
&\Leftrightarrow \triangle((1\, \mathbf{t}\, \alpha \leq \beta)\ \&\ 1) && \text{by (L7)} \\
&\Leftrightarrow (\alpha \leq \beta) && \text{by LUnit}(\mathbf{t},1)
\end{aligned}$$

(V6) Since MTL$_{\triangle}$ proves $(\triangle\alpha = 1) \vee (\triangle\alpha = 0)$, the claim is proved by considering the following cases: if $\triangle\alpha = 0$, then $\triangle\alpha \wedge \beta \Leftrightarrow 0 \Leftrightarrow \triangle\alpha\, \mathbf{t}\, \beta$ by (V1), and if $\triangle\alpha = 1$, then $\triangle\alpha \wedge \beta \Leftrightarrow \beta \Leftrightarrow \triangle\alpha\, \mathbf{t}\, \beta$ by LUnit($\mathbf{t}$, 1).

(V7) If $\triangle\alpha = 0$, then $\neg\triangle\alpha \vee \beta \Leftrightarrow 1 \Leftrightarrow (\triangle\alpha \rightarrow_{\mathbf{t}} \beta)$ by (V4), and if $\triangle\alpha = 1$, then $\neg\triangle\alpha \vee \beta \Leftrightarrow \beta \Leftrightarrow (\triangle\alpha \rightarrow_{\mathbf{t}} \beta)$ by (V2).
(V8) The claim is proved in FCT as follows:

$$\begin{aligned} \mathbf{t}_1 \subseteq \mathbf{t}_2 &\Rightarrow \gamma\,\mathbf{t}_1\,\alpha \rightarrow \gamma\,\mathbf{t}_2\,\alpha && \text{by definition} \\ &\Rightarrow (\gamma\,\mathbf{t}_2\,\alpha \rightarrow \beta) \rightarrow (\gamma\,\mathbf{t}_1\,\alpha \rightarrow \beta) && \text{by suffixing} \end{aligned}$$

whence by $\triangle$-necessitation we obtain

$$\begin{aligned} \mathbf{t}_1 \sqsubseteq \mathbf{t}_2 &\Rightarrow (\gamma\,\mathbf{t}_2\,\alpha \leq \beta) \rightarrow (\gamma\,\mathbf{t}_1\,\alpha \leq \beta) && \text{by } \triangle\text{-distribution} \\ &\Rightarrow ((\gamma\,\mathbf{t}_2\,\alpha \leq \beta)\,\&\,\gamma) \rightarrow ((\gamma\,\mathbf{t}_1\,\alpha \leq \beta)\,\&\,\gamma) && \text{by combination with } \gamma \rightarrow \gamma \end{aligned}$$

Generalization and distribution of $\forall$ as $\exists$ then yields $(\exists\gamma)((\gamma\,\mathbf{t}_2\,\alpha \leq \beta)\,\&\,\gamma) \rightarrow (\exists\gamma)((\gamma\,\mathbf{t}_1\,\alpha \leq \beta)\,\&\,\gamma)$, i.e., $(\alpha \rightarrow_{\mathbf{t}_2} \beta) \rightarrow (\alpha \rightarrow_{\mathbf{t}_1} \beta)$.
(V11) The claims follow easily from the monotony and antitony of connectives in Definition 7.4.
(V12) By (C59), (A29), (A31), and (N14), FCT proves TNorm(&). $\mathrm{LC}^{\triangle}(\&)$ follows from the MTL-theorem

$$\alpha \rightarrow ((\beta \in A)\,\&\,\beta \rightarrow (\beta \in A)\,\&\,\alpha\,\&\,\beta)$$

by generalization, shifting and distributing the quantifier as $\exists$, and residuation. The claim that $\rightarrow_{\&} = \rightarrow$ is proved by the following chain of equivalences:

$$(\exists\gamma)((\gamma\,\&\,\alpha \leq \beta)\,\&\,\gamma) \;\Leftrightarrow\; (\exists\gamma)((\gamma \leq (\alpha \rightarrow \beta))\,\&\,\gamma) \;\Leftrightarrow\; (\alpha \rightarrow \beta),$$

where the first equivalence holds by residuation and the second by (L8).
(V13) By (C59), (A29), (A31), and (N10), FCT proves TNorm($\wedge$). $\mathrm{LC}^{\triangle}(\wedge)$ is proved as follows:

$$\alpha \wedge (\exists\beta \in A)\beta \;\Leftrightarrow\; \alpha \wedge (\exists\beta)((\beta \in A) \wedge \beta) \;\Leftrightarrow\; (\exists\beta)((\beta \in A) \wedge \alpha \wedge \beta) \;\Leftrightarrow\; (\exists\beta \in A)(\alpha \wedge \beta),$$

using the crispness of A in the first and last step and the distributivity of $\wedge$ over $\exists$ in the middle step. Finally, by LUnit($\mathbf{t}$, 1) and Com($\mathbf{t}$) we have $\alpha\,\mathbf{t}\,\beta \rightarrow \alpha$ and $\alpha\,\mathbf{t}\,\beta \rightarrow \beta$, whence $\alpha\,\mathbf{t}\,\beta \rightarrow \alpha \wedge \beta$; thus $\mathbf{t} \sqsubseteq \wedge$.

The claims (V9)–(V10) follow directly from (V8) and the claim (V5) from (V4). □

The definitions of graded properties can now be made relative to a ground left-continuous t-norm $\mathbf{t}$ used for their evaluation. This is done by replacing the logical connectives of the ground logic in their definitions by their $\mathbf{t}$-based counterparts.[19] The $\mathbf{t}$-relativized property Φ will be denoted by ${}^{\mathbf{t}}\Phi$. Thus we obtain, for example, the following $\mathbf{t}$-relativized graded properties of fuzzy connectives:

$$\begin{aligned} {}^{\mathbf{t}}\mathrm{Idem}(\mathbf{c}) &\equiv (\forall\alpha)(\alpha\,\mathbf{c}\,\alpha \leftrightarrow_{\mathbf{t}} \alpha) \\ {}^{\mathbf{t}}\mathrm{Cng}(\mathbf{u}) &\equiv (\forall\alpha\beta)((\alpha \leftrightarrow_{\mathbf{t}} \beta) \rightarrow_{\mathbf{t}} (\mathbf{u}\alpha \leftrightarrow_{\mathbf{t}} \mathbf{u}\beta)) \\ \mathbf{c}\ {}^{\mathbf{t}}\subseteq \mathbf{d} &\equiv (\forall\alpha\beta)(\alpha\,\mathbf{c}\,\beta \rightarrow_{\mathbf{t}} \alpha\,\mathbf{d}\,\beta), \text{ etc.} \end{aligned}$$

Now we can start studying the dependence of the values of graded properties on the underlying t-norm. First we shall observe that the values of some graded properties do not depend on the underlying t-norm at all; we shall call them *t-absolute*. Formally we define:

Definition 7.6. We shall say that Φ is *t-absolute* iff FCT proves: $(\forall\mathbf{t} \in \mathrm{LCTNorm}^{\triangle})(\Phi \leftrightarrow {}^{\mathbf{t}}\Phi)$.

Formulae of some forms are easily recognizable as t-absolute. The following theorem gives some recursive conditions that ensure the t-absoluteness of a formula.

[19] The replacement need of course be done recursively also in all defined notions and comprehension terms that occur in the definitions.

Theorem 7.7. *Φ is t-absolute if* FCT *proves that Φ is equivalent to a formula of some of the following forms*:

(V14) *An atomic formula of* FCT, *in which all* (*topmost in nesting*) *comprehension terms $\{x \mid \varphi\}$ have φ t-absolute.*
(V15) *A formula composed of t-absolute subformulae by means of $\wedge, \vee, \triangle, 0, 1, \leq, =, \exists, \forall, \&_{\mathbf{t}}, \rightarrow_{\mathbf{t}}, \leftrightarrow_{\mathbf{t}}, \neg_{\mathbf{t}}$.*
(V16) *A formula composed of crisp t-absolute subformulae by means of $\&, \rightarrow, \leftrightarrow, \neg$, and the connectives and quantifiers mentioned in* (V15).
(V17) *$\triangle\varphi \rightarrow \psi$ or $\triangle\varphi \,\&\, \psi$, where $\triangle\varphi$ and ψ are t-absolute formulae.*

Proof.

(V14) The only non-trivial part of the claim is the one regarding comprehension terms. By Lemma 7.2 we can without loss of generality assume that the comprehension terms only occur in atomic subformulae of the form $A{=}\{x \mid \varphi\}$; observe that the construction eliminating nested function symbols (see the proof of Lemma 7.2) preserves t-absoluteness of the original subformula by other claims of this theorem. The formula $A = \{x \mid \varphi\}$ is in FCT equivalent to $(\forall x)(Ax = \varphi)$, which is again t-absolute due to other claims of the present theorem if φ is t-absolute.
(V15) For $\wedge, \vee, \triangle, 0, 1, \exists, \forall$ the claim is trivial and for $\leq, =$ it follows from (V4)–(V5). For the $\mathbf{t}$-based connectives $\&_{\mathbf{t}}, \rightarrow_{\mathbf{t}}, \leftrightarrow_{\mathbf{t}}, \neg_{\mathbf{t}}$ it is sufficient to observe that their definitions are t-absolute by other claims of this theorem (see Remark 7.8 below, which makes this observation easier).
(V16) The fact that $\mathbf{t}$-based connectives behave classically on the crisp values 0, 1 follows from (V1), LUnit($\mathbf{t}$), Com($\mathbf{t}$), (V2), (V4), and the definitions of $\leftrightarrow_{\mathbf{t}}, \neg_{\mathbf{t}}$.
(V17) The claim follows by (V15) from the fact that $\mathrm{MTL}_{\triangle}$ proves $(\triangle\varphi \rightarrow \psi) \leftrightarrow \triangle\varphi \vee \psi$ and $\triangle\varphi \,\&\, \psi \leftrightarrow \triangle\varphi \wedge \psi$. □

Remark 7.8. Observe that the quantifiers occurring in claims (V15) and (V16) can as well be restricted to any crisp t-absolute domain, as the connectives & and $\rightarrow$ occurring in $(\forall x)(\varphi \rightarrow \psi)$ and $(\exists x)(\varphi \,\&\, \psi)$ can be handled by (V17) if φ is crisp.

Moreover, even though (V14) is formulated just for the primitive language of FCT, it clearly applies as well to FCT expanded by explicitly defined notions, provided that the defining formulae of all defined notions involved are t-absolute, too. In order to prove this claim for defined predicate and function symbols it is sufficient to observe that expanding their definitions preserves t-absoluteness similarly as in the proof of (V14). The claim is also easily proved for defined sorts x^{φ} of variables delimited from a primitive sort of variables x by a *crisp* t-absolute formula φ, as their free occurrences can be eliminated by using the defining formula $\varphi(x) \rightarrow \Phi(x)$ for $\Phi(x^{\varphi})$ and applying (V17), while their bound occurrences $(\forall x^{\varphi})\Phi(x^{\varphi}) \equiv (\forall x)(\varphi(x) \rightarrow \Phi(x))$ and $(\exists x^{\varphi})\Phi(x^{\varphi}) \equiv (\exists x)(\varphi(x) \,\&\, \Phi(x))$ just instantiate quantification relativized to a crisp t-absolute domain, which preserves t-absoluteness as described above. It can be observed that the variables $\alpha, \beta, \ldots$ for internal truth values, the variables $\mathbf{u}, \mathbf{c}, \mathbf{f}, \ldots$ for fuzzy connectives, as well as the variables $\mathbf{t}, \ldots$ for left-continuous t-norms do range over crisp t-absolute domains,[20] and thus (V14) applies to atomic formulae containing such variables.

Finally, observe that (V14) applies as well to terms yielding inner truth values, as $\alpha \in \mathrm{L} = \mathrm{Ker}\,\mathrm{Pow}\{\mathrm{a}\}$ represents the value of the atomic formula $\mathrm{a} \in \alpha$ (see footnote 5).

T-absoluteness of various properties of fuzzy connectives can now be easily verified by inductive application of (V14)–(V17) to the subformulae of their defining formulae.

Corollary 7.9. *The following properties of fuzzy connectives are t-absolute, for $i, j = \pm 1$:*
$\mathrm{Hgt}, \mathrm{Plt}, \sqsubseteq, \triangle\,\mathrm{Cng}_0, \triangle\,\mathrm{Cng}_{0,0}, \triangle\,\mathrm{Mon}_i, \triangle\,\mathrm{Mon}_{i,j}, \triangle\,\mathrm{LUnit}, \triangle\,\mathrm{RUnit}, \triangle\,\mathrm{LNull}, \triangle\,\mathrm{RNull}, \triangle\,\mathrm{Com}, \triangle\,\mathrm{Ass}, \triangle\,\mathrm{Idem}, \triangle\,\mathrm{SIdem}_i, \mathrm{TNorm}^{\triangle}, \mathrm{LC}^{\triangle}, \mathrm{LCTNorm}^{\triangle}$

Moreover, since all logical connectives are in ${}^{\mathbf{t}}\Phi$ replaced by their $\mathbf{t}$-based counterparts (whose definitions are t-absolute), all $\mathbf{t}$-relativized properties ${}^{\mathbf{t}}\Phi$ are t-absolute as well, and thus their values are independent of the ground

[20] Since the class L of internal truth values is defined (see footnote 5) as $\mathrm{Ker}\,\mathrm{Pow}\{\mathrm{a}\}$, it is delimited by the formula $A \in \mathrm{L} \equiv \triangle(\forall x)(Ax \rightarrow (x = \mathrm{a}))$, which is crisp and t-absolute by Theorem 7.7. The t-absoluteness of the crisp class $\mathrm{Ker}\,\mathrm{Pow}(\mathrm{L}^n)$ of n-ary fuzzy connectives then follows from the t-absoluteness of crisp inclusion and the Cartesian product of crisp classes by (V16), and the t-absoluteness of the crisp property $\mathrm{LCTNorm}^{\triangle}$ by the recursive application of Theorem 7.7 to its defining formula.

logic used for their evaluation. In other words, since the dependence on the ground t-norm has been made explicit in **t**-relativized definitions, their value is the same in all Zadeh models of FCT.

Besides t-absoluteness, we are also able to prove the monotony or antitony of the values of many graded properties with respect to the strength of the underlying t-norm. First let us define:

Definition 7.10. We shall say that Φ is *t-monotone* iff FCT proves: $(\forall \mathbf{t}_1, \mathbf{t}_2 \in \mathrm{LCTNorm}^{\triangle})(\mathbf{t}_1 \sqsubseteq \mathbf{t}_2 \Rightarrow {}^{\mathbf{t}_1}\Phi \to {}^{\mathbf{t}_2}\Phi)$.
We shall say that Φ is *t-antitone* iff FCT proves: $(\forall \mathbf{t}_1, \mathbf{t}_2 \in \mathrm{LCTNorm}^{\triangle})(\mathbf{t}_1 \sqsubseteq \mathbf{t}_2 \Rightarrow {}^{\mathbf{t}_2}\Phi \to {}^{\mathbf{t}_1}\Phi)$.

Thus t-monotone properties have (non-strictly) larger values if **t** is pointwise larger, and t-antitone properties have (non-strictly) smaller values under the same condition. It can be observed that t-absoluteness is the conjunction of t-monotony and t-antitony:

Theorem 7.11. *Φ is t-absolute iff it is both t-monotone and t-antitone.*

Proof. The left-to-right direction is trivial. To prove the converse direction, take any $\mathbf{t} \in \mathrm{LCTNorm}^{\triangle}$. Since $\& \sqsubseteq \wedge$ and $\mathbf{t} \sqsubseteq \wedge$ by (V13), we obtain ${}^{\&}\Phi \to {}^{\wedge}\Phi$ by the t-monotony of Φ and ${}^{\wedge}\Phi \to {}^{\mathbf{t}}\Phi$ by the t-antitony of Φ. Similarly ${}^{\mathbf{t}}\Phi \to {}^{\wedge}\Phi$ by the t-monotony of Φ and ${}^{\wedge}\Phi \to {}^{\&}\Phi$ by the t-antitony of Φ. Thus ${}^{\&}\Phi \leftrightarrow {}^{\mathbf{t}}\Phi$. Finally observe that $\Phi \leftrightarrow {}^{\&}\Phi$ by (V12) and Definition 7.4. □

The following observation gives some recursive conditions that ensure the t-monotony or t-antitony of a formula.

Theorem 7.12. *Let φ, φ_1, φ_2 be t-monotone, ψ, ψ_1, ψ_2 t-antitone, and χ_1, χ_2 t-absolute. Then the formulae*

(V18) $\triangle\varphi$, $\varphi_1 \wedge \varphi_2$, $\varphi_1 \vee \varphi_2$, $\varphi_1 \&_{\mathbf{t}} \varphi_2$, $\psi \to_{\mathbf{t}} \varphi$, $\neg_{\mathbf{t}}\psi$, $(\forall x)\varphi$, $(\exists x)\varphi$, $\varphi_1 \,\&\, \varphi_2$ *are t-monotone.*
(V19) $\triangle\psi$, $\psi_1 \wedge \psi_2$, $\psi_1 \vee \psi_2$, $\psi_1 \&_{\mathbf{t}} \psi_2$, $\varphi \to_{\mathbf{t}} \psi$, $\neg_{\mathbf{t}}\varphi$, $(\forall x)\psi$, $(\exists x)\psi$, $\varphi \to \psi$, $\varphi \leq \psi$, $\neg\varphi$, $\chi_1 \leftrightarrow \chi_2$, $\chi_1 = \chi_2$ *are t-antitone.*

Proof. We shall only prove the t-antitony of $\varphi \to \psi$. Proofs of the other claims (by Theorem 7.5 and the properties of logical connectives given in the previous sections) are similar or even easier. The t-antitony of $\varphi \to \psi$ follows from the following chain of implications, for $\mathbf{t}_1 \sqsubseteq \mathbf{t}_2$:

$$({}^{\mathbf{t}_2}\varphi \to_{\mathbf{t}_2} {}^{\mathbf{t}_2}\psi) \Rightarrow ({}^{\mathbf{t}_2}\varphi \to_{\mathbf{t}_1} {}^{\mathbf{t}_2}\psi) \Rightarrow ({}^{\mathbf{t}_1}\varphi \to_{\mathbf{t}_1} {}^{\mathbf{t}_2}\psi) \Rightarrow ({}^{\mathbf{t}_1}\varphi \to_{\mathbf{t}_1} {}^{\mathbf{t}_1}\psi)$$

where the first implication follows from (V8), the second from the t-monotony of φ and $\mathrm{LAnt}(\to_{\mathbf{t}_1})$, and the third from the t-antitony of ψ and $\mathrm{RMon}(\to_{\mathbf{t}_1})$. □

Corollary 7.13. *By Theorems* 7.7 *and* 7.12, *the following graded properties of fuzzy connectives are t-antitone, for $i, j = \pm 1$, besides those listed in Corollary* 7.9:

$\subseteq, \approx, \cong, \mathrm{Cng}_0, \mathrm{Cng}_{0,0}, \mathrm{Mon}_i, \mathrm{Mon}_{i,j}, \mathrm{Surj}_1$, LUnit, RUnit, LNull, RNull, Idem, SubIdem, SupIdem, Com, Ass.

Thus, e.g., the degree of monotony of **c** evaluated by the Łukasiewicz t-norm is always larger than or equal to the degree of monotony of **c** evaluated by the product t-norm; and the minimal degree of monotony is obtained when using the minimum for the underlying left-continuous t-norm.[21]

It can be observed that the graded properties Cng and MonCng (unary as well as binary, both single-sided and both-sided, with any multiplicity parameters larger than 0), or even their non-graded variants $\triangle$ Cng and $\triangle$ MonCng, are neither t-monotone nor t-antitone (counterexamples to both meta-properties are easy to find). The values of Cng and MonCng evaluated under one t-norm thus cannot in general be estimated from their values under stronger or weaker t-norms.

[21] Interestingly, no graded property studied in this paper is t-monotone, unless it is already t-absolute. This is caused by the presence of equivalence and implication connectives in their defining formulae, which arise as natural fuzzifications of equalities and inequalities in the definitions of the corresponding non-graded notions. An example of a t-monotone graded notion which is not t-absolute is the compatibility relation $A \parallel B \equiv_{\mathrm{df}} (\exists x)(Ax \,\&\, Bx)$.

The notions of t-absoluteness, t-monotony, and t-antitony are defined for any FCT formula, and thus are applicable not only to the graded properties of fuzzy connectives studied in this paper, but to any defined notion of FCT. Thus, for instance, by Theorems 7.7 and 7.12, the graded relation of dominance studied in [12,13],

$$\mathbf{c} \ll \mathbf{d} \equiv_{\mathrm{df}} (\forall \alpha\beta\gamma\delta)((\alpha\ \mathbf{d}\ \gamma)\ \mathbf{c}\ (\beta\ \mathbf{d}\ \delta) \to (\alpha\ \mathbf{c}\ \beta)\ \mathbf{d}\ (\gamma\ \mathbf{c}\ \delta)),$$

is t-antitone, and its non-graded variant $\triangle(\mathbf{c} \ll \mathbf{d})$ is t-absolute. The theorems are applicable as well to basic graded properties of fuzzy relations studied in [19,21,15,11], such as the following ones:

$$\begin{aligned}
\mathrm{Refl}\ R &\equiv_{\mathrm{df}} (\forall x) Rxx \\
\mathrm{Sym}\ R &\equiv_{\mathrm{df}} (\forall xy)(Rxy \to Ryx) \\
\mathrm{Trans}\ R &\equiv_{\mathrm{df}} (\forall xyz)(Rxy\ \&\ Ryz \to Rxz) \\
\mathrm{AntiSym}_E\ R &\equiv_{\mathrm{df}} (\forall xy)(Rxy\ \&\ Ryx \to Exy) \\
\mathrm{Ext}_E\ A &\equiv_{\mathrm{df}} (\forall xy)(Exy\ \&\ Ax \to Ay)
\end{aligned}$$

Corollary 7.14. *By Theorems* 7.7 *and* 7.12, *the properties* Refl *and* $\triangle$ Sym *are t-absolute*, *and the properties* Sym, Trans, $\mathrm{AntiSym}_E$, *and* Ext_E *are t-antitone*.

It can be observed that unlike non-graded reflexivity $\triangle$ Refl and symmetry $\triangle$ Sym, the non-graded notions of transitivity $\triangle$ Trans, antisymmetry $\triangle\,\mathrm{AntiSym}_E$, and extensionality $\triangle\,\mathrm{Ext}_E$ are not t-absolute, but only t-antitone—which is why even in the non-graded theory of fuzzy relations they have to be parameterized by a t-norm (cf. the notion of T-transitivity, introduced in [28]). Application of Theorems 7.7 and 7.12 to other graded notions of FCT is left to the interested reader.

8. Future work

In this paper, the graded theory of fuzzy connectives has been elaborated in several directions, as discussed and motivated in the Introduction. It can be noticed that only a few sample graded properties of fuzzy connectives have been considered in the present paper—mainly the gradual versions of algebraic or order-theoretic properties. Other algebraic properties, such as various distributivity conditions, divisibility, or the existence of inverse elements, have been left aside here. The main aim of the present paper was to demonstrate the feasibility of such calculations in the framework of FCT and to give a comprehensive list of provable estimates for the above t-norm-related graded properties.

Another important topic for future work is an elaboration of graded variants of analytic properties of fuzzy connectives (regarded as functions), especially the graded variants of continuity (and left-continuity), as the latter property appears in the definition of the semantics of t-norm connectives.[22] The related notion of graded residuation, and the graded interplay between residuation and left-continuity, could make it possible to investigate a graded internalization of the metamathematics of t-norm logics, and show how graded validity of the axioms of t-norm logics depends on graded satisfaction of semantic properties.

Finally, since &-multiplicity exponents are so abundant in formulae of graded mathematics (as could be seen in the present paper), a general theory describing the 'arithmetic' of these exponents in fuzzy logics might be expedient for future work in logic-based fuzzy mathematics.

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[22] Note that in Section 7, only the *non-graded* notion of left-continuity has been considered. Some observations related to the right-continuity of unary functions $\mathbb{R} \to \mathrm{L}$ (which can readily be transferred to the left-continuity of functions $\mathrm{L} \to \mathrm{L}$) were presented in [3].

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