Position Paper
Proof theory and proof systems for projective and affine geometry

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Abstract. Purpose of this research area is the development of formal methods and tools to deal with various geometries. Furthermore we aim at a formal description of sketches in these geometries, which seem to be a basic concept and useful tool. We develop Gentzen style calculi, prove various properties of these calculi, and exhibit the equivalence of sketches in the found formalization and proofs in the calculus. Further development in the area of automatic deduction and proving with sketches and the automated translation to proofs in a calculus are discussed.

1 Introduction

Learning geometry is one of the first things you do in school, and indeed it is one of the most intuitive areas where proofs can be supported and explained using sketches. But still geometries are defined by axioms and are therefore open for a proof theoretic analysis. Furthermore sketches as they are used in geometry are a very interesting area of research. Several computer programs have been developed to work as "proving with sketches" (Dr. Genius, Geometers SketchPad, et al.) and those programs are readily available on the Internet. So why would anyone deal with sketches?

To cite from [Pol54a]:

> We secure our mathematical knowledge by *demonstrative reasoning*, but we support our conjectures by *plausible reasoning.* A mathematical proof is demonstrative reasoning, but the inductive evidence of the physicist, the circumstantial evidence of the lawyer, the documentary evidence of the historian, and the statistical evidence of the economist belong to plausible reasoning.
>
> The difference between the two kinds of reasoning is great and manifold. Demonstrative reasoning is safe, beyond controversy, and final. Plausible reasoning is hazardous, controversial, and provisional. Demonstrative reasoning penetrates the sciences just as far as mathematics does, but it is in itself (as mathematics is in itself) incapable of yielding essentially new knowledge about the world around us. Anything new that we learn about the world involves plausible reasoning [...]

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Accompanying the introduction of formal methods within mathematics and logic less and less methods of proving where founded on a formal base and therefor accepted as valid. One particular interesting example, and at the same time one of the last ones, since it is from 1953, is the use of physical notions in a proof of Schütte [SvdW53], which would be impossible today. Nevertheless it is of fundamental proof theoretic interest to understand the relation between formal proofs and informal descriptions. If we improve our understanding of these connections we could on the one hand better understand old proofs using these methods and on the other hand really use this methods in new proofs.

The use of sketches in geometry resembles exactly the description given above. They were used for plausible reasoning [Pol54a,Pol54b]. Ever since ancient civilizations like the chinese, arian or greek cultures started to prove geometrical theorems there where different forms of proving, different ways of assuring the truth of statements. All these different forms where accepted as complete proving tools. The distinction between plausible and demonstrative reasoning has not been made. With the beginning of strict formalization of mathematics and the use of formal methods within mathematics, some of these proving techniques have vanished or have changed the way they are used, because they couldn't stand the requirements of current science for a valid proving method. Some of these methods still exist but have lost there importance because they where not considered "timely" or formal enough to be used.

In our work we want to investigate the strength of non-standard proof methods in comparison with standard proof techniques via calculi. The purpose of this paper is a short presentation on how far we have come with our development of proof theory for geometries, which tools are available and what we can expect and what we hope to achieve in the future.

2 Affine and projective geometry

The root of projective geometry is the parallel postulate introduced by Euclid (*c.* 300 B.C.). The believe in the absolute truth of this postulate remains unshakable till the 19th century when the founders of non-Euclidean geometry—Carl Friedrich Gauss (1777–1855), Nicolai Ivanovitch Lobachevsky (1793–1856), and Johann Bolyai (1802–1860)—concluded independently that a consistent geometry denying Euclid's parallel postulate could be set up.

Affine Geometry is an extension of projective geometry with a new predicate $\parallel$ and some new axioms. It should resemble the geometry of space we are living in.

Both geometries deal with points and lines. These two elements are primitives, which are not further defined. Two predicates are connecting these objects. One predicate defines a relation between Points and Lines, called *Incidence*, written $P \mathcal{I} g$ meaning the Point P incides with the line g. The second predicate expresses the parallel property, written $g \parallel h$.

Projective geometry only uses the incidence (besides the equality), while affine geometry uses both predicates. The axioms for affine geometry are the following: (AG1) For every two distinct Points there is one and only one Line, so that these two Points incide with this Line. (AG2) For a Point P and a line l such that $P \not\mathcal{I}\ g$ there exists one and only one line m such that $P\ \mathcal{I}\ m$ and $l \parallel m$. (AG3) There are three noncollinear Points.

The axioms for projective geometry are (PG1) For every two distinct Points there is one and only one Line, so that these two Points incide with this Line. (PG2) For every two distinct Lines there is one and only one Point, so that this Point incides with the two Lines. (PG3) There are four Points, which never incide with a Line defined by any of the three other Points.

These are very basic geometries and we are a long way from Euclidean geometry, where higher concepts like angle, distance etc. are introduced. But this reduction to a few basic concepts really allows the proof theoretic analysis, while the introduction of higher concepts complicates the analysis to the point of impossibility.

3 Calculus $\mathbf{L_{PG}}$K for projective geometry

We have developed a calculus $\mathbf{L_{PG}K}$ for projective geometry ([Pre96,Pre97]) which is suitable for doing projective geometry, and with some adaptions to the notations also affine geometry ([Pre01]). This calculus has been analyzed and various theorems have been proven. Most important is that the cut elimination theorem from Gentzens **LK** can be extended to $\mathbf{L_{PG}K}$. A more general analyzation of proof theory for theories of this kind is done in [Neg01], where some very interesting results are shown.

The language of the calculus is a typed with two types for points and lines, otherwise it is similar to the basic **LK**. The constants $A_0,\ldots,D_0$ are used to denote the four Points obeying (PG3). The notation $[PQ]$ is used for the connection $\mathrm{con}(P,Q)$ of two Points and the notation (gh) for the intersection $\mathrm{intsec}(g,h)$ of two Lines to agree with the classical notation in projective and affine geometries. Finally $\mathcal{I}\ (P,g)$ will be written $P\ \mathcal{I}\ g$. The formalization of terms, atomic formulas and formulas is a standard technique and can be found in [Tak87].

The initial sequents are the logical ones ($A \to A$ with A is atomic) and the mathematical initial sequents are formulas of one of the following forms:

1. $\to P\ \mathcal{I}\ [PQ]$ and $\to Q\ \mathcal{I}\ [PQ]$.
2. $\to (gh)\ \mathcal{I}\ g$ and $\to (gh)\ \mathcal{I}\ h$.
3. $\to x = x$ where x is a free variable.

The rules for the calculus are the usual logical rules, the usual rules for equality and the following mathematical rules

$$\frac{\Gamma \to \Delta, P\ \mathcal{I}\ g \quad \Gamma \to \Delta, Q\ \mathcal{I}\ g \quad P = Q, \Gamma \to \Delta}{\Gamma \to \Delta, [PQ] = g}\ (\text{PG1-ID})$$

$$\frac{\Gamma \to \Delta, X\ \mathcal{I}\ [YZ]}{\Gamma \to \Delta}\ (\text{Erase})$$

where $\neq(X,Y,Z)$ and $X,Y,Z \in \{A_0, B_0, C_0, D_0\}$

3.1 Cut elimination theorem for $\mathbf{L_{PG}K}$

We will not give a formal proof but the idea behind the proof: The process of eliminating the cuts is an extension of Gentzens method. For this the proofs are first brought into a normal form where all the mathematical rules are in front of the logical rules (and both are interspersed with structural rules). In the first part geometry is practiced in the sense that in this part the knowledge about projective planes is used. The second part is a logical part connecting the statements from the geometric part to more complex statements with logical connectives. It is easy to see, that for every proof in $\mathbf{L_{PG}K}$ there is a proof in normal form of the same endsequent.

The central lemma for the elimination of cuts is the following which eliminates one cut from a proof:

Lemma 1. *For every proof in normal form with only one cut there is a proof in normal form of the same endsequent without a cut.*

PROOF (Sketch, detailed exposition in [Pre96]):

STEP 1: We will start with the cut-elimination procedure as usual in **LK** as described e.g. in [Tak87]. This procedure shifts a cut higher and higher till the cut is at an axiom where it can be eliminated trivially. Since in our case above all the logical rules there is the (atom)-part, the given procedure will only shift the cut in front of this part.

STEP 2: The cut is already in front of the (atom)-part: First all the inferences not operating on the cut-formulas or one of its predecessors are shifted under the cut-rule. Then the rule from the right branch over the cut-rule are shifted on the left side by applying the dual rules[1] in inverse order. Finally we get on the right side either a logical axiom or a mathematical axiom. The case of a logical axiom is trivial, in case of the mathematical axiom the rules from the left side are applied in inverse order on the antecedent of the mathematical axiom which yields a cut-free proof.

EXAMPLE: A trivial example should explain this method: The proof

$$\dfrac{\dfrac{x_2 = x_3 \to x_2 = x_3 \qquad \dfrac{x_1 = x_2 \to x_1 = x_2 \quad x_1 = u \to x_1 = u}{x_1 = x_2, x_1 = u \to x_2 = u}}{x_2 = x_3, x_1 = x_2, x_1 = u \to x_3 = u} \qquad x_3 = u \to}{x_2 = x_3, x_1 = x_2, x_1 = u \to}\ (\mathrm{Cut})$$

will be transformed to

$$\dfrac{x_1 = x_2 \to x_1 = x_2 \qquad \dfrac{x_2 = x_3 \to x_2 = x_3 \quad x_3 = u \to}{x_2 = x_3, x_2 = u \to}}{x_1 = x_2, x_2 = x_3, x_1 = u \to}$$

Theorem 1 (Cut Elimination for $\mathbf{L_{PG}K}$). *If there is a proof of a sequent $\Pi \to \Gamma$ in $\mathbf{L_{PG}K}$, then there is also a proof without a cut.*

[1] E.g. (trans:left) and (trans:right) are dual rules

EXAMPLE: We will present an example proof and the corresponding proof without a cut. We want to prove that for every line there is a point not on that line, in formula: $(\forall g)(\exists X)(X \not\mathcal{I}\ g)$.

We will first give the proof in words and then in $\mathbf{L_{PG}K}$.

PROOF: (Words) When $A_0 \not\mathcal{I}\ g$ then take A_0 for X. Otherwise $A_0 \mathcal{I}\ g$. Next if $B_0 \not\mathcal{I}\ g$ take B_0 for X. If also $B_0 \mathcal{I}\ g$ then take C_0, since when A_0 and B_0 lie on g, then $g = [A_0 B_0]$ and $C_0 \not\mathcal{I}\ [A_0 B_0] = g$ by (PG3).

PROOF: ($\mathbf{L_{PG}K}$)

$$
\dfrac{
\dfrac{\dfrac{A_0 \mathcal{I}\, g \to A_0 \mathcal{I}\, g}{\to A_0 \mathcal{I}\, g, A_0 \not\mathcal{I}\, g}}{\to A_0 \mathcal{I}\, g \vee A_0 \not\mathcal{I}\, g}
\qquad
\dfrac{
\dfrac{\dfrac{A_0 \mathcal{I}\, g \to A_0 \mathcal{I}\, g}{A_0 \not\mathcal{I}\, g \to A_0 \not\mathcal{I}\, g}}{A_0 \not\mathcal{I}\, g \to (\exists X)(X \not\mathcal{I}\, g)}
\quad
\dfrac{
\dfrac{\dfrac{B_0 \mathcal{I}\, g \to B_0 \mathcal{I}\, g}{\to B_0 \mathcal{I}\, g, B_0 \not\mathcal{I}\, g}}{\to B_0 \mathcal{I}\, g \vee B_0 \not\mathcal{I}\, g}
\quad
\dfrac{
\dfrac{\dfrac{B_0 \mathcal{I}\, g \to B_0 \mathcal{I}\, g}{B_0 \not\mathcal{I}\, g \to B_0 \not\mathcal{I}\, g}}{B_0 \not\mathcal{I}\, g \to (\exists X)(X \not\mathcal{I}\, g)}
\quad
\begin{matrix}\vdots\ \Pi_1 \\ A_0 \mathcal{I}\, g, B_0 \mathcal{I}\, g \to (\exists X)(X \not\mathcal{I}\, g)\end{matrix}
}{B_0 \mathcal{I}\, g \vee B_0 \not\mathcal{I}\, g, A_0 \mathcal{I}\, g \to (\exists X)(X \not\mathcal{I}\, g)}
}{A_0 \mathcal{I}\, g \to (\exists X)(X \not\mathcal{I}\, g)}\,(\mathrm{Cut})
}{A_0 \mathcal{I}\, g \vee A_0 \not\mathcal{I}\, g \to (\exists X)(X \not\mathcal{I}\, g)}
}{\dfrac{\to (\exists X)(X \not\mathcal{I}\, g)}{\to (\forall g)(\exists X)(X \not\mathcal{I}\, g)}}\,(\mathrm{Cut})
$$

Π_1 :

$$
\dfrac{
\dfrac{\dfrac{A_0 \mathcal{I}\, g \to A_0 \mathcal{I}\, g \quad B_0 \mathcal{I}\, g \to B_0 \mathcal{I}\, g \quad A_0 = B_0 \to}{A_0 \mathcal{I}\, g, B_0 \mathcal{I}\, g \to g = [A_0 B_0]} \qquad \dfrac{C_0 \mathcal{I}\, [A_0 B_0] \to C_0 \mathcal{I}\, [A_0 B_0]}{C_0 \mathcal{I}\, [A_0 B_0] \to}\,(\mathrm{Erase})}{A_0 \mathcal{I}\, g, B_0 \mathcal{I}\, g, C_0 \mathcal{I}\, g \to}
}{A_0 \mathcal{I}\, g, B_0 \mathcal{I}\, g \to (\exists X)(X \not\mathcal{I}\, g)}
$$

The cut-elimination procedure[2] yields a cut-free proof of the same end-sequent:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A_0 \mathcal{I}\, g \to B_0 \mathcal{I}\, g \quad B_0 \mathcal{I}\, g \to B_0 \mathcal{I}\, g \quad A_0 = B_0 \to}{A_0 \mathcal{I}\, g, B_0 \mathcal{I}\, g \to g = [A_0 B_0]} \qquad \dfrac{C_0 \mathcal{I}\, [A_0 B_0] \to C_0 \mathcal{I}\, [A_0 B_0]}{C_0 \mathcal{I}\, [A_0 B_0] \to}\,(\mathrm{Erase})}{A_0 \mathcal{I}\, g, B_0 \mathcal{I}\, g, C_0 \mathcal{I}\, g \to}}{\to A_0 \not\mathcal{I}\, g, B_0 \not\mathcal{I}\, g, C_0 \not\mathcal{I}\, g}}{\to (\exists X)(X \not\mathcal{I}\, g), (\exists X)(X \not\mathcal{I}\, g), (\exists X)(X \not\mathcal{I}\, g)}}{\to (\exists X)(X \not\mathcal{I}\, g)}}{\to (\forall g)(\exists X)(X \not\mathcal{I}\, g)}
$$

♡

4 Sketches in affine and projective geometry

Most of the proofs in projective geometry are illustrated by a sketch. But this method of a graphical representation of the maybe abstract facts is not only used in areas like projective geometry, but also in other fields of mathematics. The

[2] or a close look

difference between these sketches and the sketches used in projective geometry (and similar fields) is the fact, that proofs in projective geometry deal with geometric objects like Points and Lines, which are indeed objects we can imagine and draw on a piece of paper.

So the sketch in projective geometry has a more concrete task than only illustrating the facts, since it exhibits the incidences, which is the only predicate constant besides equality really needed in the formalization of projective geometry. It is a sort of proof by itself and so potentially interesting for a proof-theoretic analysis.

If we are interested in the concept of the sketch in mathematics in general and in projective geometry in special then we must set up a formal description of what we mean by a sketch. This is necessary if we want to be more concrete on facts on sketches.

We will evolve the definition of sketches and constructions for the affine case, the ones for the projective case can be obtained by dropping all occurrences of rules, cases etc regarding $\|$

All Points and Lines are combined in the sets called $\tau_{\mathcal{P}}$ and $\tau_{\mathcal{L}}$, respectively.

To ensure consistency inside a set of starting objects, they must obey one rule, namely that if a compound term is in the set, than all of its subterms are also in the set. This is the reason for the next definition.

Definition 1 (admissible set of terms). *Let $\mathcal{M}$ be a subset of $\mathcal{T}(\mathcal{C})$, the set of terms over $\mathcal{C}$, a set of constants, then $\mathcal{M}$ is called admissible if it obeys the following rules:*

- $(\forall[XY] \in \mathcal{M})(X, Y \in \mathcal{M})$
- $(\forall(gh) \in \mathcal{M})(g, h \in \mathcal{M})$

The idea is to define a set of Points, Lines and certain combinations of them (the intersection points and connection lines) and to let the sketch be a subset of all possible atomic formulas over these terms.

The universe of formulas $\mathcal{FU}_{\mathcal{P}}(\mathcal{M})$ is simple all well formed formulas with predicates in $\mathcal{P}$ and terms in $\mathcal{M}$. $\mathcal{P}$ will only be $\{\mathcal{I}, \|, =\}$ or $\{\mathcal{I}, \|\}$. The set $\mathcal{FU}$ contains all the possible positive statements which can be made over the termset $\mathcal{M}$.

We wish to approximate real sketches as close as possible, and therefore we should not allow multiple instances of the same object, i.e. we require that a object (Point, Line) has a unique name and does not have different names in different parts. We require a proper state within our construction and therefore do not allow ambiguous information, which can arise from the following situation, called *critical constellation*:

Definition 2 (Critical Constellation). *Let P and Q be terms in $\tau_{\mathcal{P}}$ and g and h terms in $\tau_{\mathcal{L}}$. Than we call the appearance of the following four formulas a critical constellation:*

$$\begin{array}{c|c} P\,\mathcal{I}\,g & P\,\mathcal{I}\,h \\ \hline Q\,\mathcal{I}\,g & Q\,\mathcal{I}\,h \end{array}$$

We will denote such critical constellations by $(P, Q; g, h)$.

Such a constellation is called critical, because from these four formulas it follows that either $P = Q$ or $g = h$ (or both), but we cannot determine which one of these alternatives without supplementary information.

When constructing any sketch we start from some assumptions over a set of constants and then construct new objects and deduce new relations. From a proof-theoretic point of view these first assumptions are the left side of the deduced sequent, i.e. the assumptions from which you deduce the fact.

Now we come to the definition of the sketch. We require a sketch to be a set describing all the incidences in the sketch. But we also require that this subset is closed under trivial incidences, which means that if we talk about a Line which is the connection of Points, then we require that the trivial formulas express that these two Points lie on the corresponding Line.

Further we require that no critical constellations occur in a sketch. That arises from the fact that we wish that every geometric object is described only by one logical object, i.e. one term. Since a critical constellation implies the equality of two logical objects, which we cannot determine automatically, we want to exclude such cases.

Definition 3 (Sketch). *Let $\mathcal{M}$ be an admissible termset over a set of constants $\mathcal{C}$, $\{A_0, B_0, C_0, [A_0B_0], \ldots, [B_0C_0]\} \subset \mathcal{M}$,*
let $\mathcal{E}$ be a subset of $\mathcal{FU}_{\{\mathcal{I},\|\}}(\mathcal{M}) \cup \overline{\mathcal{FU}_{\{\mathcal{I},\|,=\}}(\mathcal{M})}$ with $A_0 \neq B_0$, $\ldots$, $B_0 \neq C_0$, $A_0 \not\mathcal{I}\ [B_0C_0], \ldots,\ B_0 \not\mathcal{I}\ [A_0C_0] \in \mathcal{E}$,
let Q be a set of equalities and let the triple $(\mathcal{M}, \mathcal{E}, Q)$ obey the following requirements:

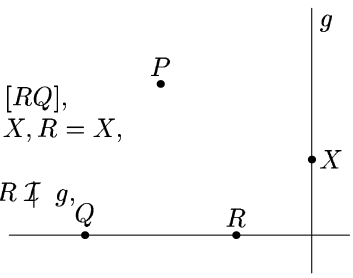
$$(\forall X, Y \in \mathcal{M}, \tau_{\mathcal{P}})([XY] \in \mathcal{M} \supset (X\ \mathcal{I}\ [XY]) \in \mathcal{E} \wedge (Y\ \mathcal{I}\ [XY]) \in \mathcal{E})$$
$$(\forall g, h \in \mathcal{M}, \tau_{\mathcal{L}})((gh) \in \mathcal{M} \supset ((gh)\ \mathcal{I}\ g) \in \mathcal{E} \wedge ((gh)\ \mathcal{I}\ h) \in \mathcal{E}) \tag{S.1}$$
$$(\neg\exists x, y \in \mathcal{M})(P(x,y) \in \mathcal{E} \wedge \neg P(x,y) \in \mathcal{E})$$
$$(\neg\exists x \in \mathcal{M})((x \neq x) \in \mathcal{E}) \tag{S.2}$$
$$\textit{there are no critical constellations in } \mathcal{E} \tag{S.3}$$
$$(\forall x \in \mathcal{M})((x = x) \in Q) \tag{S.4}$$

Then we call the triple $\mathcal{S} = (\mathcal{M}, \mathcal{E}, Q)$ a sketch.

We will call the violation of S.2 a direct contradiction.

A small example should aid understanding of the concepts: In this sketch the different sets are (where the incidences of the constants are lost!):

$$
\begin{aligned}
\mathcal{C} &= \{P, Q, R, X, g\} \\
\mathcal{M} &= \{P, Q, R, X, g, [RQ]\} \\
\mathcal{FU}_{\{\mathcal{I},=\}} &= \{P\ \mathcal{I}\ g, Q\ \mathcal{I}\ g, R\ \mathcal{I}\ g, X\ \mathcal{I}\ g, \\
&\qquad P\ \mathcal{I}\ [RQ], Q\ \mathcal{I}\ [RQ], R\ \mathcal{I}\ [RQ], X\ \mathcal{I}\ [RQ], \\
&\qquad P = Q, P = R, P = X, Q = R, Q = X, R = X, \\
&\qquad g = [RQ]\} \\
\mathcal{E} &= \{Q\ \mathcal{I}\ [RQ], R\ \mathcal{I}\ [RQ], X\ \mathcal{I}\ g, P \not\mathcal{I}\ g, Q \not\mathcal{I}\ g, R \not\mathcal{I}\ g, \\
&\qquad P \not\mathcal{I}\ [QR], X \not\mathcal{I}\ [QR]\}
\end{aligned}
$$



Note that one sketch is only one stage in the process of a construction, which starting from some initial assumptions forming a sketch deduces more and more facts and so constructs more and more complex sketches.

4.1 Actions on Sketches

Till now a sketch is only a static concept, nothing could happen, you cannot "construct". So we want to give some actions on a sketch, which construct a new sketch with more information. The new sketch may not have the properties S.1–S.3, but it must be a semisketch:

Definition 4 (Semisketch). *A* semisketch *is a sketch that need not obey to S.2 and S.3.*

These actions should correspond to similar actions in real world, i.e. actions taken when one draws a sketch. After these actions are defined we can explain what we mean by a construction in this calculus for construction.

The following list defines the allowed actions and what controls have to be executed. The following list describes the changes that have to be done on the quadruple of a sketch when we carry out the corresponding action.

In the following listing we will use the function *closure*(Q) on a set of equalities Q. This function deduces all equalities which are consequences of the set Q. This is a relatively easy computation. If we have $Q = \{x = x, y = y, z = z, x = y, y = z\}$, then the procedure returns $Q \cup \{x = z\}$. This function is used to update the set Q of a sketch after a substitution.

Joining of two Points X, Y**; Symbol:** $[XY]$: $\mathcal{M}' = \mathcal{M} + [XY]$, $\mathcal{E}' = \mathcal{E} + \{X \, \mathcal{I} \, [XY], Y \, \mathcal{I} \, [XY]\}$, $Q' = Q + ([XY] = [XY])$. The requirements (S.1) and (S.4) are fulfilled since the necessary formulas are added to $\mathcal{E}$ and Q. This action can produce a semisketch from a sketch.

Intersection of two Lines g, h**; Symbol** (gh): Dual to the joining of two points, but this action will only be allowed if in $\mathcal{E}$ there is $g \nparallel h$.

Assuming a new Object C **in general position, Symbol** $\{C\}$: $\mathcal{M}' = \mathcal{M} + C$, $\mathcal{E}' = \mathcal{E}$, $Q' = Q + (C = C)$. That $\mathcal{S}'$ is a sketch is trivial, since C is a completely new constant. C must be a constant of type $\tau_{\mathcal{P}}$ or $\tau_{\mathcal{L}}$.

Giving the Line $[XY]$ **a new name** $g := [XY]$**; Symbol** $g := [XY]$: $\mathcal{M}' = \mathcal{M}[[XY]/g]$, $\mathcal{E}' = \mathcal{E}[[XY]/g]$, $Q' = Q[[XY]/g]$. $\mathcal{S}'$ is a sketch since this operation is only a name-change. Note that g must not be in $\mathcal{M}$.

Drawing a line parallel to g **through** P**; Symbol** $h :=\parallel (g, P)$ $\mathcal{M}' = \mathcal{M} + h$, $cE' = \mathcal{E} \cup \{g \parallel h, P \, \mathcal{I} \, h\}$, $Q' = Q + (h = h)$.

Giving the Point (gh) **a new name** $P := (gh)$**; Symbol** $P := (gh)$
Dual to giving an intersection-point a name. Note that P must not be in $\mathcal{M}$.

Identifying two Points u **and** t**; Symbol** $u = t$ $\mathcal{M}' = \mathcal{M} \setminus \{u\}$, $\mathcal{E}' = \mathcal{E}[u/t]$, $Q' = \text{closure}(Q \cup \{u = t\})$. Note that the set Q' can contain terms t not in $\mathcal{M}'$. This action can produce a semisketch from a sketch.

Identifying two Lines l **and** m**; Symbol** $l = m$
Dual to identifying two Points.

Using a "Lemma": Adding $t \mathcal{I} u$**; Symbol** $t \mathcal{I} u$ $\mathcal{M}' = \mathcal{M}$, $\mathcal{E}' = \mathcal{E} + (t \mathcal{I} u)$, $Q' = Q$. This action can produce a semisketch from a sketch.

Adding a negative literal $t \not\mathcal{I} u$**; Symbol** $t \not\mathcal{I} u$ $\mathcal{M}' = \mathcal{M}$, $\mathcal{E}' = \mathcal{E} + (t \not\mathcal{I} u)$, $Q' = Q$.

Adding a negative literal $t \neq u$**; Symbol** $t \neq u$ $\mathcal{M}' = \mathcal{M}$, $\mathcal{E}' = \mathcal{E} + (t \neq u)$, $Q' = Q$.

To deduce a fact with sketches we connect the concept of the sketch and the concept of the actions into a new concept called construction. This construction will deduce the facts.

Definition 5 (Construction). *A* construction *is a rooted and directed tree with a semisketch attached to each node and an action attached to each vertex and satisfying the following conditions: If a vertex with action A leads from node N_1 to node N_2, then N_2 is obtained from N_1 by carrying out the action on N_1. If from a node N there is a vertex labeled ...*

- *with $[XY]$ or (gh), then $X \neq Y$ or $\{g \neq h, g \not\parallel h\}$ is contained in $\mathcal{E}^N$.*
- *with $h :=\parallel (g, P)$, $\{C\}$, $g := [XY]$, $P := (gh)$, then there is no other vertex from N.*
- *with $P \mathcal{I} g$, $X = Y$, $g \parallel h$, then there is exactly one other vertex labeled with the negate of the other formula. This is called a case-distinction.*

Furthermore if $\mathcal{E}$ attached to a node ...

- *yields a direct contradiction, then it has no successor,*
- *contains formulas $P \mathcal{I} g$, $P \mathcal{I} h$, $g \parallel h$ and $g \neq h$ then there is no successor, the node is contradictory.*
- *is a semisketch but not a sketch, i.e. that there are critical constellations, let $(P, Q; g, h)$ be one of them, then there are exactly two successors, one labeled with the action $P = Q$ and one labeled with the action $g = h$.*

What is deduced by a construction: A formula is true when it is true in all the models of the given calculus. The distinct models in a construction are achieved by case-distinctions. So if a formula should be deduced by a construction, it must be in all the leafs of the tree. But since some leafs end with contradictions and from the logical principle "ex falso quodlibet" we only require that a formula, which should be deduced, has to be in all leafs which are not contradictory.

We also have to pay attention to the way a construction handles identities. Since in a construction an identity is carried out in the way that all occurrences of one term are substituted for the other, we not only prove an atomic formula, but also all the formulas which are variants with respect to the corresponding set Q. This notion will now be defined.

Definition 6. *Two atomic formulas $P(t_1, u_1)$ and $P(t_2, u_2)$ are said to be* equivalent with respect to Q*, where Q is a set of equalities, in symbols $P(t_1, u_1) \equiv_{Q_N} P(t_2, u_2)$, when $(t_1 = t_2), (u_1 = u_2) \in Q_N$ (or the symmetric one).*

Now we can define the notion of what a construction deduces:

Definition 7. *A construction deduces a set of atomic formulas Δ iff for all $A \in \Delta$ there is a not contradictory leaf, where either $A \in Q_N$ or $(\exists B \in \mathcal{E}(N)) A \equiv_{Q(N)} B$.*

The meaning of this definition is that if a construction deduces Δ then the disjunction of all formulas in Δ is proved by this construction.

4.2 An example for an affine construction

We will prove the following sentence of affine geometry with a construction:

$$(\neq (l, m, n) \wedge m \parallel n \wedge P \mathcal{I}\, l \wedge P \mathcal{I}\, m) \supset (\exists Q)(Q \mathcal{I}\, l \wedge Q \mathcal{I}\, n)$$

I.e. if two lines (m, n) are parallel and another line (l) intersects with one of them, then it intersects also with the other one.

The construction is given in fig. 1. First the assumptions are build up by simple case distinction, this is an automatic process, then the proof by sketch follows closely any other proof by distinction whether $l \parallel n$ or not. In case it is parallel node 9 is reached and closed because it yields a contradiction ($P \mathcal{I}\, l$, $P \mathcal{I}\, m$, $l \parallel m$, $l \neq m$). If they are not parallel we can construct the intersection point (nl) and are finished. Therefore the construction proves the above formula (if the implication is transformed into a disjunction).

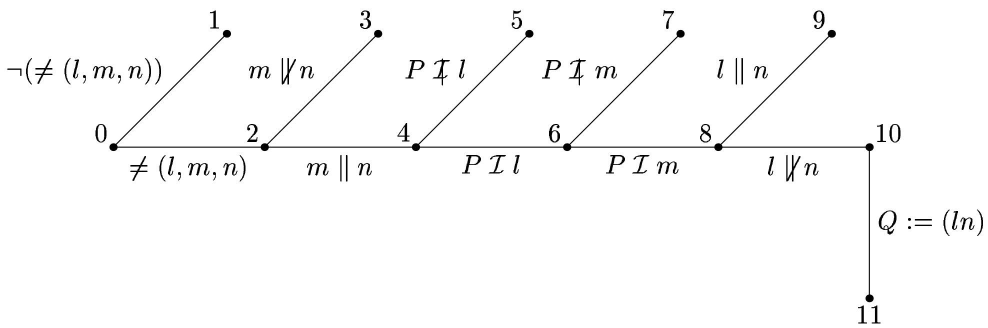


Fig. 1. An affine construction

5 Translation of Sketches to Proofs and back

We will explain how to go from sketches in affine geometry as defined above to proofs in a formal calculus and how to obtain a set of sketches that prove the same as a proof in a formal calculus.

The translation is done by transforming a given affine construction into a projective one, which in turn can be translated into a proof according to [BP02]. A similar route is used for the reverse direction: If we have a proof of a formula in affine geometry we transform it into a formula in projective geometry, build the corresponding projective sketch and translate it into an affine sketch.

Both translations are based on the well known fact that you can complete an affine geometry with one line at infinity to a projective geometry, and that you can strip down a projective geometry by deletion of one line to an affine geometry. In these cases parallel means that the intersection point of two lines incides with the taken out or added line.

It is easy to prove the axioms of affine geometry are valid in this new structure: Axiom (AG1) is the same in projective geometry, for axiom (AG2) just define $m := [P(lu)]$. It is obvious that $P \mathcal{I} m$ and that this line is unique. Finally the last axiom (AG3) is a trivial consequence from the last axiom of projective geometry.

On the other hand we can extend any affine geometry by adding a line u which holds all the meeting points of parallel lines to a projective geometry.

Lemma 2. *By adding/deleting one Line to/from an affine/projective geometry we obtain a projective/affine geometry.*

So we can define parallel by incidence and the added line as

$$g \parallel h := (g = h) \vee ((gh) \mathcal{I} u).$$

The proof of the following lemmas can be found in [Pre01].

Lemma 3. *Any construction in affine geometry can be translated into a construction in projective geometry.*

Now we show that we can do the reverse process, too. We start with a formula of affine geometry. Then we rewrite the axioms and add the line u to obtain a formula in projective geometry. This one can be proven by sketches (again according to [BP02]). Finally we have to transform the sketches in projective geometry back into affine geometry.

Lemma 4. *The construction in projective geometry of a Herbrand disjunction of an affine geometry formula can be transformed into a construction in affine geometry.*

Citing from [BP02] and as a consequence of the above lemmata we can state:

Theorem 2. *For any formula proven in any reasonable*[3] *and sound calculus for projective geometry, there is a construction which deduces this formula.*

Theorem 3. *Affine sketches and proofs are equivalent in the sense that any proof of an affine sentence can be translated into a (set of) construction(s) which deduce the same sentence, and any affine construction can be transformed into a proof.*

[3] i.e. a calculus comparable to $\mathbf{L_{PG}K}$

6 Future directions

There are three main directions of further development:

- Implementing a sketching tool and translate these sketches to proofs.
- Refine the notion of sketching.
- Extend the calculus and the notion of sketching to higher geometries.

The first one could be very rewarding in analyzing the ways proofs by sketches are carried out. With this information at hand we could refine the notion of sketches to better fit the real process of sketching. With this new informations we could go on to the inclusion of higher notions like angle, distance etc to cover parts of Euclidean geometry.

References

[BP02] M. Baaz and N. Preining. Herbrand disjunctions and extraction of proofs from sketches. *Annals of Pure and Applied Logic*, 2002. In print.

[Nee59] J. Needham. *Science and Civilization in China*, volume 3. Cambridge University Press, 1959.

[Neg01] S. Negri Contraction-free sequent calculi for geometric theories, with an application to Barr's theorem. *Institut Mittag-Leffler*. Report No. 22, 2000/2001.

[Ore79] V. P. Orevkov. Lower bounds for increasing complexity of derivations after cut elimination. *Zapiski Nauchnykh Seminarov Leningradskogo Otdeleniya Matematicheskogo Instituta*, 88:137–161, 1979.

[Pol54a] G. Polya. *Mathematics and plausible reasoning, Volume I: Induction and Analogy in Mathematics.* Princeton University Press, Princeton, New Jersey, 1954.

[Pol54b] G. Polya. *Mathematics and plausible reasoning, Volume II: Patterns of Plausible Inference.* Princeton University Press, Princeton, New Jersey, 1954.

[Pre96] Norbert Preining. Sketch-as-proof, a proof-theoretic analysis of axiomatic projective geometry. Master's thesis, University of Technology, Vienna, Austria, 1996.

[Pre97] Norbert Preining. Sketch-as-proof. In G. Gottlob, A. Leitsch, and D. Mundici, editors, *Computational Logic and Proof Theory, Proc. 5^{th} Kurt Gödel Colloquium KGC'97*, Lecture Notes in Computer Science 1289, pages 264–277, Vienna, Austria, 1997. Springer.

[Pre01] N. Preining. Sketches in affine geometry. Lecture at the CALCULEMUS'2001 in Siena, 2001.

[SvdW53] K. Schütte and B. L. van der Waerden. Das Problem der dreizehn Kugeln. *Math. Annalen*, 125:325–334, 1953.

[Tak87] Gaisi Takeuti. *Proof Theory.* North Holland, 1987.