Improving the Treatment of Negation in Propositional Dummett Logic

Guido Fiorino

CRII, Universitá dell'Insubria, Via Ravasi, 2, 21100 - Varese
guido@gongolo.usr.dsi.unimi.it

Abstract. We present an efficient tableau calculus for Dummett Logic. The object language and the rules are tailored to improve the treatment of the negation with respect to the known calculi. Both the object language and the rules have been inspired by the Kripke semantics of Dummett Logic. To prove that the calculus characterizes the logic at hand, we give soundness and completeness theorems by using machinery derived from the Fitting's techniques.

Keywords: Tableaux, Non-Classical Logic, Interpolable Logic, Intermediate Logic.

1 Introduction

In this paper we present a tableau calculus, we call it $\mathbb{T}$, for propositional Dummett Logic. The object language of [13] is extended with a new sign that takes into account the semantics of the negation in the Kripke frames characterizing Dummett Logic.

Dummett Logic, introduced in [10], is also known as Gödel Logic, since Gödel studied finite approximations of Dummett Logic ([15]), and **Lc**, because semantically characterized by linearly ordered Kripke frames. Dummett Logic is perhaps one of the most studied intermediate logics in both predicate and propositional versions. It has been considered by people interested in intermediate logics [11, 16, 9, 1, 2, 12], many-valued and fuzzy logics [5, 7, 6] and computer science [3, 4].

Many calculi for this logic have been given in recent years. In particular, in [12] a deterministic calculus for the implicative language is provided and in [7], by a semantical approach, a calculus of hypersequents in which the axioms express the linearity of the models for the logic is given. In contrast to [7], in [1], [13] and here it is not used the fact that, if a wff H is not in the logic then it is falsified on a *model* with a number of elements that is at most the number of propositional variables in H plus one. We emphasize that the calculi of [12, 1, 7] do not take into account computational complexity and the proofs with such calculi may have exponential depth.

The calculus $\mathbb{T}$ we present in this paper and the tools we use, are derived from those of [1] and [13]; to prove that our calculus characterizes Dummett Logic, we give a semantical proof of Soundness and Completeness theorems. Finally, we point out that the rules and the object language of $\mathbb{T}$ can be explained taking into account that the Kripke semantics of Dummett Logic is based on linearly ordered Kripke frames.

2 Basic Definitions

In this section we give notions and notation we will use in the paper. A detailed presentation of all notions regarding intermediate logics and Kripke models can be found in [14] and [8].

Given an enumerable set of propositional variables and the connectives $\neg, \wedge, \vee$, $\rightarrow$, a *well formed formula (wff* for short) is defined as usual. Given a wff A, we say that $\neg A$ is a *negated wff*. We use the term *atom* as synonym of propositional variable.

In the sequel **Int** denotes both an Hilbert-style calculus for Intuitionistic Propositional Logic and the set of intuitionistically valid wffs.

In this paper we are interested in propositional *Dummett Logic* (**Dum**), also known as LC (*Linear Chain*), which can be axiomatized by adding to any axiom system for **Int** the axiom scheme $(p \rightarrow q) \vee (q \rightarrow p)$.

A well known semantical characterization of **Dum** is by *linearly ordered Kripke models*. We call *model* any Kripke model $\underline{K} = \langle P, \leq, \Vdash \rangle$, where $\langle P, \leq \rangle$ is a linearly ordered set and $\Vdash$ is the *forcing relation*, defined between elements of P and propositional variables with the property that, for every $\Gamma, \Delta \in P$ such that $\Gamma \leq \Delta$, and every propositional variable p, if $\Gamma \Vdash p$ then $\Delta \Vdash p$. The *forcing relation* is extended to the wffs as follows:

- $\Gamma \Vdash A \wedge B$ iff $\Gamma \Vdash A$ and $\Gamma \Vdash B$;
- $\Gamma \Vdash A \vee B$ iff $\Gamma \Vdash A$ or $\Gamma \Vdash B$;
- $\Gamma \Vdash A \rightarrow B$ iff, for every $\Delta \in P$ such that $\Gamma \leq \Delta$, $\Delta \nVdash A$ or $\Delta \Vdash B$;
- $\Gamma \Vdash \neg A$ iff for every $\Delta \in P$ such that $\Gamma \leq \Delta$, $\Delta \nVdash A$.

From the above definition it is easy to prove that if $\Gamma \Vdash A$ and $\Gamma \leq \Delta$ then $\Delta \Vdash A$.

Let Γ be a world of P and let A be a wff, if $\Gamma \Vdash A$ we say that A *is forced in* Γ (or in a world of $\underline{K}$). A wff A is *valid* in a *model* $\underline{K} = \langle P, \leq, \Vdash \rangle$ if $\Gamma \Vdash A$ for all $\Gamma \in P$. **Dum** is the set of wffs that are *valid* in every *model*.

Let $\underline{K} = \langle P, \leq, \Vdash \rangle$ be a *model* and let Γ, Δ be members of P. By $\Gamma < \Delta$ we mean that $\Gamma \leq \Delta$ and $\Gamma \neq \Delta$, and by $\Delta \geq \Gamma$ we mean $\Gamma \leq \Delta$. Moreover, Δ is the *immediate successor* of Γ in $\underline{K} = \langle P, \leq, \Vdash \rangle$ iff $\Gamma < \Delta$ and, for all $\Theta \in P$ such that $\Gamma \leq \Theta \leq \Delta$, $\Gamma = \Theta$ or $\Delta = \Theta$. Finally, we call *root of* $\underline{K} = \langle P, \leq, \Vdash \rangle$ an element Υ (if it exists) such that, for every $\Gamma \in P$, $\Upsilon \leq \Gamma$.

In Section 4 we present the tableau calculus $\mathbb{T}$ for **Dum** whose object language is built on the set of signs $\{\mathbf{T}, \mathbf{F}, \mathbf{F_c}, \mathbf{T_{cl}}\}$ and on the set of wffs. Every member of the object language is a *signed wff* (*swff* for short) whose syntax is $\mathcal{S}A$, with $\mathcal{S} \in \{\mathbf{T}, \mathbf{F}, \mathbf{F_c}, \mathbf{T_{cl}}\}$ and A wff.

The *length* of a wff A (respectively swff $\mathcal{S}A$), denoted by $|A|$ (respectively $|\mathcal{S}A|$), is the number of symbols in A (respectively the number of symbols in A plus one). The length of a set S of wffs or swffs, denoted by $|S|$, is the sum of the lengths of its elements. Given two wffs or two swffs A, B with $A \equiv B$ we mean that A and B are syntactically identical. Finally, given a set S of wffs or swffs, by $Pv(S)$ we mean the set of the atoms appearing in the members of S.

We recall the following theorem (see [14]) that we will use in Section 4:

Theorem 1. *Let V a set of propositional variables and let $\underline{K} = \langle P, \leq, \Vdash \rangle$ and $\underline{K}' = \langle P, \leq, \Vdash' \rangle$ be two Kripke models such that for every $p \in V$ and for every $\Gamma \in P$, $\Gamma \Vdash p$ iff $\Gamma \Vdash' p$. Then, for every wff A such that $Pv(A) \subseteq V$, $\Gamma \Vdash A$ iff $\Gamma \Vdash' A$.*

3 Related Works

The main difference between $\mathbb{T}$ and the calculus **Dum**-T given in [13] concerns the rules to treat the negated formulas. In $\mathbb{T}$ we have introduced the sign $\mathbf{T_{cl}}$ that appears in the conclusion of the rules that treat the swffs of the kind $\mathbf{F}(\neg A)$, $\mathbf{F_c}(A \to B)$ and $\mathbf{F_c}(\neg A)$. Depth and width of the deductions of $\mathbb{T}$ are less than those of the deductions obtained with the calculus **Dum**-T. We emphasize that one of the calculi for Dummett Logic presented in [1] treats the case $\mathbf{F}(\neg A)$ with the rule

$$\frac{S, \mathbf{F}(\neg A)}{S, \mathbf{F_c}(\neg A)}\text{NEW-}\mathbf{F}\neg$$

where the sign $\mathbf{F}$ is changed in $\mathbf{F_c}$ but the wff is left unchanged, and the rules for the swffs $\mathbf{F_c}(A \to B)$ and $\mathbf{F_c}(\neg A)$ are

$$\frac{S, \mathbf{F_c}(A \to B)}{S_c, \mathbf{T}A, \mathbf{F_c}B}\text{NEW-}\mathbf{F_c}\to \quad \frac{S, \mathbf{F_c}(\neg A)}{S_c, \mathbf{T}A}\text{NEW-}\mathbf{F_c}\neg.$$

The calculus in [1] is not efficient, but these rules could be put in the calculus given in [13]. The resulting calculus, we call it **Dum**-T', would not be as efficient as $\mathbb{T}$ when the negated formulas are treated in the deductions. By introducing the sign $\mathbf{T_{cl}}$ the depth of the deductions obtained with $\mathbb{T}$ is less than the depth of the deductions obtained with **Dum**-T. A further advantage of using $\mathbf{T_{cl}}$ in $\mathbb{T}$ is that in the conclusions of the rules treating the swffs $\mathbf{F_c}(\neg A)$ and $\mathbf{F_c}(A \to B)$ appear the sign $\mathbf{T_{cl}}$ whereas in the rules above appear the sign $\mathbf{T}$; the rules for $\mathbf{T_{cl}}$ are *simpler* than the corresponding rules for the sign $\mathbf{T}$, where *simpler* formally means that, given a degree measure on swffs, the former lower the degree of the premise more than the latter (e.g. compare the rule $\mathbf{T_{cl}} \to$ with the rules in Table 3 and in particular with $\mathbf{T} \to\to$).

The rule $\mathbf{F} \to$ in Table 4 is derived from the rule $\mathbf{F} \to \neg$ of Table 1 introduced in [1] for calculi using the signs $\mathbf{T}$, $\mathbf{F}$ and $\mathbf{F_c}$, and also used in the calculus given in [13]. The rule $\mathbf{F} \to \neg$ treats both the swffs of the kind $\mathbf{F}(A \to B)$ and the swffs of the kind $\mathbf{F}(\neg C)$. The number of sets of swffs in the conclusion of $\mathbf{F} \to \neg$ depends on the cardinality of $S_{\mathbf{F}\to\neg}$. Our calculus $\mathbb{T}$ has one more sign and five more rules than the calculi given in [1] and [13]. The new sign $\mathbf{T_{cl}}$ is introduced to treat the swffs of the kind $\mathbf{F}(\neg C)$ with the new rule $\mathbf{F}\neg$, which conclusion is a configuration with one set of swffs. The other four new rules handle the $\mathbf{T_{cl}}$-swffs. In this way we avoid to treat the swffs of the kind $\mathbf{F}(\neg C)$ together

$$\frac{S, \mathbf{F}(A_1 \to B_1), \ldots, \mathbf{F}(A_n \to B_n), \mathbf{F}(\neg C_1), \ldots, \mathbf{F}(\neg C_m)}{S_c, S^1_{\mathbf{F}\to\neg}, \mathbf{T}A_1, \mathbf{F}B_1 | \ldots | S_c, S^n_{\mathbf{F}\to\neg}, \mathbf{T}A_n, \mathbf{F}B_n | S_c, S^{n+1}_{\mathbf{F}\to\neg}, \mathbf{T}C_1 | \ldots | S_c, S^{n+m}_{\mathbf{F}\to\neg}, \mathbf{T}C_m} \mathbf{F}\to\neg$$

$$S_c = \{\mathbf{T}A | \mathbf{T}A \in S\} \cup \{\mathbf{F_c}A | \mathbf{F_c}A \in S\} \text{ is the certain part of } S;$$

$$S_{\mathbf{F}\to\neg} = \{\mathbf{F}(A_1 \to B_1), \ldots, \mathbf{F}(A_n \to B_n), \mathbf{F}(\neg C_1), \ldots, \mathbf{F}(\neg C_m)\} \text{ is not empty}$$

$$S^i_{\mathbf{F}\to\neg} = S_{\mathbf{F}\to\neg} \backslash \{\mathbf{F}(A_i \to B_i)\} \text{ for } i = 1, \ldots, n \text{ and } S^{n+i}_{\mathbf{F}\to\neg} = S_{\mathbf{F}\to\neg} \backslash \{\mathbf{F}(\neg C_i)\} \text{ for } i = 1, \ldots, m$$

Table 1. The rule $\mathbf{F}\to\neg$ of the calculi of [1] and [13]

with the swffs of the kind $\mathbf{F}(A \to B)$. As a result we get that when the $\mathbf{F}\neg$-swffs occur in the deduction, the tableaux of $\mathbb{T}$ are smaller than the tableaux of the calculus given in [13]. The rule $\mathbf{F}\neg$ uses the fact that the "possible worlds" of every Kripke model $\underline{K}$ for **Dum** are linearly ordered. If a world Γ does not force the wff $\neg C$, then there exists a world $\Delta \geq \Gamma$ such that $\Delta \Vdash C$. Now, the possible worlds of $\underline{K}$ are linearly ordered, thus no world Θ of $\underline{K}$ can force $\neg C$, because (i) if $\Theta \geq \Delta$ then $\Theta \Vdash \neg C$ and $\Theta \Vdash C$, absurd; (ii) if $\Theta \leq \Delta$, then $\Delta \Vdash C$ and $\Delta \Vdash \neg C$, absurd. Thus, when the possible worlds of a Kripke model $\underline{K}$ are linearly ordered, if a world Δ forces a wff A, then no world of $\underline{K}$ forces $\neg A$ and we can conclude that $\neg\neg A$ is forced everywhere in $\underline{K}$. In the Completeness Theorem we show that with a careful construction of the counter model, the information given by the conclusion of $\mathbf{F}\neg$ is sufficient to get the completeness of $\mathbb{T}$.

4 The Calculus

The rules of the calculus $\mathbb{T}$ are given in Tables 2-4. The meaning of the signs $\mathbf{T}, \mathbf{F}$ $\mathbf{F_c}$ and $\mathbf{T_{cl}}$ used in the tableau calculus $\mathbb{T}$ is explained in terms of *realizability* as follows: given a *model* $\underline{K} = \langle P, \leq, \Vdash \rangle$ and a swff H, we say that $\Gamma \in P$ *realizes* H, and we write $\Gamma \rhd H$, if the following conditions hold:

1. if $H \equiv \mathbf{T}A$ then $\Gamma \Vdash A$;
2. if $H \equiv \mathbf{F}A$ then $\Gamma \nVdash A$;
3. if $H \equiv \mathbf{F_c}A$ then $\Gamma \Vdash \neg A$;
4. if $H \equiv \mathbf{T_{cl}}A$ then $\Gamma \Vdash \neg\neg A$.

We call *main set of swffs* of a rule the set of swffs that are in evidence in the premise (e.g., the *main set of swffs* of the rule $\mathbf{T} \to$ *Atom* is $\{\mathbf{T}A, \mathbf{T}(A \to B)\}$ and the *main set of swffs* of the rule $\mathbf{F} \to$ is $S_{\mathbf{F}\to} = \{\mathbf{F}(A_1 \to B_1), \ldots, \mathbf{F}(A_n \to$

$B_n)\}$, $n \geq 1$). When a *main set of swffs* of a rule is of the kind $\{H\}$, then we call H *main swff* of the rule.

Now we introduce some notions that will be used throughout this paper.

Definition 1. *1. A world Γ of a* model $\underline{K}$ realizes *a set S of swffs (and we write $\Gamma \triangleright S$) iff Γ* realizes *every swff in S. A set S of swffs is* realizable *iff there exists a world Γ of a* model $\underline{K}$ *such that $\Gamma \triangleright S$.*

2. *A* configuration *is any finite sequence $S_1|\dots|S_j|\dots|S_n$ ($n \geq 1$ and $1 \leq j \leq n$), where every S_j is a set of swffs; a configuration is* realizable *iff at least a S_j is* realizable*; we refer to S_j as an* element *of $S_1|\dots|S_n$.*
3. *A set S is* contradictory *if one of the following conditions holds:*
   - $\mathbf{T}A \in S$ *and* $\mathbf{F}A \in S$;
   - $\mathbf{T}A \in S$ *and* $\mathbf{F_c}A \in S$;
   - $\mathbf{F_c}A \in S$ *and* $\mathbf{T_{cl}}A \in S$.
4. *A* proof table of $\mathbb{T}$ *is a finite sequence of configurations $\mathcal{C}_1, \dots, \mathcal{C}_n$ ($n \geq 1$), where the configuration $\mathcal{C}_{i+1}$ is obtained from $\mathcal{C}_i = S_1|\dots|S_j|\dots|S_k$ ($k \geq 1$ and $1 \leq j \leq k$) by applying to each non-*contradictory *S_j of $\mathcal{C}_i$ a rule of the calculus $\mathbb{T}$ and taking in $\mathcal{C}_{i+1}$ every S_j of $\mathcal{C}_i$ which is* contradictory *or containing signed atoms only. Moreover, a* proof table of $\mathbb{T}$ *is* closed *iff all the sets S_j in its final configuration $\mathcal{C}_n$ are* contradictory. *Finally, the* depth *of a* proof table of $\mathbb{T}$ *is the number of its configurations.*
5. *A proof of a wff A in $\mathbb{T}$ is a closed* proof table of $\mathbb{T}$ *starting from the configuration $\{\mathbf{F}A\}$.*
6. *A finite set of swffs S in the object language of $\mathbb{T}$ is* consistent *iff no* proof table of $\mathbb{T}$ *starting from S is* closed.
7. *Let U be a* main set of swffs. *We call* extension(s) *of U the set(s) of swffs $\mathcal{R}^1_U, \dots, \mathcal{R}^n_U$ ($n \geq 1$) such that $\mathcal{R}^1_U|\dots|\mathcal{R}^n_U$ is the configuration obtained by applying the rule of $\mathbb{T}$ related to U to the set of swffs U, where S in the premise of the rule is taken as the empty set.*

We emphasize that for our purposes the initial configuration of every *proof table of* $\mathbb{T}$ has one *element.* Moreover, all the rules are applied in a *duplication-free* style: a rule $\mathcal{R}$ with *main set of swffs* $\{H_1, \dots, H_n\}$ applies to a set U of swffs if it is possible to choose S and $H_1, \dots, H_n$ in such a way that $U = S \cup \{H_1, \dots, H_n\}$, with $S = U \setminus \{H_1, \dots, H_n\}$. This implies that the *main set of swffs* does not reappear in the conclusion of the rule. In the Completeness Theorem our construction of the counter model uses non closed *proof tables of* $\mathbb{T}$ which are built by applying the rules in the *duplication-free* style. To emphasize the choice of $\{H_1, \dots, H_n\}$ we say that $\mathcal{R}$ is applied to $\{H_1, \dots, H_n\}$ and that $\{H_1, \dots, H_n\}$ is treated by $\mathcal{R}$.

In Table 5 we give an example of deduction with $\mathbb{T}$ (where ambiguity can arise the swffs treated by the rules are underlined). We emphasize that this deduction is smaller than the one obtained with the calculus **Dum**-T given in [13].

Now let us define the complexity measure *deg* on wffs and swffs we will use in this paper to study the computational complexity properties of $\mathbb{T}$.

$\dfrac{S,\mathbf{T}(A\wedge B)}{S,\mathbf{T}A,\mathbf{T}B}\mathbf{T}\wedge$	$\dfrac{S,\mathbf{F}(A\wedge B)}{S,\mathbf{F}A\mid S,\mathbf{F}B}\mathbf{F}\wedge$	$\dfrac{S,\mathbf{F_c}(A\wedge B)}{S,\mathbf{F_c}A\mid S,\mathbf{F_c}B}\mathbf{F_c}\wedge$	$\dfrac{S,\mathbf{T_{cl}}(A\wedge B)}{S,\mathbf{T_{cl}}A,\mathbf{T_{cl}}B}\mathbf{T_{cl}}\wedge$
$\dfrac{S,\mathbf{T}(A\vee B)}{S,\mathbf{T}A\mid S,\mathbf{T}B}\mathbf{T}\vee$	$\dfrac{S,\mathbf{F}(A\vee B)}{S,\mathbf{F}A,\mathbf{F}B}\mathbf{F}\vee$	$\dfrac{S,\mathbf{F_c}(A\vee B)}{S,\mathbf{F_c}A,\mathbf{F_c}B}\mathbf{F_c}\vee$	$\dfrac{S,\mathbf{T_{cl}}(A\vee B)}{S,\mathbf{T_{cl}}A\mid S,\mathbf{T_{cl}}B}\mathbf{T_{cl}}\vee$
see Table 3	see Table 4	$\dfrac{S,\mathbf{F_c}(A\to B)}{S,\mathbf{T_{cl}}A,\mathbf{F_c}B}\mathbf{F_c}\to$	$\dfrac{S,\mathbf{T_{cl}}(A\to B)}{S,\mathbf{F_c}A\mid S,\mathbf{T_{cl}}B}\mathbf{T_{cl}}\to$
$\dfrac{S,\mathbf{T}(\neg A)}{S,\mathbf{F_c}A}\mathbf{T}\neg$	$\dfrac{S,\mathbf{F}(\neg A)}{S,\mathbf{T_{cl}}A}\mathbf{F}\neg$	$\dfrac{S,\mathbf{F_c}(\neg A)}{S,\mathbf{T_{cl}}A}\mathbf{F_c}\neg$	$\dfrac{S,\mathbf{T_{cl}}(\neg A)}{S,\mathbf{F_c}A}\mathbf{T_{cl}}\neg$

Table 2. The $\mathbb{T}$ calculus

$$\frac{S,\mathbf{T}A,\mathbf{T}(A\to B)}{S,\mathbf{T}A,\mathbf{T}B}\mathbf{T}\to \mathit{Atom} \text{ with } A \text{ an atom}$$

$$\frac{S,\mathbf{T}((A\wedge B)\to C)}{S,\mathbf{T}(A\to(B\to C))}\mathbf{T}\to\wedge \quad \frac{S,\mathbf{T}(\neg A\to B)}{S,\mathbf{T_{cl}}A)\mid S,\mathbf{T}B}\mathbf{T}\to\neg$$

$$\frac{S,\mathbf{T}((A\vee B)\to C)}{S,\mathbf{T}(A\to p),\mathbf{T}(B\to p),\mathbf{T}(p\to C)}\mathbf{T}\to\vee \text{ with } p \text{ a new atom}$$

$$\frac{S,\mathbf{T}((A\to B)\to C)}{S,\mathbf{F}(A\to p),\mathbf{T}(p\to C),\mathbf{T}(B\to p)\mid S,\mathbf{T}C}\mathbf{T}\to\to \text{ with } p \text{ a new atom}$$

Table 3. The rules $\mathbf{T}\to$

Definition 2. *1. The degree of A, denoted by $deg(A)$, where A is a wff, is defined as follows: if $A\equiv p$, where p is an atom, then $deg(p)=0$; if $A\equiv\alpha\to\beta$, then $deg(\alpha\to\beta)=deg(\alpha)+deg(\beta)+1$; if $A\equiv\alpha\wedge\beta$, then $deg(\alpha\wedge\beta)=deg(\alpha)+deg(\beta)+2$; if $A\equiv\alpha\vee\beta$, then $deg(\alpha\vee\beta)=deg(\alpha)+deg(\beta)+3$; if $A\equiv\neg\alpha$, then $deg(\neg\alpha)=deg(\alpha)+1$;*

2. The degree of $\mathcal{S}A$ (where $\mathcal{S}\in\{\mathbf{T},\mathbf{F},\mathbf{F_c},\mathbf{T_{cl}}\}$), denoted by $deg(\mathcal{S}A)$, coincides with the degree of A.

3. The degree of a set S of swffs is the sum of the degrees of its elements.

By the following two propositions we get that the depth of the deductions of the calculus $\mathbb{T}$ is linearly bounded by the length of the formula to be proven.

Proposition 1. *Let S be a set of swffs, $deg(S)\leq 3|S|$.*

Proof. We prove that the complexity of every wff A is bounded by the length of A. The proof is by induction on the number of connectives in A.
Basis: if the number of connectives is zero then $A\equiv p$, with p an atom, and we

$$\frac{S, \mathbf{F}(A_1 \to B_1), \ldots, \mathbf{F}(A_n \to B_n)}{S_c, S^1_{\mathbf{F}\to}, \mathbf{T}A_1, \mathbf{F}B_1 | \ldots | S_c, S^n_{\mathbf{F}\to}, \mathbf{T}A_n, \mathbf{F}B_n} \mathbf{F}\to$$

$S_c = \{\mathbf{T}A | \mathbf{T}A \in S\} \cup \{\mathbf{F_c}A | \mathbf{F_c}A \in S\} \cup \{\mathbf{T_{cl}}A | \mathbf{T_{cl}}A \in S\}$ is the *certain part* of S

$S_{\mathbf{F}\to} = \{\mathbf{F}(A_1 \to B_1), \ldots, \mathbf{F}(A_n \to B_n)\}$ is not empty;

$S^i_{\mathbf{F}\to} = S_{\mathbf{F}\to} \setminus \{\mathbf{F}(A_i \to B_i)\}$ for $i = 1, \ldots, n$.

Table 4. The rule $\mathbf{F}\to$

$$\begin{array}{c l}
\mathbf{F}((((\neg\neg D \to \neg C) \to B) \vee \neg C) \vee \neg D) & \\
\hline
\mathbf{F}(((\neg\neg D \to \neg C) \to B) \vee \neg C), \mathbf{F}(\neg D) & \mathbf{F}\vee \\
\hline
\mathbf{F}((\neg\neg D \to \neg C) \to B), \underline{\mathbf{F}(\neg C)}, \mathbf{F}(\neg D) & \mathbf{F}\vee \\
\hline
\mathbf{F}((\neg\neg D \to \neg C) \to B), \mathbf{T_{cl}}(C), \underline{\mathbf{F}(\neg D)} & \mathbf{F}\neg \\
\hline
\mathbf{F}((\neg\neg D \to \neg C) \to B), \mathbf{T_{cl}}C, \mathbf{T_{cl}}D & \mathbf{F}\neg \\
\hline
\mathbf{T}(\neg\neg D \to \neg C), \mathbf{F}B, \mathbf{T_{cl}}C, \mathbf{T_{cl}}D & \mathbf{F}\to \\
\hline
\mathbf{F}(\neg\neg D), \mathbf{F}B, \mathbf{T_{cl}}C, \mathbf{T_{cl}}D \mid \mathbf{T}(\neg C), \mathbf{F}B, \mathbf{T_{cl}}C, \mathbf{T_{cl}}D & \mathbf{T}\to\neg \\
\hline
\mathbf{T_{cl}}(\neg D), \mathbf{F}B, \mathbf{T_{cl}}C, \mathbf{T_{cl}}D \mid \mathbf{F_c}C, \mathbf{F}B, \mathbf{T_{cl}}C, \mathbf{T_{cl}}D & \mathbf{F}\neg \mid \mathbf{T}\neg \\
\hline
\mathbf{F_c}D, \mathbf{F}B, \mathbf{T_{cl}}C, \mathbf{T_{cl}}D \mid \text{Closed} & \mathbf{T_{cl}}\neg \\
\hline
\text{Closed} \mid \text{Closed} &
\end{array}$$

Table 5. an example of *proof table of* $\mathbb{T}$

have $deg(p) = 0 < 3|p| = 3$;
Step: suppose the proposition holds for the swffs A' with $n-1$ connectives at most. Let us consider the case where $A \equiv B \vee C$, then $deg(B \vee C) = deg(B) + deg(C) + 3$. By induction hypothesis $deg(B) \leq 3|B|$ and $deg(C) \leq 3|C|$, thus $deg(B \vee C) \leq 3|B| + 3|C| + 3 = 3(|B| + |C| + 1) = 3|B \vee C|$; the other cases are similar. □

Proposition 1 establishes that function *deg* linearly grows in the length of the argument.

It easy to check that for every rule of the calculus $\mathbb{T}$ different from $\mathbf{T} \to\to$ with *main set of swffs* U, for every *extension* $\mathcal{R}^i_U$ of U, $deg(\mathcal{R}^i_U) < deg(U)$ holds. On the other hand, the degree of the *extension* $\mathcal{R}^1_U = \{\mathbf{F}(A \to p), \mathbf{T}(p \to C), \mathbf{T}(B \to p)\}$ of the *main set of swffs* $U = \{\mathbf{T}((A \to B) \to C)\}$ of $\mathbf{T} \to\to$ is equal to $deg(U) + 1$. Hence, by applying $\mathbf{T} \to\to$ to a set of swffs S, the degree of one of the sets in the obtained configuration is greater than the degree of S. In the following proposition we show that in every branch of every *proof table of* $\mathbb{T}$ the number of times that the rule $\mathbf{T} \to\to$ can be applied is linearly bounded by the degree of the first configuration and thus the number of times that the degree of a set of swffs is increased by one is linearly bounded by the length of the first configuration and this implies that in every *proof table of* $\mathbb{T}$ the degree of every set of swffs is always linearly bounded by the length of the wff to be proved.

Proposition 2. *Let S be a set of swffs. In every branch of every* proof table of $\mathbb{T}$ *starting from the configuration S, the rule $\mathbf{T} \to\to$ is applicable $deg(S)$ times, at most.*

Proof. The proof is by induction on $deg(S)$.
Basis: If $deg(S) = 0$, then $S = \{\mathcal{S}A\}$, with $\mathcal{S}A$ a signed atom, and the proposition trivially holds.
Step: Let us suppose the proposition holds for every set U of swffs such that $deg(U) < n$, and let $deg(S) = n$. The only non trivial case occurs when the rule $\mathbf{T} \to\to$ is applied to $S = S' \cup \{\mathbf{T}((A \to B) \to C)\}$. In this case we get a configuration with two sets of swffs, namely $S' \cup \{\mathbf{F}(A \to p), \mathbf{T}(B \to p), \mathbf{T}(p \to C)\}$ and $S' \cup \{\mathbf{T}C\}$. From the latter set the assertion immediately follows by the induction hypothesis. From the former set we have

$$deg(S) = deg(S') + deg(A) + deg(B) + deg(C) + 2 = n$$

and the number of times the rule $\mathbf{T} \to\to$ is applied to $S' \cup \{\mathbf{F}(A \to p), \mathbf{T}(B \to p), \mathbf{T}(p \to C)\}$ coincides with the number of times it is applied to $S' \cup \{\mathbf{T}A, \mathbf{F}p, \mathbf{T}(B \to p), \mathbf{T}C\}$ and by induction hypothesis this number is bounded by

$$\begin{aligned} deg(S' \cup \{\mathbf{T}A, \mathbf{F}p, \mathbf{T}(B \to p), \mathbf{T}C\}) &= deg(S') + deg(A) + \\ &\quad + deg(B \to p) + deg(C) \\ &= deg(S') + deg(A) + \\ &\quad + deg(B) + deg(C) + 1 \\ &< n. \end{aligned}$$

Thus the number of times that the rule $\mathbf{T} \to\to$ is applied to every *proof table of* $\mathbb{T}$ starting from the configuration S is $deg(S) = n$, at most. □

Now, using the last two propositions, the fact $deg(S_c) \leq deg(S)$ and the fact that for every *main set of swffs U* different from $\{\mathbf{T}((A \to B) \to C)\}$, $deg(\mathcal{R}^i_U) < deg(U)$, for every *extension* $\mathcal{R}^i_U$ of U, we get:

Theorem 2. *The depth of the deductions in $\mathbb{T}$ is linearly bounded by the length of the wff to be proved.*

We start to discuss the soundness of $\mathbb{T}$. To this aim we give the following proposition:

Proposition 3. *If a set of swffs S is* contradictory*, then S is not* realizable.

Proof. Let us suppose that given world Γ of a *model* $\underline{K}$, $\Gamma \rhd S$ holds. If $\mathbf{T_{cl}}A \in S$ and $\mathbf{F_c}A \in S$, then by definition of *realizability* and by the meaning of the signs $\mathbf{T_{cl}}$ and $\mathbf{F_c}$ it follows $\Gamma \Vdash \neg\neg A$ and $\Gamma \Vdash \neg A$, and we get a contradiction. The other cases are similar. □

The following proposition is the main step towards the soundness of $\mathbb{T}$ and and it states that for every rule, if the premise is *realized*, then the *configuration* in the conclusion of the rule is *realized*.

Proposition 4. *Let S be a set of swffs and let Γ be a world of the* model $\underline{K} = \langle P, \leq, \Vdash \rangle$. *If $\Gamma \rhd S$, then the configuration obtained by applying to S one of the rules of $\mathbb{T}$ is* realized *in a world of a, possibly different,* model $\underline{K}^*$.

Proof. The proof requires an analysis of the rules of $\mathbb{T}$. We will consider a few significant rules.

- The rule $\mathbf{F}\!\rightarrow$: Let us suppose that S contains at least a swff of the kind $\mathbf{F}(A \rightarrow B)$. Now, let $U = \{\mathbf{F}(A \rightarrow B) \mid \mathbf{F}(A \rightarrow B) \in S\}$. Moreover, let $S_{\mathbf{F}\rightarrow} = \{\mathbf{F}(A_1 \rightarrow B_1), \ldots, \mathbf{F}(A_n \rightarrow B_n)\}$ (with $n \geq 1$) be any subset of U. Since the worlds of $\underline{K}$ are linearly ordered, by definition of *realizability*, there exists a permutation ρ of $\{1, \ldots, n\}$ and a sequence (with possible repetitions) $\Delta_1, \ldots, \Delta_n$ of worlds of $\underline{K}$ such that, for $i = 1, \ldots, n-1$, $\Delta_i \leq \Delta_{i+1}$ and for $i = 1, \ldots, n$, $\Delta_i \Vdash A_{\rho(i)}$ and $\Delta_i \nVdash B_{\rho(i)}$. Thus Δ_1, the minimum world of the sequence $\Delta_1, \ldots, \Delta_n$, *realizes* $S_{\mathbf{F}\rightarrow}$, and hence every its subset $S^{\rho(i)}_{\mathbf{F}\rightarrow}$. Moreover Δ_1 *realizes* $\mathbf{T}A_{\rho(1)}$, $\mathbf{F}B_{\rho(1)}$ and the *certain part* S_c of S. Since S_c, $S^{\rho(1)}_{\mathbf{F}\rightarrow}$, $\mathbf{T}A_{\rho(1)}$, $\mathbf{F}B_{\rho(1)}$ is one of the *elements* in the conclusion of the rule $\mathbf{F}\rightarrow$, we get the assertion.
- The rule $\mathbf{T} \rightarrow\rightarrow$: let us suppose that $\mathbf{T}((A \rightarrow B) \rightarrow C) \in S$; by definition of $\rhd$, $\Gamma \Vdash (A \rightarrow B) \rightarrow C$. Now, let us consider the *model* $\underline{K}^* = \langle P, \leq, \Vdash' \rangle$ with the forcing relation $\Vdash'$ done as follows:
  1. for every propositional variable $q \in Pv(S)$ and for every world $\Lambda \in P$, $\Lambda \Vdash' q$ iff $\Lambda \Vdash q$;
  2. Let p be a propositional variable such that $p \notin Pv(S)$; for every $\Lambda \in P$, $\Lambda \Vdash' p$ iff $\Lambda \Vdash B$;
  3. for every variable r, such that $r \neq p$ and $r \notin Pv(S)$, and for every world $\Lambda \in P$, $\Lambda \nVdash' r$.

  Let $\rhd'$ be the *realizability* relation defined with respect to $\Vdash'$ like we did for $\rhd$ with respect to $\Vdash$.
  By Theorem 1 it follows that $\Gamma \Vdash' C$ or $\Gamma \nVdash' A \rightarrow B$. If $\Gamma \nVdash' A \rightarrow B$ then
  1. $\Gamma \rhd' \mathbf{F}(A \rightarrow p)$, because there exists $\Delta \in P$ such that $\Delta \geq \Gamma$, $\Delta \Vdash' A$, $\Delta \nVdash' B$ and, by definition of $\Vdash'$, $\Delta \nVdash' p$;
  2. $\Gamma \rhd' \mathbf{T}(p \rightarrow C)$, because if $\Delta \in P$ is such that $\Delta \geq \Gamma$ and $\Delta \Vdash' p$ then, by definition of $\Vdash'$, $\Delta \Vdash' B$ and, since $\Delta \Vdash' (A \rightarrow B) \rightarrow C$, $\Delta \Vdash' C$.
  3. $\Gamma \rhd' \mathbf{T}(B \rightarrow p)$, by definition of $\Vdash'$.
- The rule $\mathbf{F}\neg$: let us suppose that $\mathbf{F}(\neg A) \in S$; by definition of $\rhd$, $\Gamma \nVdash \neg A$. This implies that there exists a world $\Delta \in P$ such that $\Delta \geq \Gamma$ and $\Delta \Vdash A$. Now, by the linearity of the *models* we get that for every $\Lambda \in P$ such that $\Lambda \geq \Gamma$, $\Lambda \nVdash \neg A$ and this implies that $\Gamma \Vdash \neg\neg A$; hence $\Gamma \rhd \mathbf{T_{cl}}A$.
- The rule $\mathbf{F_c}(A \rightarrow B)$: let us suppose $\mathbf{F_c}(A \rightarrow B) \in S$; by definition of $\rhd$, $\Gamma \Vdash \neg(A \rightarrow B)$. This implies that for every world $\Delta \in P$ such that $\Delta \geq \Gamma$, $\Delta \nVdash A \rightarrow B$. Hence for every world $\Delta \in P$ such that $\Delta \geq \Gamma$, there exists a world $\Lambda \in P$ such that $\Lambda \geq \Delta$, $\Lambda \Vdash A$ and $\Lambda \nVdash B$. By the linearity of

the *models* we get that for every world $\Delta \in P$ such that $\Delta \geq \Gamma$, $\Delta \nVdash \neg A$ and $\Delta \nVdash B$; by definition of negation we have $\Gamma \Vdash \neg\neg A$ and $\Gamma \Vdash \neg B$ that implies $\Gamma \rhd \mathbf{T_{cl}}A$ and $\Gamma \rhd \mathbf{F_c}B$.

- The rule $\mathbf{T_{cl}}(A \to B)$: let us suppose $\mathbf{T_{cl}}(A \to B) \in S$; by definition of $\rhd$, $\Gamma \Vdash \neg\neg(A \to B)$. This implies that that for every world $\Delta \in P$ such that $\Delta \geq \Gamma$, $\Delta \nVdash \neg(A \to B)$. Hence for every world $\Delta \in P$ such that $\Delta \geq \Gamma$, there exists a world $\Lambda \in P$ such that $\Lambda \geq \Delta$, $\Lambda \Vdash A \to B$ and this implies that for every world $\Theta \in P$ such that $\Theta \geq \Lambda$, $\Theta \nVdash A$ or $\Theta \Vdash B$. Now, if $\Theta \Vdash B$ holds in some world Θ, then by the linearity of the *models* we get that for every world $\Delta \in P$ such that $\Delta \geq \Gamma$, $\Delta \nVdash \neg B$ and by definition of negation we get $\Gamma \Vdash \neg\neg B$ and $\Gamma \rhd \mathbf{T_{cl}}B$. On the other hand, if $\Theta \Vdash B$ never holds, then $\Theta \nVdash A$ holds for every world Θ and by the linearity we get $\Gamma \Vdash \neg A$ and $\Gamma \rhd \mathbf{F_c}A$.
- The rule $\mathbf{T_{cl}}(\neg A)$: let us suppose $\mathbf{T_{cl}}(\neg A) \in S$; by definition of $\rhd$, $\Gamma \Vdash \neg\neg\neg A$. This implies that for every world $\Delta \in P$ such that $\Delta \geq \Gamma$, $\Delta \nVdash \neg\neg A$. Hence for every world $\Delta \in \Gamma$ such that $\Delta \geq \Gamma$, there exists a world $\Lambda \in P$ such that $\Lambda \geq \Delta$, $\Lambda \Vdash \neg A$. Now, by using the linearity of the *models* and by definition of negation, we get $\Delta \Vdash \neg A$ and $\Gamma \Vdash \neg A$. This implies $\Gamma \rhd \mathbf{F_c}A$. □

Now we can give the Soundness Theorem for the calculus $\mathbb{T}$ with respect to the Kripke semantics for the logic **Dum**:

Theorem 3 (Soundness of $\mathbb{T}$). *If there exists proof of a wff A in $\mathbb{T}$, then A is* valid *in every* model.

Proof. Let us assume that the assertion does not hold, then there exist a *model* $\underline{K} = \langle P, \leq, \Vdash \rangle$ and a world $\Gamma \in P$ such that $\Gamma \nVdash A$. Then $\Gamma \rhd \mathbf{F}A$ and there exists a closed *proof table of* $\mathbb{T}$ starting from the configuration $\{\mathbf{F}A\}$; hence, by the previous proposition, its final configuration is realizable. But this means that a *contradictory* set is *realizable*, contradicting Proposition 3. □

Now we start to discuss the completeness of $\mathbb{T}$. The proof of the Completeness Theorem consists of showing that there exists a procedure allowing to build a *model* that *realizes* S for every *consistent* set of swffs S, that is, having no closed *proof table of* $\mathbb{T}$. Thus the proof emphasizes that the set of all non closed *proof tables of* $\mathbb{T}$ starting from S has enough information to build a *model* that realizes S.

Given a *consistent set* S, we are going to present a method to build a *model* using a single *proof table of* $\mathbb{T}$. In the last configuration of such a *proof table of* $\mathbb{T}$ the sets either are *contradictory* or contain signed atoms only. Our technique to build the Kripke model $\underline{K}(S)$ is similar to the one used in [1], which is an adaptation of Fitting's one described in [14]. The construction has two stages. In the first stage we construct the sequence $\{S_i\}_{i\in\omega}$ of sets of swffs and the set of swffs $\overline{S}$, called the *node set of* S. $\overline{S}$ will be the root of the model $\underline{K}(S)$ and the signed atoms belonging to it will determine the forcing relation in $\overline{S}$. In the second stage we construct the *successor set* of $\overline{S}$. The model $\underline{K}(S)$ will be constructed by iterating the two steps on the new element, and so on.

In the following we call *regular* rule any rule of $\mathbb{T}$ different from $\mathbf{F} \rightarrow$. If U is a set of swffs and a *regular* rule applies to U by taking $S = \emptyset$ in the premise of the rule, then we call U *regular main set of swffs*; if $U = \{H\}$ with H swff, and U is a *regular main set of swffs*, then we call H *regular* swff. Now we start the construction.

First stage: Let $A_1, \ldots, A_n$ be any listing of swffs of S (without repetitions of swffs). Starting from this enumeration we construct the following sequence $\{S_i\}_{i \in \omega}$ of sets of swffs.

- $S_0 = S$;
- Let $S_i = \{H_1, \ldots, H_u\}$; then $S_{i+1} = \bigcup_{H_j \in S_i} \mathcal{U}(H_j, i)$, where, setting $S_j^{'} = \mathcal{U}(H_1, i) \bigcup \cdots \bigcup \mathcal{U}(H_{j-1}, i) \bigcup \{H_j, \ldots, H_u\}$, we have:

1. if H_j is a *regular* swff, then $\mathcal{U}(H_j, i)$ is any *extension* $\mathcal{R}_{H_j}$ of H_j such that $(S_j^{'} - \{H_j\}) \bigcup \mathcal{R}_{H_j}$ is *consistent*;
2. if H_j is of the kind $\mathbf{T}(A \rightarrow B)$, with A an atom, and $\mathbf{T}A \in S_j^{'}$ then $\mathcal{U}(H_j, i) = \{\mathbf{T}B\}$;
3. if H_j is a signed atom or of the kind $\mathbf{F}(A \rightarrow B)$ then $\mathcal{U}(H_j, i) = \{H_j\}$.

Now, by induction on $i \geq 0$, it is easy to prove that if S is *consistent* then any S_i is *consistent*. Moreover, since S is finite there exists an index j such that $S_i = S_j$ for any $i \geq j$. Let u be the first index such that $S_u = S_{u+1}$. We call S_u the *node set of* S and we denote it with $\overline{S}$. Moreover we call $\{S_0, \ldots, S_u\}$ the *sequence generating* $\overline{S}$.

Second stage: If $\overline{S}$ contains a swff of the kind $\mathbf{F}(A \rightarrow B)$, we get the *successor set* U *of* $\overline{S}$ by applying the rule $\mathbf{F}\rightarrow$ to $\overline{S}$, choosing as U any consistent set in the resulting configuration and taking $S_{\mathbf{F}\rightarrow} = \{H | H \in \overline{S}$ and H is of the kind $\mathbf{F}(A \rightarrow B)\}$.

Given a *consistent* set of swffs S, we define a structure $\underline{K}(S) = \langle P, \leq, \Vdash \rangle$ as follows:

1. The root of $\underline{K}(S)$ is a node set $\overline{S}$ of S;
2. For any $\overline{\Gamma} \in P$, if $\overline{\Gamma}$ has the *successor set* U then $\overline{U}$, the *node set of* U, is a member of P and $\overline{U}$ is the only immediate successor of $\overline{\Gamma}$ in $\underline{K}(S)$; if $\overline{\Gamma}$ has no *successor set* then let $\Phi = \{\mathcal{S}A | \mathcal{S}A \in \overline{\Gamma},$ with $\mathcal{S} \in \{\mathbf{T}, \mathbf{F_c}, \mathbf{T_{cl}}\}\}$; $\overline{\Phi} = \Phi$ is the *node set* related to Φ, it is member of P and the immediate successor of $\overline{\Gamma}$;
3. $\leq$ is the transitive and reflexive closure of the immediate successor relation;
4. For every world $\overline{\Gamma} \in P$ and for any propositional variable p, $\overline{\Gamma} \Vdash p$ iff $\mathbf{T}p \in \overline{\Gamma}$ or $\overline{\Gamma}$ is the final world of $\underline{K}(S)$ and $\mathbf{T_{cl}}p \in \overline{\Gamma}$.

From the above definition it follows that $\underline{K}(S) = \langle P, \leq, \Vdash \rangle$ is a finite *model* if S is finite. Moreover, we point out that $\overline{\Phi}$ contains signed atoms only, none signed with $\mathbf{F}$, and $\overline{\Phi}$ is the final world of $\underline{K}(S)$.

The following lemma is the main step towards the proof of the Completeness Theorem:

Lemma 1. *Let S be a* consistent *and finite set of swffs and let* $\underline{K}(S) = \langle P, \leq, \Vdash \rangle$ *be defined as above. Then, for every* $\overline{\Gamma} \in P$ *and for every swff* $H \in \Gamma_i$*, with* Γ_i *any element of the sequence* $\Gamma_0, \ldots, \Gamma_u$ *generating* $\overline{\Gamma}$*,* $\overline{\Gamma} \rhd H$ *in* $\underline{K}(S)$*.*

Proof. The proof is by induction on the degree of the swffs, measured with respect to the function *deg*.
Basis: If $deg(H) = 0$, then $H \equiv \mathcal{S}p$, with p an atom and $\mathcal{S} \in \{\mathbf{T}, \mathbf{F}, \mathbf{F_c}, \mathbf{T_{cl}}\}$ and, by construction of $\underline{K}(S)$, $\mathcal{S}p \in \overline{\Gamma}$. If $\mathcal{S} \equiv \mathbf{T}$ then, by definition of forcing in $\overline{\Gamma}$, $\overline{\Gamma} \Vdash p$, therefore $\overline{\Gamma} \rhd \mathbf{T}p$. If $\mathcal{S} \equiv \mathbf{F}$ then $\overline{\Gamma}$ is not the final world of $\underline{K}(S)$ and since $\overline{\Gamma}$ is *consistent*, $\mathbf{T}p \notin \overline{\Gamma}$; by definition of forcing, $\overline{\Gamma} \nVdash p$, which implies $\overline{\Gamma} \rhd \mathbf{F}p$. If $\mathcal{S} \equiv \mathbf{F_c}$ then, by construction of $\underline{K}(S)$, $\mathbf{F_c}p$ belongs to every world $\overline{\Delta}$ of $\underline{K}(S)$ such that $\overline{\Gamma} \leq \overline{\Delta}$; since every $\overline{\Delta}$ is *consistent* then $\mathbf{T}p \notin \overline{\Delta}$ and $\mathbf{T_{cl}}p \notin \overline{\Delta}$, thus $\overline{\Delta} \nVdash p$ and this implies $\overline{\Delta} \rhd \mathbf{F_c}p$. If $\mathcal{S} \equiv \mathbf{T_{cl}}$ then, by construction of $\underline{K}(S)$, $\mathbf{T_{cl}}p$ belongs to every world $\overline{\Delta}$ of $\underline{K}(S)$ such that $\overline{\Gamma} \leq \overline{\Delta}$. Thus, by definition of forcing in the final world, $\overline{\Phi} \Vdash p$; this implies that $\overline{\Delta} \nVdash \neg p$ for every $\overline{\Delta} \geq \overline{\Gamma}$ and hence $\overline{\Gamma} \Vdash \neg\neg p$, that is $\overline{\Gamma} \rhd \mathbf{T_{cl}}p$.
Step: Let us suppose the assertion holds for every swff H' such that $deg(H') < deg(H)$. The proof goes on by cases according to the structure of H. We give only some illustrative examples.

- Case $H \equiv \mathbf{F}(A \to B)$; if $H \in \Gamma_i$, then, by construction of $\underline{K}(S)$, $H \in \overline{\Gamma}$; we point out that if $H \in \overline{\Theta}$, with $\overline{\Theta} \in P$, then $H \in \Psi$ or $\mathbf{T}A \in \Psi$ and $\mathbf{F}B \in \Psi$, with Ψ *successor set of* $\overline{\Theta}$. Thus, since $H \in \overline{\Gamma}$, there exists a world $\overline{\Delta} \in P$ such that $\overline{\Delta} \geq \overline{\Gamma}$, $H \in \overline{\Delta}$, $\mathbf{T}A \in \Lambda$ and $\mathbf{F}B \in \Lambda$, with Λ *successor set of* $\overline{\Delta}$. Since Λ is the first element in the sequence generating $\overline{\Lambda}$, by induction hypothesis $\overline{\Lambda} \rhd \mathbf{T}A$ and $\overline{\Lambda} \rhd \mathbf{F}B$ hold, thus we get $\overline{\Gamma} \rhd \mathbf{F}(A \to B)$.
- Case $H \equiv \mathbf{F_c}(A \to B)$; if $H \in \Gamma_i$ then, by construction of $\underline{K}(S)$, $\{\mathbf{T_{cl}}A, \mathbf{F_c}B\} \subseteq \Gamma_{i+1}$ and, by induction hypothesis, $\overline{\Gamma} \rhd \mathbf{T_{cl}}A$ and $\overline{\Gamma} \rhd \mathbf{F_c}B$; this implies that in the final world $\overline{\Phi}$, $\overline{\Phi} \Vdash \neg\neg A$ and $\overline{\Phi} \nVdash B$. From $\overline{\Phi} \Vdash \neg\neg A$ we get $\overline{\Phi} \Vdash A$ hence $\overline{\Phi} \nVdash A \to B$ and $\overline{\Gamma} \Vdash \neg(A \to B)$; this means $\overline{\Gamma} \rhd \mathbf{F_c}(A \to B)$.
- Case $H \equiv \mathbf{T}((A \to B) \to C)$; we must prove that, for every $\overline{\Delta} \in P$ such that $\overline{\Delta} \geq \overline{\Gamma}$, if $\overline{\Delta} \Vdash A \to B$ then $\overline{\Delta} \Vdash C$. If $H \in \Gamma_i$ then by construction of $\underline{K}(S)$ either (i) $\mathbf{T}C \in \Gamma_{i+1}$ or (ii) $\mathbf{F}(A \to p)$, $\mathbf{T}(B \to p)$, $\mathbf{T}(p \to C) \in \Gamma_{i+1}$. In the former case, by induction hypothesis, we immediately get $\overline{\Gamma} \rhd \mathbf{T}((A \to B) \to C)$. In the latter case, by induction hypothesis, we get $\overline{\Gamma} \nVdash A \to p$; then there exists $\overline{\Delta} \in P$ such that $\overline{\Delta} \geq \overline{\Gamma}$, $\overline{\Delta} \Vdash A$ and $\overline{\Delta} \nVdash p$. Since $\overline{\Gamma} \Vdash B \to p$, $\overline{\Delta} \nVdash B$ follows and thus $\overline{\Delta} \nVdash A \to B$. If there exists $\overline{\Lambda} \in P$ such that $\overline{\Lambda} > \overline{\Delta}$ and $\overline{\Lambda} \Vdash B$ then, since $\overline{\Lambda} \Vdash B \to p$ and $\overline{\Lambda} \Vdash p \to C$, we get $\overline{\Lambda} \Vdash C$; this implies $\overline{\Gamma} \rhd \mathbf{T}((A \to B) \to C)$.
- Case $H \equiv \mathbf{F}(\neg A)$; if $H \in \Gamma_i$ then, by construction of $\underline{K}(S)$, $\mathbf{T_{cl}}A \in \Gamma_{i+1}$. By induction hypothesis $\overline{\Gamma} \Vdash \neg\neg A$, hence $\overline{\Gamma} \nVdash \neg A$ that means $\overline{\Gamma} \rhd \mathbf{F}\neg A$.
- Case $H \equiv \mathbf{T_{cl}}(A \to B)$; if $H \in \Gamma_i$ then, by construction of $\underline{K}(S)$, $\mathbf{F_c}A \in \Gamma_{i+1}$ or $\mathbf{T_{cl}}B \in \Gamma_{i+1}$. If $\mathbf{F_c}A \in \Gamma_{i+1}$ then, by induction hypothesis, $\overline{\Gamma} \Vdash \neg A$ and thus $\overline{\Gamma} \Vdash A \to B$; this implies that for every $\overline{\Delta} \in P$ such that $\overline{\Delta} \geq \overline{\Gamma}$, $\overline{\Delta} \nVdash \neg(A \to B)$ that is $\overline{\Gamma} \Vdash \neg\neg(A \to B)$ and $\overline{\Gamma} \rhd \mathbf{T_{cl}}(A \to B)$. On the other hand, if $\mathbf{T_{cl}}B \in \Gamma_{i+1}$ then, by induction hypothesis, $\overline{\Gamma} \rhd \mathbf{T_{cl}}B$ and $\overline{\Gamma} \Vdash \neg\neg B$;

this implies that the final world $\overline{\Phi}$ of $\underline{K}(S)$ forces $\overline{\Phi} \Vdash \neg\neg B$ and hence $\overline{\Phi} \Vdash B$; this implies $\overline{\Phi} \Vdash A \to B$. Thus, for every $\overline{\Delta} \in P$ such that $\overline{\Delta} \geq \overline{\Gamma}$, $\overline{\Delta} \nVdash \neg(A \to B)$ and, by definition of negation, we get $\overline{\Gamma} \Vdash \neg\neg(A \to B)$, that is $\overline{\Gamma} \rhd \mathbf{T_{cl}}(A \to B)$.

□

Theorem 4 (Completeness of $\mathbb{T}$). *If A is* valid *in every* model*, then there exists a closed* proof table of $\mathbb{T}$ *starting from the configuration* $\{\mathbf{F}A\}$.

Proof. Suppose the assertion is not true, then $\{\mathbf{F}A\}$ is a *consistent* set of swffs. By Lemma 1 this implies that $\mathbf{F}A$ is *realizable* and we get a contradiction. □

We highlight that in the construction of the model $\underline{K}(S)$ we use the following strategy: a *successor set* U is built by the rule $\mathbf{F}{\to}$ from a *node* $\overline{V}$ considering all the $\mathbf{F}{\to}$*-swffs* in $\overline{V}$. This means that if there exists a closed *table* $\mathbb{T}$ for a set S of *swffs*, then there exists a closed *table* $\mathbb{T}'$ where the rule $\mathbf{F}{\to}$ is always applied considering all the $\mathbf{F}{\to}$*-swffs* in its premise. This implies that we can test the *consistency* of a finite set S of swffs building only one *proof table of* $\mathbb{T}$; namely the one where the rule $\mathbf{F}{\to}$ is applied to a set S' of swffs only when no other rule is applicable to S' and considering all the $\mathbf{F}{\to}$*-swffs* in S'.

Hence the construction related to the Completeness Theorem suggests that, to decide if a set of swffs Γ is *consistent*, a decision procedure can shrink the search space of all *proof tables of* $\mathbb{T}$ starting from Γ to the search space containing just one *proof table of* $\mathbb{T}$, the one built using the following strategy:

(a) The procedure picks a *regular main set of swffs* $U \subseteq \Gamma$, if any, and applies to Γ the *regular* rule related to U. The *regular* rules are invertible, thus the procedure does not need a backtracking mechanism. Hence, if it is not possible to find a closed *proof table of* $\mathbb{T}$ starting from Γ by applying the *regular* rule related to U, then a closed *proof table of* $\mathbb{T}$ for Γ does not exist;

(b) If Step (a) cannot be applied, then, if Γ contains a swff of the kind $\mathbf{F}(A \to B)$ the procedure applies the rule $\mathbf{F}{\to}$ to Γ taking as $S_{\mathbf{F}\to}$ all the swffs of the kind $\mathbf{F}(A \to B)$ occurring in Γ. If it is not possible to find a closed *proof table of* $\mathbb{T}$ starting from Γ by applying $\mathbf{F}{\to}$ to S, then there is no closed *proof table of* $\mathbb{T}$ for Γ.

It is straightforward to implement the above strategy both on a *Deterministic Turing Machine* running within a $O(n \log n)$-bound on space and on a *Nondeterministic Turing Machine* running in polynomial time.

The following are the main properties of $\mathbb{T}$ that allow to obtain procedures with such a complexity:

(i) deductions in the calculus $\mathbb{T}$ have depth which is linearly bounded by the length of the wff to be proved;

(ii) every *element* in the conclusion of every rule of $\mathbb{T}$ has a number of symbols which is bounded by the number of symbols in the premise plus a constant.

(iii) the number of *elements* in the conclusion of every rule of $\mathbb{T}$ is bounded by the length of the premise of the rule.

Property (i) is proved in Theorem 2, while Properties (ii) and (iii) can be easily checked by inspecting the rules of $\mathbb{T}$. We point out that the number of *elements* in the configuration in the conclusion of $\mathbf{F}\!\rightarrow$ is the number of swffs of the kind $\mathbf{F}(A \rightarrow B)$ in the premise at most, whereas for the other rules the number of elements is constant. By Properties (i) and (ii) it follows that, given any set S in any configuration of any *proof table of* $\mathbb{T}$, the number of swffs of S is linearly bounded by the length of the wff to be proved. Hence, by using Property (iii), every application of $\mathbf{F}\!\rightarrow$ gives rise to a number of branches which is linearly bounded in the length of the wff to be proved. Finally, we remark that in a set S new atoms may appear but their number is linearly bounded by the length of the wff to be proved; thus every new propositional variable may be coded using a logarithmic number of bits. The number of bits to codify the other symbols of S does not depend on the length of the wff to be proved.

5 Conclusions

The calculus $\mathbb{T}$ presented in this paper has an object language with one more sign and five more rules than the calculus **Dum**-T given in [13]. By the new sign a more efficient treatment of the negated formulas is possible. Thus, when negated formulas are involved, the proofs of $\mathbb{T}$ are smaller than those of **Dum**-T.

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