Overview of Gödel logics


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Foundations of Many-Valued Logics

*Fuzzy* introduced by Lotfi Zadeh in the sixties, at this time no formalization

The mathematical basis of many-valued logics is the definition via *t-norms*

Algebraization by Hajek in 1998 (*Metamathematics of Fuzzy Logic*), *BL algebras* and extensions (Gödel algebras, Łukasiewicy algebras, Product algebras), Basic Logic

Representation of all *t*-norm based logics by *McNaughton's Theorem* as an ordinal sum of three basic logics, Gödel logic *G*, Product logic *Π*, Łukasiewicz logic *Ł*. 

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Gödel logics

Propositional finite-valued Gödel logics were introduced by Gödel in 1933 to show that intuitionistic logic does not have a characteristic finite matrix.

Dummett (1955) was the first to study infinite valued Gödel logics, axiomatizing the set of tautologies over infinite truth-value sets by intuitionistic logic extended by the linearity axiom \((A \supset B) \lor (B \supset A)\).

Gödel logics are projective, setting them apart from the other basic logics.

An interesting fact is, that if you want a logic with deduction theorem, where the interpretation of an implication is 1 whenever the interpretation of the left side is less or equal the interpretation of the right side, you automatically arrive at Gödel logics. Thus, Gödel logics are the only one with (classical) deduction theorems.
Definition of Gödel logics

Syntax: usual propositional or first order language, \( \neg A \) is defined as \( A \supset \bot \).

Semantics:

\[
I(A \land B) = \min\{I(A), I(B)\} \\
I(A \lor B) = \max\{I(A), I(B)\} \\
I(A \supset B) = \begin{cases} 
I(B) & \text{if } I(A) > I(B) \\
1 & \text{if } I(A) \leq I(B) 
\end{cases}
\]

This yields the following definition of the semantics of \( \neg \):

\[
I(\neg A) = \begin{cases} 
0 & \text{if } I(A) > 0 \\
1 & \text{otherwise}
\end{cases}
\]
Definition of Gödel logics (cont.)

In the first order case the quantifiers are defined as follows:

\[ \mathcal{I}(\forall x A(x)) = \inf \text{Distr}_\mathcal{I}(A(a)) \]
\[ \mathcal{I}(\exists x A(x)) = \sup \text{Distr}_\mathcal{I}(A(a)) \]

Take a truth value set which is closed under inf and sup and contains 0 and 1, then the Gödel logic based on \( V \) is defined as the set of formulas valid under all interpretations into \( V \):

\[ A \in \mathcal{G}_V \iff \mathcal{G}_V \models A \iff \forall \mathcal{I}_V : \mathcal{I}_V(A) = 1 \]

Comments: only the relative order of the truth values are important, not the absolute value.

Gödel logics are defined extensionally, problematic in characterization
A bit of Proof Theory

Gentzen’s Sequent Calculus LK:

Sequents: $\Gamma \vdash \Delta$

Initial sequents (axioms): $A \vdash A$

Rules (eg):

\[
\begin{align*}
\Gamma \vdash \Delta, A & \quad \Gamma \vdash \Delta, B \\
\hline
\Gamma \vdash \Delta, A \land B & \quad \Gamma \vdash \Delta, A, A \\
\hline
\Gamma \vdash \Delta, A &
\end{align*}
\]

cut rule:

\[
\begin{align*}
\Gamma \vdash \Delta, A & \quad A, \Gamma \vdash \Delta \\
\hline
\Gamma \vdash \Delta &
\end{align*}
\]

There is a variant Ll for intuitionistic logic where the only restriction is that on the right hand side not more than one formula can appear.
Gentzen’s LK (cont.)

Interpretation of sequents as $\forall G \supset \exists D$

Cut-elimination and its consequences (Gentzen, Tait, Leitsch/Baaz, etc)

Midsequent theorem, interpolation, Herbrand’s theorem

Consistency proof

Extraction of programs
Hypersequent Calculus for Gödel logics

Introduced by Avron in 1991, analytic calculus for Gödel logics (propositional and first order)

Hypersequent: \( \Gamma_1 \vdash \Delta_1 | ... | \Gamma_n \vdash \Delta_n \)

Internal rules from LK, external rules: weakening and contraction

Communication rule:

\[
\frac{H \ | \ \Pi_1, \Gamma_1 \vdash A \quad H \ | \ \Pi_2, \Gamma_2 \vdash B}{H \ | \ \Pi_1, \Pi_2 \vdash A \ | \ \Gamma_1, \Gamma_2 \vdash B}
\]

Same properties as LK: cut elimination and consequences, complete for Gödel logics
The communication rule

\[
\begin{array}{c}
\text{(com)} \\
\hline
\frac{H \mid \Pi_1, \Gamma_1 \vdash A \quad H \mid \Pi_2, \Gamma_2 \vdash B}{H \mid \Pi_1, \Pi_2 \vdash A \mid \Gamma_1, \Gamma_2 \vdash B}
\end{array}
\]

Avron idea was that this rule expresses communication in parallel programs (proofs as programs).

Fermüller (2003) gave a Lorenzen style semantics for Gödel logics where the communication rule allows exchange of information of parallelized games.
Model Theory and Axiomatizability

\[ G_V = \{ A : (\forall I) I(A) = 1 \} \]

How many different Gödel logics exist? The extensional definition allows different truth values sets to create the same logics, if the order structure of the truth value sets are the same.

Which of the Gödel logics are axiomatizable (complete recursive axiomatizability)?

Expressive power of Gödel logics

Old result from Takeuti and Titani (1984), Horn (1969) etc, that the Gödel logic on \( V = [0, 1] \) is axiomatizable by \( \text{IL} + \text{LIN} \):

\[(\text{LIN}) \quad (A \supset B) \lor (B \supset A)\]
Results on Axiomatizability of Gödel logics

Propositional logic: (Dummett) finite and infinite cases are axiomatizable. All infinite cases are the same, axiomatized by $\text{IL} + \text{LIN}$. Finite cases all are different, additional axiom on the number of truth values.

Propositional entailment: (Baaz (?)) finite and uncountable truth value sets give compact Gödel logics, countable truth value sets gives not compact Gödel logics.

First order (finite and countable): (Baaz, Zach, P.) finite case is axiomatizable, countable logics are all not recursively enumerable

First order (uncountable): (Horn, Takeuti/Titani for the $[0, 1]$, P. for the general case) If 0 is in the perfect kernel or 0 is isolated, then the logic is axiomatizable, in all other cases there are not.
Sketch of Takano’s completeness proof

Equivalence classes of formulas according to $\leftrightarrow$ gives a countable linearly ordered set which can be embedded into $[0, 1]_\mathbb{Q}$ preserving order, infima and suprema.

The existence of a dense linear subordering is essential for the proof

$$h : F/\equiv \rightarrow \langle [0, 1] \cap \mathbb{Q}, \leq \rangle \quad h(a_n) = \frac{h(a_i) + h(a_j)}{2}$$

Evaluation:

$$\mathcal{I}(A) = h(|A|)$$
Takano’s completeness proof (cont.)

Quantifiers:

\[ s = \mathcal{I}(\forall x A(x)) = h(\inf_{L} |A(t)| : t \in T) \]
\[ q = \mathcal{I}(\forall x A(x)) = \inf_{\mathbb{R}} h(|A(t)|) : t \in T \]

![Diagram of mappings](image)

Figure 1: The mapping \( h \) from \( \langle L, \leq \rangle \) to \( \langle [0, 1] \cap \mathbb{R}, \leq \rangle \)
Cantor-Bendixon Derivatives and Ranks

Polish spaces, i.e. separable, completely metrizable topological spaces. \( \mathbb{R} \) is a Polish space.

\[
X' = \{ x \in X : x \text{ is limit point of } X \} \\
X^0 = X \\
X^{\alpha+1} = (X^\alpha)' \\
X^\lambda = \bigcap_{\alpha<\lambda} X^\alpha, \text{ if } \lambda \text{ is limit ordinal.}
\]

Theorem (Cantor, Bendixon) Let \( X \) be a polish space. For some countable ordinal \( \alpha_0 \), \( X^\alpha = X^{\alpha_0} \) for all \( \alpha \geq \alpha_0 \) (\( X^{\alpha_0} \) is the perfect kernel).
DLO and Perfect Sets

Dense linear order: $\forall a \forall b (a < b \supset \exists c (a < c < b))$

Perfect Set: all its points are limit points

THEOREM (Winkler) For any perfect set there is a unique partition of the real line into countable intervals such that the intersections of the perfect set with these intervals are either empty, the full interval or isomorphic to the Cantor set.

THEOREM A truth-value set contains a dense linear subordering if and only if it is uncountable.
Extending Takano’s proof to perfect sets

Problem: Mapping the set of equivalence classes into the perfect kernel preserving order, infima and suprema. The problem is the potential existence of parts within the perfect kernel which are isomorphic to the Cantor set, which is nowhere dense, but still perfect.

Solution (idea): Using the ‘original’ value (the one from Horn’s function) as starting point we shift the value such that it is in the perfect kernel, while ensuring that infima are preserved.
Non axiomatizability results

**Countable truth value sets** Axiomatize (small part) of natural numbers, divide a subset of the domain into levels, in every level add points strictly between to elements of the previous level. If this is not possible, all following levels are empty. This process must terminate (since the truth value set is countable, no dense in itself subset), axiomatizing finiteness by restricting quantifiers to non-empty levels, not axiomatizable due to Trachtenbrot’s Theorem.

**Uncountable truth value set**, $0 \notin V^\infty$, 0 not isolated Same construction in parallel in every neighborhood of 0, if one level is empty in one neighborhood, it is empty in all neighborhoods.
Conclusions, future plans, other stuff

(Some) Advantages of Gödel logics: nice proof theoretic features, even for first order (this is not the case for other MV-logics), ‘easy’ semantics without computations, existence of deduction theorem

(Some) Disadvantages: No intensional definition, involution of conjunction can be viewed as counterintuitive (pile of sand corns)

Future Work: Characterising countable Gödel logics (separating various classes, developing of a more fine-grain order-analysis than the Cantor-Bendixon analysis for countable truth value sets)

Other stuff: \( \Delta \) operator, quantified propositional Gödel logics, Sequent of Relation calculi, interpolation, …