CHARACTERIZATION OF GÖDEL LOGICS WITH COUNTABLE TRUTH VALUE SETS

Work in Progress

The Logic of Soft Computing III

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Setting the stage — Why to deal with countable Gödel logics

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- Expressive power of Gödel logics

- Topological definition of Gödel logic
Axiomatizability of f.o. Gödel logics

- $V$ is finite – axiomatizable
- $V$ has a perfect kernel and 0 is either in the perfect kernel or isolated – axiomatizable
- $V$ is countable or uncountable and 0 is not in the perfect kernel and not isolated – not axiomatizable
Cantor-Bendixon Derivatives and Ranks

Polish spaces, i.e. separable, completely metrizable topological spaces. $\mathbb{R}$ is a Polish space.

\[ X' = \{ x \in X : x \text{ is limit point of } X \} \]

\[ X^0 = X \]
\[ X^{\alpha+1} = (X^\alpha)' \]
\[ X^\lambda = \bigcap_{\alpha<\lambda} X^\alpha, \text{ if } \lambda \text{ is limit ordinal.} \]

**Theorem 1** Let $X$ be a polish space. For some countable ordinal $\alpha_0$, $X^\alpha = X^{\alpha_0}$ for all $\alpha \geq \alpha_0$ ($X^{\alpha_0}$ is the perfect kernel).
Countable compact topological spaces

If the space $X$ is countable then $X^\infty = \emptyset$, since every nonempty perfect set has at least cardinality of the continuum.

$$\text{rk}(x) = \sup\{\alpha : x \in X^\alpha\}$$

$$|X|_{CB} = \sup\{\text{rk}(x) : x \in X\}$$

If $X$ is countable we call

$$\tau(X) = (\alpha, n), \text{ with } \alpha = \alpha(X) = |X|_{CB}, \text{ } n = n(X) = |X|_{X|CB}$$

the topological type of $X$. 
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Here we present the case of topological type \( \tau = (n, 1) \):
From $V_↓$ to $V_{↓*}$

The set $V_↓$ is well known:

$$V_↓ = \{1/n : n \geq 1\} \cup \{0\}$$

Extending this we use

$$\mathbb{N}' = \mathbb{N} \setminus \{0, 1\}, \quad S_n = \mathbb{N}'^n, \quad S = \bigcup_{n \geq 0} S_n \quad S_{↓n} = \bigcup_{k=0}^n S_k$$

**Lemma 1** There is a function $f : S \rightarrow \mathbb{R}$ such that the following properties hold:

1. $f(\overrightarrow{s_n}) < f(\overrightarrow{s_n, s_{n+1}}) < f(\overrightarrow{s_{n-1}, s_n - 1})$

2. $\inf_{s_{n+1} \in \mathbb{N}'} f(\overrightarrow{s_{n+1}}) = f(\overrightarrow{s_n})$
From $V_\downarrow$ to $V_{\downarrow^*}$ (cont.)

\[
f(()) = 0 \quad f(s_n, s_{n+1}) = f(s_n') + \frac{1}{\prod_{i=1}^{n} s_i^2} \cdot \frac{1}{s_{n+1}}
\]
From $V_1$ to $V_{\downarrow^*}$ (cont.)

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Gödel Logics with Countable Truth Value Set
The class $G_{\downarrow*}$

Let $I_n = f(S_{\downarrow n}) \cup \{0, 1\}$.

Properties of $I_n$: closed set, $I'_n = I_{n-1}$

$$G_{\downarrow n} = \{G_V : \exists f : (V, \leq, \inf, \sup) \mapsto (I_n, \leq, \inf, \sup)\}$$

and

$$G_{\downarrow*} = \bigcup_{n>0} G_{\downarrow n}$$

Theorem 2 There are formulas $A_n$ such that

1. $\forall k \geq n \forall G_V \in G_{\downarrow k} : A_n \notin G_V$

2. $\forall k < n \forall G_V \in G_{\downarrow k} : A_n \in G_V$
The separation formulas

Basic principle is the usage of a special case of *tertium non datur*:

\[
\exists x (A(x) \supset A(y)) \lor (\exists x (A(x) \supset A(y)) \supset A(y))
\]

This can be used in the following way

\[
L_0(X, x_i) = \exists x_i (X(x_i) \supset \forall x'_i X(x'_i)) \supset \forall x_i X(x_i)
\]

where \(X\) can be any formula with a designated variable occurrence,

\[
L_{i,n}(P) = \forall x_1 \ldots \forall x_{i-1} L_0(\forall x_{i+1} \ldots \forall x_n P(x_1, \ldots, x_i, \cdot, x_{i+1}, \ldots, x_n), x_i)
\]

with \(P\) a predicate symbol and

\[
A_n = \bigwedge_{i=1}^{n} L_{i,n}(P, x_i) \supset \forall x_1 \ldots \forall x_n P(x_1, \ldots, x_n)
\]
The countermodel for $A_n \notin G_{\downarrow k}$ for $k \geq n$

We will use the domain of $\mathbb{N}'$ and define the valuation of the atomic formulas as follows:

$$I(P(s_1, \ldots, s_n)) = f(s_1, \ldots, s_n).$$

Consider again the formula $A_n$

$$A_n = \bigwedge_{i=1}^{n} (\text{inf} \neq \text{min} \ (\text{level} \ n)) \supset \forall x_1 \ldots \forall x_n P(x_1, \ldots, x_n)$$

The $\text{inf} \neq \text{min}$ part is valid under the above evaluation, but only for large, i.e. deep enough nested models. For $k < n$ there is at least one level where it collapses, rendering the whole formula true.
Plans for the future

- Extension to include suprema and mixed truth value sets (partly done)
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- Characterization of Gödel logics not extensionally in terms of ‘sentences valid in $V$’, but intensionally via topological properties (CB rank, type of limit points, etc)
- Settle the question on the numbers of different Gödel logics