

An analytic calculus for **MALL** extended with the axiom

$$((A \otimes A \otimes B) \oplus (B \otimes B \otimes A) \wp A^\perp) \wp B^\perp$$

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Abstract

This paper introduces an analytic calculus for **MALL** extended with the (Hilbert) axiom $((A \otimes A \otimes B) \oplus (B \otimes B \otimes A) \wp A^\perp) \wp B^\perp$. The calculus is generated by the PROLOG-program *InvAxiomCalc*, which implements the procedure in [2].

1 Introduction

Non-classical logics are often defined by adding Hilbert axioms to known systems. The usefulness of these logics, however, heavily depends on the availability of calculi which admit cut-elimination (i.e., analytic calculi). These calculi, in which proof search proceeds by stepwise decomposition of the formulas to be proved, are indeed a prerequisite for the development of automated reasoning methods, and also the key to establish essential properties of the formalized logics.

We introduce an analytic calculus for the logic obtained by extending Multiplicative Additive Linear Logic (**MALL**) with the axiom $((A \otimes A \otimes B) \oplus (B \otimes B \otimes A) \wp A^\perp) \wp B^\perp$. The calculus is obtained via a PROLOG-implementation of the procedure in [2].

2 Preliminaries

The formulas of **MALL** are built over the language of classical linear logic without exponentials [3]. The language consists of propositional variables $\mathcal{V} = \{a, b, c, \dots\}$, their duals $\mathcal{V}^\perp = \{a^\perp, b^\perp, c^\perp, \dots\}$, the constants $\{\perp, \top, 1, 0\}$ and the logical connectives $\{\&, \wp, \otimes, \oplus\}$.

The base calculus we will deal with is the sequent system **MALL** (see Table 1).

Metavariables A, B, C, \dots denote formulas, and Γ, Δ, \dots denote finite (possibly empty) multisets of formulas. We only consider single-sided multi-conclusion sequents i.e., sequents whose left-hand side (LHS) is empty and right-hand side

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$ax \frac{}{\vdash A, A^\perp}$	$cut \frac{\vdash \Gamma, A \quad \vdash A^\perp, \Delta}{\vdash \Gamma, \Delta}$	$1 \frac{}{\vdash 1}$
$\oplus_1 \frac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B}$	$\oplus_2 \frac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B}$	$\perp \frac{\vdash \Gamma}{\vdash \Gamma, \perp}$
$\top \frac{}{\vdash \Gamma, \top}$	$\otimes \frac{\vdash \Gamma, A \quad \vdash B, \Delta}{\vdash \Gamma, A \otimes B, \Delta}$	
$\wp \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \wp B}$	$\& \frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \& B}$	

Table 1: The sequent calculus **MALL**

(RHS) consists of a (possibly empty) set of formulas. In inference rules we will refer to $\Gamma, \Delta, \Sigma, \dots$ as *multiset variables* and A, B, C, \dots as *formula variables*.

The notion of a proof in **MALL** is defined as usual. Let R be a set of rules. If there is a proof in **MALL** extended with R (**MALL** + R , for short) of a sequent S_0 from a set of sequents \mathcal{S} , we say that S_0 is derivable from \mathcal{S} in **MALL** + R and write $\mathcal{S} \vdash_{\mathbf{MALL}+R} S_0$. We write $\vdash_{\mathbf{MALL}+R} \alpha$ if $\emptyset \vdash_{\mathbf{MALL}+R} \alpha$.

Definition 1 (Equivalent Rules) *Given two sets of inference rules R_1 and R_2 , we say that R_1 and R_2 are equivalent in **MALL** iff $\mathbf{MALL}+R_1$ and $\mathbf{MALL}+R_2$ prove the same set of sequents.*

If $R = \{r\}$ is a singleton, we write **MALL** + r instead of **MALL** + R . An axiom φ is a rule without premises. Thus, the above definition applies also to (sets of) axioms.

Definition 2 (Structural Rules) *A sequent structural rule r is a rule of the form:*

$$\frac{\vdash \Psi_1 \quad \dots \quad \vdash \Psi_n}{\vdash \Phi} \quad (1)$$

where each Ψ_i and Φ contains only multiset variables and formula variables.

Definition 3 (Acyclic Rules) *The cut-closure $CUT(r)$ of a structural rule r is the minimal set which contains the premises of r and is closed under the application of the cut rule. A rule r is said to be cyclic if for some formula variable A , we have $\vdash \Gamma, A, A^\perp \in CUT(r)$. Otherwise, r is acyclic.*

2.1 Substructural Hierarchy

The substructural hierarchy is a syntactic classification of Hilbert axioms. It has been introduced in [1] for formulas of Full Lambek calculus with exchange (**FLew**). In [2], the classification was adapted for formulas of **MALL** as follows:

Definition 4 (Substructural Hierarchy) [2] *Let \mathcal{A} be a set of atomic formulas. For $n \geq 0$, the sets $\mathcal{P}_n, \mathcal{N}_n$ of formulas are defined via the following grammar:*

$$\begin{aligned} \mathcal{P}_0 &::= \mathcal{A} & \mathcal{P}_{n+1} &::= \mathcal{N}_n \mid \mathcal{P}_{n+1} \otimes \mathcal{P}_{n+1} \mid \mathcal{P}_{n+1} \oplus \mathcal{P}_{n+1} \mid 1 \mid 0 \\ \mathcal{N}_0 &::= \mathcal{A} & \mathcal{N}_{n+1} &::= \mathcal{P}_n \mid \mathcal{N}_{n+1} \& \mathcal{N}_{n+1} \mid \mathcal{N}_{n+1} \wp \mathcal{N}_{n+1} \mid \top \mid \perp \end{aligned} \quad (2)$$

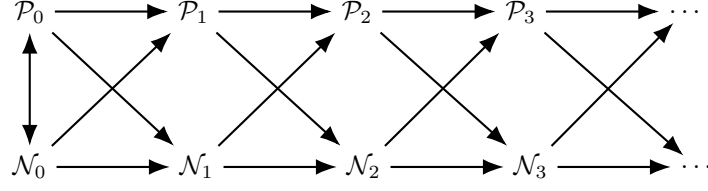


Figure 1: The substructural hierarchy [2]

A graphical representation of the substructural hierarchy is depicted in Figure 1. Note that the arrows \rightarrow stand for inclusions \subseteq of the classes.

3 From axioms to analytic rules

The axiom $\phi = ((A \otimes A \otimes B) \oplus (B \otimes B \otimes A) \wp A^\perp) \wp B^\perp$ is within the class \mathcal{N}_2 of the substructural hierarchy [2]. The transformation of ϕ into equivalent analytic sequent rule is performed in two steps:

1) The algorithm in [2, Section 4] is used to transform ϕ into the following equivalent structural rule:

$$\frac{\begin{array}{c} \vdash B^\perp, B^\perp, A^\perp, \Gamma \quad \vdash \Delta, A \\ \vdash A^\perp, A^\perp, B^\perp, \Gamma \quad \vdash \Sigma, B \end{array}}{\vdash \Gamma, \Delta, \Sigma} \quad (3)$$

Note that the acyclicity condition is satisfied by the structural rule (3), which is the prerequisite for the second step.

2), The Rule Completion algorithm from [2, Section 6] is used to transform (3) into the following analytic sequent rule:

$$\frac{\vdash \Sigma, \Sigma, \Delta, \Gamma \quad \vdash \Delta, \Delta, \Sigma, \Gamma}{\vdash \Gamma, \Delta, \Sigma} \quad (4)$$

Theorem 5 (Soundness and Completeness.) *The axiom $((A \otimes A \otimes B) \oplus (B \otimes B \otimes A) \wp A^\perp) \wp B^\perp$ is equivalent to the newly generated rule (4).*

Proof. See [2].

Theorem 6 (Cut-Elimination.) *MALL extended with the newly generated rule (4) admits cut elimination.*

Proof. See [2], which contains a general, syntactic cut-elimination procedure.

References

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