A labelled calculus for **G3I** extended with the frame condition $\forall x, y(x \leq y \rightarrow y \leq x)$

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Abstract

We introduce a cut-free labelled calculus for intermediate logics defined by extending **G3I** with the frame condition $\forall x, y(x \leq y \rightarrow y \leq x)$. The calculus is generated by the PROLOG-program *Framous*, which implements the procedure in [1].

1 Introduction

Intermediate logics, i.e., logics between intuitionistic and classical logic, have a natural Kripke semantics defined by imposing conditions on the standard intuitionistic frame. Cut-free labelled systems [3, 5, 4], which internalize Kripke semantics, have been provided for a large class of intermediate logics in a modular way in [2]. The resulting calculi are indeed defined by adding to the base labelled calculus for intuitionistic logic extra structural rules corresponding to the frame conditions — that are formulas of first-order classical logic — characterizing the considered logic.

In this paper, we introduce a cut-free labelled calculus for the logic obtained by extending **G3I** with the frame condition $\forall x, y(x \leq y \rightarrow y \leq x)$. The calculus is obtained via a PROLOG-implementation of the procedure in [1], where a classification of the frame conditions according to their quantifier alternation and an algorithm to automatically create structural rules out of them are introduced.

2 Preliminaries

The language of propositional intermediate logics consists of infinitely many propositional variables $p, q \ldots$, the connectives & (conjunction), $\vee$ (disjunction), $\supset$ (implication), and the constant $\perp$ for falsity. $\varphi, \psi, \alpha, \beta \ldots$ are formulas built from atoms by using connectives and $\perp$. As usual, $\sim \varphi$ abbreviates $\varphi \supset \perp$.

An intuitionistic frame is a pair $\mathfrak{F} = \langle W, \leqslant \rangle$ where W is a non-empty set, and $\leqslant$ is a reflexive and transitive (accessibility) relation on W. An intuitionistic model $\mathfrak{M} = \langle \mathfrak{F}, \Vdash \rangle$ is a frame $\mathfrak{F}$ together with a relation $\Vdash$ (called the forcing) between elements of W and atomic formulas. Intuitively, $x \Vdash p$ means that the atom p is true at x. Forcing is assumed to be monotonic w.r.t. the relation $\leqslant$,

*`http://www.logic.at/people/lara/tinc/framous/`

$$
x \leqslant y, x : p, \Gamma \Rightarrow \Delta, y : p
\qquad
\frac{x : \varphi, x : \psi, \Gamma \Rightarrow \Delta}{x : \varphi \& \psi, \Gamma \Rightarrow \Delta}\ L\&
\qquad
\frac{\Gamma \Rightarrow \Delta, x : \varphi \quad \Gamma \Rightarrow \Delta, x : \psi}{\Gamma \Rightarrow \Delta, x : \varphi \& \psi}\ R\&
$$

$$
\frac{}{x : \bot, \Gamma \Rightarrow \Delta}\ L\bot
\qquad
\frac{\Gamma \Rightarrow \Delta, x : \varphi, x : \psi}{\Gamma \Rightarrow \Delta, x : \varphi \vee \psi}\ R\vee
\qquad
\frac{x : \varphi, \Gamma \Rightarrow \Delta \quad x : \psi, \Gamma \Rightarrow \Delta}{x : \varphi \vee \psi, \Gamma \Rightarrow \Delta}\ L\vee
$$

$$
\frac{x \leqslant x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathit{Ref}
\qquad
\frac{x \leqslant y, y : \varphi, \Gamma \Rightarrow \Delta, y : \psi}{\Gamma \Rightarrow \Delta, x : \varphi \supset \psi}\ R\supset
\qquad
\frac{x \leqslant z, x \leqslant y, y \leqslant z, \Gamma \Rightarrow \Delta}{x \leqslant y, y \leqslant z, \Gamma \Rightarrow \Delta}\ \mathit{Trans}
$$

$$
\frac{x \leqslant y, x : \varphi \supset \psi, \Gamma \Rightarrow \Delta, y : \varphi \quad x \leqslant y, x : \varphi \supset \psi, y : \psi, \Gamma \Rightarrow \Delta}{x \leqslant y, x : \varphi \supset \psi, \Gamma \Rightarrow \Delta}\ L\supset
$$

Table 1: Labelled calculus **G3I** for intuitionistic logic [2]

namely, if $x \leqslant y$ and $x \Vdash p$ then also $y \Vdash p$. It is defined inductively on arbitrary formulas as follows:

$(\Vdash \bot)$	$x \Vdash \bot$	for no x	
$(\Vdash \&)$	$x \Vdash \varphi \& \psi$	iff	$x \Vdash \varphi$ and $x \Vdash \psi$
$(\Vdash \vee)$	$x \Vdash \varphi \vee \psi$	iff	$x \Vdash \varphi$ or $x \Vdash \psi$
$(\Vdash \supset)$	$x \Vdash \varphi \supset \psi$	iff	$x \leqslant y$ and $y \Vdash \varphi$ implies $y \Vdash \psi$.

Intermediate logics are obtained from intuitionistic logic by imposing on intuitionistic frames additional conditions on the relation $\leqslant$. The latter conditions are usually expressed as formulas of first-order classical logic in which variables are interpreted as elements of W, and the binary predicate $\leqslant$ denotes the accessibility relation of $\mathfrak{F}$. Atomic formulas are *relational atoms* of the form $x \leqslant y$. Compound formulas are built from relational atoms using the propositional connectives $\wedge$, $\vee$, $\rightarrow$, $\neg$, and the quantifiers $\forall$ and $\exists$.

Labelled systems are a variant of sequent calculus in which the relational semantics of the formalized logics is made explicit part of the syntax [3, 5, 4]. In a labelled system, each formula φ receives a label x, indicated by $x : \varphi$. The labels are interpreted as possible worlds, and a labelled formula $x : \varphi$ corresponds to $x \Vdash \varphi$. Moreover, labels may occur also in expressions for accessibility relation (relational atoms) like, e.g., $x \leqslant y$ of intuitionistic and intermediate logics.

Definition 1 *A labelled sequent is a sequent consisting of labelled formulas and relational atoms.*

Table 1 depicts the labelled calculus **G3I** for intuitionistic logic. Note that its logical rules are obtained directly from the inductive definition of forcing. The rule $R \supset$ must satisfy the *eigenvariable* condition (y does not occur in the conclusion). The structural rules Ref and $Trans$ for relational atoms correspond to the assumptions of reflexivity and transitivity of $\leqslant$ in $\mathfrak{F}$.

3 From frame conditions to labelled rules

Using the algorithm described in [1], the set $*$ of frame conditions contains:

- $\forall x, y(x \leq y \rightarrow y \leq x)$

The frame condition is transformed into the following structural rule:

$$\frac{x \leq y, y \leq x, \Gamma \Rightarrow \Delta}{x \leq y, \Gamma \Rightarrow \Delta}$$

Let **G3SI*** be the labelled calculus obtained by adding to **G3I** initial sequents of the form $x \leq y, \Gamma \Rightarrow \Delta, x \leq y$[1] and the rule stated above.

Let $\mathfrak{F}_{SI^*} = \langle W, \leqslant \rangle$ be a frame with the properties of the accessibility relation expressed as formulas in $*$. Let $L = \{x, y, z \ldots\}$ be the labels occurring in a **G3SI***-derivation. An *interpretation* I of L in $\mathfrak{F}_{SI^*}$ is a function $I : L \to W$.

Definition 2 *Let $\mathfrak{M}_{SI^*} = \langle \mathfrak{F}_{SI^*}, \Vdash \rangle$ be a model and I an interpretation. A labelled sequent $\Gamma \Rightarrow \Delta$ is* valid in $\mathfrak{M}_{SI^*}$ *if for every interpretation I we have: if for all labelled formulas $x : \varphi$ and relational atoms $y \leqslant z$ in Γ, $x^I \Vdash \varphi$ and $y^I \leqslant z^I$ hold, then for some $w : \psi$, $u \leqslant v$ in Δ we have $w^I \Vdash \psi$ or $u^I \leqslant v^I$. A sequent $\Gamma \Rightarrow \Delta$ is* valid *in a frame $\mathfrak{F}_{SI^*}$ when it is valid in every model $\mathfrak{M}_{SI^*}$.*

Theorem 3 (Soundness and Completeness) *For any sequent $\Gamma \Rightarrow \Delta$*

$$\vdash_{\mathbf{G3SI}^*} \Gamma \Rightarrow \Delta \textit{ iff } \Gamma \Rightarrow \Delta \textit{ is valid in every frame } \mathfrak{F}_{SI^*}.$$

Proof. See [1].

Theorem 4 (Cut elimination) *The cut rule (Z is either $u : \varphi$ or $x \leqslant y$)*

$$\frac{\Gamma \Rightarrow \Delta, Z \quad Z, \Gamma' \Rightarrow \Delta}{\Gamma, \Gamma' \Rightarrow \Delta', \Delta} \text{CUT}$$ *can be eliminated from* **G3SI****-derivations.*

Proof. See [1].

References

[1] A. Ciabattoni, P. Maffezioli, and L. Spendier. Hypersequent and Labelled Calculi for Intermediate Logics. In *Proceedings of TABLEAUX 2013*, D. Galmiche and D. Larchey-Wendling (Eds.), LNCS 8123, pp. 81–96, 2013.

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[1] Note that these sequents were first introduced for **G3I** and later removed as they were not needed in the labelled systems for intermediate logics presented in [2]; the reason being that in these systems no rule contains atoms $x \leqslant y$ in the succedent.