

# An analytic calculus for **HFLe<sub>w</sub>** extended with the axiom $(\alpha\beta) \vee ((\alpha\neg\beta) \wedge (\neg\beta\alpha)) \vee (\beta\alpha)$

AxiomCalc\*

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## Abstract

This paper introduces a cut-free hypersequent calculus for **FLe<sub>w</sub>** extended with the (Hilbert) axiom  $(\alpha\beta) \vee ((\alpha\neg\beta) \wedge (\neg\beta\alpha)) \vee (\beta\alpha)$ . The calculus is generated by the PROLOG-program *AxiomCalc*, which implements the procedure in [1].

## 1 Introduction

Non-classical logics are usually defined by adding Hilbert axioms to known systems. The usefulness of these logics, however, heavily depends on the availability of calculi which admit cut-elimination (i.e., analytic calculi). These calculi are indeed a prerequisite for the development of automated reasoning methods, and also the key to establish essential properties of the formalized logics.

We introduce a cut-free hypersequent calculus for the logic obtained by extending with the axiom  $(\alpha\beta) \vee ((\alpha\neg\beta) \wedge (\neg\beta\alpha)) \vee (\beta\alpha)$ . The calculus is obtained via a PROLOG-implementation of the procedure in [1].

## 2 Preliminaries

The formulas of **HFLe<sub>w</sub>** are built from propositional variables  $p, q, r, \dots$  and the 0-ary connectives (constants) 1 and 0 by using the binary logical connectives  $\cdot$  (fusion),  $\rightarrow$  (implication),  $\wedge$  (conjunction) and  $\vee$  (disjunction).  $\neg\alpha$  will be used as abbreviation for  $\alpha 0$ .

The base calculus we will deal with is the hypersequent system **HFLe<sub>w</sub>** (see Table 1).

Metavariables  $\alpha, \beta, \dots$  denote formulas,  $\Pi$  stands for stoups, i.e., either a formula or the empty set, and  $\Gamma, \Delta, \dots$  for finite (possibly empty) multisets of formulas. We only consider single-conclusion sequents i.e., sequents whose right-hand side (RHS) contains at most one formula.

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\*<http://www.logic.at/people/lara/axiomcalc.html>

Table 1: The hypersequent calculus **HFLe<sub>w</sub>**

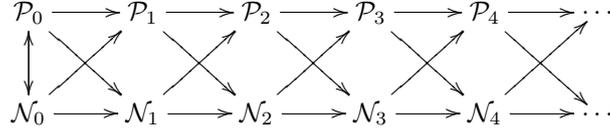


Figure 1: The substructural hierarchy [1]

## 2.1 Substructural Hierarchy

The substructural hierarchy is a novel classification of Hilbert axioms based on the logical connectives of **FLew**.

**(Substructural Hierarchy)** [1] Let  $\mathcal{A}$  be a set of atomic formulas. For  $n \geq 0$ , the sets  $\mathcal{P}_n, \mathcal{N}_n$  of formulas are defined as follows:

$$\begin{aligned} \mathcal{P}_0 &::= \mathcal{N}_0 ::= \mathcal{A} \\ \mathcal{P}_{n+1} &::= \mathcal{N}_n \mid \mathcal{P}_{n+1} \cdot \mathcal{P}_{n+1} \mid \mathcal{P}_{n+1} \vee \mathcal{P}_{n+1} \mid \\ \mathcal{N}_{n+1} &::= \mathcal{P}_n \mid \mathcal{P}_{n+1} \mathcal{N}_{n+1} \mid \mathcal{N}_{n+1} \wedge \mathcal{N}_{n+1} \mid \end{aligned}$$

A graphical representation of the substructural hierarchy is depicted in Figure 1. Note that the arrows stand for inclusions  $\subseteq$  of the classes.

## 3 From axioms to analytic rules

The axiom  $(\alpha\beta) \vee ((\alpha\neg\beta) \wedge (\neg\beta\alpha)) \vee (\beta\alpha)$  is within the class  $\mathcal{P}_3$  of the substructural hierarchy [1]. Using the algorithm in [1], the axiom

$$(\alpha\beta) \vee ((\alpha\neg\beta) \wedge (\neg\beta\alpha)) \vee (\beta\alpha)$$

can be transformed into the following rules to be added to the hypersequent calculus **HFLe<sub>w</sub>**:

$$\frac{\begin{array}{cc} G\Gamma_1, \Delta_2\Pi_2 & G\Gamma_3, \Delta_2\Pi_2 \\ G\Gamma_2, \Delta_1\Pi_1 & G\Gamma_4, \Delta_1\Pi_1 \end{array}}{\begin{array}{cc} G\Gamma_3, \Gamma_4\Gamma_2, \Delta_2\Pi_2\Gamma_1, \Delta_1\Pi_1 & \\ G\Gamma_1, \Delta_2\Pi_2 & G\Gamma_2, \Delta_1\Pi_1 \end{array}} \\ \frac{G\Gamma_2, \Delta_3\Pi_3 \quad G\Gamma_1, \Gamma_3}{G\Gamma_3, \Delta_3\Pi_3\Gamma_2, \Delta_2\Pi_2\Gamma_1, \Delta_1\Pi_1}$$

**(Soundness and Completeness.)** The axiom  $(\alpha\beta) \vee ((\alpha\neg\beta) \wedge (\neg\beta\alpha)) \vee (\beta\alpha)$  is equivalent to the newly generated rules. *Proof.* See [1].

**(Cut-Admissibility.)** The cut rule is admissible in the calculus **HFLe<sub>w</sub>** extended with the newly generated rules. *Proof.* See [1].

A cut-elimination procedure can be found in [2].

## References

- [1] A. Ciabattoni, N. Galatos, and K. Terui. From axioms to analytic rules in nonclassical logics. In *IEEE Symposium on Logic in Computer Science (LICS 08)*, pages 229–240, 2008.
- [2] A. Ciabattoni, L. Straßburger, and K. Terui. Expanding the realm of systematic proof theory. In *Proceedings of Computer Science Logic (CSL 09), LNCS*, 2009.