An analytic calculus for **HFLew** extended with the axiom $(\alpha\beta) \vee ((\alpha\neg\beta) \wedge (\neg\beta\alpha)) \vee (\beta\alpha)$

AxiomCalc[*]

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Abstract

This paper introduces a cut-free hypersequent calculus for **FLew** extended with the (Hilbert) axiom $(\alpha\beta) \vee ((\alpha\neg\beta) \wedge (\neg\beta\alpha)) \vee (\beta\alpha)$. The calculus is generated by the PROLOG-program *AxiomCalc*, which implements the procedure in [1].

1 Introduction

Non-classical logics are usually defined by adding Hilbert axioms to known systems. The usefulness of these logics, however, heavily depends on the availability of calculi which admit cut-elimination (i.e., analytic calculi). These calculi are indeed a prerequisite for the development of automated reasoning methods, and also the key to establish essential properties of the formalized logics.

We introduce a cut-free hypersequent calculus for the logic obtained by extending with the axiom $(\alpha\beta) \vee ((\alpha\neg\beta) \wedge (\neg\beta\alpha)) \vee (\beta\alpha)$. The calculus is obtained via a PROLOG-implementation of the procedure in [1].

2 Preliminaries

The formulas of **HFLew** are built from propositional variables $p, q, r, \ldots$ and the 0-ary connectives (constants) 1 and 0 by using the binary logical connectives $\cdot$ (fusion), (implication), $\wedge$ (conjunction) and $\vee$ (disjunction). $\neg\alpha$ will be used as abbreviation for $\alpha 0$.

The base calculus we will deal with is the hypersequent system **HFLew** (see Table 1).

Metavariables $\alpha, \beta, \ldots$ denote formulas, Π stands for stoups, i.e., either a formula or the empty set, and $\Gamma, \Delta, \ldots$ for finite (possibly empty) multisets of formulas. We only consider single-conclusion sequents i.e., sequents whose right-hand side (RHS) contains at most one formula.

[*]`http://www.logic.at/people/lara/axiomcalc.html`

Table 1: The hypersequent calculus **HFLew**

$$\begin{tikzcd}
\mathcal{P}_0 \arrow[r] \arrow[d, leftrightarrow] \arrow[dr] & \mathcal{P}_1 \arrow[r] \arrow[dr] & \mathcal{P}_2 \arrow[r] \arrow[dr] & \mathcal{P}_3 \arrow[r] \arrow[dr] & \mathcal{P}_4 \arrow[r] \arrow[dr] & \cdots \\
\mathcal{N}_0 \arrow[r] \arrow[ur] & \mathcal{N}_1 \arrow[r] \arrow[ur] & \mathcal{N}_2 \arrow[r] \arrow[ur] & \mathcal{N}_3 \arrow[r] \arrow[ur] & \mathcal{N}_4 \arrow[r] \arrow[ur] & \cdots
\end{tikzcd}$$

Figure 1: The substructural hierarchy [1]

2.1 Substructural Hierarchy

The substructural hierarchy is a novel classification of Hilbert axioms based on the logical connectives of **FLew**.

(Substructural Hierarchy) *[1]* Let $\mathcal{A}$ be a set of atomic formulas. For $n \geq 0$, the sets $\mathcal{P}_n, \mathcal{N}_n$ of formulas are defined as follows:

$\mathcal{P}_0 ::= \mathcal{N}_0 ::= \mathcal{A}$
$\mathcal{P}_{n+1} ::= \mathcal{N}_n \mid \mathcal{P}_{n+1} \cdot \mathcal{P}_{n+1} \mid \mathcal{P}_{n+1} \vee \mathcal{P}_{n+1} \mid$
$\mathcal{N}_{n+1} ::= \mathcal{P}_n \mid \mathcal{P}_{n+1}\mathcal{N}_{n+1} \mid \mathcal{N}_{n+1} \wedge \mathcal{N}_{n+1} \mid$

A graphical representation of the substructural hierarchy is depicted in Figure 1. Note that the arrows stand for inclusions $\subseteq$ of the classes.

3 From axioms to analytic rules

The axiom $(\alpha\beta) \vee ((\alpha\neg\beta) \wedge (\neg\beta\alpha)) \vee (\beta\alpha)$ is within the class $\mathcal{P}_3$ of the substructural hierarchy [1]. Using the algorithm in [1], the axiom

$$(\alpha\beta) \vee ((\alpha\neg\beta) \wedge (\neg\beta\alpha)) \vee (\beta\alpha)$$

can be transformed into the following rules to be added to the hypersequent calculus **HFLew**:

$$\dfrac{\begin{array}{cc} G\Gamma_1, \Delta_2\Pi_2 & G\Gamma_3, \Delta_2\Pi_2 \\ G\Gamma_2, \Delta_1\Pi_1 & G\Gamma_4, \Delta_1\Pi_1 \end{array}}{G\Gamma_3, \Gamma_4\Gamma_2, \Delta_2\Pi_2\Gamma_1, \Delta_1\Pi_1}$$

$$\dfrac{\begin{array}{cc} G\Gamma_1, \Delta_2\Pi_2 & G\Gamma_2, \Delta_1\Pi_1 \\ G\Gamma_2, \Delta_3\Pi_3 & G\Gamma_1, \Gamma_3 \end{array}}{G\Gamma_3, \Delta_3\Pi_3\Gamma_2, \Delta_2\Pi_2\Gamma_1, \Delta_1\Pi_1}$$

(Soundness and Completeness.) The axiom $(\alpha\beta) \vee ((\alpha\neg\beta) \wedge (\neg\beta\alpha)) \vee (\beta\alpha)$ is equivalent to the newly generated rules. *Proof.* See [1].

(Cut-Admissibility.) The cut rule is admissible in the calculus **HFLew** extended with the newly generated rules. *Proof.* See [1].

A cut-elimination procedure can be found in [2].

References

[1] A. Ciabattoni, N. Galatos, and K. Terui. From axioms to analytic rules in nonclassical logics. In *IEEE Symposium on Logic in Computer Science (LICS 08)*, pages 229–240, 2008.

[2] A. Ciabattoni, L. Straßburger, and K. Terui. Expanding the realm of systematic proof theory. In *Proceedings of Computer Science Logic (CSL 09), LNCS*, 2009.