Standard completeness for **MTL** extended with the axiom $\neg(\alpha \wedge \neg\alpha)$

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Abstract

This paper introduces a cut-free hypersequent calculus for **MTL** extended with the (Hilbert) axiom $\neg(\alpha \wedge \neg\alpha)$. The calculus is generated by the PROLOG-program *AxiomCalc*, which implements the procedure in [4]. Moreover, it shows that the resulting logic is standard complete. This is done by checking the conditions in [1, 2] on the generated calculus, which guarantee standard completeness for the considered logic.

1 Introduction

We introduce a cut-free hypersequent calculus for Monoidal t-norm logic **MTL** extended with the axiom $\neg(\alpha \wedge \neg\alpha)$. The analytic calculus for this logic is obtained via a PROLOG-implementation of the procedure in [4]. Moreover, we check whether the newly generated rule is convergent. This ensures standard completeness for **MTL** extended with $\neg(\alpha \wedge \neg\alpha)$, that is, completeness of the logic with respect to algebras based on the truth values in $[0, 1]$.

2 Preliminaries

The basic system we will deal with is Monoidal t-norm logic **MTL** which is the logic of left-continuous t-norms[1]. It is obtained by adding the prelinearity axiom $(\alpha \to \beta) \vee (\beta \to \alpha)$ to intuitionistic logic without contraction, see Table 1 for the corresponding hypersequent calculus *HMTL*. **MTL** is standard complete.

Formulas of **MTL** are built from propositional variables and the constants 0 and 1 by using $\to$ (implication), $\wedge$ (additive conjunction), $\cdot$ (multiplicative conjunction), and $\vee$ (disjunction). We use $\neg\alpha$ as an abbreviation for $\alpha \to 0$.

Metavariables $\alpha, \beta, \ldots$ denote formulas, Π stands for stoups, i.e., either a formula or the empty set, and $\Gamma, \Delta, \ldots$ for finite (possibly empty) multisets of formulas.

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*`http://www.logic.at/tinc/webaxiomcalc`

[1] A *t*-norm is a commutative, associative, increasing function $* : [0, 1]^2 \to [0, 1]$ with identity element 1. $*$ is *left continuous* iff whenever $\{x_n\}, \{y_n\}$ $(n \in N)$ are increasing sequences in $[0, 1]$ s.t. their suprema are x and y, then $\sup\{x_n * y_n : n \in N\} = x * y$. The residuum of $*$ is a function $\to^*$ where $x \to^* y = \max\{z \mid x * z \leq y\}$.

$$\frac{G \mid \Gamma \Rightarrow \alpha \quad G \mid \alpha, \Delta \Rightarrow \Pi}{G \mid \Gamma, \Delta \Rightarrow \Pi}\ (cut) \qquad \frac{}{G \mid \alpha \Rightarrow \alpha}\ (init) \qquad \frac{}{G \mid 0 \Rightarrow}\ (0l)$$

$$\frac{G \mid \Gamma \Rightarrow \alpha \quad G \mid \Delta \Rightarrow \beta}{G \mid \Gamma, \Delta \Rightarrow \alpha \cdot \beta}\ (\cdot\, r) \qquad \frac{G \mid \alpha, \beta, \Gamma \Rightarrow \Pi}{G \mid \alpha \cdot \beta, \Gamma \Rightarrow \Pi}\ (\cdot\, l) \qquad \frac{G \mid \Gamma \Rightarrow \Pi}{G \mid 1, \Gamma \Rightarrow \Pi}\ (1l)$$

$$\frac{G \mid \Gamma \Rightarrow \alpha \quad G \mid \beta, \Delta \Rightarrow \Pi}{G \mid \Gamma, \alpha \rightarrow \beta, \Delta \Rightarrow \Pi}\ (\rightarrow l) \qquad \frac{G \mid \alpha, \Gamma \Rightarrow \beta}{G \mid \Gamma \Rightarrow \alpha \rightarrow \beta}\ (\rightarrow r) \qquad \frac{G \mid \Gamma \Rightarrow}{G \mid \Gamma \Rightarrow 0}\ (0r)$$

$$\frac{G \mid \Gamma \Rightarrow \alpha \quad G \mid \Gamma \Rightarrow \beta}{G \mid \Gamma \Rightarrow \alpha \wedge \beta}\ (\wedge r) \qquad \frac{G \mid \alpha_i, \Gamma \Rightarrow \Pi}{G \mid \alpha_1 \wedge \alpha_2, \Gamma \Rightarrow \Pi}\ (\wedge l) \qquad \frac{}{G \mid \Rightarrow 1}\ (1r)$$

$$\frac{G \mid \alpha, \Gamma \Rightarrow \Pi \quad G \mid \beta, \Gamma \Rightarrow \Pi}{G \mid \alpha \vee \beta, \Gamma \Rightarrow \Pi}\ (\vee l) \qquad \frac{G \mid \Gamma \Rightarrow \alpha_i}{G \mid \Gamma \Rightarrow \alpha_1 \vee \alpha_2}\ (\vee r) \qquad \frac{G \mid \Gamma \Rightarrow \Pi}{G \mid \Gamma, \alpha \Rightarrow \Pi}\ (wl)$$

$$\frac{G \mid \Gamma \Rightarrow \Pi \mid \Gamma \Rightarrow \Pi}{G \mid \Gamma \Rightarrow \Pi}\ (EC) \qquad \frac{G}{G \mid \Gamma \Rightarrow \Pi}\ (EW) \qquad \frac{G \mid \Gamma \Rightarrow}{G \mid \Gamma \Rightarrow \Pi}\ (wr)$$

$$\frac{G \mid \Gamma_1, \Delta_1 \Rightarrow \Pi_1 \quad G \mid \Gamma_2, \Delta_2 \Rightarrow \Pi_2}{G \mid \Gamma_1, \Gamma_2 \Rightarrow \Pi_1 \mid \Delta_1, \Delta_2 \Rightarrow \Pi_2}\ (com)$$

Table 1: Hypersequent calculus *HMTL* for **MTL**

Definition 1 *A* hypersequent G *is a multiset* $S_1 \mid \cdots \mid S_n$ *where each* S_i *for* $i = 1, \ldots, n$ *is a sequent, called a* component *of the hypersequent. A hypersequent is called* single-conclusion *if all its components are single-conclusion.*

The symbol "|" is intended to denote disjunction at the meta-level. In this paper, we only consider single-conclusion (hyper)sequents. Given a sequent S henceforth we will denote by $LHS(S)$ its left hand side and by $RHS(S)$ its right hand side. Let $S := \Gamma_1, \Gamma_2 \Rightarrow \Pi$, we indicate by $S[^{\Gamma_1}/_{\Sigma}]^l$ the sequent $\Sigma, \Gamma_2 \Rightarrow \Pi$.

As in the case of sequent calculus, the hypersequent calculus consists of initial axioms, logical rules, the cut-rule and structural rules. Initial axioms, logical rules and the cut-rule are essentially the same as in the sequent calculus. The only difference is that a (possibly empty) side hypersequent G may occur in hypersequents. The structural rules are divided into two groups: *internal* structural rules and *external* structural rules. The former are applied to formulas within sequents. External rules instead manipulate the components of a hypersequent and therefore increase the expressive power of hypersequent calculus with respect to sequent calculus.

The notion of proof in *HMTL* is defined as usual. Let R be a set of rules. If there is a proof in *HMTL* extended with R (*HMTL*+R, for short) of a sequent S_0 from a set of sequents $\mathcal{S}$, we say that S_0 is derivable from $\mathcal{S}$ in *HMTL*+R and write $\mathcal{S} \vdash_{HMTL+R} S_0$. We write $\vdash_{HMTL+R} \alpha$ if $\emptyset \vdash_{HMTL+R} \Rightarrow \alpha$.

Two hypersequent rules (hr_0) and (hr_1) are equivalent (in *HMTL*) if the relations $\vdash_{HMTL+(hr_0)}$ and $\vdash_{HMTL+(hr_1)}$ coincide when restricted to sequents.

2.1 Substructural Hierarchy

The substructural hierarchy is a novel classification of Hilbert axioms based on the logical connectives of **MTL**.

$$
\begin{tikzcd}
\mathcal{P}_0 \arrow[r] \arrow[d, leftrightarrow] \arrow[dr] & \mathcal{P}_1 \arrow[r] \arrow[dr] & \mathcal{P}_2 \arrow[r] \arrow[dr] & \mathcal{P}_3 \arrow[r] \arrow[dr] & \mathcal{P}_4 \arrow[r] \arrow[dr] & \dots \\
\mathcal{N}_0 \arrow[r] \arrow[ur] & \mathcal{N}_1 \arrow[r] \arrow[ur] & \mathcal{N}_2 \arrow[r] \arrow[ur] & \mathcal{N}_3 \arrow[r] \arrow[ur] & \mathcal{N}_4 \arrow[r] \arrow[ur] & \dots
\end{tikzcd}
$$

Figure 1: The substructural hierarchy [4]

Definition 2 (Substructural Hierarchy) [4] *Let $\mathcal{A}$ be a set of atomic formulas. For $n \geq 0$, the sets $\mathcal{P}_n, \mathcal{N}_n$ of formulas are defined as follows:*

$$
\begin{aligned}
&\mathcal{P}_0 ::= \mathcal{N}_0 ::= \mathcal{A} \\
&\mathcal{P}_{n+1} ::= \mathcal{N}_n \mid \mathcal{P}_{n+1} \cdot \mathcal{P}_{n+1} \mid \mathcal{P}_{n+1} \vee \mathcal{P}_{n+1} \mid 1 \\
&\mathcal{N}_{n+1} ::= \mathcal{P}_n \mid \mathcal{P}_{n+1} \to \mathcal{N}_{n+1} \mid \mathcal{N}_{n+1} \wedge \mathcal{N}_{n+1} \mid 0
\end{aligned}
$$

A graphical representation of the substructural hierarchy is depicted in Figure 1. Note that the arrows $\to$ stand for inclusions $\subseteq$ of the classes.

2.2 From axioms to analytic rules

The axiom $\neg(\alpha \wedge \neg\alpha)$ is within the class $\mathcal{N}_2$ of the substructural hierarchy [4]. Using the algorithm in [4], the axiom

$$\neg(\alpha \wedge \neg\alpha)$$

can be transformed into the following rule to be added to the hypersequent calculus **HMTL**:

$$\frac{G \mid \Gamma_1, \Gamma_1 \Rightarrow}{G \mid \Gamma_1 \Rightarrow}$$

Theorem 3 (Soundness and Completeness.) *The axiom $\neg(\alpha \wedge \neg\alpha)$ is equivalent (in presence of the axiom $(\alpha \to \beta) \vee (\beta \to \alpha)$) to the newly generated rule.*

Proof. See [4, 3].

Theorem 4 (Cut-Admissibility.) *The cut rule is admissible in the calculus **HMTL** extended with the newly generated rule.*

Proof. See [4].

A cut-elimination procedure can be found in [6].

3 Standard completeness for MTL+$\neg(\alpha \wedge \neg\alpha)$

Let (r) be any hypersequent rule generated by the procedure in [4] where S_i, C_j denote sequents

$$\frac{G \mid S_1 \quad \dots \quad G \mid S_m}{G \mid C_1 \mid \dots \mid C_q}$$

Definition 5 *Let $G|S_i$ and $G|S_j$ be among the premises of* (r).

(0-pivot) $G|S_i$ *is a* 0-pivot *if there is an* $s \in \{1, \dots, q\}$ *such that* $RHS(S_i) = RHS(C_s)$ *and the different metavariables in the* $LHS(S_i)$ *are contained in those of* $LHS(C_s)$.

(n-pivot) $G|S_j$ *is an* n-pivot for $G|S_i$, *for* $n > 0$, *if the following conditions hold:*

- $G|S_j$ *is a* 0-pivot
- $RHS(S_i) = RHS(S_j)$
- $LHS(S_j) = LHS(S_i[^{\Gamma_1}/_{\Delta_1}, \dots {}^{\Gamma_n}/_{\Delta_n}]^l)$ *for* $\Gamma_1, \dots \Gamma_n \in LHS(S_i)$ *and* $\Delta_1, \dots \Delta_n \in LHS(S_j)$
- $G|S_j$ *is a* (n-1)-pivot *for n premises* $G|S_{j_1}, \dots, G|S_{j_n}$, *and for* $i = 1..n$ $LHS(S_j) = LHS(S_{j_i}[^{\Gamma_1}/_{\Delta_1}, \dots, {}^{\Gamma_{i-1}}/_{\Delta_{i-1}}, \dots, {}^{\Gamma_{i+1}}/_{\Delta_{i+1}}, {}^{\Gamma_n}/_{\Delta_n}]^l)$

Definition 6 *A completed hypersequent rule (r) is* convergent *if for each premise* $G|S_i$ *one of the following conditions holds: (1)* $RHS(S_i) = \emptyset$*, (2)* $G|S_i$ *is a 0-pivot, or (3) there is a premise* $G|S_j$ *which is an* n-pivot *for* $G|S_i$, *with* $n > 0$.

Lemma 7 *The rule equivalent to the axiom* $\neg(\alpha \wedge \neg\alpha)$ *is convergent.*

Proof. Consider again the generated rule:

$$\frac{G \mid \Gamma_1, \Gamma_1 \Rightarrow}{G \mid \Gamma_1 \Rightarrow}$$

The premise(s) of the rule satisfy condition (2) in Definition 6.

Theorem 8 *The logic formalized by the calculus HMTL extended with any convergent rule is standard complete.*

Proof. See [1].

Hence, **MTL** extended with $\neg(\alpha \wedge \neg\alpha)$ is standard complete.

References

[1] P. Baldi, A. Ciabattoni, and L. Spendier. Standard completeness for extensions of MTL: an automated approach. In *Proceedings of Int. Workshop on Logic, Language, Information and Computation (WoLLIC 2012), LNCS*, pages 154–167, 2012.

[2] P. Baldi and K. Terui. Densification of FL chains via residuated frames. *Algebra Universalis*, accepted for publication.

[3] P. Baldi. Standard completeness: proof-theoretic and algebraic methods. PhD Thesis, Vienna University of Technology, 2015.

[4] A. Ciabattoni, N. Galatos, and K. Terui. From axioms to analytic rules in nonclassical logics. In *IEEE Symposium on Logic in Computer Science (LICS 08)*, pages 229–240, 2008.

[5] A. Ciabattoni and L. Spendier. Tools for the Investigation of Substructural and Paraconsistent Logics. In *Proceedings of JELIA 2014, LNAI*, pages 18–32, 2014.

[6] A. Ciabattoni, L. Straßburger, and K. Terui. Expanding the realm of systematic proof theory. In *Proceedings of Computer Science Logic (CSL 09), LNCS*, pages 163–178, 2009.