Lectures on the Epsilon Calculus

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Contents

Contents **1**

1 Syntax **1**
- 1.1 Languages . . . 1
- 1.2 ε-Types, Degree, and Rank . . . 3
- 1.3 Axioms and Proofs . . . 5

2 Semantics **7**
- 2.1 Semantics for $\mathrm{EC}^{\mathrm{ext}}_{\varepsilon\forall}$ . . . 7
- 2.2 Soundness for $\mathrm{EC}^{\mathrm{ext}}_{\varepsilon\forall}$ . . . 9
- 2.3 Completeness for $\mathrm{EC}^{\mathrm{ext}}_{\varepsilon\forall}$ . . . 10
- 2.4 Semantics for $\mathrm{EC}_{\varepsilon\forall}$ . . . 11

3 The First Epsilon-Theorem **12**
- 3.1 The Case Without Identity . . . 12
- 3.2 The Case With Identity . . . 14

Exercise 1. Find the mistakes in these notes.

1 Syntax

1.1 Languages

Definition 1.1. The language of of the elementary calculus L_{EC} contains the following symbols:

1. Variables Var: x_0, x_1, …
2. Function symbols Fct^n of arity n, for each $n \geq 0$: f^n_0, f^n_1, …
3. Predicate symbols Pred^n of arity n, for each $n \geq 0$: P^n_0, P^n_1, …

4. Identity: $=$
5. Propositional constants: $\bot$, $\top$.
6. Propositional operators: $\neg$, $\wedge$, $\vee$, $\to$, $\leftrightarrow$.
7. Punctuation: parentheses: (,); comma: ,

For any language L, we denote by L^- the language L without the identity symbol, by L_ε the language L plus the symbol ε, and by $L_\forall$ the language L plus the quantifiers $\forall$ and $\exists$. We will usually leave out the subscript EC, and write $L_\forall$ for the language of the predicate calculus, L_ε for the language of the ε-calculus, and $L_{\varepsilon\forall}$ for the language of the extended epsilon calculus.

Definition 1.2. The *terms* Trm and *formulas* Frm of $L_{\varepsilon\forall}$ are defined as follows.

1. Every variable x is a term, and x is free in it.
2. If $t_1, \ldots, t_n$ are terms, then $f_i^n(t_1, \ldots, t_n)$ is a term, and x occurs free in it wherever it occurs free in $t_1, \ldots, t_n$.
3. If $t_1, \ldots, t_n$ are terms, then $P_i^n(t_1, \ldots, t_n)$ is an (atomic) formula, and x occurs free in it wherever it occurs free in $t_1, \ldots, t_n$.
4. $\bot$ and $\top$ are formulas.
5. If A is a formula, then $\neg A$ is a formula, with the same free occurrences of variables as A.
6. If A and B are formulas, then $(A \wedge B)$, $(A \vee B)$, $(A \to B)$, $(A \leftrightarrow B)$ are formulas, with the same free occurrences of variables as A and B.
7. If A is a formula in which x has a free occurrence but no bound occurrence, then $\forall x\, A$ and $\exists x\, A$ are formulas, and all occurrences of x in them are bound.
8. If A is a formula in which x has a free occurrence but no bound occurrence, then $\varepsilon_x\, A$ is a term, and all occurrences of x in it are bound.

The terms $\mathrm{Trm}(L)$ and formulas $\mathrm{Frm}(L)$ of a langauge L are those terms and formulas of $L_{\varepsilon\forall}$ in the vocabulary of L.

If E is an expression (term or formula), then $\mathrm{FV}(E)$ is the set of variables which have free occurrences in E. E is called *closed* if $\mathrm{FV}(E) = \emptyset$. A closed formula is also called a *sentence*.

When E, E' are expressions (terms or formulas), we write $E \equiv E'$ iff E and E' are syntactically identical up to a renaming of bound variables. We say that a term t is *free for x in E* iff x does not occur free in the scope of an ε-operator ε_y or quantifier $\forall y$, $\exists y$ for any $y \in \mathrm{FV}(t)$.

If E is an expression and t is a term, we write $E[x/t]$ for the result of substituting every free occurrence of x in E by t, provided t is free for x in E, and renaming bound variables in t if necessary.

If t is not free for x in E, $E[x/t]$ is any formula $E['/x]t$ where $E' \equiv E$ and t is free for x in E'. If $E' \equiv E[x_1/t_1]\dots[x_n/t_n]$, E' it is called an *instance of* E.

We write $E(x)$ to indicate that $x \in \mathrm{FV}(E)$, and $E(t)$ for $E[x/t]$. It will be apparent from the context which variable x is substituted for.

Definition 1.3. A term t is a *subterm* of an expression (term or formula) E, if for some $E'(x)$, $E \equiv E'(x)[x/t]$. It is a *proper subterm* of a term u if it is a subterm of u but $t \not\equiv u$.

A term t is an *immediate subterm* of an expression E if t is a subterm of E, but not a subterm of a proper subterm of E.

Definition 1.4. If t is a subterm of E, i.e., for some E' we have $E \equiv E'[x/t]$, then $E\{t/u\}$ is $E'[x/u]$.

We intend $E\{t/u\}$ to be the result of replacing every occurrence of t in E by u. But, the "brute-force" replacement of every occurrence of t in u may not be what we have in mind here. (a) We want to replace not just every occurrence of t by u, but every occurrence of a term $t' \equiv t$. (b) t may have an occurrence in E where a variable in t is bound by a quantifier or ε outside t, and such occurrences shouldn't be replaced (they are not subterm occurrences). (c) When replacing t by u, bound variables in u might have to be renamed to avoid conflicts with the bound variables in E' and bound variables in E' might have to be renamed to avoid free variables in u being bound.

Definition 1.5 (ε-Translation)**.** If E is an expression, define E^ε by:

1. $E^\varepsilon = E$ if E is a variable, a constant symbol, or $\bot$.
2. If $E = f^n_i(t_1, \dots, t_n)$, $E^\varepsilon = f^n_i(t^\varepsilon_1, \dots, t^\varepsilon_n)$.
3. If $E = P^n_i(t_1, \dots, t_n)$, $E^\varepsilon = P^n_i(t^\varepsilon_1, \dots, t^\varepsilon_n)$.
4. If $E = \neg A$, then $E^\varepsilon = \neg A^\varepsilon$.
5. If $E = (A \land B)$, $(A \lor B)$, $(A \to B)$, or $(A \leftrightarrow B)$, then $E^\varepsilon = (A^\varepsilon \land B^\varepsilon)$, $(A^\varepsilon \lor B^\varepsilon)$, $(A^\varepsilon \to B^\varepsilon)$, or $(A^\varepsilon \leftrightarrow B^\varepsilon)$, respectively.
6. If $E = \exists x\, A(x)$ or $\forall x\, A(x)$, then $E^\varepsilon = A^\varepsilon(\varepsilon_x\, A(x)^\varepsilon)$ or $A^\varepsilon(\varepsilon_x\, \neg A(x)^\varepsilon)$.
7. If $E = \varepsilon_x\, A(x)$, then $E^\varepsilon = \varepsilon_x\, A(x)^\varepsilon$.

1.2 ε-Types, Degree, and Rank

Definition 1.6. An ε-term $p \equiv \varepsilon_x\, B(x; x_1, \dots, x_n)$ is a *type of an* ε*-term* $\varepsilon_x\, A(x)$ iff

1. $p \equiv \varepsilon_x\, A(x)[x_1/t_1]\dots[x_n/t_n]$ for some terms $t_1, \dots, t_n$.

2. $\mathrm{FV}(p) = \{x_1, \ldots, x_n\}$.

3. $x_1, \ldots, x_n$ are all immediate subterms of p.

4. Each x_i has exactly one occurrence in p.

5. The occurrence of x_i is left of the occurrence of x_j in p if $i < j$.

We denote the set of types of a langauge as Typ.

Proposition 1.7. *The type of an epsilon term $\varepsilon_x\, A(x)$ is unique up to renaming of bound, and disjoint renaming of free variables, i.e., if $p = \varepsilon_x\, B(x; x_1, \ldots, x_n)$, $p' = \varepsilon_y\, B'(y; y_1, \ldots, y_m)$ are types of $\varepsilon_x\, A(x)$, then $n = m$ and $p' \equiv p[x_1/y_1] \ldots [x_n/y_n]$*

Proof. Exercise. □

Definition 1.8. An ε-term e is *nested in* an ε-term e' if e is a proper subterm of e.

Definition 1.9. The *degree* $\deg(e)$ of an ε-term e is defined as follows:

1. $\deg(e) = 1$ iff e contains no nested ε-terms.

2. $\deg(e) = \max\{\deg(e_1), \ldots, \deg(e_n)\} + 1$ if $e_1, \ldots, e_n$ are all the ε-terms nested in e.

For convenience, let $\deg(t) = 0$ if t is not an ε-term.

Definition 1.10. An ε-term e is *subordinate to* an ε-term $e' = \varepsilon_x\, A(x)$ if some $e'' \equiv e$ occurs in e' and $x \in \mathrm{FV}(e'')$.

Note that if e is subordinate to e' it is *not* a subterm of e', because x is free in e and so the occurrence of e (really, of the variant e'') in e' is in the scope of ε_x. One might think that replacing e in $\varepsilon_x\, A(x)$ by a new variable y would result in an ε-term $\varepsilon_x\, A'(y)$ so that $e' \equiv \varepsilon_x\, A'(y)[y/e]$. But (a) $\varepsilon_x\, A'(y)$ is not in general a term, since it is not guaranteed that x is free in $A'(y)$ and (b) e is not free for y in $\varepsilon_x\, A'(y)$.

Definition 1.11. The *rank* $\mathrm{rk}(e)$ of an ε-term e is defined as follows:

1. $\mathrm{rk}(e) = 1$ iff e contains no subordinate ε-terms.

2. $\mathrm{rk}(e) = \max\{\mathrm{rk}(e_1), \ldots, \mathrm{rk}(e_n)\} + 1$ if $e_1, \ldots, e_n$ are all the ε-terms subordinate to e.

Proposition 1.12. *If p is the type of e, then $\mathrm{rk}(p) = \mathrm{rk}(e)$.*

Proof. Exercise. □

1.3 Axioms and Proofs

Definition 1.13. The axioms of the *elementary calculus* EC are

$$A \quad \text{for any tautology } A \tag{Taut}$$
$$t = t \quad \text{for any term } t \tag{$=_1$}$$
$$t = u \to (A[x/t] \leftrightarrow A[x/u]) \tag{$=_2$}$$

and its only rule of inference is

$$\frac{A \quad A \to B}{A} \text{ MP}$$

The axioms and rules of the (intensional) *ε-calculus* EC_ε are those of EC plus the *critical formulas*

$$A(t) \to A(\varepsilon_x\, A(x)). \tag{crit}$$

The axioms and rules of the *extensional ε-calculus* $\mathrm{EC}_\varepsilon^{\mathrm{ext}}$ are those of EC_ε plus

$$(\forall x(A(x) \leftrightarrow B(x)))^\varepsilon \to \varepsilon_x\, A(x) = \varepsilon_x\, B(x) \tag{ext}$$

that is,

$$A(\varepsilon_x \neg(A(x) \leftrightarrow B(x))) \leftrightarrow B(\varepsilon_x \neg(A(x) \leftrightarrow B(x))) \to \varepsilon_x\, A(x) = \varepsilon_x\, B(x)$$

The axioms and rules of $\mathrm{EC}_\forall$, $\mathrm{EC}_{\varepsilon\forall}$, $\mathrm{EC}_{\varepsilon\forall}^{\mathrm{ext}}$ are those of EC, EC_ε, $\mathrm{EC}_\varepsilon^{\mathrm{ext}}$, respectively, together with the axioms

$$A(t) \to \exists x\, A(x) \tag{Ax$\exists$}$$
$$\forall x\, A(x) \to A(t) \tag{Ax$\forall$}$$

and the rules

$$\frac{A(x) \to B}{\exists x\, A(x) \to B}\ R\exists \qquad \frac{B \to A(x)}{B \to \forall x\, A(x)}\ R\forall$$

Applications of these rules must satisfy the *eigenvariable condition*, viz., the variable x must not appear in the conclusion or anywhere below it in the proof.

Definition 1.14. If Γ is a set of formulas, a *proof of A from Γ in* $\mathrm{EC}_{\varepsilon\forall}^{\mathrm{ext}}$ is a sequence π of formulas $A_1, \ldots, A_n = A$ where for each $i \leq n$, one of the following holds:

1. $A_i \in \Gamma$.
2. A_i is an instance of an axiom.
3. A_i follows from some A_k, A_l (k, $l < i$) by (MP), i.e., $A_i \equiv C$, $A_k \equiv B$, and $A_l \equiv B \to C$.
4. A_i follows from some A_j ($j < i$) by (R$\exists$), i.e., i.e., $A_i \equiv \exists x\, B(x) \to C$, $A_j \equiv B(x) \to C$, and x is an eigenvariable, i.e., it satisfies $x \notin \mathrm{FV}(A_k)$ for any $k \geq i$ (this includes $k = i$, so $x \notin \mathrm{FV}(C)$).

5. A_i follows from some A_j $(j < i)$ by (R$\forall$), i.e., i.e., $A_i \equiv C \to \forall x\, B(x)$, $A_j \equiv C \to B(x)$, and the eigenvariable condition is satisfied.

If π only uses the axioms and rules of EC, EC_ε, $\mathrm{EC}_\varepsilon^{\mathrm{ext}}$, etc., then it is a proof of A from Γ in EC, EC_ε, $\mathrm{EC}_\varepsilon^{\mathrm{ext}}$, etc., and we write $\Gamma \vdash^\pi A$, $\Gamma \vdash_\varepsilon^\pi A$, $\Gamma \vdash_{\varepsilon\mathrm{ext}}^\pi A$, etc.

We say that A is provable from Γ in EC, etc. ($\Gamma \vdash A$, etc.), if there is a proof of A from Γ in EC, etc.

Note that our definition of proof, because of its use of $\equiv$, includes a tacit rule for renaming bound variables. Note also that substitution into members of Γ is *not* permitted. However, we can simulate a provability relation in which substitution into members of Γ is allowed by considering Γ^{inst}, the set of all substitution instances of members of Γ. If Γ is a set of sentences, then $\Gamma^{\mathrm{inst}} = \Gamma$.

Proposition 1.15. *If $\pi = A_1, \ldots, A_n \equiv A$ is a proof of A from Γ and $x \notin \mathrm{FV}(\Gamma)$ is not an eigenvariable in π, then $\pi[x/t] = A_1[x/t], \ldots, A_n[x/t]$ is a proof of $A[x/t]$ from Γ^{inst}.*

In particular, if Γ is a set of sentences and π is a proof in EC, EC_ε, *or* $\mathrm{EC}_\varepsilon^{\mathrm{ext}}$, *then $\pi[x/t]$ is a proof of $A[x/t]$ from Γ in* EC, EC_ε, *or* $\mathrm{EC}_\varepsilon^{\mathrm{ext}}$

Proof. Exercise. □

Lemma 1.16. *If π is a proof of B from $\Gamma \cup \{A\}$, then there is a proof $\pi[A]$ of $A \to B$ from Γ, provided A contains no eigenvariables of π free.*

Proof. Construct $\pi[A]_0 = \emptyset$. Let $\pi_{i+1}[A] = \pi_i[A]$ plus additional formulas, depending on A_i:

1. If $A_i \in \Gamma$, add $A \to A$, if $A_i \equiv A$, or else add A_i, the tautology $A_i \to (A \to A_i)$, and $A \to A_i$. The last formula follows from the previous two by (MP).

2. If A_i is a tautology, add $A \to A_i$, which is also a tautology.

3. If A_i follows from A_k and A_l by (MP), i.e., $A_i \equiv C$, $A_k \equiv B$ and $A_l \equiv B \to C$, then $\pi[A]_i$ contains $A \to B$ and $A \to (B \to C)$. Add the tautology $(A \to B) \to ((B \to C) \to (A \to C)$ and $A \to C$. The latter follows from the former by two applications of (MP).

4. If A_i follows from A_j by (R$\exists$), i.e., $A_i \equiv \exists x\, B(x) \to C$ and $A_j \equiv B(x) \to C$, then $\pi[A]_i$ contains $A \to (B(x) \to C)$. $\pi[A]_{i+1}$ is

$$\begin{array}{lr}
\pi[A]_i & \\
(A \to (B(x) \to C)) \to (B(x) \to (A \to C)) & \text{(taut)} \\
B(x) \to (A \to C) & \text{(MP)} \\
\exists x\, B(x) \to (A \to C) & \text{(R}\exists\text{)} \\
(\exists x\, B(x) \to (A \to C)) \to (A \to (\exists x\, B(x) \to C)) & \text{(taut)} \\
A \to (\exists x\, B(x) \to C) & \text{(MP)}
\end{array}$$

Since $x \notin \mathrm{FV}(A)$, the eigenvariable condition is satisfied.

5. Exercise: A_i follows by (R$\forall$).

Now take $\pi[A] = \pi[A]_i$. □

Theorem 1.17 (Deduction Theorem)**.** *If $\Sigma \cup \{A\}$ is a set of sentences, $\Sigma \vdash A \to B$ iff $\Sigma \cup \{A\} \vdash B$.*

Corollary 1.18. *If $\Sigma \cup \{A\}$ is a set of sentences, $\Sigma \vdash A$ iff $\Sigma \cup \{\neg A\} \vdash \bot$.*

Lemma 1.19 (ε-Embedding Lemma)**.** *If $\Gamma \vdash^{\pi}_{\varepsilon\forall} A$, then there is a proof π^ε so that $\Gamma^{\varepsilon\, inst} \vdash^{\pi^\varepsilon}_{\varepsilon} A^\varepsilon$*

Proof. Exercise. □

2 Semantics

2.1 Semantics for $\mathrm{EC}^{\mathrm{ext}}_{\varepsilon\forall}$

Definition 2.1. A *structure* $\mathfrak{M} = \langle |\mathfrak{M}|, (\cdot)^{\mathfrak{M}} \rangle$ consists of a nonempty *domain* $|\mathfrak{M}| \neq \emptyset$ and a maping $(\cdot)^{\mathfrak{M}}$ on function and predicate symbols where:

$$(f^0_i)^{\mathfrak{M}} \in |\mathfrak{M}|$$
$$(f^n_i)^{M} \in \mathfrak{M}^{\mathfrak{M}^n}$$
$$(P^n_i)^{\mathfrak{M}} \subseteq \mathfrak{M}^n$$

Definition 2.2. An *extensional choice function Φ on $\mathfrak{M}$* is a function $\Phi \colon \wp(|\mathfrak{M}|) \to |\mathfrak{M}|$ where $\Phi(X) \in X$ whenever $X \neq \emptyset$.

Note that Φ is total on $\wp(|\mathfrak{M}|)$, and so $\Phi(\emptyset) \in |\mathfrak{M}|$.

Definition 2.3. An *assignment s on $\mathfrak{M}$* is a function $s \colon \mathrm{Var} \to |\mathfrak{M}|$.

If $x \in \mathrm{Var}$ and $m \in |\mathfrak{M}|$, $s[x/m]$ is the assignment defined by

$$s[x/m](y) = \begin{cases} m & \text{if } y = x \\ s(y) & \text{otherwise} \end{cases}$$

Definition 2.4. The *value* $\mathrm{val}_{\mathfrak{M},\Phi,s}(t)$ *of a term* and the *satisfaction relation* $\mathfrak{M}, \Phi, s \models A$ are defined as follows:

1. $\mathrm{val}_{\mathfrak{M},\Phi,s}(x) = s(x)$

2. $\mathfrak{M}, \Phi, s \models \top$ and $\mathfrak{M}, \Phi, s \not\models \bot$

3. $\mathrm{val}_{\mathfrak{M},\Phi,s}(f^n_i(t_1, \ldots, t_n)) = (f^n_i)^{\mathfrak{M}}(\mathrm{val}_{\mathfrak{M},\Phi,s}(t_1), \ldots, \mathrm{val}_{\mathfrak{M},\Phi,s}(t_n))$

4. $\mathfrak{M}, \Phi, s \models P^n_i(t_1, \ldots, t_n)$ iff $\langle \mathrm{val}_{\mathfrak{M},\Phi,s}(t_1), \ldots, \mathrm{val}_{\mathfrak{M},\Phi,s}(t_n) \rangle \in (P^n_i)^{\mathfrak{M}}$

5. $\text{val}_{\mathfrak{M},\Phi,s}(\varepsilon_x\, A(x)) = \Phi(\text{val}_{\mathfrak{M},\Phi,s}(A(x)))$ where

$$\text{val}_{\mathfrak{M},\Phi,s}(A(x)) = \{m \in |\mathfrak{M}| : \mathfrak{M}, \Phi, s[x/m] \models A(x)\}$$

6. $\mathfrak{M}, \Phi, s \models \exists x\, A(x)$ iff for some $m \in |\mathfrak{M}|$, $\mathfrak{M}, \Phi, s[x/m] \models A(x)$

7. $\mathfrak{M}, \Phi, s \models \forall x\, A(x)$ iff for all $m \in |\mathfrak{M}|$, $\mathfrak{M}, \Phi, s[x/m] \models A(x)$

Proposition 2.5. *If $s(x) = s'(x)$ for all $x \notin \text{FV}(t) \cup \text{FV}(A)$, then $\text{val}_{\mathfrak{M},\Phi,s}(t) = \text{val}_{\mathfrak{M},\Phi,s'}(t)$ and $\mathfrak{M}, \Phi, s \models A$ iff $\mathfrak{M}, \Phi, s' \models A$.*

Proof. Exercise. □

Proposition 2.6 (Substitution Lemma)**.** *If $m = \text{val}_{\mathfrak{M},\Phi,s}(u)$, then $\text{val}_{\mathfrak{M},\Phi,s}(t(u)) = \text{val}_{\mathfrak{M},\Phi,s[x/m]}(t(x))$ and $\mathfrak{M}, \Phi, s \models A(u)$ iff $\mathfrak{M}, \Phi, s[x/m] \models A(x)$*

Proof. Exercise. □

Definition 2.7.

1. A is *locally true* in $\mathfrak{M}$ with respect to Φ and s iff $\mathfrak{M}, \Phi, s \models A$.

2. A is *true* in $\mathfrak{M}$ with respect to Φ, $\mathfrak{M}, \Phi \models A$, iff for all s on $\mathfrak{M}$: $\mathfrak{M}, \Phi, s \models A$.

3. A is *generically true* in $\mathfrak{M}$ with respect to s, $\mathfrak{M}, s \models^g A$, iff for all choice functions Φ on $\mathfrak{M}$: $\mathfrak{M}, \Phi, s \models A$.

4. A is *generically valid* in $\mathfrak{M}$, $\mathfrak{M} \models A$, if for all choice functions Φ and assignments s on $\mathfrak{M}$: $\mathfrak{M}, \Phi, s \models A$.

Definition 2.8. Let $\Gamma \cup \{A\}$ be a set of formulas.

1. A is a *local consequence* of Γ, $\Gamma \models^l A$, iff for all $\mathfrak{M}$, Φ, and s:
   if $\mathfrak{M}, \Phi, s \models \Gamma$ then $\mathfrak{M}, \Phi, s \models A$.

2. A is a *truth consequence* of Γ, $\Gamma \models A$, iff for all $\mathfrak{M}$, Φ:
   if $\mathfrak{M}, \Phi \models \Gamma$ then $\mathfrak{M}, \Phi \models A$.

3. A is a *generic consequence* of Γ, $\Gamma \models^g A$, iff for all $\mathfrak{M}$ and s:
   if $\mathfrak{M}, s \models^g \Gamma$ then $\mathfrak{M} \models A$.

4. A is a *generic validity consequence* of Γ, $\Gamma \models^v A$, iff for all $\mathfrak{M}$:
   if $\mathfrak{M} \models^v \Gamma$ then $\mathfrak{M} \models A$.

Exercise 2. What is the relationship between these consequence relations? For instance, if $\Gamma \models^l A$ then $\Gamma \models A$ and $\Gamma \models^g A$, and if eiter $\Gamma \models A$ or $\Gamma \models^g A$, then $\Gamma \models^v A$. Are these containments strict? Are they identities (in general, and in cases where the language of Γ, A is restricted, or if Γ, A are sentences)? For instance:

Proposition 2.9. *If $\Sigma \cup \{A\}$ is a set of sentences, $\Sigma \models^l A$ iff $\Sigma \models A$*

Proposition 2.10. *If $\Sigma \cup \{A, B\}$ is a set of sentences, $\Sigma \cup \{A\} \models B$ iff $\Sigma \models A \to B$.*

Proof. Exercise. □

Corollary 2.11. *If $\Sigma \cup \{A\}$ is a set of sentences, $\Sigma \models A$ iff for no $\mathfrak{M}$, Φ, $\mathfrak{M} \models \Sigma \cup \{\neg A\}$*

Proof. Exercise. □

Exercise 3. For which of the other consequence relations, if any, do these results hold?

2.2 Soundness for $\mathrm{EC}_{\varepsilon\forall}^{\mathrm{ext}}$

Theorem 2.12. *If $\Gamma \vdash_{\varepsilon\forall} A$, then $\Gamma \models^l A$.*

Proof. Suppose $\Gamma, \Phi, s \models \Gamma$. We show by induction on the length n of a proof π that $\mathfrak{M}, \Phi, s \models' A$ for all s' which agree with s on $\mathrm{FV}(\Gamma)$. We may assume that no eigenvariable x of π is in $\mathrm{FV}(\Gamma)$ (if it is, let $y \notin \mathrm{FV}(\pi)$ and not occurring in π; consider $\pi[x/y]$ instead of pi).

If $n = 0$ there's nothing to prove. Otherwise, we distinguish cases according to the last line A_n in π:

1. $A_n \in \Gamma$. The claim holds by assumption.

2. A_n is a tautology. Obvious.

3. A_n is an identity axiom. Obvious.

4. A_n is a critical formula, i.e., $A_n \equiv A(t) \to A(\varepsilon_x\, A(x))$. Then either $\mathfrak{M}, \Phi, s \models A(t)$ or not (in which case there's nothing to prove). If yes, $\mathfrak{M}, \Phi, s[x/m] \models A(x)$ for $m = \mathrm{val}_{\mathfrak{M},\Phi,s}(t)$, and so $Y = \mathrm{val}_{\mathfrak{M},\Phi,s}(A(x)) \neq \emptyset$. Consequently, $\Phi(Y) \in Y$, and hence $\mathfrak{M}, \Phi, s \models A(\varepsilon_x\, A(x))$.

5. A_n is an extensionality axiom. Exercise.

6. A_n follows from B and $B \to C$ by (MP). By induction hypothesis, $\mathfrak{M}, \Phi, s \models B$ and $\mathfrak{M}, \Phi, s \models B \to C$.

7. A follows from $B(x) \to C$ by (R∃), and x satisfies the eigenvariable condition. Exercise.

8. A follows from $C \to B(x)$ by (R∀), and x satisfies the eigenvariable condition. Exercise.

□

Exercise 4. Complete the missing cases.

2.3 Completeness for $\mathrm{EC}^{\mathrm{ext}}_{\varepsilon\forall}$

Lemma 2.13. *If Γ is a set of sentences in L_ε and $\Gamma \nvdash_\varepsilon \bot$, then there are $\mathfrak{M}$, Φ so that $\mathfrak{M}, \Phi \models \Gamma$.*

Theorem 2.14 (Completeness)**.** *If $\Gamma \cup \{A\}$ are sentences in L_ε and $\Gamma \models A$, then $\Gamma \vdash_\varepsilon A$.*

Proof. Suppose $\Gamma \not\models A$. Then for some $\mathfrak{M}$, Φ we have $\mathfrak{M}, \Phi \models \Gamma$ but $\mathfrak{M}, \Phi \not\models A$. Hence $\mathfrak{M}, \Phi \models \Gamma \cup \{\neg A\}$. By the Lemma, $\Gamma \cup \{\neg A\} \vdash_\varepsilon \bot$. By Corollary 1.18, $\Gamma \vdash_\varepsilon A$. □

The proof of the Lemma comes in several stages. We have to show that if Γ is consistent, we can construct $\mathfrak{M}$, Φ, and s so that $\mathfrak{M}, \Phi, s \models \Gamma$. Since $\mathrm{FV}(\Gamma) = \emptyset$, we then have $\mathfrak{M}, \Phi \models \Gamma$.

Lemma 2.15. *If $\Gamma \nvdash_\varepsilon \bot$, there is $\Gamma^* \supseteq \Gamma$ with (1) $\Gamma^* \nvdash_\varepsilon \bot$ and (2) for all formulas A, either $A \in \Gamma^*$ or $\neg A \in \Gamma^*$.*

Proof. Let A_1, A_2, ... be an enumeration of Frm_ε. Define $\Gamma_0 = \Gamma$ and

$$\Gamma_{n+1} = \begin{cases} \Gamma_n \cup \{A_n\} & \text{if } \Gamma_n \cup \{A_n\} \nvdash_\varepsilon \bot \\ \Gamma_n \cup \{\neg A_n\} & \text{if } \Gamma_n \cup \{\neg A_n\} \nvdash_\varepsilon \bot \text{ otherwise} \end{cases}$$

Let $\Gamma^* = \bigcup_{n \geq 0} \Gamma_n$. Obviously, $\Gamma \subseteq \Gamma^*$. For (1), observe that if $\Gamma^* \vdash^\pi_\varepsilon \bot$, then π contains only finitely many formulas from Γ^*, so for some n, $\Gamma_n \vdash^\pi_\varepsilon \bot$. But Γ_n is consistent by definition.

To verify (2), we have to show that for each n, either $\Gamma_n \cup \{A_n\} \nvdash_\varepsilon \bot$ or $\Gamma_n \cup \{\neg A\} \nvdash_\varepsilon \bot$. For $n = 0$, this is the assumtion of the lemma. So suppose the claim holds for $n - 1$. Suppose $\Gamma_n \cup \{A\} \vdash^\pi_\varepsilon \bot$ and $\Gamma_n \cup \{\neg A\} \vdash^{\pi'}_\varepsilon \bot$. Then by the Deduction Theorem, we have $\Gamma_n \vdash^{\pi[A]}_A \to \bot$ and $\Gamma_n \vdash^{\pi'[A']}_\neg A \to \bot$. Since $(A \to \bot) \to ((\neg A \to \bot) \to \bot)$ is a tautology, we have $\Gamma_n \vdash_\varepsilon \bot$, contradicting the induction hypothesis. □

Lemma 2.16. *If $\Gamma^* \vdash \varepsilon B$, then $B \in \Gamma^*$.*

Proof. If not, then $\neg B \in \Gamma^*$ by maximality, so Γ^* would be inconsistent. □

Definition 2.17. Let $\approx$ be the relation on Trm_ε defined by

$$t \approx u \text{ iff } t = u \in \Gamma^*$$

It is easily seen that $\approx$ is an equivalence relation. Let $\widetilde{t} = \{u : u \approx t\}$ and $\widetilde{\mathrm{Trm}} = \{\widetilde{t} : t \in \mathrm{Trm}\}$.

Definition 2.18. A set $T \in \widetilde{\mathrm{Trm}}$ is *represented by* $A(x)$ if $T = \{\widetilde{t} : A(t) \in \Gamma^*\}$.

Let Φ_0 be a fixed choice function on $\widetilde{\mathrm{Trm}}$, and define

$$\Phi(T) = \begin{cases} \widetilde{\varepsilon_x A(x)} & \text{if } T \text{ is represented by } A(x) \\ \Phi_0(T) & \text{otherwise.} \end{cases}$$

Proposition 2.19. *Φ is a well-defined choice function on $\widetilde{\mathrm{Trm}}$.*

Proof. Exercise. Use (ext) for well-definedness and (crit) for choice function. □

Now let $\mathfrak{M} = \langle \widetilde{\mathrm{Trm}}, (\cdot)^{\mathfrak{M}} \rangle$ with $c^{\mathfrak{M}} = \widetilde{c}$, $(P_i^n)^{\mathfrak{M}} = \{\langle \widetilde{t_1}, \dots, \widetilde{t_1} \rangle : P_i^n(t_1, \dots, t_n)\}$, and let $s(x) = \widetilde{s}$.

Proposition 2.20. $\mathfrak{M}, \Phi, s \models \Gamma^*$.

Proof. We show that $\mathrm{val}_{\mathfrak{M},\Phi,s}(t) = \widetilde{t}$ and $\mathfrak{M}, \Phi, s \models A$ iff $A \in \Gamma^*$ by simultaneous induction on the complexity of t and A.

If $t = c$ is a constant, the claim holds by definition of $(\cdot)^{\mathfrak{M}}$. If $A = \bot$ or $= \top$, the claim holds by Lemma 2.16.

If $A \equiv P^n(t_1, \dots, t_n)$, then by induction hypothesis, $\mathrm{val}_{\mathfrak{M},\Phi,s}(t)_i = \widetilde{t_i}$. By definition of $(\cdot)^{\mathfrak{M}}$, $\langle \widetilde{t_1}, \dots, \widetilde{t_n} \rangle \in (P_i^n)(t_1, \dots, t_n)$ iff $P_i^n(t_1, \dots, t_n) \in \Gamma^*$.

If $A \equiv \neg B$, $(B \wedge C)$, $(B \vee C)$, $(B \to C)$, $(B \leftrightarrow C)$, the claim follows immediately from the induction hypothesis and the definition of $\models$ and the closure properties of Γ^*. For instance, $\mathfrak{M}, \Phi, s \models (B \wedge C)$ iff $\mathfrak{M}, \Phi, s \models B$ and $\mathfrak{M}, \Phi, s \models C$. By induction hypothesis, this is the case iff $B \in \Gamma^*$ and $C \in \Gamma^*$. But since $B, C \vdash_\varepsilon B \wedge C$ and $B \wedge C \vdash_\varepsilon B$ and $\vdash_\varepsilon C$, this is the case iff $(B \wedge C) \in \Gamma^*$. Remaining cases: Exercise.

If $t \equiv \varepsilon_x A(x)$, then $\mathrm{val}_{\mathfrak{M},\Phi,s}(t) = \Phi(\mathrm{val}_{\mathfrak{M},\Phi,s}(A(x)))$. Since $\mathrm{val}_{\mathfrak{M},\Phi,s}(A(x))$ is represented by $A(x)$ by induction hypothesis, we have $\mathrm{val}_{\mathfrak{M},\Phi,s}(t) = \widetilde{\varepsilon_x A(x)}$ by definition of Φ. □

Exercise 5. Complete the proof.

Exercise 6. Generalize the proof to $L_{\varepsilon\forall}$ and $\mathrm{EC}_{\varepsilon\forall}$.

Exercise 7. Show EC_ε without $(=_1)$ and $(=_2)$, (ext), and the additional axiom

$$(\forall x(A(x) \leftrightarrow B(x)))^\varepsilon \to (C(\varepsilon_x A(x)) \leftrightarrow C(\varepsilon_x B(x))) \qquad (\mathrm{ext}^-)$$

is complete for $\models$ in the language $L^-_{\varepsilon\forall}$.

2.4 Semantics for $\mathrm{EC}_{\varepsilon\forall}$

In order to give a complete semantics for $\mathrm{EC}_{\varepsilon\forall}$, i.e., for the calculus without the extensionality axion (ext), it is necessary to chnage the notion of choice function so that two ε-terms $\varepsilon_x A(x)$ and $\varepsilon_x B(x)$ may be assigned different representatives even when $\mathfrak{M}, \Phi, s \models \forall x(A(x) \leftrightarrow B(x))$, since then the negation of (ext) is consistent in the resulting calculus. The idea is to add the ε-term itself as an additional argument to the choice function. However, in order for this semantics to be sound for the calculus—specifically, in order for $(=_2)$ to be valid—we have to use not ε-terms but ε-types.

Definition 2.21. An *intensional choice operator* is a mapping $\Psi\colon \mathrm{Typ} \times |\mathfrak{M}|^{<\omega} \to |\mathfrak{M}|^{\wp(|\mathfrak{M}|)}$ such that for every type $p = \varepsilon_x\, A(x; y_1, \dots, y_n)$ is a type, and $m_1, \dots, m_n \in |\mathfrak{M}|$, $\Psi(p, m_1, \dots, m_n)$ is a choice function.

Definition 2.22. If $\mathfrak{M}$ is a structure, Ψ an intensional choice operator, and s an assignment, $\mathrm{val}_{\mathfrak{M},\Psi,s}(t)$ and $\mathfrak{M}, \Psi, s \models A$ is defined as before, except (5) in Definition 2.4 is replaced by:

(5′) $\mathrm{val}_{\mathfrak{M},\Psi,s}(\varepsilon_x\, A(x)) = \Psi(p, m_1, \dots, m_n)(\mathrm{val}_{\mathfrak{M},\Phi,s}(A(x)))$ where

a) $p = \varepsilon_x\, A'(x; x_1, \dots, x_n)$ is the type of $\varepsilon_x\, A(x)$,

b) $t_1, \dots, t_n$ are the subterms corresponding to $x_1, \dots, x_n$, i.e., $\varepsilon_x\, A(x) \equiv \varepsilon_x\, A'(x; t_1, \dots, t_n)$,

c) $m_i = \mathrm{val}_{\mathfrak{M},\Psi,s}(t)_1$, and

d) $\mathrm{val}_{\mathfrak{M},\Phi,s}(A(x)) = \{m \in |\mathfrak{M}| : \mathfrak{M}, \Psi, s[x/m] \models A(x)\}$

Exercise 8. Prove the substitution lemma for this semantics.

Exercise 9. Prove soundness.

Exercise 10. Prove completeness of $\mathrm{EC}_{\varepsilon\forall}$ for this semantics.

Exercise 11. Define a semantics for the language without = where the choice operatore takes ε-terms as arguments. Is the semantics sound and complete for $\mathrm{EC}^-_{\varepsilon\forall}$?

3 The First Epsilon Theorem

3.1 The Case Without Identity

Theorem 3.1. *If E is a formula not containing any ε-terms and $\vdash_{\mathrm{EC}_{\varepsilon\forall}} E$, then $\vdash_{\mathrm{EC}} E$.*

Definition 3.2. An ε-term e is *critical in* π if $A(t) \to A(e)$ is one of the critical formulas in π. The *rank* $\mathrm{rk}(\pi)$ *of a proof* π is the maximal rank of its critical ε-terms. The *r-degree* $\deg(\pi, r)$ of π is the maximum degree of its critical ε-terms of rank r. The *r-order* $o(\pi, r)$ of π is the number of different (up to renaming of bound variables) critical ε-terms of rank r.

Lemma 3.3. *If $e = \varepsilon_x\, A(x)$, $\varepsilon_y\, B(y)$ are critical in π, $\mathrm{rk}(e) = \mathrm{rk}(\pi)$, and $B^* \equiv B(u) \to B(\varepsilon_y\, B(y))$ is a critical formula in π. Then, if e is a subterm of B^*, it is a subterm of $B(y)$ or a subterm of u.*

Proof. Suppose not. Then, since e is a subterm of B^*, we have $B(y) \equiv B'(\varepsilon_x\, A'(x, y), y)$ and either $e \equiv \varepsilon_x\, A'(x, u)$ or $e \equiv \varepsilon_x\, A'(x, \varepsilon_y\, B(y))$. In each case, we see that $\varepsilon_x\, A'(x, y)$ and e have the same rank, since the latter is an instance of the former (and so have the same type). On the other hand, in either case, $\varepsilon_y\, B(y)$ would be

$$\varepsilon_y\, B'(\varepsilon_x\, A'(x,y), y)$$

and so would have a higher rank than $\varepsilon_x\, A'(x,y)$ as that ε-term is subordinate to it. This contradicts $\mathrm{rk}(e) = \mathrm{rk}(\pi)$. □

Lemma 3.4. *Let e, B^* be as in the lemma, and t be any term. Then*

1. *If e is not a subterm of $B(y)$, $B^*\{e/t\} \equiv B(u') \to B(\varepsilon_y\, B(y))$.*
2. *If e is a subterm of $B(y)$, i.e., $B(y) \equiv B'(e,y)$, $B^*\{e/t\} \equiv B'(t,u') \to B'(t, \varepsilon_y\, B'(t,y))$.*

Proof. By inspection. □

Lemma 3.5. *If $\vdash^{\pi}_{\mathrm{EC}_\varepsilon} E$ and E does not contain ε, then there is a proof π' such that $\vdash^{\pi'}_{\mathrm{EC}_\varepsilon} E$ and $\mathrm{rk}(\pi') \leq \mathrm{rk}(pi) = r$ and $o(\pi', r) < o(\pi, r)$.*

Proof. Let e be an ε-term critical in π and let $A(t_1) \to A(e)$, dots, $A(t_n) \to A(e)$ be all its critical formulas in π.

Consider $\pi\{e/t\}_i$, i.e., π with e replaced by t_i throughout. Each critical formula belonging to e now is of the form $A(t'_j) \to A(t_i)$, since e obviously cannot be a subterm of $A(x)$ (if it were, e would be a subterm of $\varepsilon_x\, A(x)$, i.e., of itself!). Let $\hat{\pi}_i$ be the sequence of tautologies $A(t_i) \to (A(t'_j) \to A(t_i))$ for $i = 1, \ldots, n$, followed by $\pi\{e/t\}_i$. Each one of the formulas $A(t'_j) \to A(t_i)$ follows from one of these by (MP) from $A(t_i)$. Hence, $A(t_i) \vdash^{\hat{\pi}_i}_{\mathrm{EC}_\varepsilon} E$. Let $\pi_i = \hat{\pi}_i[A_i]$ as in Lemma 1.16. We have $\vdash^{\pi_i}_{\mathrm{EC}_\varepsilon} A_i \to E$.

The ε-term e is not critical in π_i: Its original critical formulas are replaced by $A(t_i) \to (A(t'_j) \to A(t_i))$, which are tautologies. By (1) of the preceding Lemma, no critical ε-term of rank r was changed at all. By (2) of the preceding Lemma, no critical ε-term of rank $< r$ was replaced by a critical ε-term of rank $\geq r$. Hence, $o(\pi_i, r) = o(\pi) - 1$.

Let π'' be the sequence of tautologies $\neg \bigvee_{i=1}^{n} A(t_i) \to (A(t_i) \to A(e))$ followed by π. Then $\bigvee_{i=1}^{n} A(t_i) \vdash^{\pi''}_{E}$, e is not critical in π'', and otherwise π and π'' have the same critical formulas. The same goes for $\pi''[\neg \bigvee A(t_i)]$, a proof of $\neg \bigvee A(t_i) \to E$.

We now obtain π' as the π_i, $i = 1, \ldots, n$, followed by $\pi[\neg \bigvee_{i=1}^{n}]$, followed by the tautology

$$(\neg \bigvee A(t_i) \to E) \to (A(t_1) \to E) \to \cdots \to (A(t_n) \to E) \to E) \ldots)$$

from which E follows by $n+1$ applications of (MP). □

of the first ε-Theorem. By induction on $o(\pi, r)$, we have: if $\vdash^{\pi}_{\mathrm{EC}_\varepsilon} E$, then there is a proof π^* of E with $\mathrm{rk}(\pi^-) < r$. By induction on $\mathrm{rk}(()\pi)$ we have a proof π^{**} of E with $\mathrm{rk}(\pi^{**}) = 0$, i.e., without critical formulas at all. □

Exercise 12. Check these proofs. Can you think of ways to improve the proofs?

Exercise 13. If E contains ε-terms, the replacement of ε-terms in the construction of π_i may change E—but of course only the ε-terms appearing as subterms in it. Use this fact to prove: If $\vdash_{\mathrm{EC}_{\varepsilon\forall}} E(e)$, then $\vdash_{\mathrm{EC}} \bigvee_{i=1}^{m} E(t_j)$ for some terms t_j. Can you guarantee that t_j are ε-free.

NEW 3.2 The Case With Identity

In the presence of the identity (=) predicate in the language, things get a bit more complicated. The reason is that instances of the $(=_2)$ axiom schema,

$$t = u \to (A(t) \to A(u))$$

may also contain ε-terms, and the replacement of an ε-term e by a term t_i in the construction of π_i may result in a formula which no longer is an instance of $(=_2)$. For instance, suppose that t is a subterm of $e = e'(t)$ and $A(t)$ is of the form $A'(e'(t))$. Then the original axiom is

$$t = u \to (A'(e'(t)) \to A'(e'(u))$$

which after replacing $e = e'(t)$ by t_i turns into

$$t = u \to (A'(t_i) \to A'(e'(u)).$$

So this must be avoided. In order to do this, we first observe that just as in the case of the predicate calculus, the instances of $(=_2)$ can be derived from restricted instances. In the case of the predicate calculus, the restricted axioms are

$$t = u \to (P^n(s_1, \ldots, t, \ldots s_n) \to P^n(s_1, \ldots, u, \ldots, s_n) \qquad (=_2')$$
$$t = u \to f^n(s_1, \ldots, t, \ldots, s_n) = f^n(s_1, \ldots, u, \ldots, s_n) \qquad (=_2'')$$

to which we have to add the *ε-identity axiom schema:*

$$t = u \to \varepsilon_x A(x; s_1, \ldots, t, \ldots s_n) = \varepsilon_x A(x; s_1, \ldots, u, \ldots s_n) \qquad (=_\varepsilon)$$

where $\varepsilon_x A(x; x_1, \ldots, x_n)$ is an ε-type.

Proposition 3.6. *Every instance of $(=_2)$ can be derived from $(=_2')$, $(=_2'')$, and $(=_\varepsilon)$.*

Proof. Exercise. □

Now replacing every occurrence of e in an instance of $(=_2')$ or $(=_2'')$—where e obviously can only occur inside one of the terms t, u, s_1, …, s_n—results in a (different) instance of $(=_2')$ or $(=_2'')$. The same is true of $(=_\varepsilon)$, *provided* *that* the e is neither $\varepsilon_x A(x; s_1, \dots, t, \dots s_n)$ nor $\varepsilon_x A(x; s_1, \dots, u, \dots s_n)$. This would be guaranteed if the type of e is not $\varepsilon_x A(x; x_1, \dots, x_n)$, in particular, if the rank of e is higher than the rank of $\varepsilon_x A(x; x_1, \dots, x_n)$. Moreover, the result of replacing e by t_i in any such instance of $(=_\varepsilon)$ results in an instance of $(=_\varepsilon)$ which belongs to the same ε-type. Thus, in order for the proof of the first ε-theorem to work also when $=$ and axioms $(=_1)$, $(=_2')$, $(=_2'')$, and $(=_\varepsilon)$ are present, it suffices to show that the instances of $(=_\varepsilon)$ with ε-terms of rank $\mathrm{rk}(\pi)$ can be removed. Call an ε-term e *special* in π, if π contains an occurrence of $t = u \to e' = e$ as an instance of $(=_\varepsilon)$.

Theorem 3.7. *If* $\vdash^{\pi}_{\mathrm{EC}_\varepsilon} E$*, then there is a proof* $\pi^=$ *so that* $\vdash^{\pi^=}_{\mathrm{EC}_\varepsilon} E$*,* $\mathrm{rk}(\pi^=) = \mathrm{rk}(pi)$*, and the rank of the special* ε*-terms in* $\pi^=$ *has rank* $< \mathrm{rk}(\pi)$*.*

The basic idea is simple: Suppose $t = u \to e' = e$ is an instance of $(=_\varepsilon)$, with $e' \equiv \varepsilon_x A(x; s_1, \dots, t, \dots s_n)$ and $e \equiv \varepsilon_x A(x; s_1, \dots, u, \dots s_n)$. Replace e everywhere in the proof by e'. Then the instance of $(=_\varepsilon)$ under consideration is removed, since it is now provable from $e' = e'$. This potentially interferes with critical formulas belonging to e, but this can also be fixed: we just have to show that by a judicious choice of e it can be done in such a way that the other $(=_\varepsilon)$ axioms are still of the required form.

Let $p = \varepsilon_x A(x; x_1, \dots, x_n)$ be an ε-type of rank $\mathrm{rk}(\pi)$, and let $e_1, \dots, e_l$ be all the ε-terms of type p which have a corresponding instance of $(=_\varepsilon)$ in π. Let T_i be the set of all immediate subterms of $e_1, \dots, e_l$, in the same position as x_i, i.e., the smallest set of terms so that if $e_i \equiv \varepsilon_x A(x; t_1, \dots, t_n)$, then $t_i \in T$. Now let let T^* be all instances of p with terms from T_i substituted for the x_i. Obviously, T and thus T^* are finite (up to renaming of bound variables). Pick a strict order $\prec$ on T which respects degree, i.e., if $\deg(t) < \deg(u)$ then $t \prec u$. Extend $\prec$ to T^* by

$$\varepsilon_x A(x; t_1, \dots, t_n) \prec \varepsilon_x A(x; t'_1, \dots, t'_n)$$

iff

1. $\max\{\deg(t_i) : i = 1, \dots, n\} < \max\{\deg(t_i) : i = 1, \dots, n\}$ or
2. $\max\{\deg(t_i) : i = 1, \dots, n\} = \max\{\deg(t_i) : i = 1, \dots, n\}$ and
    a) $t_i \equiv t'_i$ for $i = 1, \dots, k$.
    b) $t_{k+1} \prec t'_{k+1}$

Lemma 3.8. *Suppose* $\vdash^{\pi}_{\mathrm{EC}_\varepsilon} E$*,* e *a special* ε*-term in* π *with* $\mathrm{rk}(e) = \mathrm{rk}(\pi)$*,* $\deg(e)$ *maximal among the special* ε*-terms of rank* $\mathrm{rk}(\pi)$*, and* e *maximal with respect to* $\prec$ *defined above. Let* $t = u \to e' = e$ *be an instance of* $(=_\varepsilon)$ *in* π*. Then there is a proof* π'*,* $\vdash^{\pi'}_{\mathrm{EC}_\varepsilon} E$ *such that*

1. $\mathrm{rk}(\pi') = \mathrm{rk}(\pi)$
2. π' *does not contain* $t = u \to e' = e$ *as an axiom*

3. Every special ε-term e'' of π' with the same type as e is so that $e'' \prec e$.

Proof. Let $\pi_0 = \pi\{e/e'\}$.

Suppose $t' = u' \to e''' = e''$ is an $(=_\varepsilon)$ axiom in π.

If $\mathrm{rk}(e'') < \mathrm{rk}(e)$, then the replacement of e by e' can only change subterms of e'' and e'''. In this case, the uniform replacement results in another instance of $(=_\varepsilon)$ with ε-terms of the same ε-type, and hence of the same rank $< \mathrm{rk}(\pi)$, as the original.

If $\mathrm{rk}(e'') = \mathrm{rk}(e)$ but has a different type than e, then this axiom is unchanged in π_0: Neither e'' nor e''' can be $\equiv e$, bcause they have different ε-types, and neither e'' nor e''' (nor t' or u', which are subterms of e'', e''') can contain e as a subterm, since then e wouldn't be degree-maximal among the special ε-terms of π of rank $\mathrm{rk}(\pi)$.

If the type of e'', e''' is the same as that of e, e cannot be a proper subterm of e'' or e''', since otherwise e'' or e''' would again be a special ε-term of rank $\mathrm{rk}(\pi)$ but of higher degree than e. So either $e \equiv e''$ or $e \equiv e'''$, without loss of generality suppose $e \equiv e''$. Then the $(=_\varepsilon)$ axiom in question has the form

$$t' = u' \to \underbrace{\varepsilon_x\, A(x; s_1, \ldots t', \ldots s_n)}_{e'''} = \underbrace{\varepsilon_x\, A(x; s_1, \ldots u', \ldots s_n)}_{e'' \equiv e}$$

and with e replaced by e':

$$t' = u' \to \underbrace{\varepsilon_x\, A(x; s_1, \ldots t', \ldots s_n)}_{e'''} = \underbrace{\varepsilon_x\, A(x; s_1, \ldots t, \ldots s_n)}_{e'}$$

which is no longer an instance of $(=_\varepsilon)$, but can be proved from new instances of $(=_\varepsilon)$. We have to distinguish two cases according to whether the indicated position of t and t' in e', e''' is the same or not. In the first case, $u \equiv u'$, and the new formula

$$t' = u \to \underbrace{\varepsilon_x\, A(x; s_1, \ldots t', \ldots s_n)}_{e'''} = \underbrace{\varepsilon_x\, A(x; s_1, \ldots t, \ldots s_n)}_{e'}$$

can be proved from $t = u$ together with

$$t' = t \to \underbrace{\varepsilon_x\, A(x; s_1, \ldots t', \ldots s_n)}_{e'''} = \underbrace{\varepsilon_x\, A(x; s_1, \ldots t, \ldots s_n)}_{e'} \qquad (=_\varepsilon)$$

$$t = u \to (t' = u \to t' = t) \qquad (=_2')$$

Since e' and e''' already occured in π, by assumption e', $e''' \prec e$.

In the second case, the original formulas read, with terms indicated:

$$t = u \to \underbrace{\varepsilon_x\, A(x; s_1, \ldots t, \ldots, u', \ldots, s_n)}_{e'} = \underbrace{\varepsilon_x\, A(x; s_1, \ldots u, \ldots, u', \ldots, s_n)}_{e}$$

$$t' = u' \to \underbrace{\varepsilon_x\, A(x; s_1, \ldots u, \ldots, t', \ldots, s_n)}_{e'''} = \underbrace{\varepsilon_x\, A(x; s_1, \ldots u, \ldots, u', \ldots, s_n)}_{e'' \equiv e}$$

and with e replaced by e' the latter becomes:

$$t' = u' \to \underbrace{\varepsilon_x A(x; s_1, \dots u, \dots, t', \dots s_n)}_{e'''} = \underbrace{\varepsilon_x A(x; s_1, \dots t, \dots, u', \dots, s_n)}_{e'}$$

This new formula is provable from $t = u$ together with

$$u = t \to \underbrace{\varepsilon_x A(x; s_1, \dots u, \dots, t', \dots s_n)}_{e'''} = \underbrace{\varepsilon_x A(x; s_1, \dots t, \dots, t', \dots, s_n)}_{e''''}$$
$$t' = u' \to \underbrace{\varepsilon_x A(x; s_1, \dots t, \dots, t', \dots s_n)}_{e''''} = \underbrace{\varepsilon_x A(x; s_1, \dots t, \dots, u', \dots, s_n)}_{e'}$$

and some instances of $(=_2')$. Hence, π' contains a (possibly new) special ε-term e''''. However, $e'''' \prec e$ (Exercise: prove this.)

In the special case where $e = e''$ and $e' = e'''$, i.e., the instance of $(=_\varepsilon)$ we started with, then replacing e by e' results in $t = u \to e' = e'$, which is provable from $e' = e'$, an instance of $(=_1)$.

Let π_1 be π_0 with the necessary new instances of $(=_\varepsilon)$, added. The instances of $(=_\varepsilon)$ in π_1 satisfy the properties required in the statement of the lemma.

However, the results of replacing e by e' may have impacted some of the critical formulas in the original proof. For a critical formula to which $e \equiv \varepsilon_x A(x, u)$ belongs is of the form

$$A(t', u) \to A(\varepsilon_x A(x, u), u) \tag{1}$$

which after replacing e by e' becomes

$$A(t'', u) \to A(\varepsilon_x A(x, t), u) \tag{2}$$

which is no longer a critical formula. This formula, however, can be derived from $t = u$ together with

$$A(t'', u) \to A(\varepsilon_x A(x, t), u) \qquad (\varepsilon)$$
$$t = u \to (A(\varepsilon_x A(x, t), t) \to A(\varepsilon_x A(x, t), u)) \qquad (=_2)$$
$$u = t \to (A(t'', u) \to A(t'', t)) \qquad (=_2)$$

Let π_2 be π_1 plus these derivations of (2) with the instances of $(=_2)$ themselves proved from $(=_2')$ and $(=_\varepsilon)$. The rank of the new critical formulas is the same, so the rank of π_2 is the same as that of π. The new instances of $(=_\varepsilon)$ required for the derivation of the last two formulas only contain ε-terms of lower rank that that of e. (Exercise: verify this.)

π_2 is thus a proof of E from $t = u$ which satisfies the conditions of the lemma. From it, we obtain a proof $\pi_2[t = u]$ of $t = u \to E$ by the deduction theorem. On the other hand, the instance $t = u \to e' = e$ under consideration can also be proved trivially from $t \neq u$. The proof $\pi[t \neq u]$ thus is also a proof, this time of $t \neq u \to E$, which satisfies the conditions of the lemma. We obtain π' by combining the two proofs. □

Proof. Proof of the Theorem By repeated application of the Lemma, every instance of $(=_\varepsilon)$ involving ε-terms of a given type p can be eliminated from the proof. The Theorem follows by induction on the number of different types of special ε-terms of rank $\mathrm{rk}(\pi)$ in π. □

Exercise 14. Prove Proposition 3.6.

Exercise 15. Verify that $\prec$ is a strict total order.

Exercise 16. Complete the proof of the Lemma.