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Dissertation

CERES in Higher-Order Logic

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Univ.Prof. Dr.phil. Alexander Leitsch
Institut für Computersprachen
Theoretische Informatik und Logik

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Technischen Universität Wien
Fakultät für Informatik

von

Dipl.-Ing.(FH) Daniel Weller
Matrikelnummer: 0627522
Schrekergasse 16/2
A-1160 Wien

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Daniel Weller

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Doctoral Thesis

CERES in Higher-Order Logic

carried out at the

Institute of Computer Languages
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under the supervision of

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Kurzfassung

Der Sequentkalkül, eingeführt von Gerhard Gentzen, ist ein wohlstudiertes Modell des mathematischen Beweises. Die Benutzung von Lemmata entspricht dem Einsatz der Schnittregel im Sequentkalkül. Schon Gentzen bewies dass, falls ein Theorem mittels der Schnittregel bewiesen werden kann, dies auch ohne ihre Verwendung möglich ist. Dieses metamathematische Resultat wird *Schnitteliminationssatz* genannt. Die ursprüngliche Motivation für dieses Resultat war es, die Konsistenz des Sequentkalküls zu zeigen: Es ist leicht zu sehen, dass ein Widerspruch nur mittels der Schnittregel hergeleitet werden kann. Wenn nun so ein Widerspruch existieren würde, hätte er — nach Gentzens Resultat — auch eine Herleitung ohne Schnittregel, was unmöglich ist. Die ursprüngliche Verwendung des Schnitteliminationssatzes betraf also einen hypothetischen Beweis.

Seitdem wurde die Formalisierung der Mathematik in Form von Beweisen in logischen Systemen durch den Computer ermöglicht. Es wurde also möglich (Implementationen von) Schnitteliminationssätze auf (Formalisierungen von) mathematische Beweise anzuwenden. Das Resultat ist dann ein neuer, schnittfreier Beweis, von dem interessante Information abgelesen werden kann. Informell gesprochen ist der Beweis direkt: er enthält keine Umwege in der Form von Lemmata.

Das Thema dieser Dissertation ist die Erweiterung der wohlbekannten Schnitteliminationsmethode CERES (*cut-elimination by resolution*) von der Logik erster Stufe auf die Logik höherer Stufe. Sowohl theoretisch als auch praktisch wurde der Wert der Methode (in der Logik erster Stufe) nachgewiesen. Aber die Logik erster Stufe hat Einschränkungen, die ihre Verwendung für die Formalisierung von Mathematik behindern: oft kann die intuitive Beschreibung mathematischer Objekte nicht direkt formalisiert werden, sondern die Objekte müssen kodiert werden.

In der Logik höherer Stufe gelten gewisse grundlegenden syntaktischen Eigenschaften von Beweisen, die für die Definition von CERES verwendet werden, nicht. Deshalb wird ein flexibler Sequentkalkül für die Verwendung mit CERES entwickelt und seine Eigenschaften untersucht. Ausserdem

wird ein Resolutionskalkül definiert und formal mit dem Standardresolutionskalkül von Peter B. Andrews verbunden. Auf diesen Systemen aufbauend wird die Schnitteliminationsmethode CERES$^{\omega}$ definiert. Sie wird mittels Übersetzungen, die direkt implementiert werden können, mit den Standardsystemen verbunden.

Abstract

A well-studied logical model of mathematical proofs is the sequent calculus introduced by Gerhard Gentzen. In the sequent calculus, the use of lemmas corresponds to the application of the cut rule. Already Gentzen proved that if a theorem can be proved using the cut rule, it can also be proved without using it. This metamathematical result is called *cut-elimination theorem*. The original motivation for obtaining it was to establish the consistency of the sequent calculus: It is easy to observe that a contradiction can only be derived by use of the cut rule. But if such a derivation of a contradiction would exist, by Gentzen's result there would also be a derivation of it without the use of cut, which is impossible. Hence the original application of the cut-elimination theorem was to a hypothetical proof.

Since then, the formalization of mathematics in the form of proofs in logical systems has been made possible by the computer. Hence it has become possible to apply (implementations of) cut-elimination theorems to (formalizations of) mathematical proofs. The result is a new, cut-free proof, from which interesting information can be extracted. Informally, the proof will be direct: it will not contain detours in the form of lemmas.

The subject of this thesis is to extend a well-known method of cut-elimination, CERES (*cut-elimination by resolution*), from first-order logic to higher-order logic. In first-order logic, both theoretical and practical results have established the value of the method. But first-order logic has some limitations which restrict its usefulness for the formalization of mathematics: often, the intuitive description of mathematical objects cannot be formalized in a straightforward way, but rather needs to be encoded.

In higher-order logic, some of the basic syntactical properties of proofs in first-order logic on which CERES depends fail to hold. Hence a more flexible sequent calculus, suitable for CERES, is defined in the thesis, and its properties investigated. Furthermore, a resolution calculus is defined and formally linked to the standard higher-order resolution calculus of Peter B. Andrews. Building on the defined systems, the cut-elimination by resolution method for higher-order logic, CERES$^\omega$, is defined.

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Contents

Chapter 1

Introduction

One of the main objectives of the working mathematician is to demonstrate the validity of sentences. Therefore, it is not surprising that there exists a subfield of mathematical logic, *proof theory*, devoted to the study of proofs as mathematical objects. An important aspect in the beginnings of proof theory was to establish the consistency of logical theories using purely "finitistic" methods. As proofs can be regarded as finite objects (i.e. finite strings of symbols), it seemed reasonable to use proof-theoretic methods to attain this goal. Unfortunately, by the seminal results of Gödel, it became clear that it is not possible to prove the consistency of interesting mathematical theories using *purely* finitistic means. Still, partial success has been obtained: For example, Gentzen in [25] and Gödel in [29] proved the consistency of arithmetic in a *rather* "finitistic" way, isolating the need for stronger assumptions (induction up to ϵ_0 in the former, normalization of functional terms in the latter case).

Towards this aim, Gentzen invented the *sequent calculus* in [23, 24]. This formalism has turned out to have a great variety of uses, and in particular the *cut-elimination theorem*, which by itself establishes the consistency of pure logic, has a variety of applications. One of those uses is in the relatively new field of *applied proof theory* or *proof mining*. The idea is to use proof-theoretic methods to extract information from formal proofs, in particular as an application to formalized proofs from "real-world" mathematics. Examples of proof mining include the extraction of algorithms, bounds, and new proofs from existing proofs. Some concrete studies can be found in [54, 53, 50, 39, 41, 4, 5, 6, 33].

There exist different methods to prove the cut-elimination theorem (and hence different ways of doing cut-elimination-based proof mining). The original idea by Gentzen can be regarded as a rewrite system on proofs that is applied according to a specific strategy, hence it has been called *reductive* *cut-elimination*. The point is to go from complex cuts to less complex ones, maybe creating new cuts in the process, but in such a way that the procedure must eventually terminate and produce a cut-free proof. Methods based on this rewrite system (by supplying a particular strategy) are called *reductive methods*.

An alternative method is *cut-elimination by resolution*, which was introduced in [9]. The technique is novel because it relies on the resolution method from automated-theorem proving (introduced in [51]). It was shown in [3, 9] that in first-order logic, CERES outperforms reductive methods in the sense that CERES simulates them up to an exponential, and that there exists a non-elementary speed-up of CERES over the reductive methods. Apart from this theoretical advantage, the application of CERES in practice differs essentially from applying reductive methods: The kernel of CERES is a certain set of clauses that is extracted from a proof with cuts. The main point in the cut-elimination process is the refutation of this set of clauses via resolution, which can be performed semi-automatically using a resolution theorem prover. Contrast this with the situation of applying reductive cut-elimination, where the process is guided solely by fixing a particular strategy of applying the reduction rules.

1.1 Moving CERES to higher-order logic

The CERES method was originally defined as a cut-elimination method for first-order logic. In this framework, it is not clear how to best handle induction (which is essential to many mathematical proofs): It is represented by an *axiom scheme*, and standard first-order logic does not have any object-level tools to handle schemes. In second-order logic, induction can be represented by a single axiom, which can be handled during proof search using standard tools like higher-order unification. In particular, the second-order theory of arithmetic ACA_0 (see [55]) is interesting because a good portion of mathematics is formalizable in it, but it is much weaker than (say) full second-order arithmetic.

The main objective of this thesis is therefore the extension of the CERES method to (suitable subsets of) higher-order logic. A first attempt is presented in Chapter 3, where a CERES method for the class of **QFC**-proofs (roughly second-order logic with quantifier-free comprehension) is presented. Here, one of the main tools used in the first-order CERES method can still be applied successfully: proof Skolemization. Still, the class of **QFC**-proofs can be considered too weak for our purposes: for one, it does not contain ACA_0.

Hence in Chapter 4, the CERES$^\omega$ method for full (intensional) higher-order logic is defined. There are several problems that need to be solved to this end, resulting in significant technical effort. To give the reader an overview, we will now explain which features are essential in the first-order CERES method, and how they make the extension to higher-order logic non-trivial.

1.2 Proof Skolemization

In first-order logic, the CERES method is restricted to work on Skolemized proofs, i.e. proofs of theorems which do not contain strong quantifiers (that is, $\forall$ in a positive or $\exists$ in a negative context). This is not problematic: It is well-known that for every formula F there exists such a formula F', its *Skolemization*, such that F and F' are validity-equivalent. Moreover, for every **LK**-proof π of a sequent S, there exists a proof ψ of the Skolemization of S whose size is polynomial in π (see [7, 8]). Reversing this transformation is possible too: Given a proof ψ of the Skolemization of S, there exists a proof π of S such that the size of π is bounded by an elementary function (of comparatively low complexity) in the size of ψ.

Skolemized proofs have the nice property that they do not contain strong quantifier inferences operating on end-sequent ancestors. The strong quantifier rules are the only rules of **LK** which impose restrictions on the context of the rule. Their absence gives more flexibility in defining proof transformations, because these restrictions then do not have to be taken into account.

This is exactly the property exploited by the CERES method: Roughly, so-called proof projections are constructed by leaving out inferences operating on ancestors of cuts. If there where strong quantifier inferences operating on end-sequent ancestors, their eigenvariable conditions may be violated and the extracted trees fail to be proofs. For example, consider the following proof:

$$
\dfrac{
\dfrac{\dfrac{P(\alpha) \vdash P(\alpha)}{(\forall x)P(x) \vdash P(\alpha)}\,\forall_l}{(\forall x)P(x) \vdash (\forall x)P(x)}\,\forall_r
\qquad
\dfrac{\dfrac{P(\beta) \vdash P(\beta)}{(\forall x)P(x) \vdash P(\beta)}\,\forall_l}{(\forall x)P(x) \vdash (\forall x)P(x)}\,\forall_r
}{(\forall x)P(x) \vdash (\forall x)P(x)}\,cut
$$

Its Skolemization is

$$\dfrac{\dfrac{\dfrac{P(\alpha) \vdash P(\alpha)}{(\forall x)P(x) \vdash P(\alpha)}\ \forall_l}{(\forall x)P(x) \vdash (\forall x)P(x)}\ \forall_r \qquad \dfrac{P(c) \vdash P(c)}{(\forall x)P(x) \vdash P(c)}\ \forall_l}{(\forall x)P(x) \vdash P(c)}\ cut$$

From this, CERES extracts the proof-projection

$$P(c) \vdash P(c)$$

by collecting the inferences operating on ancestors of $P(c)$ in the end-sequent (which are none except the axiom). On the other hand, CERES would extract the tree

$$\dfrac{P(\beta) \vdash P(\beta)}{P(\beta) \vdash (\forall x)P(x)}\ \forall_r$$

from the un-Skolemized proof, and this proof-projection is not a proof because the eigenvariable condition is violated.

In higher-order logic, the usual notion of a Skolemized proof (a proof of a Skolemized sequent) does not provide this crucial property. Consider the proof π:

$$\dfrac{\dfrac{\dfrac{\dfrac{P(\alpha) \vdash P(\alpha)}{(\forall x)P(x) \vdash P(\alpha)}\ \forall_l}{(\forall x)P(x) \vdash (\forall x)P(x)}\ \forall_r \qquad \dfrac{\dfrac{P(\beta) \vdash P(\beta)}{(\forall x)P(x) \vdash P(\beta)}\ \forall_l}{(\forall x)P(x) \vdash (\forall x)P(x)}\ \forall_r}{(\forall x)P(x) \vdash (\forall x)P(x)}\ cut}{(\forall x)P(x) \vdash (\exists Z)Z}\ \exists_r$$

π is a Skolemized proof in this sense, but contains strong quantifier inferences operating on end-sequent ancestors (here, the $\exists_r$ inference is a second-order quantifier inference introducing a 0-ary predicate variable).

An idea which is immediate is to also Skolemize the auxiliary formulas of higher-order quantifier inferences. Strictly syntactically speaking, this is not always possible — a characterization of when it is possible is given in Section 3.1.2. An intermediary solution is to restrict attention to proofs using only quantifier-free comprehension. As mentioned above, a CERES method for this class of proofs — the class of **QFC**-proofs — is defined in Section 3.2.2.

A first approach towards a complete solution, initiated by Stefan Hetzl in [31], was to analyse how the eigenvariable violations where introduced by CERES. As the above example indicates, such violations are introduced in the construction of the proof projections. In CERES, these projections are combined using atomic cuts to form a proof of the original end-sequent, and the occurrences causing the eigenvariable violations are such that they will be cut-ancestors in this proof. This lead to the idea of allowing eigenvariable violations only in cut-ancestors, and using a particular strategy of eliminating atomic cuts reductively to obtain a cut-free proof — which would be, by definition, without eigenvariable violations. Unfortunately, a counterexample (already in first-order logic) to this approach was found by the author of this thesis, showing that in the assembled proof with atomic cuts, eigenvariable violations caused by end-sequent ancestors can in fact occur because projections are combined using cut *and* substitution.

The failure of this approach led to the development of $\mathbf{LK}_{\mathrm{sk}}$: A cut-free sequent calculus introducing quantifiers from Skolem terms. The idea of introducing quantifiers from Skolem terms is not new; it has been used to define the Skolem expansion trees of [43, 44], in higher-order resolution [1, 35, 13], in the ENAR calculus [21], and in tableaux systems [16, 30]. In $\mathbf{LK}_{\mathrm{sk}}$, this approach is applied to the sequent calculus, and properties specific to sequent calculi are investigated:

1. The violation of eigenterm conditions,
2. two notions of regularity and
3. the permutability of inferences.

The development and analysis of this calculus is the content of Section 4.1. In particular, we show soundness by giving a concrete transformation from $\mathbf{LK}_{\mathrm{sk}}$ to $\mathbf{LK}$.

1.3 Resolution

The second aspect of CERES which exhibits crucial differences between first- and higher-order logic is the resolution calculus. In first-order logic, resolution essentially consists of substitution (via unification), contraction, and atomic cut, applied to clauses. In higher-order logic, the notion of clause is not closed under substitution, and hence "logical rules" have to be added to the calculus, which perform the transformation of arbitrary sequents to clauses.

In first-order logic, CERES exploits the fact that, when applying the global substitution σ of a resolution refutation γ of a set of clauses $\mathcal{C}$ to γ, one obtains an $\mathbf{LK}$-refutation of $\mathcal{C}\sigma$ which uses only atomic cut and contraction. The desired proof of the original end-sequent is then obtained by combining the proof-projections with this refutation.

In higher-order logic, the correspondence between the rules of the resolution calculus and the rules of **LK** is not so easy anymore. To be able to cope with the interplay between quantifier rules in $\mathbf{LK}_{\mathrm{sk}}$ and quantifier rules in the resolution calculus, we use labels in both $\mathbf{LK}_{\mathrm{sk}}$ and our resolution calculus $\mathcal{R}_{al}$. Labels such as these are often used to add (syntactic) information to formulas (see [22]). They have been used in a setting very similar to ours in [21].

1.4 Cut-elimination by resolution

Using this machinery, we are able to define the CERES method for higher-order logic in Section 4.3: Starting with an **LK**-proof π, we transform π into an $\mathbf{LK}_{\mathrm{skc}}$-proof ψ ($\mathbf{LK}_{\mathrm{skc}}$ is obtained by combining the rules of **LK** and $\mathbf{LK}_{\mathrm{sk}}$). Next, we extract a set $\mathcal{P}(\psi)$ of so-called CERES-*projections* (which are $\mathbf{LK}_{\mathrm{sk}}$-trees) and a set of labelled sequents $\mathrm{CS}(\psi)$, the *characteristic sequent set*. We modify the $\mathbf{LK}_{\mathrm{sk}}$-trees from $\mathcal{P}(\psi)$ according to a $\mathcal{R}_{al}$-refutation of $\mathrm{CS}(\pi)$, resulting in an $\mathbf{LK}_{\mathrm{sk}}$-proof of the original end-sequent. By application of the results from the previous sections, we obtain a cut-free **LK**-proof of the original end-sequent, and have thereby reproved cut-elimination in higher-order logic, not by reduction, but by resolution.

1.5 Methods of cut-elimination

Apart from cut-elimination by resolution and the reductive methods mentioned above, there still exist other ways to prove cut-elimination. The following methods are similar to the aforementioned reductive methods: the core of the arguments is to show normalization (strong or weak) of a rewrite system. In the context of intutionistic logic, cut-elimination (or rather, normalization of proof terms) has been proved using a notion of *computability predicate* [57, 42]. Using a similar technique due to [26], in the context of classical second-order logic cut-elimination has been proved in [47] (via normalization of terms in the $\lambda\mu$-calculus).

A different approach is to use semantics to establish cut-free completeness, and thereby cut-elimination. Again in intuitionistic logic, cut-elimination can be shown via the definition of *algebraic semantics* as in [14, 2]. Another approach via non-standard semantics is the use of *phase semantics* due to [27]. In [46] this approach is applied to a range of systems. Last but not least, cut-elimination in classical higher-order logic (also known as *Takeuti's conjecture*) was originally proved independently in [58] and [49], the latter being based on [52].

We can now state informally how cut-elimination by resolution is related to these approaches: the problem of finding a cut-free proof of S is reduced to finding a resolution refutation of $\mathrm{CS}(\pi)$, where $\mathrm{CS}(\pi)$ is constructed from a proof π of S *which contains cuts*. Hence $\mathrm{CS}(\pi)$ contains information from π, and refuting $\mathrm{CS}(\pi)$ is in general easier than proving S from scratch in a cut-free way. Completeness for a resolution system is usually proved using semantic methods. Hence cut-elimination by resolution can be seen as a strengthening of the latter constructions, allowing information from cuts to be used in cut-free proof search.

Chapter 2

Preliminaries

2.1 Types, language and Skolem terms

We now introduce the higher-order language we will use. It is a version of Church's simple theory of types [18]. We define the set of base types $BT = \{\iota, o\}$ where the intended interpretation of ι, o is the type of individuals and booleans, respectively.

Definition 2.1.1 (Types)**.** We define the set $\mathcal{T}$ of *types* along with their *order* o inductively

1. $BT \subseteq \mathcal{T}$. For all $t \in BT$, $\mathrm{o}(t) = 1$.

2. If $t_1, t_2 \in \mathcal{T}$ then $t = t_1 \to t_2 \in \mathcal{T}$, and $\mathrm{o}(t) = \max(\mathrm{o}(t_1), \mathrm{o}(t_2) + 1)$.

Notationally $\to$ associates to the right (i.e. $t_1 \to t_2 \to t_3$ is $t_1 \to (t_2 \to t_3)$). We assume given, for each type α, denumerable pairwise disjoint sets of variable symbols $\mathcal{V}_\alpha$ and constant symbols $\mathcal{C}_\alpha$, and a function τ such that for all $x \in \mathcal{V}_\alpha$ and $c \in \mathcal{C}_\alpha$: $\tau(x) = \tau(c) = \alpha$. We assume that the constant symbols $\vee$, $\neg$, $\forall_\alpha$ (the *logical constants*) with $\tau(\vee) = o \to o \to o$, $\tau(\neg) = o \to o$ and $\tau(\forall_\alpha) = (\alpha \to o) \to o$ for all types α are included in the given symbols. Hence our language can be considered as a higher-order logic, as universal and existential quantification over all finite types can be expressed, as well as all propositional connectives (as will be done below). We will drop the subscript α in $\forall_\alpha$ if the type is clear from the context. A quantifier $\forall$ occurring positively is called a *strong* quantifier, one occurring negatively is called a *weak* quantifier.

Definition 2.1.2 (Expressions)**.** We define the set $\mathcal{E}$ of *expressions* inductively and extend τ to expressions:

1. Constant symbols and variable symbols s are expressions: $s \in \mathcal{E}$.

2. If $f \in \mathcal{E}$ and $\tau(f) = \alpha_1 \to \alpha_2$ and $\tau(t) = \alpha_1$, then ft is an expression of type α_2.

3. If x is a variable symbol, $\tau(x) = t_1$, and $e \in \mathcal{E}$, $\tau(e) = t_2$, then $\lambda x.e$ is an expression of type $t_1 \to t_2$.

If $e \in \mathcal{E}$, we set $o(e) = o(\tau(e))$.

Notationally, application associates to the left (i.e. ft_1t_2 is $(ft_1)t_2$). Expressions of type o are called *formulas*. If the uppermost symbol of a formula $\mathbf{F}$ is not one of $\forall_\alpha$, $\vee$ or $\neg$, then $\mathbf{F}$ is called *atomic*. We will henceforth refer to variable and constant symbols as variables and constants, respectively. As metavariables for expressions we use $\mathbf{T}, \mathbf{S}, \mathbf{R}, \ldots$, for variables we use $\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \ldots$, for formulas we use $\mathbf{F}, \mathbf{G}, \mathbf{H}, \ldots$, for sequences of formulas we use $\Gamma, \Pi, \Lambda, \ldots$ (possibly with subscripts).

For convenience, we write $\mathbf{F} \vee \mathbf{G}$ for $\vee \mathbf{F}\mathbf{G}$ and define some logical operators by settings

$$\begin{aligned}
(\forall \mathbf{X})\mathbf{F} &= \forall_\alpha(\lambda \mathbf{X}.\mathbf{F}) \\
(\exists \mathbf{X})\mathbf{F} &= \neg(\forall \mathbf{X})\neg \mathbf{F} \\
\mathbf{F} \wedge \mathbf{G} &= \neg((\neg \mathbf{F}) \vee (\neg \mathbf{G})) \\
\mathbf{F} \to \mathbf{G} &= (\neg \mathbf{F}) \vee \mathbf{G}
\end{aligned}$$

for all types α. We will drop parentheses inside expressions when they are redundant with respect to the usual strongness-of-binding order on the logical connectives.

Definition 2.1.3 (Subexpression relation)**.** Define the relation $<_1 \subset \mathcal{E} \times \mathcal{E}$ by induction on $\mathbf{E} \in \mathcal{E}$:

1. If $\mathbf{E} = \mathbf{fT}$, then $\mathbf{T} <_1 \mathbf{E}$ and $\mathbf{f} <_1 \mathbf{E}$.

2. If $\mathbf{E} = \lambda \mathbf{X}.\mathbf{F}$, then $\mathbf{F} <_1 \mathbf{E}$.

Define $\leq$ as the reflexive and transitive closure of $<_1$, and define $<$ as the transitive closure of $<_1$.

Proposition 2.1.4. *The subexpression relation $\leq$ is a partial order.*

Proof. $\leq$ is transitive and reflexive by definition. It is antisymmetric: Assume $\mathbf{E} \leq \mathbf{F}$, $\mathbf{F} \leq \mathbf{E}$ and $\mathbf{F} \neq \mathbf{E}$. Then there exist $\mathbf{E}', \mathbf{F}'$ such that $\mathbf{E} <_1 \mathbf{E}' \leq \mathbf{F}$ and $\mathbf{F} <_1 \mathbf{F}' \leq \mathbf{E}$ and therefore $\mathbf{E} < \mathbf{F} < \mathbf{E}$, but $<$ is irreflexive! □

If $\mathbf{E} \leq \mathbf{F}$ we say that $\mathbf{F}$ *contains* $\mathbf{E}$, if $\mathbf{E} < \mathbf{F}$ we say that $\mathbf{F}$ *properly contains* $\mathbf{E}$.

Definition 2.1.5 (Free variables)**.** Let $\mathbf{T} \in \mathcal{E}$. We define the set of *free variables* $\mathrm{FV}(\mathbf{T})$ by induction on $\mathbf{T}$:

1. $\mathbf{T} = \mathbf{X}$. Then $\mathrm{FV}(\mathbf{T}) = \{\mathbf{X}\}$.
2. $\mathbf{T} = \mathbf{C}$ a constant. Then $\mathrm{FV}(\mathbf{T}) = \emptyset$.
3. $\mathbf{T} = \mathbf{RS}$. Then $\mathrm{FV}(\mathbf{T}) = \mathrm{FV}(\mathbf{R}) \cup \mathrm{FV}(\mathbf{S})$.
4. $\mathbf{T} = \lambda\mathbf{X}.\mathbf{R}$. Then $\mathrm{FV}(\mathbf{T}) = \mathrm{FV}(\mathbf{R}) \setminus \{\mathbf{X}\}$.

In the following, if we do not explicitly state otherwise, equality between expressions is taken to be α*-equality*, i.e. equality modulo renaming of bound variables. Another important basic notion is the notion of replacement. Due to our convention of regarding expressions modulo α-equality, we assume in the following definition that no bound variable of $\mathbf{R}$ is an element of $\mathrm{FV}(\mathbf{S}) \cup \mathrm{FV}(\mathbf{T})$.

Definition 2.1.6 (Replacement)**.** If $\mathbf{S}, \mathbf{T} \in \mathcal{E}$ and $\tau(\mathbf{S}) = \tau(\mathbf{T})$, then $\sigma = [\mathbf{S} \leftarrow \mathbf{T}]$ is a *replacement*. If $\mathbf{S}$ is a variable, then σ is a called a *substitution*. Let $\mathbf{R} \in \mathcal{E}$, and $\sigma = [\mathbf{S} \leftarrow \mathbf{T}]$ be a replacement. Then define $\mathbf{R}\sigma$ inductively:

1. If $\mathbf{R} = \mathbf{S}$ then $\mathbf{R}\sigma = \mathbf{T}$.
2. If $\mathbf{R} \neq \mathbf{S}$ is a constant or variable then $\mathbf{R}\sigma = \mathbf{R}$.
3. If $\mathbf{R} = \mathbf{UV}$ then $\mathbf{R}\sigma = \mathbf{U}\sigma\mathbf{V}\sigma$.
4. If $\mathbf{R} = (\lambda\mathbf{X}.\mathbf{G})$ then $\mathbf{R}\sigma = (\lambda\mathbf{X}.\mathbf{G}\sigma)$.

As usual, we define a notion of reduction on expressions.

Definition 2.1.7 (β-reduction)**.** Let $\mathbf{T} = (\lambda\mathbf{X}.\mathbf{R})\mathbf{S}$ be an expression such that $\mathbf{S}$ is free for $\mathbf{R}$. Then $\mathbf{T} \to_\beta \mathbf{R}\,[\mathbf{X} \leftarrow \mathbf{S}]$. The relation $\twoheadrightarrow_\beta$ is the transitive and reflexive closure of the compatible closure of the relation of β-reduction. The relation $=_\beta$ is the symmetric closure of $\twoheadrightarrow_\beta$.

It is well known that β-reduction in the simple theory of types is strongly normalizing and confluent:

Proposition 2.1.8. *Let* $\mathbf{E}$ *be an expression. Then there exists a unique expression* $\overline{\mathbf{E}}$ *such that* $\mathbf{E} \twoheadrightarrow_\beta \overline{\mathbf{E}}$ *and* $\overline{\mathbf{E}}$ *is irreducible. Furthermore, there does not exist an infinite* $\twoheadrightarrow_\beta$*-chain starting at* $\mathbf{E}$.

If $\mathbf{E}$ is an expression, $\overline{\mathbf{E}}$ is called *the β-normal form* of $\mathbf{E}$. Our expressions will contain Skolem terms. To obtain sound proof systems, we will need to restrict the expressions that can be used: we follow the approach of Miller [44]:

Definition 2.1.9 (Skolem symbols)**.** Let $\alpha, \beta_1, \ldots, \beta_n$ be types. Then the list $\sigma = \beta_1, \ldots, \beta_n, \alpha$ is called a *signature* (for a Skolem symbol). For each signature σ, let $\sigma^T = \beta_1 \to \ldots \to \beta_n \to \alpha$, and let $\mathcal{K}_\sigma \subseteq \mathcal{C}_{\sigma^T}$ be a denumerable set of constant symbols of type σ^T such that if σ_1 and σ_2 are different signatures then $\mathcal{K}_{\sigma_1}$ and $\mathcal{K}_{\sigma_2}$ are disjoint. $\mathbf{C} \in \mathcal{K}_\sigma$ is called a *Skolem symbol of signature σ with arity n*. Then define the *Herbrand Universe* as the set of all $\mathbf{T} \in \mathcal{E}$ such that whenever a Skolem symbol of arity n has an occurrence in $\mathbf{T}$, it is applied to at least n arguments. Furthermore, if a variable has a free occurrence in any of these arguments, that occurrence is also free in $\mathbf{T}$.

From now on, by $\mathcal{E}$ we denote the Herbrand Universe (i.e. we only consider expressions contained in the Herbrand Universe).

2.2 The sequent calculus LK

The starting point of our investigations will be the following formulation of a sequent calculus **LK**. A *sequent* is a pair of lists of formulas, written $\Gamma \vdash \Delta$. While we define sequents as lists to be able to define occurrences in sequents and proofs, we will treat them as multisets most of the time. Hence we do not explicitly include exchange or permutation rules in our calculi. For simplicity, we restrict ourselves to prooftrees in which all formulas are in β-normal form. Hence we note that the quantifier rules below include an implicit β-reduction.

Definition 2.2.1 (**LK** rules and proofs)**.** The following figures are the rules of inference of **LK**:

Propositional rules:

$$\frac{\Gamma \vdash \Delta, \mathbf{F}}{\neg \mathbf{F}, \Gamma \vdash \Delta}\ \neg\colon l \qquad \frac{\mathbf{F}, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg \mathbf{F}}\ \neg\colon r \qquad \frac{\mathbf{F}, \Gamma \vdash \Delta \quad \mathbf{G}, \Pi \vdash \Lambda}{\mathbf{F} \vee \mathbf{G}, \Gamma, \Pi \vdash \Delta, \Lambda}\ \vee\colon l$$

$$\frac{\Gamma \vdash \Delta, \mathbf{F}}{\Gamma \vdash \Delta, \mathbf{F} \vee \mathbf{G}}\ \vee\colon r^1 \qquad \frac{\Gamma \vdash \Delta, \mathbf{G}}{\Gamma \vdash \Delta, \mathbf{F} \vee \mathbf{G}}\ \vee\colon r^2$$

Structural rules:

$$\frac{\Gamma \vdash \Delta, \mathbf{F}, \mathbf{F}}{\Gamma \vdash \Delta, \mathbf{F}}\ \text{contr}\colon r \qquad \frac{\mathbf{F}, \mathbf{F}, \Gamma \vdash \Delta}{\mathbf{F}, \Gamma \vdash \Delta}\ \text{contr}\colon l$$

$$\frac{\Gamma \vdash \Delta}{\mathbf{F}, \Gamma \vdash \Delta}\ \text{weak:}\, l \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \mathbf{F}}\ \text{weak:}\, r \qquad \frac{\Gamma \vdash \Delta, \mathbf{F} \quad \mathbf{F}, \Pi \vdash \Lambda}{\Gamma, \Pi \vdash \Delta, \Lambda}\ \text{cut}$$

Quantifier rules:

$$\frac{\mathbf{RT}, \Gamma \vdash \Delta}{\forall \mathbf{R}, \Gamma \vdash \Delta}\ \forall\colon l \qquad \frac{\Gamma \vdash \Delta, \mathbf{RX}}{\Gamma \vdash \Delta, \forall \mathbf{R}}\ \forall\colon r$$

In $\forall\colon r$, $\mathbf{X}$ must not occur free in $\Gamma, \Delta, \mathbf{R}$. $\mathbf{X}$ is called the *eigenvariable* of this rule. In $\forall\colon l$, $\mathbf{T}$ is called the *substitution term* of the rule.

An **LK**-*proof* is a tree formed according to the rules of **LK** such that all leafs are of the form $\mathbf{F} \vdash \mathbf{F}$. The formulas in $\Gamma, \Delta, \Pi, \Lambda$ are called *context formulas*. The formulas in the upper sequents which are not context formulas are called *auxiliary formulas*, those in the lower sequents are called *main formulas*. If π is an **LK**-proof, by $|\pi|$ we denote the number of sequent occurrences in π.

If $\mathcal{S}$ is a set of sequents, then an **LK**-*refutation of* $\mathcal{S}$ is an **LK**-tree π where the end-sequent of π is the empty sequent, and the leaves of π are either axioms $\mathbf{F} \vdash \mathbf{F}$ or sequents in $\mathcal{S}$.

A formula *occurrence* in a sequent or prooftree is a formula together with its position in the sequent or prooftree. Formula occurrences in prooftrees come equipped with an *ancestor* and *descendant* relation which is defined in the usual way. An inference ρ in a prooftree is said to *operate* on an occurrence ω if ω is the auxiliary or main formula of ρ. An **LK**-proof π is called *regular* if for all $\forall\colon r$ inferences ρ with eigenvariable $\mathbf{X}$ in π, $\mathbf{X}$ only occurs in the subproof ending in ρ. It is well-known that every **LK**-proof of a closed sequent S can be transformed into a regular **LK**-proof of S by renaming eigenvariables.

Example 2.2.2. We prove the first-order theorem

$$(\exists y)(\forall x)P(x, y) \vdash (\forall x)(\exists y)P(x, y),$$

where $\tau(y) = \tau(x) = \iota$ and $\tau(P) = \iota \to \iota \to o$. Expanding the definitions, this is the sequent

$$\neg\forall_\iota \lambda y.\neg\forall_\iota \lambda x.P(x, y) \vdash \forall_\iota \lambda x.\neg\forall_\iota \lambda y.\neg P(x, y)$$

The following is a cut-free **LK** proof of this theorem:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{P(u,v) \vdash P(u,v)}{\neg P(u,v), P(u,v) \vdash}\neg\colon l}{\forall\lambda y.\neg P(u,y), P(u,v) \vdash}\forall\colon l}{P(u,v) \vdash \neg\forall\lambda y.\neg P(u,y)}\neg\colon r}{\forall\lambda x.P(x,v) \vdash \neg\forall\lambda y.\neg P(u,y)}\forall\colon l}{\vdash \neg\forall\lambda y.\neg P(u,y), \neg\forall\lambda x.P(x,v)}\neg\colon r}{\vdash \neg\forall\lambda y.\neg P(u,y), \forall\lambda y.\neg\forall\lambda x.P(x,y)}\forall\colon r}{\neg\forall\lambda y.\neg\forall\lambda x.P(x,y) \vdash \neg\forall\lambda y.\neg P(u,y)}\neg\colon l}{\neg\forall\lambda y.\neg\forall\lambda x.P(x,y) \vdash \forall\lambda x.\neg\forall\lambda y.\neg P(x,y)}\forall\colon r$$

Note that, as usual, one can define the rules $\exists\colon l$, $\exists\colon r$, $\rightarrow\colon r$, $\rightarrow\colon l$, $\wedge\colon r$ and $\wedge\colon l$ from the rules of **LK**. For example, the rule

$$\cfrac{\Gamma \vdash \Delta, \mathbf{F}\,[\mathbf{X} \leftarrow \mathbf{T}]}{\Gamma \vdash \Delta, (\exists \mathbf{X})\mathbf{F}}\;\exists\colon r$$

can be defined as

$$\cfrac{\cfrac{\cfrac{\Gamma \vdash \Delta, \mathbf{F}\,[\mathbf{X} \leftarrow \mathbf{T}]}{\neg\mathbf{F}\,[\mathbf{X} \leftarrow \mathbf{T}]\,, \Gamma \vdash \Delta}\neg\colon l}{(\forall \mathbf{X})\neg\mathbf{F}, \Gamma \vdash \Delta}\forall\colon l}{\Gamma \vdash \Delta, \neg(\forall \mathbf{X})\neg\mathbf{F}}\neg\colon r$$

Recall the system $\mathcal{T}$ introduced in [18] and used in [1]. $\mathcal{T}$ is a Hilbert-type system for higher-order logic. Using the well-known transformations from sequent calculi to Hilbert-type systems (see [24, 60]), we can prove a relative soundness result. If $S = \Gamma \vdash \Delta$ is a sequent, then $F(S) = \bigvee \neg\Gamma \bigvee \Delta$. If $\mathcal{S}$ is a set of sequents, then $F(\mathcal{S}) = \{F(S) \mid S \in \mathcal{S}\}$.

Proposition 2.2.3. *If there exists an* **LK***-refutation of* $\mathcal{S}$*, then there exists a* $\mathcal{T}$*-refutation of* $F(\mathcal{S})$*.*

Finally we introduce the following useful sequent-merge notation:

Definition 2.2.4. Let $S_1 = \Gamma_1 \vdash \Delta_1$, $S_2 = \Gamma_2 \vdash \Delta_2$. Then we define $S_1 \circ S_2 = \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2$. Let $\mathcal{C} = \{C_1, \ldots, C_m\}$, $\mathcal{D} = \{D_1, \ldots, D_n\}$ be sets of sequents, then $\mathcal{C} \times \mathcal{D} = \{C_i \circ D_j \mid i \leq m, j \leq n\}$.

2.3 The higher-order resolution calculus $\mathcal{R}$

We follow Benzmüller's presentation in [12] of Andrews' resolution calculus $\mathcal{R}$ originally introduced in [1]. We consider Andrews' calculus instead of more recent calculi such as [35, 13, 11, 40, 48, 38] since Andrews' calculus can be regarded as the most simple formulation of a resolution calculus for higher-order logic. Note that in particular, we do not consider unification (see [36, 56, 20, 37]) but that in principle, calculi which use more advanced features during proof search can be used provided that their deductions can be transformed into deductions in our format.

As in the previous Section, we assume that all formulas occuring in $\mathcal{R}$-deductions are in β-normal form, so the substitution and quantifier rules will include an implicit β-reduction step. In contrast to Benzmüller, we present clauses as atomic sequents (and pre-clauses as sequents).

Definition 2.3.1 ($\mathcal{R}$ rules and deductions)**.** The rules of $\mathcal{R}$ are:

$$\frac{\Gamma \vdash \Delta, \neg \mathbf{A}}{\mathbf{A}, \Gamma \vdash \Delta}\ \neg^T \qquad \frac{\neg \mathbf{A}, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \mathbf{A}}\ \neg^F \qquad \frac{\Gamma \vdash \Delta, \mathbf{A} \vee \mathbf{B}}{\Gamma \vdash \Delta, \mathbf{A}, \mathbf{B}}\ \vee^T$$

$$\frac{\mathbf{A} \vee \mathbf{B}, \Gamma \vdash \Delta}{\mathbf{A}, \Gamma \vdash \Delta}\ \vee^F_l \qquad \frac{\mathbf{A} \vee \mathbf{B}, \Gamma \vdash \Delta}{\mathbf{B}, \Gamma \vdash \Delta}\ \vee^F_r$$

$$\frac{\Gamma \vdash \Delta, \forall \mathbf{A}}{\Gamma \vdash \Delta, \mathbf{A}\mathbf{X}}\ \forall^T \qquad \frac{\forall \mathbf{A}, \Gamma \vdash \Delta}{\mathbf{A}(\mathbf{f}\mathbf{X}_1 \dots \mathbf{X}_n), \Gamma \vdash \Delta}\ \forall^F \qquad \frac{S}{S\,[\mathbf{X} \leftarrow \mathbf{T}]}\ \text{Sub}$$

$$\frac{\mathbf{A}, \mathbf{A}, \Gamma \vdash \Delta}{\mathbf{A}, \Gamma \vdash \Delta}\ \text{Sim}^F \qquad \frac{\Gamma \vdash \Delta, \mathbf{A}, \mathbf{A}}{\Gamma \vdash \Delta, \mathbf{A}}\ \text{Sim}^T \qquad \frac{\Gamma \vdash \Delta, \mathbf{A} \quad \mathbf{A}, \Pi \vdash \Lambda}{\Gamma, \Pi \vdash \Delta, \Lambda}\ \text{Cut}$$

where in $\forall^F$, $\mathbf{X}_1, \dots, \mathbf{X}_n$ are all the free variables occuring in $\mathbf{A}$, and if $\tau(\mathbf{X}_i) = t_i$ for $1 \leq i \leq n$ and $\tau(\mathbf{A}) = t \to o$, then $\mathbf{f} \in \mathcal{K}_{t_1,\dots,t_n,t}$.

Let $\mathcal{C}$ be a set of sequents. Then a sequence of sequents $S_1, \dots, S_m$ is an *$\mathcal{R}$-deduction of S_m from $\mathcal{C}$* if for all $1 \leq i \leq m$, either

1. $S_i \in \mathcal{C}$, or
2. S_i is derived from S_j (and S_k) by a rule of $\mathcal{R}$, where $j, k < i$. If S_i is derived by $\forall^F$, then the Skolem symbol introduced by the inference must not occur in any S_ℓ with $\ell < i$, and must not occur in $\mathcal{C}$. If S_i is derived by $\forall^T$, then the same must hold for the variable $\mathbf{X}$ that is introduced.

An $\mathcal{R}$-deduction of the empty sequent from $\mathcal{C}$ is called an *$\mathcal{R}$-refutation* of $\mathcal{C}$.

Chapter 3

CERES for QFC-proofs

3.1 Towards CERES in higher-order logic

We intend to extend the first-order cut-elimination method CERES introduced in [9] to higher-order logic. As we have noted in Section 1.2, the method of proof Skolemization is a crucial aspect in the definition of the CERES method: it allows to reduce the problem of cut-elimination on arbitrary **LK**-proofs to the problem of cut-elimination on **LK**-proofs of sequents without strong quantifiers.

In first-order logic, cut-free proofs of sequents without strong quantifiers do not contain strong quantifier inferences, and hence there are no eigenvariable conditions to observe. It is exactly this aspect which is exploited in the definition of the proof projections in the CERES method: from a proof with cuts, cut-free proofs are extracted by "leaving out" inferences — in the general setting, such a construction would be unsound due to eigenvariable violations.

Unfortunatly, in higher-order logic cut-free proofs of sequents without strong quantifiers may still contain strong quantifier inferences due to comprehension.

Example 3.1.1. The following **LK**-proof proves a sequent that does not contain strong quantifiers, but the proof contains a strong quantifier inference:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(\beta,a) \vdash P(\beta,a)}{(\forall x)P(x,a) \vdash P(\beta,a)}\ \forall\colon l}{(\forall x)P(x,a) \vdash (\forall z)P(z,a)}\ \forall\colon r \quad \dfrac{P(c,b) \vdash P(c,b)}{(\forall z)P(z,b) \vdash P(c,b)}\ \forall\colon l}{(\forall x)P(x,a), (\forall z)P(z,a) \to (\forall z)P(z,b) \vdash P(c,b)}\ \to\colon l}{(\forall x)P(x,a), (\forall X)(X(a) \to X(b)) \vdash P(c,b)}\ \forall\colon l\ \lambda x.(\forall z)P(z,x)}{(\forall X)(X(a) \to X(b)) \vdash (\forall x)P(x,a) \to P(c,b)}\ \to\colon r
$$

3.1.1 Structural Skolemization in higher-order logic

A first attempt to solve this problem would be to Skolemize the auxiliary formulas of the weak quantifier inferences. First, we introduce a useful normal form for expressions and proofs, which makes quantified variables explicit. This normal form will be useful for defining Skolemization of formulas later on.

Definition 3.1.2. An expression $\mathbf{E}$ is in *$q\eta$*-normal form if all its subexpressions of the form $\forall \mathbf{G}$ are such that $\mathbf{G} = \lambda \mathbf{X}.\mathbf{F}$ for some $\mathbf{X}, \mathbf{F}$. We say that a sequent is in *$q\eta$*-normal form if all its formulas are in *$q\eta$*-normal form, and an **LK**-proof π of S is *$q\eta$*-normal if

1. all formulas in S are in $q\eta$-normal form and
2. for all $\forall\colon l$ applications in π, their substitution terms are in $q\eta$-normal form.

It is easy to see that every expression can be converted into $q\eta$-normal form by some η-expansion steps and that furthermore, every **LK**-proof of a sequent S can be converted into an **LK**-proof π of a $q\eta$-normal form of S thus obtained such that π is in $q\eta$-normal form. Hence for the rest of this chapter, we will only treat $q\eta$-normal proofs.

Example 3.1.3. The proof

$$
\dfrac{\forall P \vdash \forall P}{(\forall X)X \vdash \forall P}\ \forall\colon l\ \forall P
$$

is not in $q\eta$-normal form, but the proof

$$
\dfrac{(\forall x)P(x) \vdash (\forall x)P(x)}{(\forall X)X \vdash (\forall x)P(x)}\ \forall\colon l\ (\forall x)P(x)
$$

is.

Proposition 3.1.4. *If π is a $q\eta$-normal* **LK***-proof, then all formula occurrences in π are in $q\eta$-normal form*

Proof. By induction on the length of π. $\square$

Now we are ready to extend the definition of structural Skolemization to the higher-order setting:

Definition 3.1.5. We define the *Skolemization operator* sk. Let $\mathbf{F}$ be a closed formula in $q\eta$-normal form containing strong quantifiers. Let $\forall_\alpha \mathbf{G}$ be a subformula of $\mathbf{F}$ occurring positively. Let $\forall_{\beta_1}(\lambda \mathbf{X}_1.\mathbf{H}_1), \ldots, \forall_{\beta_n}(\lambda \mathbf{X}_n.\mathbf{H}_n)$ be the weak quantifier occurrences dominating $\forall_\alpha$. Then let $\mathrm{sk}_0(\mathbf{F})$ be $\mathbf{F}$ where the subformula corresponding to $\forall_\alpha \mathbf{G}$ is replaced by $\overline{\mathbf{G}(\mathbf{f}\mathbf{X}_1 \ldots \mathbf{X}_n)}$, where $\mathbf{f} \in \mathcal{K}_{\beta_1,\ldots,\beta_n,\alpha}$ is a new Skolem symbol. Then we define

$$\mathrm{sk}(\mathbf{F}) = \begin{cases} \mathbf{F} & \text{if } \mathbf{F} \text{ does not contain strong quantifiers} \\ \mathrm{sk}(\mathrm{sk}_0(\mathbf{F})) & \text{otherwise} \end{cases}$$

Let S be a closed sequent $\mathbf{F}_1, \ldots, \mathbf{F}_n \vdash \mathbf{G}_1, \ldots, \mathbf{G}_m$, and let $\mathbf{F} = (\mathbf{F}_1 \wedge \ldots \wedge \mathbf{F}_n) \to (\mathbf{G}_1 \vee \ldots \vee \mathbf{G}_m)$. If $\mathrm{sk}(\mathbf{F}) = (\mathbf{F}'_1 \wedge \ldots \wedge \mathbf{F}'_n) \to (\mathbf{G}'_1 \vee \ldots \vee \mathbf{G}'_m)$, then $\mathrm{sk}(S) = \mathbf{F}'_1, \ldots, \mathbf{F}'_n \vdash \mathbf{G}'_1, \ldots, \mathbf{G}'_m$.

Example 3.1.6. Let

$$\mathbf{F} = \neg(\forall X(X(0) \wedge \forall n(X(n) \to X(s(n))) \to \forall n X(n))).$$

Then

$$\mathrm{sk}(\mathbf{F}) = \neg(\forall X(X(0) \wedge (X(f(X)) \to X(s(f(X))))) \to \forall n X(n)).$$

where $f \in \mathcal{K}_{\iota\to o,o}$.

Proposition 3.1.7. *Let* $\mathbf{F}$ *be a formula. Then* $\mathrm{sk}(\mathbf{F})$ *does not contain strong quantifiers.*

We make the notion of Skolemization of auxiliary formulas of weak quantifier inferences precise:

Definition 3.1.8. Let ρ be

$$\frac{\overline{\mathbf{AF}}, \Gamma \vdash \Delta}{\forall \mathbf{A}, \Gamma \vdash \Delta}\ \forall\colon l$$

where $\forall \mathbf{A}$ is in $q\eta$-normal form, and let $\mathbf{A}^*$ be the formula obtained from $\overline{\mathbf{AF}}$ by Skolemizing the strong quantifiers introduced by $\mathbf{F}$. ρ is called *Skolemizable* if there exists a formula $\mathbf{F}^*$ such that

$$\frac{\overline{\mathbf{AF}^*}, \Gamma \vdash \Delta}{\forall \mathbf{A}, \Gamma \vdash \Delta}\ \forall\colon l$$

is a valid inference.

Definition 3.1.9. An **LK**-proof ψ is called *Skolemizable* if all weak quantifier inferences occurring in ψ that operate on end-sequent ancestors are Skolemizable.

It is easy to see that not all **LK**-proofs are Skolemizable; in fact the proof exhibited in Example 3.1.1 is not Skolemizable. Skolemzation of the $\forall\colon l$ inference would yield $\overline{\mathbf{A}^* = P(s,a) \rightarrow (\forall z)}P(z,b)$ with $s \in \mathcal{K}_\iota$. Clearly there is no $\mathbf{F}^*$ such that $\overline{(\lambda X.X(a) \rightarrow X(b))}\mathbf{F}^* = \mathbf{A}^*$.

3.1.2 Skolemizable proofs

We will now give a syntactic characterization of the Skolemizable inferences. For this, we need some definitions.

Definition 3.1.10. Let **F** be a formula in $q\eta$-normal form. We say that **X** is *linear* in **F** if the number of occurrences of **X** in **F** is < 2. Let **X** be linear in **F**, then we call **X** *restricted* in **F** if

1. no weak quantifier dominates **X** or
2. exactly one weak quantifier $(\forall \mathbf{Y})$ dominates **X** and **X** occurs as **XY**.

Definition 3.1.11. Let $(\forall \mathbf{X})\mathbf{F}$ be a formula. The occurrence of $(\forall \mathbf{X})$ is called *non-dummy* if **F** contains **X**.

For the replacement of a subexpression at position λ by **C** in **A** we write $\mathbf{A}[\mathbf{C}]_\lambda$.

Proposition 3.1.12. *Let ρ be*

$$\frac{\overline{\mathbf{A}\left[\mathbf{X} \leftarrow \mathbf{F}\right]}, \Gamma \vdash \Delta}{(\forall \mathbf{X})\mathbf{A}, \Gamma \vdash \Delta}\ \forall\colon l$$

*such that $(\forall \mathbf{X})\mathbf{A}$ is in $q\eta$-normal form and in β-normal form, and **F** is in β-normal form. Then ρ is Skolemizable if and only if either*

1. ***X** is linear in **A** and if **F** contains non-dummy strong quantifiers w.r.t. $\overline{\mathbf{A}\left[\mathbf{X} \leftarrow \mathbf{F}\right]}$ then **X** is restricted in **A** or*
2. ***X** occurs only positively (negatively) in **A** and all non-dummy quantifier occurrences in **F** are weak (strong) quantifiers or*
3. ***F** does not contain non-dummy quantifiers.*

Proof. First, we will show that the given criteria imply Skolemizability of ρ. We will define the formula $\mathbf{F}^*$ that will be used for the inference

$$\frac{\overline{\mathbf{A}\,[\mathbf{X} \leftarrow \mathbf{F}^*]}, \Gamma \vdash \Delta}{(\forall \mathbf{X})\mathbf{A}, \Gamma \vdash \Delta}\ \forall\colon l$$

where $\mathbf{A}^* = \overline{\mathbf{A}\,[\mathbf{X} \leftarrow \mathbf{F}^*]}$ is the result of Skolemizing the strong quantifiers introduced by $\mathbf{F}$ in $\overline{\mathbf{A}\,[\mathbf{X} \leftarrow \mathbf{F}]}$. Either:

1. $\mathbf{X}$ is linear in $\mathbf{A}$ and if $\mathbf{F}$ contains non-dummy strong quantifiers w.r.t. $\overline{\mathbf{A}\,[\mathbf{X} \leftarrow \mathbf{F}]}$ then $\mathbf{X}$ is restricted in $\mathbf{A}$. If $\mathbf{X}$ does not occur in $\mathbf{A}$, then there is nothing to show, so assume $\mathbf{X}$ occurs at position ξ in $\mathbf{A}$. Then $\overline{\mathbf{A}\,[\mathbf{X} \leftarrow \mathbf{F}]} = \overline{\mathbf{A}[\mathbf{F}]_\xi}$ and $\mathbf{A}^* = \overline{\mathbf{A}[\mathbf{F}']_\xi}$ where $\mathbf{F}'$ is the Skolemization of $\mathbf{F}$ in $\overline{\mathbf{A}\,[\mathbf{X} \leftarrow \mathbf{F}]}$. If $\mathbf{F}$ does not contain non-dummy strong quantifiers w.r.t. $\overline{\mathbf{A}\,[\mathbf{X} \leftarrow \mathbf{F}]}$, then $\mathbf{F}'$ is just $\mathbf{F}$ after dropping some quantifiers, and we can use $\mathbf{F}^* = \mathbf{F}'$. Otherwise, distinguish the cases

   (a) $\mathbf{X}$ does not occur as $\mathbf{X}\mathbf{t}$ for some $\mathbf{t}$. Then, as $(\forall \mathbf{X})\mathbf{A}$ is in β-normal form, also $\mathbf{A}\,[\mathbf{X} \leftarrow \mathbf{F}]$ is in β-normal form, and so we may again take $\mathbf{F}^* = \mathbf{F}'$.

   (b) $\mathbf{X}$ occurs as $\mathbf{X}\mathbf{t}$ for some $\mathbf{t}$. Distinguish further:

      i. No weak quantifier dominates $\mathbf{X}$. Then $\mathbf{F}'\mathbf{t}$ only contains variables that occur in $\mathbf{F}\mathbf{t}$, so $\mathbf{F}'\mathbf{t}$ does not contain any variable that is bound in $\mathbf{A}$, therefore we may use $\mathbf{F}^* = \mathbf{F}'$.

      ii. Exactly one weak quantifier $(\forall \mathbf{Y})$ dominates $\mathbf{X}$ and $\mathbf{t} = \mathbf{Y}$. If $\mathbf{F}' = \lambda \mathbf{Z}.\mathbf{G}$, then set $\mathbf{F}^* = \lambda \mathbf{Z}.\mathbf{G}\,[\mathbf{Y} \leftarrow \mathbf{Z}]$, otherwise set $\mathbf{F}^* = \lambda \mathbf{Z}.\mathbf{F}'\,[\mathbf{Y} \leftarrow \mathbf{Z}]\,\mathbf{Z}$. Then $\overline{\mathbf{A}\,[\mathbf{X} \leftarrow \mathbf{F}^*]} = \overline{\mathbf{A}[\mathbf{F}^*]_\xi} = \overline{\mathbf{A}[\mathbf{F}']_\xi}$ and again $\mathbf{F}^*$ does not contain any variable that is bound in $\mathbf{A}$.

2. $\mathbf{X}$ occurs only positively in $\mathbf{A}$ and all non-dummy quantifiers in $\mathbf{F}$ are weak. Then the Skolemization of $\mathbf{F}$ in $\overline{\mathbf{A}\mathbf{F}}$, call it $\mathbf{F}'$, is just $\mathbf{F}$ after dropping some dummy strong quantifiers and we may use $\mathbf{F}^* = \mathbf{F}'$.

3. $\mathbf{X}$ occurs only negatively in $\mathbf{A}$ and all non-dummy quantifiers in $\mathbf{F}$ are strong. Analogous to the previous case.

4. $\mathbf{F}$ only contains dummy quantifiers. Analogous to the previous cases.

For the other direction, we show that if the given criteria are not fulfilled, then ρ is not Skolemizable. We proceed with a proof by contradiction. We may assume that $\mathbf{F}$ contains non-dummy quantifiers. We distinguish the cases

1. $\mathbf{X}$ is not linear in $\mathbf{A}$. To simplify the argument, we assume that there are occurrences of $\mathbf{X}$ in $\mathbf{A}$ of the form $\mathbf{Xt}_1$, $\mathbf{Xt}_2$ at position η_1, η_2 — the argument for the other type of occurrence is analogous. Then at positions η_1, η_2 in $\overline{\mathbf{A}\,[\mathbf{X} \leftarrow \mathbf{F}]}$ we have subformula occurrences of $\overline{\mathbf{Ft}_1}$, $\overline{\mathbf{Ft}_2}$. There are the following subcases:

   (a) $\mathbf{X}$ occurs positively in $\mathbf{A}$ and $\mathbf{F}$ contains non-dummy strong quantifiers. Then the occurrences of $\overline{\mathbf{Ft}_1}$, $\overline{\mathbf{Ft}_2}$ are positive. $\mathbf{F}$ contains non-dummy strong quantifiers, so at the same relative positions in the Skolemizations of $\overline{\mathbf{Ft}_1}$, $\overline{\mathbf{Ft}_2}$ we have Skolem terms with different head symbols, say $\mathbf{f}_1$, $\mathbf{f}_2$. $\mathbf{F}^*$ cannot contain two terms with different heads at the same position, so they must be introduced in $\mathbf{A}^*$ by β-reduction when applying $\mathbf{F}^*$ to $\mathbf{t}_1$, $\mathbf{t}_2$. But $\mathbf{t}_1$, $\mathbf{t}_2$ cannot contain $\mathbf{f}_1$, $\mathbf{f}_2$, because they are fresh symbols, and we arrive at a contradiction.

   (b) $\mathbf{X}$ occurs negatively in $\mathbf{A}$ and $\mathbf{F}$ contains non-dummy weak quantifiers. Analogous to the previous case.

   (c) $\mathbf{F}$ contains non-dummy strong and weak quantifiers. As $\mathbf{X}$ occurs in $\mathbf{A}$, it does so either positively or negatively, so one of the above cases applies.

   (d) $\mathbf{X}$ occurs positively and negatively in $\mathbf{A}$. As $\mathbf{F}$ contains non-dummy quantifiers, it either contains strong or weak ones, so one of the above cases applies.

2. $\mathbf{F}$ contains non-dummy strong quantifiers w.r.t. $\overline{\mathbf{A}\,[\mathbf{X} \leftarrow \mathbf{F}]}$ and there are weak quantifiers dominating $\mathbf{X}$ in $\mathbf{A}$ and either

   (a) more than one weak quantifier dominates $\mathbf{X}$ or

   (b) exactly one quantifier $(\forall \mathbf{Y})$ dominates $\mathbf{X}$ and $\mathbf{X}$ does not occur as $\mathbf{XY}$ in $\mathbf{A}$.

   Regarding (2a): Assume $\mathbf{X}$ occurs at position η as $\mathbf{Xt}$ (the argument for the other type of occurrence is analogous) in $\mathbf{A}$. Then at position η in $\overline{\mathbf{A}\,[\mathbf{X} \leftarrow \mathbf{F}]}$ we have the formula $\overline{\mathbf{Ft}}$ that is dominated by more than one weak quantifier, say among them are $(\forall \mathbf{X}_1)$, $(\forall \mathbf{X}_2)$. $\mathbf{F}$ contains strongly quantified variables, so its Skolemization will contain a Skolem term $\mathbf{f}(\ldots, \mathbf{X}_1, \ldots, \mathbf{X}_2, \ldots)$. $\mathbf{F}^*$ must not contain variables that are quantified in $\mathbf{A}$, so $\mathbf{X}_1, \mathbf{X}_2$ must be introduced in $\mathbf{f}$ by β-reduction when reducing $\mathbf{A}\,[\mathbf{X} \leftarrow \mathbf{F}^*]$. But $\mathbf{f}$ is a new function symbol, so $\mathbf{t}$ cannot contain $\mathbf{f}$, so if $\mathbf{t}$ contains both $\mathbf{X}_1$ and $\mathbf{X}_2$, then it has at the head some function symbol $\mathbf{g}$, but the function symbol in the Skolemization of $\mathbf{F}$ that is directly above $\mathbf{X}_1, \mathbf{X}_2$ is $\mathbf{f}$, so we arrive at a contradiction.

Regarding (2b): We may assume that exactly one weak quantifier dominates $\mathbf{X}$. Let $\mathbf{X}$ occur as $\mathbf{XT}$ with $\mathbf{T} \neq \mathbf{Y}$ (the argument for the other type of occurrence is analogous) at position η in $\mathbf{A}$. $\mathbf{F}$ contains non-dummy strong quantifiers w.r.t. $\overline{\mathbf{AF}}$, so in the Skolemization of $\mathbf{F}$ in $\overline{\mathbf{AF}}$, there will be a Skolem term $\mathbf{f}(\ldots, \mathbf{Y}, \ldots)$. In $\mathbf{A}$, $\mathbf{Y}$ is bound, so $\mathbf{Y}$ must be introduced in $\mathbf{f}$ by β-reduction when reducing $\mathbf{A}\,[\mathbf{X} \leftarrow \mathbf{F}^*]$. But $\mathbf{T} \neq \mathbf{Y}$, so if $\mathbf{T}$ contains $\mathbf{Y}$, it will be below some function symbol $\mathbf{g}$, but $\mathbf{Y}$ is directly below $\mathbf{f}$, so we again have a contradiction.

□

3.2 QFC-CERES

In this section, we study a formulation of CERES for a (trivially) Skolemizable class of proofs in second-order logic called **QFC**. For this purpose we will, throughout this section, restrict our attention to a subset $\mathcal{E}'$ of our set of expressions $\mathcal{E}$: $\mathcal{E}'$ is the largest subset of $\mathcal{E}$ fulfilling:

1. if $\mathbf{T} \in \mathcal{E}'$ and $\mathbf{X}$ occurs in $\mathbf{T}$ then either $\tau(\mathbf{X}) = \iota$ or $\tau(\mathbf{X}) = \iota \to o$, and
2. if $\mathbf{T} \in \mathcal{E}'$ and $\forall_\alpha$ occurs in $\mathbf{T}$, then $\alpha = \iota$ or $\alpha = \iota \to o$.

Hence, this class is slightly more liberal than the one originally treated in [34] and [32]. It can be seen as "first-order logic with monadic predicate variables and arbitrary higher-order constants". We now turn to the definition of **QFC**-proofs (proofs using *quantifier-free comprehension*):

Definition 3.2.1. An instance of the $\forall\colon l$ rule

$$\frac{\overline{\mathbf{FT}}, \Gamma \vdash \Delta}{\forall \mathbf{F}, \Gamma \vdash \Delta}\ \forall\colon l$$

is called *quantifier-free* if $\mathbf{T}$ does not contain any quantifiers. We call an **LK**-proof π a **QFC**-proof if all $\forall\colon l$ applications in π are quantifier-free.

The following is a trivial consequence of the definition:

Proposition 3.2.2. *Let π be a* **QFC***-proof. Then π is Skolemizable.*

Proposition 3.2.3. *Let π be a* **QFC***-proof of S such that S does not contain strong quantifiers. Then π does not contain strong quantifiers inferences operating on end-sequent ancestors.*

Proof. By structural induction on π. □

3.2.1 QFC-$\mathcal{R}$-deductions

To define CERES on **QFC**-proofs, we restrict the resolution calculus $\mathcal{R}$ analogously to the restriction of **LK** to **QFC**:

Definition 3.2.4. Let γ be an $\mathcal{R}$-deduction of S from $\mathcal{C}$. γ is a **QFC**-$\mathcal{R}$-deduction if

1. For all Sub applications in γ, the substitution term $\mathbf{T}$ is quantifier-free and
2. for all $C \in \mathcal{C}$, C is quantifier-free.

Proposition 3.2.5. *Let γ be a* **QFC**-*$\mathcal{R}$-deduction of S from $\mathcal{C}$. Then S is quantifier-free.*

Proof. By induction on the length of γ. □

Corollary 3.2.6. *Let γ be a* **QFC**-*$\mathcal{R}$-deduction. Then γ contains neither $\forall^T$ nor $\forall^F$ inferences.*

Definition 3.2.7. We define the application of a quantifier-free substitution σ to a set of clauses $\mathcal{C} = \{C_1, \ldots, C_n\}$, denoted $\mathcal{S}(\mathcal{C}, \sigma)$, as the clause form of the set of quantifier-free sequents $\{C_1\sigma, \ldots, C_n\sigma\}$. Note that this includes transformation to CNF, therefore $|\mathcal{S}(\mathcal{C}, \sigma)| \geq |\mathcal{C}|$.

Next we state two lemmas that show that **QFC**-$\mathcal{R}$-deductions can be transformed into **QFC**-proofs.

Lemma 3.2.8. *Let C be a clause and σ be a quantifier-free substitution. Then we can construct a cut-free* **QFC**-*proof of $C\sigma$ from $\mathcal{S}(\{C\}, \sigma)$.*

Proof. By the definition of $\mathcal{S}$, $\mathcal{S}(\{C\}, \sigma)$ is the clause form of $C\sigma$ and therefore, as σ is quantifier-free, propositionally equivalent to $C\sigma$. By completeness and the decidability of propositional logic, the desired **QFC**-proof can be constructed. □

Lemma 3.2.9. *Let R be a* **QFC**-*$\mathcal{R}$-deduction of $\Gamma \vdash \Delta$ from a set of clauses $\mathcal{C}$. Then there exists a* **QFC**-*proof ψ of $\Gamma \vdash \Delta$ from $\mathcal{D}$ containing quantifier-free cuts only, where $\mathcal{D} = \mathcal{S}(\mathcal{C}, \sigma)$ for some quantifier-free σ.*

Proof. We proceed by induction on the size of R, letting $\mathcal{C} = \{C_1, \ldots, C_n\}$. In addition to what is stated in the theorem, the proofs we construct will have the property of containing neither $\forall\colon l$ nor $\forall\colon r$ inferences.

1. $|R| = 0$. Then $R = C_i$ for some $1 \leq i \leq n$. Take ψ as the sequent C_i.

2. $|R| = m + 1$. Distinguish the last inference in R:

(a) R is

$$\frac{\Gamma \vdash \Delta, \neg \mathbf{A}}{\mathbf{A}, \Gamma \vdash \Delta} \neg^T$$

By (IH) we have a **QFC**-proof ψ of $\Gamma \vdash \Delta, \neg \mathbf{A}$. Take as the desired **QFC**-proof

$$\dfrac{\begin{array}{c}\psi \\ \Gamma \vdash \Delta, \neg \mathbf{A}\end{array} \qquad \dfrac{\mathbf{A} \vdash \mathbf{A}}{\neg \mathbf{A}, \mathbf{A} \vdash} \neg\colon l}{\mathbf{A}, \Gamma \vdash \Delta} \text{cut}$$

By Proposition 3.2.5, the cut is quantifier-free.

(b) R is

$$\frac{\neg \mathbf{A}, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \mathbf{A}} \neg^F$$

Symmetric to the previous case.

(c) R is

$$\frac{\Gamma \vdash \Delta, \mathbf{A} \vee \mathbf{B}}{\Gamma \vdash \Delta, \mathbf{A}, \mathbf{B}} \vee^T$$

By (IH) we have a **QFC**-proof of $\Gamma \vdash \Delta, \mathbf{A} \vee \mathbf{B}$. Take as the desired **QFC**-proof

$$\dfrac{\begin{array}{c}\psi \\ \Gamma \vdash \Delta, \mathbf{A} \vee \mathbf{B}\end{array} \qquad \dfrac{\mathbf{A} \vdash \mathbf{A} \qquad \mathbf{B} \vdash \mathbf{B}}{\mathbf{A} \vee \mathbf{B} \vdash \mathbf{A}, \mathbf{B}} \vee\colon l}{\Gamma \vdash \Delta, \mathbf{A}, \mathbf{B}} \text{cut}$$

By Proposition 3.2.5, the cut is quantifier-free.

(d) R is

$$\frac{\mathbf{A} \vee \mathbf{B}, \Gamma \vdash \Delta}{\mathbf{A}, \Gamma \vdash \Delta} \vee^F_l$$

By (IH) we have a **QFC**-proof of $\mathbf{A} \vee \mathbf{B}, \Gamma \vdash \Delta$. Take as the desired **QFC**-proof

$$\dfrac{\dfrac{\mathbf{A} \vdash \mathbf{A}}{\mathbf{A} \vdash \mathbf{A} \vee \mathbf{B}} \vee\colon r_1 \qquad \begin{matrix}\psi \\ \mathbf{A} \vee \mathbf{B}, \Gamma \vdash \Delta\end{matrix}}{\mathbf{A}, \Gamma \vdash \Delta} \text{cut}$$

By Proposition 3.2.5, the cut is quantifier-free.

(e) R is

$$\dfrac{\mathbf{A} \vee \mathbf{B}, \Gamma \vdash \Delta}{\mathbf{B}, \Gamma \vdash \Delta} \vee_r^F$$

Analogous to the previous case.

(f) R is

$$\dfrac{\mathbf{A}, \mathbf{A}, \Gamma \vdash \Delta}{\mathbf{A}, \Gamma \vdash \Delta} \text{Sim}^F$$

By (IH) we have a **QFC**-proof of $\mathbf{A}, \mathbf{A}, \Gamma \vdash \Delta$. Take as the desired **QFC**-proof

$$\dfrac{\mathbf{A}, \mathbf{A}, \Gamma \vdash \Delta}{\mathbf{A}, \Gamma \vdash \Delta} \text{contr}\colon l$$

(g) R is

$$\dfrac{\Gamma \vdash \Delta, \mathbf{A}, \mathbf{A}}{\Gamma \vdash \Delta, \mathbf{A}} \text{Sim}^T$$

Symmetric to the previous case.

(h) R is

$$\dfrac{\Gamma \vdash \Delta, \mathbf{A} \qquad \mathbf{A}, \Pi \vdash \Lambda}{\Gamma, \Pi \vdash \Delta, \Lambda} \text{Cut}$$

By (IH) we have **QFC**-proofs of $\Gamma \vdash \Delta, \mathbf{A}$ and $\mathbf{A}, \Pi \vdash \Lambda$. By Proposition 3.2.5, we may use quantifier-free cut to combine the two **QFC**-proofs to obtain the desired one.

(i) R is

$$\dfrac{S}{S\,[\mathbf{X} \leftarrow \mathbf{T}]} \text{Sub}$$

By (IH) we have a **QFC**-proof ψ of S from $\mathcal{D} = \mathcal{S}(\mathcal{C}, \mu)$, for some quantifier-free μ, not containing any $\forall\colon l$ or $\forall\colon r$ inferences. By applying $\sigma = [\mathbf{X} \leftarrow \mathbf{T}]$ to ψ, we obtain a **QFC**-proof of $S\,[\mathbf{X} \leftarrow \mathbf{T}]$ where every leaf is either of the form $A\sigma \vdash A\sigma$ or $D\sigma$ for $D \in \mathcal{D}$. Applying Lemma 3.2.8 to the latter leafs, we obtain the desired **QFC**-proof from $\mathcal{S}(\mathcal{D}, \sigma)$.

$\square$

3.2.2 CERES on QFC-proofs

This section will describe a version of the CERES method for **QFC**-proofs, called **QFC**-CERES. We will first prove a lemma that shows that strong quantifiers can be removed from a sequent in a **QFC**-proof by dropping quantifiers and introducing Skolem terms.

Lemma 3.2.10. *Let ψ be a* **QFC***-proof of a sequent S, then we can construct an* **QFC***-proof of* $\mathrm{sk}(S)$*.*

Proof. This proof is based on the proof of the proposition for first-order logic in [8]. Let $S = \Gamma \vdash \Delta$ and assume S contains a positive occurrence of $(\forall \mathbf{X})\mathbf{A}$, where $\tau(\mathbf{X}) = \alpha$. Then this quantifier has been introduced in one of the following ways in ψ:

(a) $$\frac{\Pi \vdash \Lambda, \mathbf{B}}{\Pi \vdash \Lambda, \mathbf{B} \lor \mathbf{C}} \lor{:}\, r_1$$

s.t. $(\forall \mathbf{X})\mathbf{A}$ occurs as a subformula of $\mathbf{C}$. Let $\rho[\mathbf{B} \lor \mathbf{C}]$ be the path connecting $\Pi \vdash \Lambda, \mathbf{B} \lor \mathbf{C}$ with S. Let $\overline{\mathbf{A}(\mathbf{f}\mathbf{X}_1 \ldots \mathbf{X}_n)}$ be the subformula in $\mathrm{sk}(S)$ corresponding to $(\forall \mathbf{X})\mathbf{A}$ in S (i.e. its Skolemization). Then define $\mathbf{C}' = \mathbf{C}[\overline{\mathbf{A}(\mathbf{f}\mathbf{X}_1 \ldots \mathbf{X}_n)}]_{\xi}$, where ξ is the position of $(\forall \mathbf{X})\mathbf{A}$ in $\mathbf{C}$ and replace $\rho[\mathbf{B} \lor \mathbf{C}]$ by $\rho[\mathbf{B} \lor \mathbf{C}']$.

(b) $$\frac{\Pi \vdash \Lambda, \mathbf{C}}{\Pi \vdash \Lambda, \mathbf{B} \lor \mathbf{C}} \lor{:}\, r_2$$

s.t. $(\forall \mathbf{X})\mathbf{A}$ occurs as a subformula of $\mathbf{B}$. Analogous to (a).

(c) $$\frac{\Pi \vdash \Lambda}{\Pi \vdash \Lambda, \mathbf{B}} \text{weak}{:}\, r$$

s.t. $(\forall \mathbf{X})\mathbf{A}$ occurs as a subformula of $\mathbf{B}$. Analogous to (a).

(d) $$\frac{\Pi \vdash \Lambda}{\mathbf{B}, \Pi \vdash \Lambda} \text{weak}{:}\, l$$

s.t. $(\forall \mathbf{X})\mathbf{A}$ occurs as a subformula of $\mathbf{B}$. Analogous to (a).

(e) $$\frac{\begin{array}{c}\varphi(\mathbf{Y}) \\ \Pi \vdash \Lambda, \overline{\mathbf{AY}}\end{array}}{\Pi \vdash \Lambda, (\forall \mathbf{X})\mathbf{A}} \forall{:}\, r$$

Let $\rho[(\forall \mathbf{X})\mathbf{A}]$ be the path connecting $\Pi \vdash \Lambda, (\forall \mathbf{X})\mathbf{A}$ with S. Let $\mathbf{t}_1, \ldots, \mathbf{t}_n$ be the substitution terms of the $\forall\colon l$ inferences on $\rho[(\forall \mathbf{X})\mathbf{A}]$ whose quantifiers dominate the occurrence of $(\forall \mathbf{X})\mathbf{A}$. Let the main formulas of these inferences be $(\forall \mathbf{X}_i)\mathbf{A}_i$, and let $\tau(\mathbf{X}_i) = \beta_i$ and $\tau(\mathbf{X}) = \alpha$. We choose a fresh $\mathbf{f} \in \mathcal{K}_{\beta_1,\ldots,\beta_n,\alpha}$ and replace $\varphi(\mathbf{Y})$ by $\varphi(\mathbf{f}(\mathbf{t}_1 \ldots \mathbf{t}_n))$. By the eigenvariable condition on $\mathbf{Y}$, this yields an $\mathbf{LK}$-proof of $\Pi \vdash \Lambda, \overline{\mathbf{Af}(\mathbf{t}_1 \ldots \mathbf{t}_n)}$ (note that $\mathbf{t}_1, \ldots, \mathbf{t}_n$ cannot contain eigenvariables from the proof $\varphi(\mathbf{Y})$, as they cannot be present in $\Pi \vdash \Lambda, (\forall \mathbf{X})$ by the eigenvariable condition and $\mathbf{t}_1, \ldots, \mathbf{t}_n$ occur below this sequent). We remove the $\forall\colon l$ inference and replace $\rho[(\forall \mathbf{X})\mathbf{A}]$ by $\rho[\overline{\mathbf{Af}(\mathbf{t}_1 \ldots \mathbf{t}_n)}]$, modifying the substitution terms of the $\forall\colon l$ inferences appropriatly. By construction, the terms $\mathbf{t}_1, \ldots, \mathbf{t}_n$ will be eliminated one-by-one by the $\forall\colon l$ inferences on $\rho[\overline{\mathbf{Af}(\mathbf{t}_1 \ldots \mathbf{t}_n)}]$, and the occurrence of $(\forall \mathbf{X})\mathbf{A}$ in S will thus become $\overline{\mathbf{Af}(\mathbf{X}_1 \ldots \mathbf{X}_n)}$, which is exactly the corresponding occurrence in $\mathrm{sk}(S)$.

In all cases, we have to take contractions into consideration: If there are two predecessors of the form $\mathbf{D} = \mathbf{C}[(\forall \mathbf{X})\mathbf{A}]$ of the occurrence of $(\forall \mathbf{X})\mathbf{A}$ in S s.t. there is a contraction

$$\frac{\mathbf{F}[\mathbf{D}], F[\mathbf{D}]\Lambda \vdash \Pi}{\mathbf{F}[\mathbf{D}], \Lambda \vdash \Pi}\ c:l$$

we have to introduce the same Skolem symbol for both predecessors (as otherwise the contraction can not be applied anymore). □

Definition 3.2.11. Let ψ be an $\mathbf{LK}$-proof. If all strong quantifier inferences in ψ operate on cut-ancestors, then ψ is said to be in *Skolem form.*

The following proposition shows that from a $\mathbf{QFC}$-proof, we can indeed obtain a proof in Skolem form. As mentioned before, such proofs allow the definition of proof projections by leaving out inferences from the proof, as no eigenvariable violations can occur by doing so. The proof projections used for $\mathbf{QFC}$-CERES are described in Definition 3.2.13.

Proposition 3.2.12. *For every* $\mathbf{QFC}$*-proof* ψ *of* S *there exists a* $\mathbf{QFC}$*-proof* ψ' *of* $\mathrm{sk}(S)$ *in Skolem form.*

Proof. From Lemma 3.2.10 we obtain a proof φ of $\mathrm{sk}(S)$. By Propositions 3.2.3 and 3.1.7, φ is in Skolem form. □

We can now define the main parts of the $\mathbf{QFC}$-CERES-method: the characteristic clause set and the set of proof projections of a proof π. The former will be always unsatisfiable and give rise to a quantifier-free $\mathcal{R}$-refutation, while the latter will allow the quantifier-free $\mathcal{R}$-refutation to be transformed into a proof of the end-sequent of π.

Definition 3.2.13. Let π be a **QFC**-proof in Skolem form. For each inference ρ in π, we define a set of cut-free **QFC**-proofs, the set of *projections* $\mathcal{P}_\rho(\pi)$ of π, and a set of clauses, the *characteristic clause set* $\mathrm{CS}_\rho(\pi)$ of π, at the position ρ.

- If ρ is a leaf, let $\Gamma_1 \vdash \Delta_1$ be the part of it which consists of ancestors of cut formulas, let $\Gamma_2 \vdash \Delta_2$ be the part which consists of ancestors of the end-sequent of π and define

$$\begin{array}{ll} \mathcal{P}_\rho(\pi) & = \{\Gamma_1, \Gamma_2 \vdash \Delta_2, \Delta_1\} \\ \mathrm{CS}_\rho(\pi) & = \{\Gamma_1 \vdash \Delta_1\}. \end{array}$$

- If ρ is a unary inference with immediate predecessor ρ' with $\mathcal{P}_{\rho'}(\pi) = \{\psi_1, \ldots, \psi_n\}$, distinguish:

  (a) The active formulas of ρ are ancestors of cut formulas. Then

  $$\mathcal{P}_\rho(\pi) = \mathcal{P}_{\rho'}(\pi)$$

  (b) The active formulas of ρ are ancestors of the end-sequent. Then

  $$\mathcal{P}_\rho(\pi) = \{\rho(\psi_1), \ldots, \rho(\psi_n)\}$$

  where $\rho(\psi)$ is the proof that is obtained from ψ by applying ρ to its end-sequent. Note that by assumption, all strong quantifier inferences go into cuts, so ρ cannot be a strong quantifier inference, so no eigenvariable violation can occur here.

  In any case, $\mathrm{CS}_\rho(\pi) = \mathrm{CS}_{\rho'}(\pi)$.

- Let ρ be a binary inference with immediate predecessors ρ_1 and ρ_2.

  (a) If the active formulas of ρ are ancestors of cut-formulas, let $\Gamma_i \vdash \Delta_i$ be the ancestors of the end-sequent in the conclusion sequent of ρ_i and define

  $$\mathcal{P}_\rho(\pi) = \mathcal{P}_{\rho_1}(\pi)^{\Gamma_2 \vdash \Delta_2} \cup \mathcal{P}_{\rho_2}(\pi)^{\Gamma_1 \vdash \Delta_1}$$

  where $P^{\Gamma \vdash \Delta} = \{\psi^{\Gamma \vdash \Delta} \mid \psi \in P\}$ and $\psi^{\Gamma \vdash \Delta}$ is ψ followed by weakenings adding $\Gamma \vdash \Delta$. For the characteristic clause set, define

  $$\mathrm{CS}_\rho(\pi) = \mathrm{CS}_{\rho_1}(\pi) \cup \mathrm{CS}_{\rho_2}(\pi)$$

(b) If the active formulas of ρ are ancestors of the end-sequent, then

$$\mathcal{P}_\rho(\pi) = \mathcal{P}_{\rho_1}(\pi) \times \mathcal{P}_{\rho_2}(\pi).$$

where

$$P \times Q = \{\rho(\psi, \chi) \mid \psi \in P, \chi \in Q\}$$

and $\rho(\psi, \chi)$ is the proof that is obtained from the proofs ψ and χ by applying the binary inference ρ. For the characteristic clause set, define

$$\mathrm{CS}_\rho(\pi) = \mathrm{CS}_{\rho_1}(\pi) \times \mathrm{CS}_{\rho_2}(\pi).$$

The *set of projections* of π, $\mathcal{P}(\pi)$ is defined as $\mathcal{P}_{\rho_0}(\pi)$, and the *characteristic clause set* of π, $\mathrm{CS}(\pi)$ is defined as $\mathrm{CS}_{\rho_0}(\pi)$, where ρ_0 is the last inference of π.

Note that for the soundness of this definition, we need the assumption that π is in Skolem form: if this were not the case, violations of eigenvariable conditions could appear in the projections.

Example 3.2.14. Consider the proof ψ:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Theta(a) \vdash \Theta(a)}{\vdash \Theta(a), \neg\Theta(a)}\,\neg\colon r \qquad \dfrac{\Theta(b) \vdash \Theta(b)}{\neg\Theta(b), \Theta(b) \vdash}\,\neg\colon l}{\Theta(b), \neg\Theta(a) \to \neg\Theta(b) \vdash \Theta(a)}\,\to\colon l}{\Theta(b), (\forall X)(X(a) \to X(b)) \vdash \Theta(a)}\,\forall\colon l}{(\forall X)(X(a) \to X(b)) \vdash \Theta(b) \to \Theta(a)}\,\to\colon r}{(\forall X)(X(a) \to X(b)) \vdash (\forall X)(X(b) \to X(a))}\,\forall\colon r \qquad \dfrac{\dfrac{\dfrac{P(b) \vdash P(b) \qquad P(a) \vdash P(a)}{P(b) \to P(a), P(b) \vdash P(a)}\,\to\colon l}{P(b) \to P(a) \vdash P(b) \to P(a)}\,\to\colon r}{(\forall X)(X(b) \to X(a)) \vdash P(b) \to P(a)}\,\forall\colon l}{(\forall X)(X(a) \to X(b)) \vdash P(b) \to P(a)}\,\mathrm{cut}$$

where X, Θ are variables of type $\iota \to o$, a, b are constants of type ι, and P is a constant of type $\iota \to o$. Then

$$\begin{aligned} \mathrm{CS}(\psi) &= (\{\vdash \Theta(a)\} \times \{\Theta(b) \vdash\}) \cup \{\vdash P(b)\} \cup \{P(a) \vdash\} \\ &= \{\Theta(b) \vdash \Theta(a);\ \vdash P(b);\ P(a) \vdash\} \end{aligned}$$

and $\mathcal{P}(\psi)$ consists of the proofs

$$\dfrac{\dfrac{\dfrac{\dfrac{\Theta(a) \vdash \Theta(a)}{\vdash \Theta(a), \neg\Theta(a)}\,\neg\colon r \qquad \dfrac{\Theta(b) \vdash \Theta(b)}{\neg\Theta(b), \Theta(b) \vdash}\,\neg\colon l}{\Theta(b), \neg\Theta(a) \to \neg\Theta(b) \vdash \Theta(a)}\,\to\colon l}{\Theta(b), (\forall X)(X(a) \to X(b)) \vdash \Theta(a)}\,\forall\colon l}{\Theta(b), (\forall X)(X(a) \to X(b)) \vdash P(b) \to P(a), \Theta(a)}\,\mathrm{weak}\colon r$$

and

$$\cfrac{\cfrac{\cfrac{P(a) \vdash P(a)}{P(b), P(a) \vdash P(a)}\ \text{weak:}\,l}{P(a) \vdash P(b) \to P(a)}\ \to:r}{P(a), (\forall X)(X(a) \to X(b)) \vdash P(b) \to P(a)}\ \text{weak:}\,l$$

and

$$\cfrac{\cfrac{\cfrac{P(b) \vdash P(b)}{P(b) \vdash P(b), P(a)}\ \text{weak:}\,r}{\vdash P(b) \to P(a), P(b)}\ \to:r}{(\forall X)(X(a) \to X(b)) \vdash P(b) \to P(a), P(b)}\ \text{weak:}\,l$$

We will want to combine the **QFC**-proofs from $\mathcal{P}(\pi)$ with the **QFC**-proof constructed in Lemma 3.2.9 from the **QFC**-$\mathcal{R}$-refutation of CS(π). For this purpose, we will need to instantiate the projections; the following Lemma shows how this can be achieved.

Lemma 3.2.15. *Let $C = \Gamma \vdash \Delta$ be a clause, $\mathcal{D}$ be a set of clauses, ψ be a* **QFC***-proof of $\Gamma, \Pi \vdash \Lambda, \Delta$ from $\mathcal{D}$ with only quantifier-free cuts, let σ be a quantifier-free substitution whose domain contains no variable which occurs free in $\Pi \cup \Lambda$ and let $\Gamma^* \vdash \Delta^* \in \mathcal{S}(\{C\}, \sigma)$. Then we can construct a* **QFC***-proof ψ^* of $\Gamma^*, \Pi \vdash \Lambda, \Delta^*$ from $\mathcal{S}(\mathcal{D}, \sigma)$ with only quantifier-free cuts and with $|\psi^*| \leq |\psi| + \rho(|\Gamma\sigma \vdash \Delta\sigma|)$, where ρ is exponential if σ substitutes for a predicate variable in $\mathcal{D}$, and polynomial otherwise.*

Proof. The proof ψ^* has the following form:

$$\begin{array}{c}
(\psi'\sigma) \\
\Gamma\sigma, \Pi \vdash \Lambda, \Delta\sigma \\
\vdots\ (\text{a}) \\
\Pi \vdash \Lambda, \bigwedge \Gamma\sigma \to \bigvee \Delta\sigma \\
\vdots\ (\text{b}) \\
\Pi \vdash \Lambda, \mathrm{NNF}(\bigwedge \Gamma\sigma \to \bigvee \Delta\sigma) \\
\vdots\ (c) \\
\Gamma^*, \Pi \vdash \Lambda, \Delta^*
\end{array}$$

First, we obtain $\psi'\sigma$ from ψ: Let $\mathcal{D} = \{C_1, \ldots, C_n\}$, then when we apply σ to ψ, this yields a proof of $\Gamma\sigma, \Pi \vdash \Lambda, \Delta\sigma$ where every leaf is either of the form $\mathbf{A}\sigma \vdash \mathbf{A}\sigma$, or $C_i\sigma$ for some $1 \leq i \leq n$. We replace the two kinds of leafs by **QFC**-proofs to obtain the proof $\psi'\sigma$ from $\mathcal{S}(\mathcal{D}, \sigma)$:

1. We replace leafs of the form $\mathbf{A}\sigma \vdash \mathbf{A}\sigma$ by their respective proofs from atomic leafs and
2. apply Lemma 3.2.8 to leafs of the form $C_i\sigma$ to obtain proofs of $C_i\sigma$ from $\mathcal{S}(\{C_i\}, \sigma)$. We replace the leafs by the respective proofs.

This yields the desired proof $\psi'\sigma$ from $\mathcal{S}(\mathcal{D}, \sigma)$. We will now describe the steps (a-c) in detail:

(a) By a series of $\wedge\colon l$- and $\vee\colon r$-, followed by an $\to\colon r$-inference.

(b) For any formula $\mathbf{F}$ there is a proof $\chi_{\mathbf{F}}$ of $\mathbf{F} \vdash \mathrm{NNF}(\mathbf{F})$ based on the well-known rewrite rules of (i) replacing implication by negation and disjunction, (ii) the de Morgan-laws and (iii) double negation elimination. Phase (b) consists of a single cut against such a proof.

(c) For every negation normal form $\mathbf{F}$ and clause $\Gamma^* \vdash \Delta^* \in \mathrm{CNF}(\mathbf{F})$, there exists a proof $\chi_{\mathbf{F}}^{\Gamma^*\vdash\Delta^*}$ of $\mathbf{F}, \Gamma^* \vdash \Delta^*$. $\chi_{\mathbf{F}}^{\Gamma^*\vdash\Delta^*}$ is constructed as follows:

If $\mathbf{F} = \mathbf{G} \wedge \mathbf{H}$, then $\Gamma^* \vdash \Delta^* \in \mathrm{CNF}(\mathbf{G})$ or $\Gamma^* \vdash \Delta^* \in \mathrm{CNF}(\mathbf{H})$. Define

$$\chi_{\mathbf{G}\wedge\mathbf{H}}^{\Gamma^*\vdash\Delta^*} \quad := \quad \dfrac{\begin{matrix}(\chi_{\mathbf{G}}^{\Gamma^*\vdash\Delta^*})\\ \mathbf{G}, \Gamma^* \vdash \Delta^*\end{matrix}}{\mathbf{G} \wedge \mathbf{H}, \Gamma^* \vdash \Delta^*}\ \wedge\colon \mathrm{l1}$$

in the first case and use $\wedge$: l2 analogously in the second case. If $\mathbf{F} = \mathbf{G} \vee \mathbf{H}$, then $\Gamma^* = \Gamma_1^* \cup \Gamma_2^*$ and $\Delta^* = \Delta_1^* \cup \Delta_2^*$ s.t. $\Gamma_1^* \vdash \Delta_1^* \in \mathrm{CNF}(\mathbf{G})$ and $\Gamma_2^* \vdash \Delta_2^* \in \mathrm{CNF}(\mathbf{H})$. Define

$$\chi_{\mathbf{G}\vee\mathbf{H}}^{\Gamma^*\vdash\Delta^*} \quad := \quad \dfrac{\begin{matrix}(\chi_{\mathbf{G}}^{\Gamma_1^*\vdash\Delta_1^*})\\ \mathbf{G}, \Gamma_1^* \vdash \Delta_1^*\end{matrix} \quad \begin{matrix}(\chi_{\mathbf{H}}^{\Gamma_2^*\vdash\Delta_2^*})\\ \mathbf{H}, \Gamma_2^* \vdash \Delta_2^*\end{matrix}}{\mathbf{G} \vee \mathbf{H}, \Gamma^* \vdash \Delta^*}\ \vee\colon \mathrm{l}$$

If $\mathbf{F} = \neg\mathbf{G}$, then $\mathbf{G}$ is an atom, $\Gamma^* = \{\mathbf{G}\}$, $\Delta^* = \emptyset$ and define

$$\chi_{\neg\mathbf{G}}^{\mathbf{G}\vdash} \quad := \quad \dfrac{\mathbf{G} \vdash \mathbf{G}}{\neg\mathbf{G}, \mathbf{G} \vdash}\ \neg\colon \mathrm{l}$$

If $\mathbf{F}$ is an atom, then $\Gamma^* = \emptyset$, $\Delta^* = \{\mathbf{F}\}$ and therefore $\mathbf{F}, \Gamma^* \vdash \Delta^*$ is already an axiom. Phase (c) consists of a single cut against $\chi_{\mathrm{NNF}(\bigwedge\Gamma\sigma\to\bigvee\Delta\sigma)}^{\Gamma^*\vdash\Delta^*}$.

The total size of ψ^* is $|\psi\sigma| = |\psi|$ plus $O(|\Gamma\sigma \vdash \Delta\sigma|)$ for each of the three phases, plus a number exponential in the size of σ in case we have to apply Lemma 3.2.8. □

We will now prove the main properties of **QFC**-CERES. The following lemmas are used to establish that for **QFC**-proofs π in Skolem form, we can always find a **QFC**-$\mathcal{R}$-refutation of CS(π).

Lemma 3.2.16. *Let $\mathcal{C}$ be a set of clauses, π be a regular* **QFC***-proof of* $\vdash$ *from $\mathcal{C}$. Then there exists a* **QFC***-proof ψ of $\vdash$ from a set of clauses $\mathcal{S}(\mathcal{C}, \sigma)$ for some quantifier-free σ such that ψ consists of atomic cuts, contractions and permutations.*

Proof. As we know from e.g. [19], reductive cut-elimination in second-order logic terminates, so we can apply it to π to eliminate all non-atomic cuts and obtain a proof π' of $\vdash$. First, note that π' consists of atomic cut, contraction and permutation: weakening is automatically eliminated by cut-elimination. Denote the set of leafs of a proof φ by $init(\varphi)$. We will show that π' can be transformed into a proof ψ s.t. $init(\psi)$ consists of quantifier-free instances of clauses in $\mathcal{C}$. We can then take $\mathcal{D}$ as $init(\psi)$. We proceed by induction on the cut-elimination of π to obtain π'. As induction invariant, we take the following: π' can be transformed into a **QFC**-proof ψ s.t. $init(\psi)$ consists of quantifier-free instances of clauses in $\mathcal{C}$.

For the base case, we take $\psi = \pi$, so as $init(\pi) = init(\psi)$ and π uses quantifier free comprehension, the invariant holds.

1. The cut-eliminiation performs a rank reduction on π. Then the leafs of π and π' coincide, except when performing rank reduction over a contraction: Here, we perform adequate renamings of eigenvariables in π' to keep regularity and take $\psi = \pi'$. Clearly, $init(\psi)$ consists of $init(\pi)$ together with some renamed variants of clauses in $init(\pi)$, and the substitution terms of the $\forall\colon l$ applications are not changed, so the proposition holds.

2. The cut-elimination performs a grade reduction on φ. Distinguish:

   (a) Grade reduction is performed on propositional inferences. Then $init(\pi)$ and $init(\pi')$ coincide and we take $\psi = \pi'$, still the substitution terms of the $\forall\colon l$ applications are not changed.

   (b) Grade reduction is performed on $\forall_\alpha$ inferences. Let $\mathbf{X}$ be the eigenvariable of the $\forall\colon r$ inference, and let $\mathbf{F}$ be the substitution term of the $\forall\colon l$ inference. Then $\sigma = [\mathbf{X} \leftarrow \mathbf{F}]$ is the substitution that is applied by the cut-elimination. By (IH), σ is quantifier-free. Let $init(\pi) = \{\Gamma_1 \vdash \Delta_1, \ldots, \Gamma_n \vdash \Delta_n\}$. Then

$$init(\pi') = \{(\Gamma_1 \vdash \Delta_1)\sigma, \ldots, (\Gamma_n \vdash \Delta_n)\sigma\}.$$

By Lemma 3.2.8, for every $1 \leq i \leq n$, we have a proof of $(\Gamma_i \vdash \Delta_i)\sigma$ from $\mathcal{S}(\{\Gamma_i \vdash \Delta_i\}, \sigma)$. Take ψ to be π' where those leafs are replaced by the respective proofs, then

$$init(\psi) = \mathcal{S}(\{\Gamma_1 \vdash \Delta_1\}, \sigma) \cup \ldots \cup \mathcal{S}(\{\Gamma_n \vdash \Delta_n\}, \sigma)$$

and the first part of the proposition holds. For the second part, note that as σ is quantifier-free and no new $\forall_\beta$ applications are introduced in this step, all $\forall\colon l$ applications are still quantifier-free.

□

Lemma 3.2.17. *Let π be a* **QFC***-proof in Skolem form. Then there exists a* **QFC***-$\mathcal{R}$-refutation of* $\mathrm{CS}(\pi)$*.*

Proof. Analogous to the proof of unsatisfiability of $\mathrm{CS}(\pi)$ for first-order logic in [9] by removing all inferences of π except the ancestors of the cuts, and removing all formula occurrences in π except the ancestors of cuts, we construct a **QFC**-proof ψ of $\vdash$ from $\mathrm{CS}(\pi)$. We apply Lemma 3.2.16 to obtain a **QFC**-proof γ of $\vdash$ from quantifier-free instances of $\mathrm{CS}(\pi)$ using atomic cut, contraction and permutation only. γ readily gives rise to an R-refutation of $\mathrm{CS}(\pi)$: First, derive the necessary instances used in γ from $\mathrm{CS}(\pi)$ using the CNF-rules (as we only need quantifier-free instances, the restrictions of **QFC**-$\mathcal{R}$-deductions are met). Then, whenever atomic cuts are used in γ, apply Cut, and whenever contractions are used in γ, apply Sim^T or Sim^F. □

We are now ready to define the **QFC**-CERES method and state our central result.

Definition 3.2.18. Let π be a **QFC**-proof of S. Then the **QFC**-CERES method is the following algorithm:

1. Compute a **QFC**-proof π_{sk} of $\mathrm{sk}(S)$.
2. Compute $\mathrm{CS}(\pi_{sk})$, $\mathcal{P}(\pi_{sk})$.
3. Compute a quantifier-free $\mathcal{R}$-refutation γ of $\mathrm{CS}(\pi_{sk})$.
4. Convert γ into an **LK**-proof γ' of $\vdash$ from $\mathrm{CS}(\pi_{sk})$.
5. Plug instances of the proofs in $\mathcal{P}(\pi_{sk})$ into the leaves of γ' to obtain a proof ψ of $\mathrm{sk}(S)$ containing quantifier-free cuts only.
6. Perform quantifier-free cut-elimination on ψ to obtain a proof φ of $\mathrm{sk}(S)$ containing no non-atomic cuts.

Let us remark here that in step 6, any method for cut-elimination for quantifier-free cuts can be used (e.g. reductive methods, "zero-th order" CERES). Furthermore, considering that the instantiations of quantifiers are the core information in a proof, one can even leave out this step as the instantiations in φ and ψ coincide.

Theorem 3.2.19. *Let π be a* **QFC***-proof of S. Then the* **QFC**-CERES *method transforms π into an* **LK***-proof φ of* $\mathrm{sk}(S)$ *such that φ is in atomic-cut normal form.*

Proof. Using Proposition 3.2.12, we convert π to π_{sk}. By Lemma 3.2.17, we can construct a quantifier-free $\mathcal{R}$-refutation γ of $\mathrm{CS}(\pi_{sk})$. By Lemma 3.2.9, from γ we can construct an **LK**-refutation γ' of $\mathrm{CS}(\pi_{sk})$. Every leaf of γ' is either a sequent $A \vdash A$ or an instance C^* of some $C \in \mathrm{CS}(\pi_{sk})$ under a substitution σ. Let $C = \Pi \vdash \Lambda$ and $\mathrm{sk}(S) = \Gamma \vdash \Delta$, then by Definition 3.2.13 we have a cut-free **QFC**-proof ψ_C of $\Gamma, \Pi \vdash \Lambda, \Delta$. Let $C^* = \Pi^* \vdash \Lambda^*$, then by Lemma 3.2.15, we can construct **LK**-proofs ψ_{C^*} of $\Gamma, \Pi^* \vdash \Lambda^*, \Delta$ that contain quantifier-free cuts only. By plugging these proofs onto the leaves of γ' and adding contractions at the end, we obtain an **LK**-proof of $\Gamma \vdash \Delta$ containing quantifier-free cuts only. By applying cut-elimination to this proof, we obtain the desired proof φ. □

Chapter 4

CERES **for higher-order logic:** CERES^ω

In this chapter, we will generalize the CERES method to full higher-order logic. As we have seen in Section 3.1.2, it seems that the proof transformation Skolemization cannot be generalized to yield **LK**-proofs not containing strong quantifiers inferences operating on end-sequent ancestors. For this reason, we now introduce a sequent calculus without eigenvariables.

4.1 The calculus $\mathbf{LK}_{\mathrm{sk}}$

Definition 4.1.1 (Labelled sequents)**.** A *label* is a finite multiset of terms. A *labelled sequent* is a sequent $\mathbf{F}_1, \ldots, \mathbf{F}_n \vdash \mathbf{F}_{n+1}, \ldots, \mathbf{F}_m$ together with labels ℓ_i for $1 \leq i \leq m$; we write $\langle \mathbf{F}_1 \rangle^{\ell_1}, \ldots, \langle \mathbf{F}_n \rangle^{\ell_n} \vdash \langle \mathbf{F}_{n+1} \rangle^{\ell_{n+1}}, \ldots, \langle \mathbf{F}_m \rangle^{\ell_m}$. We identify labelled formulas with empty labels with the respective unlabelled formulas. If S is a labelled sequent, then the *reduct* of S is S where all labels are empty. If $\mathcal{C}$ is a set of labelled sequents, then the reduct of $\mathcal{C}$ is $\{S \mid S \text{ a reduct of some } S' \in \mathcal{C}\}$.

We extend substitutions to labelled sequents: Let σ be a substitution and $S = \langle \mathbf{F}_1 \rangle^{\ell_1}, \ldots, \langle \mathbf{F}_n \rangle^{\ell_n} \vdash \langle \mathbf{F}_{n+1} \rangle^{\ell_{n+1}}, \ldots, \langle \mathbf{F}_m \rangle^{\ell_m}$, then

$$S\sigma = \langle \mathbf{F}_1\sigma \rangle^{\ell_1\sigma}, \ldots, \langle \mathbf{F}_n\sigma \rangle^{\ell_n\sigma} \vdash \langle \mathbf{F}_{n+1}\sigma \rangle^{\ell_{n+1}\sigma}, \ldots, \langle \mathbf{F}_m\sigma \rangle^{\ell_m\sigma}.$$

Labels such as ours are often used to add (syntactic) information to formulas, see [22]. They have been used in a setting very similar to ours in [21].

The purpose of the labels will be twofold: first, they will track quantifier instantiation information throughout prooftrees (as expressed in Proposition 4.1.5). Second, they will enable us to combine resolution refutations and sequent calculus proofs in a certain way — this will be one of the main constructions of the CERES$^\omega$ method; see Lemma 4.3.19.

From now on, we will only consider labelled sequents, and therefore we will call them only sequents. Analogously, we will refer to labelled formula occurrences as formula occurrences. We will denote the union of labels ℓ_1 and ℓ_2 by ℓ_1, ℓ_2. Let $\mathbf{T}$ be a term and ℓ a label, then we denote by $\ell, \mathbf{T}$ the union $\ell \cup \{\mathbf{T}\}$.

Definition 4.1.2 ($\mathbf{LK}_{\mathrm{sk}}$ rules)**.** The following figures are the rules of $\mathbf{LK}_{\mathrm{sk}}$:
Labelled quantifier rules (or $\forall^{sk}$ rules):

$$\frac{\Gamma \vdash \Delta, \langle \mathbf{F}(\mathbf{fS}_1 \dots \mathbf{S}_n)\rangle^{\ell}}{\Gamma \vdash \Delta, \langle \forall_\alpha \mathbf{F}\rangle^{\ell}}\ \forall^{sk}\colon r$$

where $\ell = \mathbf{S}_1, \dots, \mathbf{S}_n$ and, if $\tau(\mathbf{S}_i) = \alpha_i$ for $1 \leq i \leq n$, then $\mathbf{f} \in \mathcal{K}_{\alpha_1,\dots,\alpha_n,\alpha}$ is a Skolem symbol. An application of this rule is called *source inference* of $\mathbf{fS}_1 \dots \mathbf{S}_m$, and $\mathbf{fS}_1 \dots \mathbf{S}_m$ is called the *Skolem term* of this inference. Note that we do *not* impose an eigenvariable or eigenterm restriction on this rule.

$$\frac{\langle \mathbf{FT}\rangle^{\ell,\mathbf{T}}, \Gamma \vdash \Delta}{\langle \forall_\alpha \mathbf{F}\rangle^{\ell}, \Gamma \vdash \Delta}\ \forall^{sk}\colon l$$

$\mathbf{T}$ is called the *substitution term* of this inference. The other rules of $\mathbf{LK}$ are transferred directly to $\mathbf{LK}_{\mathrm{sk}}$:
Propositional rules:

$$\frac{\langle \mathbf{F}\rangle^{\ell}, \Gamma \vdash \Delta \quad \langle \mathbf{G}\rangle^{\ell}, \Pi \vdash \Lambda}{\langle \mathbf{F} \vee \mathbf{G}\rangle^{\ell}, \Gamma, \Pi \vdash \Delta, \Lambda}\ \vee\colon l \qquad \frac{\Gamma \vdash \Delta, \langle \mathbf{F}\rangle^{\ell}}{\Gamma \vdash \Delta, \langle \mathbf{F} \vee \mathbf{G}\rangle^{\ell}}\ \vee\colon r^1$$

The rest of the propositional rules of $\mathbf{LK}$ are adapted analogously.
Structural rules:

$$\frac{\Gamma \vdash \Delta, \langle \mathbf{F}\rangle^{\ell}, \langle \mathbf{F}\rangle^{\ell}}{\Gamma \vdash \Delta, \langle \mathbf{F}\rangle^{\ell}}\ \mathrm{contr}\colon r \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \langle \mathbf{F}\rangle^{\ell}}\ \mathrm{weak}\colon r$$

and analogously for contr: l and weak: l.

$$\frac{\Gamma \vdash \Delta, \langle \mathbf{F}\rangle^{\ell_1} \quad \langle \mathbf{F}\rangle^{\ell_2}, \Pi \vdash \Lambda}{\Gamma, \Pi \vdash \Delta, \Lambda}\ \mathrm{cut}$$

Note that the labels ℓ_1, ℓ_2 are arbitrary.

In this section, we will restrict our attention to the cut-free fragment of the rules of $\mathbf{LK}_{\mathrm{sk}}$. An $\mathbf{LK}_{\mathrm{sk}}$-*tree* is a tree formed according to the rules of

$\mathbf{LK}_{\text{sk}}$ except cut, such that all leaves are of the form $\langle \mathbf{F} \rangle^{\ell_1} \vdash \langle \mathbf{F} \rangle^{\ell_2}$ for some formula $\mathbf{F}$ and some labels ℓ_1, ℓ_2. The *axiom partner* of $\langle \mathbf{F} \rangle^{\ell_1}$ is defined to be $\langle \mathbf{F} \rangle^{\ell_2}$, and vice-versa. Let π be an $\mathbf{LK}_{\text{sk}}$-tree with end-sequent S. If S does not contain Skolem terms or free variables, and all labels in S are empty, then S is called *proper*. If the end-sequent of π is proper, we say that π is proper.

Example 4.1.3. The following figure shows a proper $\mathbf{LK}_{\text{sk}}$-tree of a valid sequent:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\langle S(f(\lambda x.\neg S(x)))\rangle^{\lambda x.\neg S(x)} \vdash \langle S(f(\lambda x.\neg S(x)))\rangle^{\lambda x.\neg S(x)}}{\langle \neg S(f(\lambda x.\neg S(x)))\rangle^{\lambda x.\neg S(x)}, \langle S(f(\lambda x.\neg S(x)))\rangle^{\lambda x.\neg S(x)} \vdash}\ \neg\colon l}{\langle S(f(\lambda x.\neg S(x)))\rangle^{\lambda x.\neg S(x)} \vdash \langle \neg\neg S(f(\lambda x.\neg S(x)))\rangle^{\lambda x.\neg S(x)}}\ \neg\colon r}{\vdash \langle S(f(\lambda x.\neg S(x))) \to \neg\neg S(f(\lambda x.\neg S(x)))\rangle^{\lambda x.\neg S(x)}}\ \to\colon r}{\vdash \langle (\forall z)(S(z) \to \neg\neg S(z))\rangle^{\lambda x.\neg S(x)}}\ \forall^{sk}\colon r}{\vdash \langle (\exists Y)(\forall z)(S(z) \to \neg Y(z))\rangle}\ \exists^{sk}\colon r}{\vdash \langle (\forall X)(\exists Y)(\forall z)(X(z) \to \neg Y(z))\rangle}\ \forall^{sk}\colon r$$

where $S \in \mathcal{K}_{\iota \to o}$, $f \in \mathcal{K}_{\iota \to o, \iota}$, and the substitution term of the $\exists^{sk}\colon r$ is $\lambda x.\neg S(x)$. Note that although the labels in the axiom coincide, this is not required in general.

When writing down $\mathbf{LK}_{\text{sk}}$-trees, we will often leave out the labels to increase readability. So far, we have not called the trees built up using the rules of $\mathbf{LK}_{\text{sk}}$ *proofs*. The reason is that without further restrictions, $\mathbf{LK}_{\text{sk}}$-trees are unsound:

Example 4.1.4. Consider the following $\mathbf{LK}_{\text{sk}}$-tree of $(\exists x)P(x) \vdash (\forall x)P(x)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{P(s) \vdash P(s)}{\vdash P(s), \neg P(s)}\ \neg : r}{\vdash P(s), (\forall x)\neg P(x)}\ \forall^{sk}\colon r}{\neg(\forall x)\neg P(x) \vdash P(s)}\ \neg : l}{\neg(\forall x)\neg P(x) \vdash (\forall x)P(x)}\ \forall^{sk}\colon r$$

where $s \in \mathcal{K}_{\iota}$. The source of unsoundness in this example stems from the fact that in $\mathbf{LK}_{\text{sk}}$-trees, it is possible to use the same Skolem term for distinct and "unrelated" $\forall^{sk}\colon r$ applications.

Towards introducing our global soundness condition, which will be more general than the eigenvariable condition of $\mathbf{LK}$, we introduce some definitions and facts about occurrences in $\mathbf{LK}_{\text{sk}}$-trees.

Proposition 4.1.5. *Let ω be a formula occurrence in a proper* $\mathbf{LK}_{\text{sk}}$*-tree* π *with label* $\{\mathbf{T}_1, \ldots, \mathbf{T}_n\}$*. Then* $\mathbf{T}_1, \ldots, \mathbf{T}_n$ *are exactly the substitution terms of the* $\forall^{sk}\colon l$ *inferences operating on descendants of* ω*.*

Proof. By induction on the number of sequents between ω and the end-sequent of π. If ω occurs in the end-sequent, then it has no descendants and, as π is proper, ω has the empty label.

Assume ω occurs in the premise of an inference. Denote the direct descendant of ω by ω'. If ω occurs in the context, then ω has the same label as ω', the $\forall^{sk}\colon l$ inferences operating on descendants of ω are the same as those operating on descendants of ω', so we conclude with the induction hypothesis. If ω is the auxiliary formula of a propositional inference, a contraction inference, or a $\forall^{sk}\colon r$ inference, the argument is analogous. Finally, assume ω is the auxiliary formula of a $\forall^{sk}\colon l$ inference ρ with substitution term $\mathbf{T}$, and that the label of ω is $\mathbf{T}_1, \ldots, \mathbf{T}_n, \mathbf{T}$. Then the label of ω' is $\mathbf{T}_1, \ldots, \mathbf{T}_n$, and by (IH) these are exactly the substitution terms of the $\forall^{sk}\colon l$ inferences $\rho_1, \ldots, \rho_n$ operating on descendants of ω'. Then the $\forall^{sk}\colon l$ inferences operating on descendants of ω are $\rho_1, \ldots, \rho_n, \rho$, and hence the label of ω is as desired. □

Definition 4.1.6 (Paths)**.** Let $\mu = \mu_1, \ldots, \mu_n$ be a sequence of formula occurrences in an $\mathbf{LK}_{\text{sk}}$-tree. If for all $1 \leq i < n$, μ_i is an immediate ancestor (immediate descendant) of μ_{i+1}, then μ is called a *downwards (upwards) path*. If μ is a downwards (upwards) path ending in an occurrence in the end-sequent (a leaf), then μ is called *maximal*.

Definition 4.1.7 (Homomorphic paths)**.** If ω is a formula occurrence, then denote by $F(\omega)$ the formula at ω. If μ is a sequence of formula occurrences, we define $F(\mu)$ as μ where every formula occurrence ω is replaced by $F(\omega)$, and repetitions are ommited. Two sequences of formula occurrences μ, ν are called *homomorphic* if $F(\mu) = F(\nu)$.

Example 4.1.8. Consider the $\mathbf{LK}_{\text{sk}}$-tree π:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\langle R(a,f(a))\rangle^a \vdash \langle R(a,f(a))\rangle^a}{\vdash \langle R(a,f(a))\rangle^a, \langle \neg R(a,f(a))\rangle^a}\ \neg\colon r}{\vdash \langle R(a,f(a))\rangle^a, \langle R(a,f(a)) \vee \neg R(a,f(a))\rangle^a}\ \vee\colon r^2}{\vdash \langle R(a,f(a)) \vee \neg R(a,f(a))\rangle^a, \langle R(a,f(a)) \vee \neg R(a,f(a))\rangle^a}\ \vee\colon r^1}{\vdash \langle R(a,f(a)) \vee \neg R(a,f(a))\rangle^a}\ \text{contr}\colon r}{\vdash \langle (\forall y)(R(a,y) \vee \neg R(a,y))\rangle^a}\ \forall^{sk}\colon r}{\vdash (\exists x)(\forall y)(R(x,y) \vee \neg R(x,y))}\ \exists^{sk}\colon r$$

π contains two maximal paths μ_1, μ_2:

$$\begin{aligned}
\mu_1 = \quad & \langle R(a, f(a))\rangle^a, \langle \neg R(a, f(a))\rangle^a, \langle R(a, f(a)) \lor \neg R(a, f(a))\rangle^a, \\
& \langle R(a, f(a)) \lor \neg R(a, f(a))\rangle^a, \langle R(a, f(a)) \lor \neg R(a, f(a))\rangle^a, \\
& \langle (\forall y)(R(a, y) \lor \neg R(a, y))\rangle^a, (\exists x)(\forall y)(R(x, y) \lor \neg R(x, y))
\end{aligned}$$

$$\begin{aligned}
\mu_2 = \quad & \langle R(a, f(a))\rangle^a, \langle R(a, f(a))\rangle^a, \langle R(a, f(a))\rangle^a, \\
& \langle R(a, f(a)) \lor \neg R(a, f(a))\rangle^a, \langle R(a, f(a)) \lor \neg R(a, f(a))\rangle^a, \\
& \langle (\forall y)(R(a, y) \lor \neg R(a, y))\rangle^a, (\exists x)(\forall y)(R(x, y) \lor \neg R(x, y))
\end{aligned}$$

$$\begin{aligned}
F(\mu_1) = \quad & \langle R(a, f(a))\rangle^a, \langle \neg R(a, f(a))\rangle^a, \langle R(a, f(a)) \lor \neg R(a, f(a))\rangle^a, \\
& \langle (\forall y)(R(a, y) \lor \neg R(a, y))\rangle^a, (\exists x)(\forall y)(R(x, y) \lor \neg R(x, y))
\end{aligned}$$

$$\begin{aligned}
F(\mu_2) = \quad & \langle R(a, f(a))\rangle^a, \langle R(a, f(a)) \lor \neg R(a, f(a))\rangle^a, \\
& \langle (\forall y)(R(a, y) \lor \neg R(a, y))\rangle^a, (\exists x)(\forall y)(R(x, y) \lor \neg R(x, y))
\end{aligned}$$

Proposition 4.1.9. *Let π be a proper $\mathbf{LK}_{\mathrm{sk}}$-tree, let ρ be a $\forall^{sk}\colon r$ inference in π with Skolem term $\mathbf{S}$ and auxiliary formula α, and let μ be a maximal downwards path starting at α. Then* $\mathrm{FV}(\mathbf{S}) = \mathrm{FV}(\mu)$.

Proof. As π is proper, its end-sequent does not contain free variables. Hence all free variables in μ are contained in substitution terms of $\forall^{sk}\colon l$ inferences, and they are exactly the free variables of $\mathbf{S}$ by Proposition 4.1.5. □

Proposition 4.1.10. *Let α_1, α_2 be formula occurrences. If there exists a downwards path from α_1 to α_2, then it is unique.*

Proof. Every formula occurrence has at most one direct descendant. □

Corollary 4.1.11. *If α is a formula occurrence, then there exists a unique maximal downwards path starting at α.*

Our investigation of paths allows us to define a relation between inferences in a tree that, through paths, are connected in a strong sense.

Definition 4.1.12 (Homomorphic inferences)**.** Let α_1, α_2 be formula occurrences in an $\mathbf{LK}_{\mathrm{sk}}$-tree π. Let c be a contraction inference below both α_1, α_2 with auxiliary occurrences γ_1, γ_2. Then α_1, α_2 are *homomorphic in c* if the downwards paths $\alpha_1, \ldots, \gamma_1$ and $\alpha_2, \ldots, \gamma_2$ exist and are homomorphic. α_1, α_2 are called *homomorphic* if there exists a c such that they are homomorphic in c.

Let ρ_1, ρ_2 be inferences of the same type with auxiliary formula occurrences α_1^1 (α_1^2) and α_2^1 (α_2^2). ρ_1, ρ_2 are called *homomorphic* if there exists a contraction inference c such that α_1^1 and α_2^1 are homomorphic in c and α_1^2 and α_2^2 are homomorphic in c. Call this contraction inference the *uniting contraction* of ρ_1, ρ_2.

Example 4.1.13. Consider the following $\mathbf{LK}_{\mathrm{sk}}$-tree π:

$$
\dfrac{\dfrac{\dfrac{\langle P(s)\rangle^{s} \vdash P(s)}{(\forall x)P(x) \vdash P(s)}\;\forall^{sk}\colon l}{(\forall x)P(x) \vdash (\forall x)P(x)}\;\forall^{sk}\colon r\ (1) \qquad \dfrac{\dfrac{\langle P(s)\rangle^{s} \vdash P(s)}{\langle P(s)\rangle^{s} \vdash (\forall x)P(x)}\;\forall^{sk}\colon r\ (3)}{(\forall x)P(x) \vdash (\forall x)P(x)}\;\forall^{sk}\colon l}{\dfrac{(\forall x)P(x) \lor (\forall x)P(x) \vdash (\forall x)P(x), (\forall x)P(x)}{(\forall x)P(x) \lor (\forall x)P(x) \vdash (\forall x)P(x)}\;\mathrm{contr}\colon r\ (2)}\;\lor\colon l
$$

The inferences $(1), (3)$ in π are homomorphic, and (2) is their uniting contraction. More concretely, let μ be the path from the auxiliary formula of (1) to the auxiliary formula of (2). Let ν be the path from the auxiliary formula of (3) to the auxiliary formula of (2). Then $F(\mu) = P(s), (\forall x)P(x) = F(\nu)$.

On the other hand, consider π':

$$
\dfrac{\dfrac{\dfrac{\langle P(s_1)\rangle^{s_1} \vdash P(s_1)}{(\forall x)P(x) \vdash P(s_1)}\;\forall^{sk}\colon l}{(\forall x)P(x) \vdash (\forall x)P(x)}\;\forall^{sk}\colon r\ (1) \qquad \dfrac{\dfrac{\langle P(s_2)\rangle^{s_2} \vdash P(s_2)}{\langle P(s_2)\rangle^{s_2} \vdash (\forall x)P(x)}\;\forall^{sk}\colon r\ (3)}{(\forall x)P(x) \vdash (\forall x)P(x)}\;\forall^{sk}\colon l}{\dfrac{(\forall x)P(x) \lor (\forall x)P(x) \vdash (\forall x)P(x), (\forall x)P(x)}{(\forall x)P(x) \lor (\forall x)P(x) \vdash (\forall x)P(x)}\;\mathrm{contr}\colon r\ (2)}\;\lor\colon l
$$

In π', there are no homomorphic inferences because the auxiliary formulas of the $\forall^{sk}\colon r$ applications differ: Define μ, ν as above, then $F(\mu) = P(s_1), (\forall x)P(x) \neq P(s_2), (\forall x)P(x) = F(\nu)$.

The previous example motivates the following statement about homomorphic quantifier inferences.

Proposition 4.1.14. *If two $\forall^{sk}\colon r$ inferences are homomorphic, they have identical Skolem terms.*

Proof. Denote the $\forall^{sk}\colon r$ applications by ρ_1, ρ_2. Then there exist homomorphic paths p_1, p_2 starting at the auxiliary formulas of ρ_1, ρ_2 respectively. The second elements of p_1, p_2 are the main formula occurrences of ρ_1, ρ_2 respectively. As p_1, p_2 are homomorphic the formula lists induced by them are equal, therefore ρ_1, ρ_2 have the same auxiliary and main formulas and therefore their Skolem terms are identical. □

Proposition 4.1.15. *The homomorphism relation on inferences is a partial equivalence relation.*

Proof. The homomorphism relation on inferences is symmetric because the homomorphism relation on sequences of formula occurrences is. It is transitive: Assume ρ_1, ρ_2 are homomorphic, and ρ_2, ρ_3 are homomorphic. We assume that ρ_1, ρ_2, ρ_3 are unary inferences, the binary case is analogous. Designate the respective auxiliary formulas by $\alpha_1, \alpha_2, \alpha_3$. Then there is a contraction c on formula occurrences γ_1, γ_2 s.t. the downwards paths $\alpha_1, \ldots, \gamma_1$ and $\alpha_2, \ldots, \gamma_2$ exist and are homomorphic, and there is a contraction c' on formula occurrences γ_2', γ_3 s.t. the paths $\alpha_2, \ldots, \gamma_2'$ and $\alpha_3, \ldots, \gamma_3$ exist and are homomorphic. From the existence of these paths, it follows that c, c' cannot be parallel. W.l.o.g. assume that c is above c', then

$$\alpha_2, \ldots, \gamma_2' = \alpha_2, \ldots, \gamma_2, \gamma_2^*, \ldots, \gamma_2'$$

by Proposition 4.1.10, and there exists a path

$$\alpha_1, \ldots, \gamma_1, \gamma_2^*, \ldots, \gamma_2'.$$

For $i \in \{1, 2\}$, let ω_i be the first formula occurrence from the right in $\alpha_i, \ldots, \gamma_i$ such that $F(\omega_i) \neq F(\gamma_i)$, ρ_1, ρ_3 are homomorphic by the following chain of equalities:

$$\begin{array}{l} F(\alpha_1, \ldots, \gamma_1, \gamma_2^*, \ldots, \gamma_2') = \\ F(\alpha_1, \ldots, \omega_1), F(\gamma_2^*, \ldots, \gamma_2') = \\ F(\alpha_2, \ldots, \omega_2), F(\gamma_2^*, \ldots, \gamma_2') = \\ F(\alpha_2, \ldots, \gamma_2, \ldots, \gamma_2') = \\ F(\alpha_3, \ldots, \gamma_3) \end{array}$$

□

We can now define the notion of an $\mathbf{LK}_{\mathrm{sk}}$-proof, for which we will require the converse of the Proposition 4.1.14 to hold.

Definition 4.1.16 (Weak regularity and $\mathbf{LK}_{\mathrm{sk}}$-proofs)**.** Let π be an $\mathbf{LK}_{\mathrm{sk}}$-tree with end-sequent S. π is *weakly regular* if for all distinct $\forall^{sk}\colon r$ inferences ρ_1, ρ_2 in π: If ρ_1, ρ_2 have identical Skolem terms, then ρ_1, ρ_2 are homomorphic. We say that π is an $\mathbf{LK}_{\mathrm{sk}}$-*proof* if it is weakly regular and proper.

In ordinary **LK**, it follows directly from the definition of regularity that all $\forall\colon r$ inferences in a regular **LK**-tree π fulfill the eigenvariable condition, and thus are **LK**-proofs. Hence the name "weak regularity": inferences are allowed to use the same eigenterm, provided they are homomorphic.

Example 4.1.17. The $\mathbf{LK}_{\mathrm{sk}}$-tree from Example 4.1.3 is (trivially) an $\mathbf{LK}_{\mathrm{sk}}$-proof. Also the first $\mathbf{LK}_{\mathrm{sk}}$-tree from Example 4.1.13 is an $\mathbf{LK}_{\mathrm{sk}}$-proof: the only two $\forall^{sk}\colon r$ applications in the tree are homomorphic.

Finally, consider the following example:

$$
\begin{prooftree}
\AxiomC{$\langle R(s,f(s))\rangle^{s} \vdash \langle R(s,f(s))\rangle^{f(s)}$}
\RightLabel{$\exists^{sk}\colon l$}
\UnaryInfC{$\langle(\exists y)R(s,y)\rangle^{s} \vdash \langle R(s,f(s))\rangle^{f(s)}$}
\AxiomC{$\langle R(s,f(s))\rangle^{s} \vdash \langle R(s,f(s))\rangle^{f(s)}$}
\RightLabel{$\exists^{sk}\colon l$}
\UnaryInfC{$\langle(\exists y)R(s,y)\rangle^{s} \vdash \langle R(s,f(s))\rangle^{f(s)}$}
\RightLabel{$\neg\colon l$}
\UnaryInfC{$\langle(\exists y)R(s,y)\rangle^{s}, \langle\neg R(s,f(s))\rangle^{f(s)} \vdash$}
\RightLabel{$\to\colon l$}
\BinaryInfC{$\langle(\exists y)R(s,y)\rangle^{s}, \langle(\exists y)R(s,y)\rangle^{s}, \langle R(s,f(s)) \to \neg R(s,f(s))\rangle^{f(s)} \vdash$}
\RightLabel{$\forall^{sk}\colon l$}
\UnaryInfC{$(\forall x)(\exists y)R(x,y), \langle(\exists y)R(s,y)\rangle^{s}, \langle R(s,f(s)) \to \neg R(s,f(s))\rangle^{f(s)} \vdash$}
\RightLabel{$\forall^{sk}\colon l$}
\UnaryInfC{$(\forall x)(\exists y)R(x,y), (\forall x)(\exists y)R(x,y), \langle R(s,f(s)) \to \neg R(s,f(s))\rangle^{f(s)} \vdash$}
\RightLabel{contr$\colon l$}
\UnaryInfC{$(\forall x)(\exists y)R(x,y), \langle R(s,f(s)) \to \neg R(s,f(s))\rangle^{f(s)} \vdash$}
\RightLabel{$\forall^{sk}\colon l$}
\UnaryInfC{$(\forall x)(\exists y)R(x,y), (\forall y)(R(s,y) \to \neg R(s,y)) \vdash$}
\RightLabel{$\exists^{sk}\colon l$}
\UnaryInfC{$(\forall x)(\exists y)R(x,y), (\exists x)(\forall y)(R(x,y) \to \neg R(x,y)) \vdash$}
\RightLabel{$\neg\colon r$}
\UnaryInfC{$(\forall x)(\exists y)R(x,y), \vdash \neg(\exists x)(\forall y)(R(x,y) \to \neg R(x,y))$}
\end{prooftree}
$$

where $f \in \mathcal{K}_{\iota,\iota}$ and $s \in \mathcal{K}_{\iota}$.

Denote the upper-left $\exists^{sk}\colon l$ application by ρ_1, the upper-right $\exists^{sk}\colon l$ application by ρ_2, and the bottommost $\exists^{sk}\colon l$ application by ρ_3. ρ_3 is the only $\exists^{sk}\colon l$ application with Skolem term s, so there is nothing to check. On the other hand, ρ_1 and ρ_2 have the same Skolem term $f(s)$. They are indeed homomorphic: the contr$\colon l$ application is their uniting contraction, and the homomorphic paths are

$$
\begin{aligned}
\mu(\rho_1) = \quad & \langle R(s,f(s))\rangle^{s}, \langle(\exists y)R(s,y)\rangle^{s}, \\
& \langle(\exists y)R(s,y)\rangle^{s}, (\forall x)(\exists y)R(s,y), \\
& (\forall x)(\exists y)R(s,y)
\end{aligned}
$$

$$
\begin{aligned}
\mu(\rho_2) = \quad & \langle R(s,f(s))\rangle^{s}, \langle(\exists y)R(s,y)\rangle^{s}, \\
& \langle(\exists y)R(s,y)\rangle^{s}, \langle(\exists y)R(s,y)\rangle^{s}, \\
& \langle(\exists y)R(s,y)\rangle^{s}, (\forall x)(\exists y)R(s,y)
\end{aligned}
$$

because $F(\mu(\rho_1)) = F(\mu(\rho_2)) = \langle R(s,f(s))\rangle^{s}, \langle(\exists y)R(s,y)\rangle^{s}$.

The rest of this section will be devoted to proving that weak regularity still suffices for soundness of $\mathbf{LK}_{\mathrm{sk}}$-proofs.

Definition 4.1.18. Let π be an $\mathbf{LK}_{\mathrm{sk}}$-tree, and ρ an inference in π. Define the *height* of ρ, height(ρ), as the maximal number of sequents between ρ and an axiom in π.

Lemma 4.1.19. *Let* $\mathbf{T}$ *be a Skolem term and* π *be a* $\mathbf{LK}_{\text{sk}}$*-tree of* S *such that* π *does not contain a source inference of* $\mathbf{T}$*. Let* $\mathbf{X}$ *be a variable not occurring in* π*, then there exists an* $\mathbf{LK}_{\text{sk}}$*-tree* $\pi\,[\mathbf{T} \leftarrow \mathbf{X}]$ *of* $S\,[\mathbf{T} \leftarrow \mathbf{X}]$*. Furthermore, if* π *is weakly regular (proper) then* $\pi\,[\mathbf{T} \leftarrow \mathbf{X}]$ *is weakly regular (proper).*

Proof. Let $\sigma = [\mathbf{T} \leftarrow \mathbf{X}]$, and let ρ be an inference in π with conclusion S. By induction on height(ρ), we construct $\mathbf{LK}_{\text{sk}}$-trees π_ρ of $S\sigma$.

1. ρ is an axiom $\langle \mathbf{A} \rangle^{\ell_1} \vdash \langle \mathbf{A} \rangle^{\ell_2}$. Take for π_ρ the axiom $\langle \mathbf{A}\sigma \rangle^{\ell_1 \sigma} \vdash \langle \mathbf{A}\sigma \rangle^{\ell_2 \sigma}$.

2. ρ is a $\forall^{sk}\colon r$ inference

$$\frac{\Gamma \vdash \Delta, \langle \mathbf{FR} \rangle^{\ell}}{\Gamma \vdash \Delta, \langle \forall \mathbf{F} \rangle^{\ell}}\ \forall^{sk}\colon r$$

where $\mathbf{R}$ is the Skolem term of ρ. By (IH) we have a $\mathbf{LK}_{\text{sk}}$-tree ψ of $\Gamma\sigma \vdash \Delta\sigma, \langle \mathbf{FR}\sigma \rangle^{l\sigma}$. Note that $\mathbf{FR}\sigma =_\beta \mathbf{F}\sigma\mathbf{R}\sigma$. Hence we may take for π_ρ

$$\frac{\begin{matrix}(\psi)\\ \Gamma\sigma \vdash \Delta\sigma, \langle \mathbf{F}\sigma\mathbf{R}\sigma \rangle^{\ell\sigma}\end{matrix}}{\Gamma\sigma \vdash \Delta\sigma, \langle \forall \mathbf{F}\sigma \rangle^{\ell\sigma}}\ \forall^{sk}\colon r$$

3. ρ is a $\forall^{sk}\colon l$ inference

$$\frac{\langle \mathbf{FR} \rangle^{\ell,\mathbf{R}}, \Gamma \vdash \Delta}{\langle \forall \mathbf{F} \rangle^{\ell}, \Gamma \vdash \Delta}\ \forall^{sk}\colon l$$

By (IH) we have an $\mathbf{LK}_{\text{sk}}$-tree ψ of $\langle \mathbf{FR}\sigma \rangle^{\ell\sigma,\mathbf{R}\sigma}, \Gamma\sigma \vdash \Delta\sigma$. By the soundness assumption for Skolem terms from [44], $\mathbf{T}$ does not contain variables bound in $\mathbf{F}$, hence $\mathbf{FR}\sigma =_\beta \mathbf{F}\sigma\mathbf{R}\sigma$. Therefore we may take as π_ρ:

$$\frac{\begin{matrix}(\psi)\\ \langle \mathbf{F}\sigma\mathbf{R}\sigma \rangle^{\ell\sigma,\mathbf{R}\sigma}, \Gamma\sigma \vdash \Delta\sigma\end{matrix}}{\langle \forall \mathbf{F}\sigma \rangle^{\ell\sigma}, \Gamma\sigma \vdash \Delta\sigma}\ \forall^{sk}\colon l$$

4. ρ is a structural or propositional inference. As in the previous cases, we simply apply the rule to the tree(s) obtained by hypothesis to obtain π_ρ.

Let ρ be the last inference in π; then we set $\pi\sigma = \pi_\rho$. It remains to show that weak regularity is preserved. As we apply σ on the whole tree, every path μ in $\pi\sigma$ induces a path ν in π such that $\mu = \nu\sigma$. Hence homomorphisms of downwards paths are preserved. □

Example 4.1.20. Consider the following $\mathbf{LK}_{\mathrm{sk}}$-tree π, where $s \in \mathcal{K}_\iota$ and $f \in \mathcal{K}_{\iota,\iota}$:

$$\dfrac{\dfrac{\dfrac{\langle R(s, f(s), s)\rangle^{f(s)} \vdash \langle R(s, f(s), s)\rangle^{s}}{\langle R(s, f(s), s)\rangle^{f(s)} \vdash \langle (\forall x)R(s, x, s)\rangle^{s}}\ \forall^{sk}\colon r}{\langle R(s, f(s), s)\rangle^{f(s)} \vdash (\exists y)(\forall x)R(s, x, y)}\ \exists^{sk}\colon r}{(\forall y)R(s, y, s) \vdash (\exists y)(\forall x)R(s, x, y)}\ \forall^{sk}\colon l$$

Then $\pi\,[s \leftarrow z]$:

$$\dfrac{\dfrac{\dfrac{\langle R(z, f(z), z)\rangle^{f(z)} \vdash \langle R(s, f(z), z)\rangle^{z}}{\langle R(z, f(z), z)\rangle^{f(z)} \vdash \langle (\forall x)R(z, x, z)\rangle^{z}}\ \forall^{sk}\colon r}{\langle R(z, f(z), z)\rangle^{f(z)} \vdash (\exists y)(\forall x)R(z, x, y)}\ \exists^{sk}\colon r}{(\forall y)R(z, y, z) \vdash (\exists y)(\forall x)R(z, x, y)}\ \forall^{sk}\colon l$$

is an $\mathbf{LK}_{\mathrm{sk}}$-tree.

Lemma 4.1.21. *Let ρ, ρ' be homomorphic inferences, and c their uniting contraction. Let $\rho_1, \ldots, \rho_n$ and $\rho'_1, \ldots, \rho'_m$ be the logical inferences operating on descendants of the auxiliary formulas of ρ, ρ' above c. Then $n = m$ and for all $1 \leq i \leq n$, ρ_i and ρ'_i are homomorphic.*

Proof. By induction on n. $n = 0$ is trivial. For the induction step, let μ, μ' be the homomorphic downwards paths from ρ, ρ' respectively to c. Consider ρ_1. As it is a logical inference, its auxiliary formula is different from its main formula. As $F(\mu) = F(\mu')$, there exists the logical inference ρ'_1 of the same type (and even with the same substitution or Skolem term, if applicable), and the downwards paths from ρ_1, ρ'_1 respectively to c exist and are homomorphic. Hence ρ_1, ρ'_1 are homomorphic and we may conclude with the induction hypothesis. □

4.1.1 Sequential Pruning

To show soundness of $\mathbf{LK}_{\mathrm{sk}}$, we will transform $\mathbf{LK}_{\mathrm{sk}}$-proofs into $\mathbf{LK}$-proofs. Roughly, this will be accomplished by permuting inferences and substituting eigenvariables for Skolem terms. In $\mathbf{LK}_{\mathrm{sk}}$-proofs, a certain kind of redundancy may be present: namely, it may be the case that two $\forall^{sk}\colon r$ inferences on a common branch use the same Skolem term. This will prevent an eigenterm condition from holding, and hence in this situation we cannot substitute an eigenvariable for the Skolem term. This subsection is devoted to showing how to eliminate this redundancy.

Definition 4.1.22 (Sequential pruning)**.** Let π be an $\mathbf{LK}_{\mathrm{sk}}$-tree and ρ, ρ' inferences in π. Then ρ, ρ' are called *sequential* if they are on a common branch in π. We define the set of *sequential homomorphic pairs* as

$$\mathrm{SHP}(\pi) = \{\langle \rho, \rho' \rangle \mid \rho, \rho' \text{ homomorphic in } \pi \text{ and } \rho, \rho' \text{ sequential}\}.$$

We say that π is *sequentially pruned* if $\mathrm{SHP}(\pi) = \emptyset$.

Towards pruning sequential homomorphic pairs, we analyze the permutation of contraction inferences over independent inferences:

Definition 4.1.23. Let ρ be an inference above an inference σ. Then ρ and σ are *independent* if the auxiliary formula of σ is not a descendent of the main formula of ρ.

Definition 4.1.24 (Permuting contractions down)**.** We will now define the rewrite relation $\triangleright_c$ for $\mathbf{LK}_{\mathrm{sk}}$-trees π, π', where we assume the inferences contr: $*$ and σ to be independent:

1. If π is

$$\cfrac{\cfrac{\Pi, \Pi, \Gamma \vdash \Delta, \Lambda, \Lambda}{\Pi, \Gamma \vdash \Delta, \Lambda}\,\text{contr:}*}{\Pi, \Gamma' \vdash \Delta', \Lambda}\,\sigma$$

   and π' is

$$\cfrac{\cfrac{\Pi, \Pi, \Gamma \vdash \Delta, \Lambda, \Lambda}{\Pi, \Pi, \Gamma' \vdash \Delta', \Lambda, \Lambda}\,\sigma}{\Pi, \Gamma' \vdash \Delta', \Lambda}\,\text{contr:}*$$

   then $\pi \triangleright_c^1 \pi'$.

2. If π is

$$\cfrac{\cfrac{\Pi, \Pi, \Gamma \vdash \Delta, \Lambda, \Lambda}{\Pi, \Gamma \vdash \Delta, \Lambda}\,\text{contr:}* \qquad \Sigma \vdash \Theta}{\Pi, \Gamma' \vdash \Delta', \Lambda}\,\sigma$$

and π' is

$$\dfrac{\dfrac{\Pi, \Pi, \Gamma \vdash \Delta, \Lambda, \Lambda \qquad \Sigma \vdash \Theta}{\Pi, \Pi, \Gamma' \vdash \Delta', \Lambda, \Lambda}\ \sigma}{\Pi, \Gamma' \vdash \Delta', \Lambda}\ \text{contr}: *$$

then $\pi \rhd_c^1 \pi'$.

3. If π is

$$\dfrac{\Sigma \vdash \Theta \qquad \dfrac{\Pi, \Pi, \Gamma \vdash \Delta, \Lambda, \Lambda}{\Pi, \Gamma \vdash \Delta, \Lambda}\ \text{contr}: *}{\Pi, \Gamma' \vdash \Delta', \Lambda}\ \sigma$$

and π' is

$$\dfrac{\dfrac{\Sigma \vdash \Theta \qquad \Pi, \Pi, \Gamma \vdash \Delta, \Lambda, \Lambda}{\Pi, \Pi, \Gamma' \vdash \Delta', \Lambda, \Lambda}\ \sigma}{\Pi, \Gamma' \vdash \Delta', \Lambda}\ \text{contr}: *$$

then $\pi \rhd_c^1 \pi'$.

The $\rhd_c$ relation is then defined as the transitive and reflexive closure of the compatible closure of the $\rhd_c^1$ relation.

Lemma 4.1.25. *Let π be a weakly regular $\mathbf{LK}_{\mathrm{sk}}$-tree of S. If $\pi \rhd_c \psi$ then ψ is a weakly regular $\mathbf{LK}_{\mathrm{sk}}$-tree of S.*

Proof. By induction on the length of the $\rhd_c$-rewrite sequence. The case of $\pi = \psi$ is trivial, so assume there exists a subtree φ of π such that $\varphi \rhd_c^1 \varphi'$ and ψ is obtained from π by replacing φ by φ'. Then the end-sequent of ψ is the same as that of π. Also weak regularity is preserved: The paths in ψ and π are the same modulo some repetitions. □

Lemma 4.1.26. *Let π be a $\mathbf{LK}_{\mathrm{sk}}$-tree with end-sequent S such that π is not sequentially pruned. Then there exists a $\mathbf{LK}_{\mathrm{sk}}$-tree π' with end-sequent S such that*

$$|\mathrm{SHP}(\pi')| < |\mathrm{SHP}(\pi)|$$

Furthermore, if π is weakly regular, so is π'.

Proof. Consider a sequential homomorphic pair in π with uniting contraction c. By Lemma 4.1.21, there exists a sequential homomorphic pair ρ, ρ' with uniting contraction c such that no logical inference operates on descendants of the auxiliary formulas of ρ, ρ' above c (ρ, ρ' are the lowermost ρ_i, ρ'_j of Lemma 4.1.21, respectively). W.l.o.g. assume that ρ is above ρ'. As no logical inference operates on descendants ω of the auxiliary formula of ρ on the path to c, we can permute all contraction inferences operating on such ω below ρ' using $\triangleright_c$. By Lemma 4.1.25 the resulting tree is weakly regular and its end-sequent is S. Clearly the number of sequential homomorphic pairs stays the same.

For example, if there are two such contractions inferences between ρ and ρ', the situation is

$$\begin{array}{l} \overline{\hphantom{xxxx}}\ \rho \\ \quad\vdots \\ \overline{\hphantom{xxxx}}\ \text{contr}: l \\ \quad\vdots \\ \overline{\hphantom{xxxx}}\ \text{contr}: l \\ \quad\vdots \\ \overline{\hphantom{xxxx}}\ \rho' \end{array}$$

which is transformed to

$$\begin{array}{l} \overline{\hphantom{xxxx}}\ \rho \\ \quad\vdots \\ \overline{\hphantom{xxxx}}\ \rho' \\ \overline{\hphantom{xxxx}}\ \text{contr}: l \\ \overline{\hphantom{xxxx}}\ \text{contr}: l \end{array}$$

Hence we may assume that no inference operates on descendants of the auxiliary formula of ρ between ρ, ρ'. Now distinguish the cases

1. ρ is a unary inference. W.l.o.g. assume that the auxiliary and main formulas of ρ occur on the right. Then the situation is:

$$\dfrac{\Gamma \vdash \Delta, \langle \mathbf{F} \rangle^{\ell_1}}{\Gamma \vdash \Delta, \langle \mathbf{G} \rangle^{\ell_2}}\ \rho$$
$$\vdots$$
$$\dfrac{\Gamma' \vdash \Delta', \langle \mathbf{F} \rangle^{\ell_1}, \langle \mathbf{G} \rangle^{\ell_2}}{\Gamma' \vdash \Delta', \langle \mathbf{G} \rangle^{\ell_2}, \langle \mathbf{G} \rangle^{\ell_2}}\ \rho'$$
$$\vdots$$
$$\dfrac{\Gamma^* \vdash \Delta^*, \langle \mathbf{G} \rangle^{\ell_2}, \langle \mathbf{G} \rangle^{\ell_2}}{\Gamma^* \vdash \Delta^*, \langle \mathbf{G} \rangle^{\ell_2}}\ c$$

We replace this subtree by

$$\dfrac{\dfrac{\begin{array}{c}\Gamma \vdash \Delta, \langle \mathbf{F} \rangle^{\ell_1} \\ \vdots \\ \Gamma' \vdash \Delta', \langle \mathbf{F} \rangle^{\ell_1}, \langle \mathbf{F} \rangle^{\ell_1}\end{array}}{\Gamma' \vdash \Delta', \langle \mathbf{F} \rangle^{\ell_1}}\, c}{\begin{array}{c}\Gamma' \vdash \Delta', \langle \mathbf{G} \rangle^{\ell_2} \\ \vdots \\ \Gamma^* \vdash \Delta^*, \langle \mathbf{G} \rangle^{\ell_2}\end{array}}\, \rho'$$

2. ρ is a $\vee\colon l$ inference. W.l.o.g. the situation is

$$\dfrac{\dfrac{\begin{array}{c}\dfrac{\dfrac{\langle \mathbf{F} \rangle^{\ell}, \Gamma \vdash \Delta \qquad \langle \mathbf{G} \rangle^{\ell}, \Pi \vdash \Lambda}{\langle \mathbf{F} \vee \mathbf{G} \rangle^{\ell}, \Gamma, \Pi \vdash \Delta, \Lambda}\,\rho}{} \\ \vdots \\ \langle \mathbf{F} \rangle^{\ell}, \langle \mathbf{F} \vee \mathbf{G} \rangle^{\ell}, \Gamma^* \vdash \Delta^* \end{array} \qquad \langle \mathbf{G} \rangle^{\ell}, \Pi^* \vdash \Lambda^*}{\begin{array}{c}\langle \mathbf{F} \vee \mathbf{G} \rangle^{\ell}, \langle \mathbf{F} \vee \mathbf{G} \rangle^{\ell}, \Gamma^*, \Pi^* \vdash \Delta^*, \Lambda^* \\ \vdots \\ \langle \mathbf{F} \vee \mathbf{G} \rangle^{\ell}, \langle \mathbf{F} \vee \mathbf{G} \rangle^{\ell}, \Gamma^+ \vdash \Delta^+\end{array}}\,\rho'}{\langle \mathbf{F} \vee \mathbf{G} \rangle^{\ell}, \Gamma^+ \vdash \Delta^+}\, c$$

This is transformed to

$$\dfrac{\dfrac{\begin{array}{c}\dfrac{\langle \mathbf{F} \rangle^{\ell}, \Gamma \vdash \Delta}{\langle \mathbf{F} \rangle^{\ell}, \Gamma, \Pi \vdash \Delta, \Lambda}\,\text{weak:}* \\ \vdots \\ \langle \mathbf{F} \rangle^{\ell}, \langle \mathbf{F} \rangle^{\ell}, \Gamma^* \vdash \Delta^*\end{array}}{\langle \mathbf{F} \rangle^{\ell}, \Gamma^* \vdash \Delta^*}\, c \qquad \langle \mathbf{G} \rangle^{\ell}, \Pi^* \vdash \Lambda^*}{\begin{array}{c}\langle \mathbf{F} \vee \mathbf{G} \rangle^{\ell}, \Gamma^*, \Pi^* \vdash \Delta^*, \Lambda^* \\ \vdots \\ \langle \mathbf{F} \vee \mathbf{G} \rangle^{\ell}, \Gamma^+ \vdash \Delta^+\end{array}}\,\rho'$$

As we only permute contractions and delete inferences, weak regularity is preserved by this transformation. Furthermore, consider a sequential homomorphic pair $\langle \sigma, \sigma' \rangle$ in π' (w.l.o.g. we consider the case that ρ is $\vee\colon l$). Clearly σ, σ' also exist in π and $\langle \sigma, \sigma' \rangle$ is a homomorphic pair in π (if its uniting contraction in π' is c in the second figure, then the c in the first figure is its uniting contraction in π). It is sequential since we have not changed the branching structure of the tree (except for deleting a subtree from π to obtain π').

Hence the number of sequentially homomorphic pairs is reduced, which was to show. □

Lemma 4.1.27 (Sequential Pruning)**.** *Let π be a* $\mathbf{LK}_{\mathrm{sk}}$*-tree of S, then there exists* $\mathbf{LK}_{\mathrm{sk}}$*-tree π' of S s.t. π' is sequentially pruned. Furthermore, if π is weakly regular, so is π'.*

Proof. Repeated application of Lemma 4.1.26 does the job. □

Example 4.1.28. Consider the $\mathbf{LK}_{\mathrm{sk}}$-tree π:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(s_1) \vdash P(s_1) \qquad P(s_1) \vdash P(s_1)}{P(s_1) \vee P(s_1) \vdash P(s_1), P(s_1)}\,\vee\colon l}{P(s_1) \vee P(s_1) \vdash P(s_1), (\forall x)P(x)}\,\forall^{sk}\colon r}{P(s_1) \vee P(s_1) \vdash (\forall x)P(x), (\forall x)P(x)}\,\forall^{sk}\colon r \qquad \dfrac{Q(t_1) \vdash Q(t_1)}{(\forall x)Q(x) \vdash Q(t_1)}\,\forall^{sk}\colon l}{P(s_1) \vee P(s_1), (\forall x)P(x) \to (\forall x)Q(x) \vdash (\forall x)P(x), Q(t_1)}\,\to\colon l \qquad \dfrac{Q(t_2) \vdash Q(t_2)}{(\forall x)Q(x) \vdash Q(t_2)}\,\forall^{sk}\colon l}{P(s_1) \vee P(s_1), (\forall x)P(x) \to (\forall x)Q(x), (\forall x)P(x) \to (\forall x)Q(x) \vdash Q(t_1), Q(t_2)}\,\to\colon l}{P(s_1) \vee P(s_1), (\forall x)P(x) \to (\forall x)Q(x) \vdash Q(t_1), Q(t_2)}\,\text{contr}\colon l$$

where $s_1, s_2 \in \mathcal{K}_\iota$.

Denote the upper-left $\forall^{sk}\colon r$ application by ρ_1, the $\forall^{sk}\colon r$ application directly below ρ_1 by ρ_2, the upper $\to\colon l$ application by η_1 and the lower $\to\colon l$ application by η_2. Then

$$\mathrm{SHP}(\pi) = \{\{\rho_1, \rho_2\}, \{\eta_1, \eta_2\}\}$$

and the contr: l application is the uniting contraction of both pairs. We apply Lemma 4.1.26, removing $\{\eta_1, \eta_2\}$ and obtaining π':

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(s_1) \vdash P(s_1) \qquad P(s_1) \vdash P(s_1)}{P(s_1) \vee P(s_1) \vdash P(s_1), P(s_1)}\,\vee\colon l}{P(s_1) \vee P(s_1) \vdash P(s_1), (\forall x)P(x)}\,\forall^{sk}\colon r}{P(s_1) \vee P(s_1) \vdash (\forall x)P(x), (\forall x)P(x)}\,\forall^{sk}\colon r}{P(s_1) \vee P(s_1) \vdash (\forall x)P(x), (\forall x)P(x), Q(t_1)}\,\text{weak}\colon r}{P(s_1) \vee P(s_1) \vdash (\forall x)P(x), Q(t_1)}\,\text{contr}\colon r \qquad \dfrac{Q(t_2) \vdash Q(t_2)}{(\forall x)Q(x) \vdash Q(t_2)}\,\forall^{sk}\colon l}{P(s_1) \vee P(s_1), (\forall x)P(x) \to (\forall x)Q(x) \vdash Q(t_1), Q(t_2)}\,\to\colon l$$

such that

$$\mathrm{SHP}(\pi') = \{\{\rho_1, \rho_2\}\}$$

We apply Lemma 4.1.26 again, removing $\{\rho_1, \rho_2\}$ and obtaining the sequentially pruned π'':

$$
\begin{prooftree}
\AxiomC{$P(s_1) \vdash P(s_1)$}
\AxiomC{$P(s_1) \vdash P(s_1)$}
\RightLabel{$\vee\colon l$}
\BinaryInfC{$P(s_1) \vee P(s_1) \vdash P(s_1), P(s_1)$}
\RightLabel{contr$\colon r$}
\UnaryInfC{$P(s_1) \vee P(s_1) \vdash P(s_1)$}
\RightLabel{$\forall^{sk}\colon r$}
\UnaryInfC{$P(s_1) \vee P(s_1) \vdash (\forall x)P(x)$}
\RightLabel{weak$\colon r$}
\UnaryInfC{$P(s_1) \vee P(s_1) \vdash (\forall x)P(x), Q(t_1)$}
\AxiomC{$Q(t_2) \vdash Q(t_2)$}
\RightLabel{$\forall^{sk}\colon l$}
\UnaryInfC{$(\forall x)Q(x) \vdash Q(t_2)$}
\RightLabel{$\to\colon l$}
\BinaryInfC{$P(s_1) \vee P(s_1), (\forall x)P(x) \to (\forall x)Q(x) \vdash Q(t_1), Q(t_2)$}
\end{prooftree}
$$

4.1.2 Soundness of $\mathbf{LK}_{\mathrm{sk}}$

The main result of this subsection will be to show that $\mathbf{LK}_{\mathrm{sk}}$-proofs can be translated into **LK**-proofs. The proof will be effective, and will be based on permuting inferences and pruning. To this end, we will analyze the permutation of inferences in $\mathbf{LK}_{\mathrm{sk}}$-trees. Such an analysis is often useful, see for example [60] for the case of a first-order sequent calculus. In $\mathbf{LK}_{\mathrm{sk}}$, we have more freedom in the permutation of inferences since we do not have to consider an eigenvariable condition, although we will want to preserve weak regularity.

To ease the following case distinctions, we introduce the following notation:

$$\Gamma, A^1 = \Gamma, A$$
$$\Gamma, A^0 = \Gamma$$

and let $i, i_1, \ldots, i_4 \in \{0, 1\}$, $\bar{x} = |x - 1|$. In the following transformations, we do not display the labels of the labelled formula occurrences since we always leave them unchanged (what this means exactly will be clear from the context).

Definition 4.1.29 (The relation $\triangleright_u$)**.** This definition shows how to permute down a unary logical inference ρ over an inference σ, assuming that ρ and σ are independent. In case 1, σ is a unary logical inference, in case 2 σ is a weakening inference, in case 3 σ is a contraction inference, and in cases 4–5 σ is an $\vee\colon l$ inference. We define a relation $\triangleright_u^1$ between $\mathbf{LK}_{\mathrm{sk}}$-trees π and π':

1. If π is

$$\dfrac{\dfrac{\mathbf{F}^{i_1}, \mathbf{G}^{i_2}, \Gamma \vdash \Delta, \mathbf{G}^{\bar{i_2}}, \mathbf{F}^{\bar{i_1}}}{\mathbf{M}^{i_3}, \mathbf{G}^{i_2}, \Gamma \vdash \Delta, \mathbf{G}^{\bar{i_2}}, \mathbf{M}^{\bar{i_3}}}\ \rho}{\mathbf{M}^{i_3}, \mathbf{N}^{i_4}, \Gamma \vdash \Delta, \mathbf{N}^{\bar{i_4}}, \mathbf{M}^{\bar{i_3}}}\ \sigma$$

and π' is

$$\dfrac{\dfrac{\mathbf{F}^{i_1}, \mathbf{G}^{i_2}, \Gamma \vdash \Delta, \mathbf{G}^{\bar{i_2}}, \mathbf{F}^{\bar{i_1}}}{\mathbf{F}^{i_1}, \mathbf{N}^{i_4}, \Gamma \vdash \Delta, \mathbf{N}^{\bar{i_4}}, \mathbf{F}^{\bar{i_1}}}\ \sigma}{\mathbf{M}^{i_1}, \mathbf{N}^{i_2}, \Gamma \vdash \Delta, \mathbf{N}^{\bar{i_2}}, \mathbf{M}^{\bar{i_1}}}\ \rho$$

then $\pi \rhd_u^1 \pi'$.

2. If π is

$$\dfrac{\dfrac{\mathbf{F}^{i_1}, \Gamma \vdash \Delta, \mathbf{F}^{\bar{i_1}}}{\mathbf{M}^{i_2}, \Gamma \vdash \Delta, \mathbf{M}^{\bar{i_2}}}\ \rho}{\mathbf{N}^{i_3}, \mathbf{M}^{i_2}, \Gamma \vdash \Delta, \mathbf{M}^{\bar{i_2}}, \mathbf{N}^{\bar{i_3}}}\ \sigma\ (\text{weak:} *)$$

and π' is

$$\dfrac{\dfrac{\mathbf{F}^{i_1}, \Gamma \vdash \Delta, \mathbf{F}^{\bar{i_1}}}{\mathbf{N}^{i_3}, \mathbf{F}^{i_1}, \Gamma \vdash \Delta, \mathbf{F}^{\bar{i_1}}, \mathbf{N}^{\bar{i_3}}}\ \sigma\ (\text{weak:} *)}{\mathbf{N}^{i_3}, \mathbf{M}^{i_2}, \Gamma \vdash \Delta, \mathbf{M}^{\bar{i_2}}, \mathbf{N}^{\bar{i_3}}}\ \rho$$

then $\pi \rhd_u^1 \pi'$.

3. If π is

$$\dfrac{\dfrac{\mathbf{F}^{i_1}, \mathbf{G}^{i_2}, \mathbf{G}^{i_2}, \Gamma \vdash \Delta, \mathbf{G}^{\bar{i_2}}, \mathbf{G}^{\bar{i_2}}, \mathbf{F}^{\bar{i_1}}}{\mathbf{M}^{i_3}, \mathbf{G}^{i_2}, \mathbf{G}^{i_2}, \Gamma \vdash \Delta, \mathbf{G}^{\bar{i_2}}, \mathbf{G}^{\bar{i_2}}, \mathbf{M}^{\bar{i_3}}}\ \rho}{\mathbf{M}^{i_3}, \mathbf{G}^{i_2}, \Gamma \vdash \Delta, \mathbf{G}^{\bar{i_2}}, \mathbf{M}^{\bar{i_3}}}\ \sigma\ (\text{contr:} *)$$

and π' is

$$\dfrac{\dfrac{\mathbf{F}^{i_1}, \mathbf{G}^{i_2}, \mathbf{G}^{i_2}, \Gamma \vdash \Delta, \mathbf{G}^{\bar{i_2}}, \mathbf{G}^{\bar{i_2}}, \mathbf{F}^{\bar{i_1}}}{\mathbf{F}^{i_1}, \mathbf{G}^{i_2}, \Gamma \vdash \Delta, \mathbf{G}^{\bar{i_2}}, \mathbf{F}^{\bar{i_1}}}\ \sigma\ (\text{contr:} *)}{\mathbf{M}^{i_3}, \mathbf{G}^{i_2}, \Gamma \vdash \Delta, \mathbf{G}^{\bar{i_2}}, \mathbf{M}^{\bar{i_3}}}\ \rho$$

then $\pi \rhd_u^1 \pi'$.

4. If π is

$$\cfrac{\cfrac{\mathbf{F}^{i_1}, \mathbf{G}_1, \Gamma \vdash \Delta, \mathbf{F}^{\bar{i_1}}}{\mathbf{M}^{i_2}, \mathbf{G}_1, \Gamma \vdash \Delta, \mathbf{M}^{\bar{i_2}}}\,\rho \qquad \mathbf{G}_2, \Pi \vdash \Lambda}{\mathbf{G}_1 \vee \mathbf{G}_2, \mathbf{M}^{i_2}, \Gamma, \Pi \vdash \Delta, \Lambda, \mathbf{M}^{\bar{i_2}}}\,\sigma$$

and π' is

$$\cfrac{\cfrac{\mathbf{F}^{i_1}, \mathbf{G}_1, \Gamma \vdash \Delta, \mathbf{F}^{\bar{i_1}} \qquad \mathbf{G}_2, \Pi \vdash \Lambda}{\mathbf{G}_1 \vee \mathbf{G}_2, \mathbf{F}^{i_1}, \Gamma, \Pi \vdash \Delta, \Lambda, \mathbf{F}^{\bar{i_1}}}\,\sigma}{\mathbf{G}_1 \vee \mathbf{G}_2, \mathbf{M}^{i_2}, \Gamma, \Pi \vdash \Delta, \Lambda, \mathbf{M}^{\bar{i_2}}}\,\rho$$

then $\pi \rhd_u^1 \pi'$.

5. If π is

$$\cfrac{\mathbf{G}_1, \Gamma \vdash \Delta \qquad \cfrac{\mathbf{F}^{i_1}, \mathbf{G}_2, \Pi \vdash \Lambda, \mathbf{F}^{\bar{i_1}}}{\mathbf{M}^{i_2}, \mathbf{G}_2, \Pi \vdash \Lambda, \mathbf{M}^{\bar{i_2}}}\,\rho}{\mathbf{G}_1 \vee \mathbf{G}_2, \mathbf{M}^{i_2}, \Gamma, \Pi \vdash \Delta, \Lambda, \mathbf{M}^{\bar{i_2}}}\,\sigma$$

and π' is

$$\cfrac{\cfrac{\mathbf{G}_1, \Gamma \vdash \Delta \qquad \mathbf{F}^{i_1}, \mathbf{G}_2, \Pi \vdash \Lambda, \mathbf{F}^{\bar{i_1}}}{\mathbf{G}_1 \vee \mathbf{G}_2, \mathbf{F}^{i_1}, \Gamma, \Pi \vdash \Delta, \Lambda, \mathbf{F}^{\bar{i_1}}}\,\sigma}{\mathbf{G}_1 \vee \mathbf{G}_2, \mathbf{M}^{i_2}, \Gamma, \Pi \vdash \Delta, \Lambda, \mathbf{M}^{\bar{i_2}}}\,\rho$$

then $\pi \rhd_u^1 \pi'$.

Finally, we define the $\rhd_u$ relation as the transitive and reflexive closure of the compatible closure of the $\rhd_u^1$ relation.

Lemma 4.1.30. *Let π be a weakly regular $\mathbf{LK}_{\text{sk}}$-tree of S. If $\pi \rhd_u \psi$ then ψ is a weakly regular $\mathbf{LK}_{\text{sk}}$-tree of S.*

Proof. By induction on the length of the $\rhd_u$-rewrite sequence. The case of $\pi = \psi$ is trivial, so assume there exists a subtree φ of π such that $\varphi \rhd_u^1 \varphi'$ and ψ is obtained from π by replacing φ by φ'. Then the end-sequent of ψ is the same as that of π. Also weak regularity is preserved since the paths in ψ and π are the same modulo some repetitions. □

Definition 4.1.31 (Permuting binary inferences downwards)**.** This definition shows how to permute down a $\vee\colon l$ inference ρ (the only binary inference in $\mathbf{LK}_{\mathrm{sk}}$), together with some contractions the auxiliary formulas of which come from both permises of ρ. In the prooftrees, the indicated occurrences of $\mathbf{F}_1$ and $\mathbf{F}_2$ will be the auxiliary occurrences of ρ. We will now define the rewrite relation $\rhd_b$ on $\mathbf{LK}_{\mathrm{sk}}$-trees, where we assume ρ and σ to be independent. Cases 1–3 treat the case of σ being a unary logical inference, in case 4 σ is a weakening inference, in cases 5–6 σ is a contraction inference, and in cases 7–9 σ is $\vee\colon l$.

1. If π is

$$\dfrac{\dfrac{\dfrac{\mathbf{F}_1, \Pi, \Gamma_1, \mathbf{G}^{i_1} \vdash \Delta_1, \mathbf{G}^{\bar{i_1}}, \Lambda \qquad \mathbf{F}_2, \Pi, \Gamma_2 \vdash \Delta_2, \Lambda}{\mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \Pi, \Gamma_1, \Gamma_2, \mathbf{G}^{i_1} \vdash \Delta_1, \mathbf{G}^{\bar{i_1}}, \Delta_2, \Lambda, \Lambda}\,\rho}{\mathbf{G}^{i_1}, \mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \Lambda, \mathbf{G}^{\bar{i_1}}}\,\mathrm{contr}\colon *}{\mathbf{M}^{i_2}, \mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \Lambda, \mathbf{M}^{\bar{i_2}}}\,\sigma$$

   and π' is

$$\dfrac{\dfrac{\dfrac{\mathbf{G}^{i_1}, \mathbf{F}_1, \Pi, \Gamma_1 \vdash \Delta_1, \Lambda, \mathbf{G}^{\bar{i_1}}}{\mathbf{M}^{i_2}, \mathbf{F}_1, \Pi, \Gamma_1 \vdash \Delta_1, \Lambda, \mathbf{F}_1, \mathbf{M}^{\bar{i_2}}}\,\sigma \qquad \mathbf{F}_2, \Pi, \Gamma_2 \vdash \Delta_2, \Lambda}{\mathbf{F}_1 \vee \mathbf{F}_2, \mathbf{M}^{i_2}, \Pi, \Pi, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \Lambda, \Lambda, \mathbf{M}^{\bar{i_2}}}\,\rho}{\mathbf{F}_1 \vee \mathbf{F}_2, \mathbf{M}^{i_2}, \Pi, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \Lambda, \mathbf{M}^{\bar{i_2}}}\,\mathrm{contr}\colon *$$

   then $\pi \rhd_b^1 \pi'$.

2. If π is

$$\dfrac{\dfrac{\dfrac{\mathbf{F}_1, \Pi, \Gamma_1 \vdash \Delta_1, \Lambda \qquad \mathbf{F}_2, \Pi, \Gamma_2, \mathbf{G}^{i_1} \vdash \Delta_2, \Lambda, \mathbf{G}^{\bar{i_1}}}{\mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \Pi, \Gamma_1, \Gamma_2, \mathbf{G}^{i_1} \vdash \Delta_1, \Delta_2, \Lambda, \Lambda, \mathbf{G}^{\bar{i_1}}}\,\rho}{\mathbf{G}^{i_1}, \mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \Lambda, \mathbf{G}^{\bar{i_1}}}\,\mathrm{contr}\colon *}{\mathbf{M}^{i_2}, \mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \Lambda, \mathbf{M}^{\bar{i_2}}}\,\sigma$$

and π' is

$$\dfrac{\dfrac{\mathbf{F}_1, \Pi, \Gamma_1 \vdash \Delta_1, \Lambda \qquad \dfrac{\mathbf{G}^{i_1}, \mathbf{F}_2, \Pi, \Gamma_2 \vdash \Delta_2, \Lambda, \mathbf{G}^{\bar{i_1}}}{\mathbf{M}^{i_2}, \mathbf{F}_2, \Pi, \Gamma_2 \vdash \Delta_2, \Lambda, \mathbf{M}^{\bar{i_2}}}\,\sigma}{\mathbf{F}_1 \vee \mathbf{F}_2, \mathbf{M}^{i_2}, \Pi, \Pi, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \Lambda, \Lambda, \mathbf{M}^{\bar{i_2}}}\,\rho}{\mathbf{F}_1 \vee \mathbf{F}_2, \mathbf{M}^{i_2}, \Pi, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \Lambda, \mathbf{M}^{\bar{i_2}}}\,\text{contr}: *$$

then $\pi \triangleright_b^1 \pi'$.

3. If π is

$$\dfrac{\dfrac{\dfrac{\mathbf{F}_1, \Pi, \mathbf{G}^{i_1}, \Gamma_1 \vdash \Delta_1, \Lambda, \mathbf{G}^{\bar{i_1}} \qquad \mathbf{F}_2, \Pi, \mathbf{G}^{i_1}, \Gamma_2 \vdash \Delta_2, \Lambda, \mathbf{G}^{\bar{i_1}}}{\mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \mathbf{G}^{i_1}, \Pi, \mathbf{G}^{i_1}, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \Lambda, \mathbf{G}^{\bar{i_1}}, \Lambda, \mathbf{G}^{\bar{i_1}}}\,\rho}{\mathbf{G}^{i_1}, \mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \Lambda, \mathbf{G}^{\bar{i_1}}}\,\text{contr}: *}{\mathbf{M}^{i_2}, \mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \Lambda, \mathbf{M}^{\bar{i_2}}}\,\sigma$$

and π' is

$$\dfrac{\dfrac{\dfrac{\mathbf{G}^{i_1}, \mathbf{F}_1, \Pi, \Gamma_1 \vdash \Delta_1, \Lambda, \mathbf{G}^{\bar{i_1}}}{\mathbf{M}^{i_2}, \mathbf{F}_1, \Pi, \Gamma_1 \vdash \Delta_1, \Lambda, \mathbf{M}^{\bar{i_2}}}\,\sigma \qquad \dfrac{\mathbf{G}^{i_1}, \mathbf{F}_2, \Pi, \Gamma_2 \vdash \Delta_2, \Lambda, \mathbf{G}^{\bar{i_1}}}{\mathbf{M}^{i_2}, \mathbf{F}_2, \Pi, \Gamma_2 \vdash \Delta_2, \Lambda, \mathbf{M}^{\bar{i_2}}}\,\sigma}{\mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \mathbf{M}^{i_2}, \Pi, \mathbf{M}^{i_2}, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \Lambda, \mathbf{M}^{\bar{i_2}}, \Lambda, \mathbf{M}^{\bar{i_2}}}\,\rho}{\mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \mathbf{M}^{i_2}, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \Lambda, \mathbf{M}^{\bar{i_2}}}\,\text{contr}: *$$

then $\pi \triangleright_b^1 \pi'$.

4. If π is

$$\dfrac{\dfrac{\dfrac{\mathbf{F}_1, \Pi, \Gamma_1 \vdash \Delta_1, \Lambda \qquad \mathbf{F}_2, \Pi, \Gamma_2 \vdash \Delta_2, \Lambda}{\mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \Pi, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \Lambda, \Lambda}\,\rho}{\mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \Lambda}\,\text{contr}: *}{\mathbf{M}^{i}, \mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \Lambda, \mathbf{M}^{\bar{i}}}\,\text{weak}: *$$

and π' is

$$\dfrac{\dfrac{\dfrac{\mathbf{F}_1, \Pi, \Gamma_1 \vdash \Delta_1, \Lambda}{\mathbf{M}^{i}, \mathbf{F}_1, \Pi, \Gamma_1 \vdash \Delta_1, \Lambda, \mathbf{M}^{\bar{i}}}\,\text{weak}: * \qquad \mathbf{F}_2, \Pi, \Gamma_2 \vdash \Delta_2, \Lambda}{\mathbf{F}_1 \vee \mathbf{F}_2, \mathbf{M}^{i}, \Pi, \Pi, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \Lambda, \Lambda, \mathbf{M}^{\bar{i}}}\,\rho}{\mathbf{F}_1 \vee \mathbf{F}_2, \mathbf{M}^{i}, \Pi, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \Lambda, \mathbf{M}^{\bar{i}}}\,\text{contr}: *$$

then $\pi \triangleright_b^1 \pi'$.

5. If π is

$$\dfrac{\dfrac{\dfrac{\mathbf{F}_1, \Pi, \Gamma_1, \mathbf{G}^i, \mathbf{G}^i \vdash \Delta_1, \Lambda, \mathbf{G}^{\bar{i}}, \mathbf{G}^{\bar{i}} \qquad \mathbf{F}_2, \Pi, \Gamma_2 \vdash \Delta_2, \Lambda}{\mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \Pi, \Gamma_1, \Gamma_2, \mathbf{G}^i, \mathbf{G}^i \vdash \Delta_1, \Delta_2, \Lambda, \Lambda, \mathbf{G}^{\bar{i}}, \mathbf{G}^{\bar{i}}}\,\rho}{\mathbf{G}^i, \mathbf{G}^i, \mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \Lambda, \mathbf{G}^{\bar{i}}, \mathbf{G}^{\bar{i}}}\,\text{contr:}*}{\mathbf{G}^i, \mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \Lambda, \mathbf{G}^{\bar{i}}}\,\sigma$$

then π' is

$$\dfrac{\dfrac{\dfrac{\mathbf{F}_1, \Pi, \Gamma_1, \mathbf{G}^i, \mathbf{G}^i \vdash \Delta_1, \Lambda, \mathbf{G}^{\bar{i}}, \mathbf{G}^{\bar{i}}}{\mathbf{F}_1, \Pi, \Gamma_1, \mathbf{G}^i \vdash \Delta_1, \Lambda, \mathbf{G}^{\bar{i}}}\,\sigma \qquad \mathbf{F}_2, \Pi, \Gamma_2 \vdash \Delta_2, \Lambda}{\mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \Pi, \Gamma_1, \Gamma_2, \mathbf{G}^i \vdash \Delta_1, \Delta_2, \Lambda, \Lambda, \mathbf{G}^{\bar{i}}}\,\rho}{\mathbf{G}^i, \mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \Lambda, \mathbf{G}^{\bar{i}}}\,\text{contr:}*$$

6. If π is

$$\dfrac{\dfrac{\dfrac{\mathbf{F}_1, \Pi, \Gamma_1 \vdash \Delta_1, \Lambda \qquad \mathbf{F}_2, \Pi, \Gamma_2, \mathbf{G}^i, \mathbf{G}^i \vdash \Delta_2, \Lambda, \mathbf{G}^{\bar{i}}, \mathbf{G}^{\bar{i}}}{\mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \Pi, \Gamma_1, \Gamma_2, \mathbf{G}^i, \mathbf{G}^i \vdash \Delta_1, \Delta_2, \Lambda, \Lambda, \mathbf{G}^{\bar{i}}, \mathbf{G}^{\bar{i}}}\,\rho}{\mathbf{G}^i, \mathbf{G}^i, \mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \Lambda, \mathbf{G}^{\bar{i}}, \mathbf{G}^{\bar{i}}}\,\text{contr:}*}{\mathbf{G}^i, \mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \Lambda, \mathbf{G}^{\bar{i}}}\,\sigma$$

then π' is

$$\dfrac{\dfrac{\mathbf{F}_1, \Pi, \Gamma_1 \vdash \Delta_1, \Lambda \qquad \dfrac{\mathbf{F}_2, \Pi, \Gamma_2, \mathbf{G}^i, \mathbf{G}^i \vdash \Delta_2, \Lambda, \mathbf{G}^{\bar{i}}, \mathbf{G}^{\bar{i}}}{\mathbf{F}_2, \Pi, \Gamma_2, \mathbf{G}^i \vdash \Delta_2, \Lambda, \mathbf{G}^{\bar{i}}}\,\sigma}{\mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \Pi, \Gamma_1, \Gamma_2, \mathbf{G}^i \vdash \Delta_1, \Delta_2, \Lambda, \Lambda, \mathbf{G}^{\bar{i}}}\,\rho}{\mathbf{G}^i, \mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \Lambda, \mathbf{G}^{\bar{i}}}\,\text{contr:}*$$

7. If π is

$$\dfrac{\dfrac{\dfrac{\mathbf{F}_1, \Pi, \Gamma_1, \mathbf{G}_1 \vdash \Delta_1, \Lambda \qquad \mathbf{F}_2, \Pi, \Gamma_2 \vdash \Delta_2, \Lambda}{\mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \Pi, \Gamma_1, \mathbf{G}_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \Lambda, \Lambda}\,\rho}{\mathbf{G}_1, \mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \Lambda}\,\text{contr:}* \qquad \mathbf{G}_2, \Sigma \vdash \Theta}{\mathbf{G}_1 \vee \mathbf{G}_2, \mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \Gamma_1, \Gamma_2, \Sigma \vdash \Theta, \Delta_1, \Delta_2, \Lambda}\,\sigma$$

and π' is

$$
\dfrac{\dfrac{\dfrac{\mathbf{G}_1, \mathbf{F}_1, \Pi, \Gamma_1 \vdash \Delta_1, \Lambda \qquad \mathbf{G}_2, \Sigma \vdash \Theta}{\mathbf{G}_1 \vee \mathbf{G}_2, \mathbf{F}_1, \Pi, \Gamma_1, \Sigma \vdash \Theta, \Delta_1, \Lambda}\,\sigma \qquad \mathbf{F}_2, \Pi, \Gamma_2 \vdash \Delta_2, \Lambda}{\mathbf{F}_1 \vee \mathbf{F}_2, \mathbf{G}_1 \vee \mathbf{G}_2, \Pi, \Pi, \Gamma_1, \Gamma_2, \Sigma \vdash \Theta, \Delta_1, \Delta_2, \Lambda, \Lambda}\,\rho}{\mathbf{F}_1 \vee \mathbf{F}_2, \mathbf{G}_1 \vee \mathbf{G}_2, \Pi, \Gamma_1, \Gamma_2, \Sigma \vdash \Theta, \Delta_1, \Delta_2, \Lambda}\,\text{contr:}\ast
$$

then $\pi \triangleright_b^1 \pi'$.

8. If π is

$$
\dfrac{\dfrac{\dfrac{\mathbf{F}_1, \Pi, \Gamma_1 \vdash \Delta_1, \Lambda \qquad \mathbf{F}_2, \Pi, \Gamma_2, \mathbf{G}_1 \vdash \Delta_2, \Lambda}{\mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \Pi, \Gamma_1, \Gamma_2, \mathbf{G}_1 \vdash \Delta_1, \Delta_2, \Lambda, \Lambda}\,\rho}{\mathbf{G}_1, \mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \Lambda}\,\text{contr:}\ast \qquad \mathbf{G}_2, \Sigma \vdash \Theta}{\mathbf{G}_1 \vee \mathbf{G}_2, \mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \Gamma_1, \Gamma_2, \Sigma \vdash \Theta, \Delta_1, \Delta_2, \Lambda}\,\sigma
$$

and π' is

$$
\dfrac{\dfrac{\mathbf{F}_1, \Pi, \Gamma_1 \vdash \Delta_1, \Lambda \qquad \dfrac{\mathbf{G}_1, \mathbf{F}_2, \Pi, \Gamma_2 \vdash \Delta_2, \Lambda \qquad \mathbf{G}_2, \Sigma \vdash \Theta}{\mathbf{G}_1 \vee \mathbf{G}_2, \mathbf{F}_2, \Pi, \Gamma_2, \Sigma \vdash \Theta, \Delta_2, \Lambda}\,\sigma}{\mathbf{F}_1 \vee \mathbf{F}_2, \mathbf{G}_1 \vee \mathbf{G}_2, \Pi, \Pi, \Gamma_1, \Gamma_2, \Sigma \vdash \Theta, \Delta_1, \Delta_2, \Lambda, \Lambda}\,\rho}{\mathbf{F}_1 \vee \mathbf{F}_2, \mathbf{G}_1 \vee \mathbf{G}_2, \Pi, \Gamma_1, \Gamma_2, \Sigma \vdash \Theta, \Delta_1, \Delta_2, \Lambda}\,\text{contr:}\ast
$$

then $\pi \triangleright_b^1 \pi'$.

9. If π is

$$
\dfrac{\dfrac{\dfrac{\mathbf{F}_1, \Pi, \mathbf{G}_1, \Gamma_1 \vdash \Delta_1, \Lambda \qquad \mathbf{F}_2, \Pi, \mathbf{G}_1, \Gamma_2 \vdash \Delta_2, \Lambda}{\mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \mathbf{G}_1, \Pi, \mathbf{G}_1, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \Lambda, \Lambda}\,\rho}{\mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \mathbf{G}_1, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \Lambda}\,\text{contr:}\ast \qquad \mathbf{G}_2, \Sigma \vdash \Theta}{\mathbf{G}_1 \vee \mathbf{G}_2, \mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \Gamma_1, \Gamma_2, \Sigma \vdash \Theta, \Delta_1, \Delta_2, \Lambda}\,\sigma
$$

and π' is

$$
\dfrac{\dfrac{\dfrac{\mathbf{G}_1, \mathbf{F}_1, \Pi, \Gamma_1 \vdash \Delta_1, \Lambda \qquad \mathbf{G}_2, \Sigma \vdash \Theta}{\mathbf{G}_1 \vee \mathbf{G}_2, \mathbf{F}_1, \Pi, \Gamma_1, \Sigma \vdash \Theta, \Delta_1, \Lambda}\,\sigma \qquad (\psi)}{\mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \mathbf{G}_1 \vee \mathbf{G}_2, \Pi, \mathbf{G}_1 \vee \mathbf{G}_2, \Gamma_1, \Gamma_2, \Sigma, \Sigma \vdash \Theta, \Theta, \Delta_1, \Delta_2, \Lambda, \Lambda}\,\rho}{\mathbf{F}_1 \vee \mathbf{F}_2, \mathbf{G}_1 \vee \mathbf{G}_2, \Pi, \Gamma_1, \Gamma_2, \Sigma \vdash \Theta, \Delta_1, \Delta_2, \Lambda}\,\text{contr:}\ast
$$

where ψ is

$$\frac{\mathbf{G}_1, \mathbf{F}_2, \Pi, \Gamma_2 \vdash \Delta_2, \Lambda \qquad \mathbf{G}_2, \Sigma \vdash \Theta}{\mathbf{G}_1 \vee \mathbf{G}_2, \mathbf{F}_2, \Pi, \Gamma_2, \Sigma \vdash \Theta, \Delta_2, \Lambda} \sigma$$

then $\pi \rhd_b^1 \pi'$.

Finally, we define the $\rhd_b$ relation as the transitive and reflexive closure of the compatible closure of the $\rhd_b^1$ relation.

Lemma 4.1.32. *Let π be a weakly regular* $\mathbf{LK}_{\mathrm{sk}}$*-tree of S. If $\pi \rhd_b \psi$ then ψ is a weakly regular* $\mathbf{LK}_{\mathrm{sk}}$*-tree of S.*

Proof. By induction on the length of the $\rhd_b$-rewrite sequence. The case of $\pi = \psi$ is trivial, so assume there exists a subtree φ of π such that $\varphi \rhd_b^1 \varphi'$ and ψ is obtained from π by replacing φ by φ'. Then the end-sequent of ψ is the same as that of π. Also weak regularity is preserved:

1. In cases 1, 2 and 4–8 of Definition 4.1.31, the paths in ψ and π are the same modulo some repetitions.

2. In case 3, the paths in ψ and π are the same modulo some repetitions, but a new copy of σ is introduced. Note that the two copies are homomorphic, so we may conclude by Proposition 4.1.15.

3. In case 9, σ is duplicated together with the subtree ending in $\Sigma \vdash \Theta$. Observe that all the descendants of the two copies of $\Sigma \vdash \Theta$ are contracted, and hence all the duplicated inferences are homomorphic. Therefore we may again conclude by Proposition 4.1.15.

□

Summarizing, we obtain

Lemma 4.1.33. *Let π be a weakly regular* $\mathbf{LK}_{\mathrm{sk}}$*-tree of S. If $\pi \rhd_b \psi$, $\pi \rhd_u \psi$, or $\pi \rhd_c \psi$, then ψ is a weakly regular* $\mathbf{LK}_{\mathrm{sk}}$*-tree of S.*

Proof. By Lemmas 4.1.32, 4.1.30, and 4.1.25. □

The following definitions will be used in the algorithm translating $\mathbf{LK}_{\mathrm{sk}}$-proofs into such $\mathbf{LK}_{\mathrm{sk}}$-proofs which fulfil an eigenterm condition.

Definition 4.1.34. Let π be a $\mathbf{LK}_{\mathrm{sk}}$-tree, and let ξ be a branch in π. Let σ, ρ be inferences on ξ and w.l.o.g. let σ be above ρ. Let $\xi_1, \ldots, \xi_n$ be the binary inferences between σ and ρ. For $1 \leq i \leq n$, let λ_i be the subproofs ending in a premise sequent of ξ_i such that λ_i do not contain σ. Then $\lambda_1, \ldots, \lambda_n$ are called the *parallel trees between σ and ρ.*

Definition 4.1.35. Let σ be a $\forall^{sk}\colon r$ inference in π with Skolem term $\mathbf{S}$, and ρ be a $\forall^{sk}\colon l$ inference in π with substitution term $\mathbf{T}$. We say that ρ *blocks* σ if ρ is below σ and $\mathbf{T}$ contains $\mathbf{S}$. We call σ *correctly placed* if no $\forall^{sk}\colon l$ inference in π blocks σ.

Example 4.1.36. Consider the $\mathbf{LK}_{\text{sk}}$-proof π:

$$\dfrac{\dfrac{\langle P(c)\rangle^c \vdash P(c)}{\langle P(c)\rangle^c \vdash (\forall x)P(x)}\ \forall^{sk}\colon r}{(\forall x)P(x) \vdash (\forall x)P(x)}\ \forall^{sk}\colon l$$

Here, the $\forall^{sk}\colon l$ inference blocks the $\forall^{sk}\colon r$ inference.

As indicated before, we will rearrange the quantifier inferences in an $\mathbf{LK}_{\text{sk}}$-proof π in such a way that there are no eigenterm violations: this will allow us to convert the $\mathbf{LK}_{\text{sk}}$-proof into an $\mathbf{LK}$ proof. During this rearranging, we may have to permute binary inferences, causing duplication of subproofs. This is bad for showing termination of the rearranging algorithm because our termination measure will be based on the number of inferences in π. As Example 4.1.28 shows, sequential pruning may severly reduce the number of inferences in an $\mathbf{LK}_{\text{sk}}$-proof (especially when pruning binary inferences). In fact, this pruning will be sufficient to show termination of the rearranging procedure in the subsequent lemma. For the termination argument, we will use the notion of lexicographic order:

Definition 4.1.37 (Lexicographic order)**.** Let $X_1, \ldots, X_n$ be sets and for $i \leq n$ let $\leq_i$ be a partial order on X_i. Then the *lexicographic order on* $X_1 \times \ldots \times X_n$*:* $<_{\text{LEX}}$ is defined by

$$(x_1, \ldots, x_n) <_{\text{LEX}} (x'_1, \ldots, x'_n) \iff (\exists m)(\forall i < m)(x_i = x'_i) \land (x_m <_m x'_m)$$

Lemma 4.1.38. *Let π be an $\mathbf{LK}_{\text{sk}}$-proof of S. Then there exists a $\mathbf{LK}_{\text{sk}}$-proof π' of S such that all $\forall^{sk}\colon r$ inferences in π' are correctly placed.*

Proof. We introduce some notations that will be useful. Let π be an $\mathbf{LK}_{\text{sk}}$-tree, ρ be a $\forall^{sk}\colon r$ inference in π with Skolem term $\mathbf{S}$. Define Q_ρ as the number of inferences blocking ρ. Then define $\mathbf{BLOCK}_\pi(\mathbf{S}) = \sum_\sigma Q_\sigma$ where σ ranges over the $\forall^{sk}\colon r$ inferences in π with Skolem term $\mathbf{S}$.

Define SK_π as the set of Skolem terms occurring in π. Let $|\text{SK}_\pi| = n$, then denote the elements of SK_π by $\mathbf{S}_1, \ldots, \mathbf{S}_n$ s.t. for all $1 \leq i \leq n$ and all $j < i$: either $\mathbf{S}_j > \mathbf{S}_i$ or $\mathbf{S}_j$, $\mathbf{S}_i$ are incomparable w.r.t. the subexpression ordering. Then define the n-tupel $\alpha_\pi = \langle \mathbf{BLOCK}_\pi(\mathbf{S}_1), \ldots, \mathbf{BLOCK}_\pi(\mathbf{S}_n)\rangle$.

We show that there exists an $\mathbf{LK}_{\text{sk}}$-proof π' of S with the property that $\alpha_{\pi'} = \langle 0, \ldots, 0 \rangle$, which implies that there are no blocking inferences in π'.

We may assume that some member of α_π is not 0. We will transform π into an $\mathbf{LK}_{\text{sk}}$-proof π' of S such that $\alpha_{\pi'} <_{\text{LEX}} \alpha_\pi$ — existence of the desired $\mathbf{LK}_{\text{sk}}$-proof then follows by induction. Let k be the least integer such that $\mathbf{BLOCK}_\pi(\mathbf{S}_k) > 0$. Then there exists a lowermost $\forall^{sk}\!: r$ inference ρ with Skolem term $\mathbf{S}_k$ such that there is a $\forall^{sk}\!: l$ inference σ blocking ρ. Observe that σ does not operate on a descendant of the main formula of ρ: Assume it does, then by Proposition 4.1.5, $\mathbf{S}_k$ properly contains the substitution term of σ and, by the definition of blocking, therefore properly contains itself!

Define $\text{RR}(\pi, \xi, \sigma) = \sum_\mu Q_\mu$ where μ ranges over the inferences homomorphic to ρ in the parallel trees between ξ and σ. Define $\text{BR}(\pi, \xi, \sigma) = \mathbf{BLOCK}_\pi(\mathbf{S}_k) - \text{RR}(\pi, \xi, \sigma)$. The intuitive idea is: When we permute down inferences, new subtrees can be created which contain inferences homomorphic to ρ. $\text{RR}(\pi, \xi, \sigma)$ counts the number of "blockings" created by these inferences. The point then is that these inferences will eventually be deleted, and then $\text{BR}(\pi, \xi, \sigma) = \mathbf{BLOCK}_\pi(\mathbf{S}_k)$ and therefore $\mathbf{BLOCK}_\pi(\mathbf{S}_k)$ will properly decrease by permuting ρ below σ.

Formally, let $R_n, \ldots, R_1$ be the inferences between ρ and σ (exluding ρ and σ) operating on descendants of the main formula of ρ, i.e.:

$$
\dfrac{\vdots}{\Gamma \vdash \Delta}\,\rho
$$

$$
\dfrac{\vdots}{\Gamma_n \vdash \Delta_n}\,R_n
$$

$$
\vdots
$$

$$
\dfrac{\vdots}{\Gamma_1 \vdash \Delta_1}\,R_1
$$

$$
\dfrac{\vdots}{\Pi \vdash \Lambda}\,\sigma
$$

We construct by induction $\mathbf{LK}_{\text{sk}}$-proofs $\pi_1, \ldots, \pi_l$ where one of the inferences is permuted down below σ. The induction invariant is:

$$
\forall j < k(\mathbf{BLOCK}_{\pi_{l+1}}(\mathbf{S}_j) = 0) \land \text{BR}(\pi_l, \rho, \sigma) \geq \text{BR}(\pi_{l+1}, \rho, \sigma).
$$

Assume l inferences have been shifted, that is

$$\dfrac{\vdots}{\Gamma \vdash \Delta}\,\rho$$

$$\dfrac{\vdots}{\Gamma_n \vdash \Delta_n}\,R_n$$

$$\vdots$$

$$\dfrac{\vdots}{\Gamma_{l+1} \vdash \Delta_{l+1}}\,R_{l+1}$$

$$\dfrac{\vdots}{\Pi' \vdash \Lambda'}\,\sigma$$

$$\dfrac{\vdots}{\Gamma'_l \vdash \Delta'_l}\,R_l$$

$$\vdots$$

$$\dfrac{\vdots}{\Pi \vdash \Lambda}\,R_1$$

Depending on whether R_{l+1} is a unary, binary, or contraction inference, we use $\rhd_u$, $\rhd_b$, or $\rhd_c$ respectively to permute it below σ, obtaining π_{l+1}. By Lemma 4.1.33, π_{l+1} is an $\mathbf{LK}_{\text{sk}}$-proof of S. We verify the induction invariant by distinguishing what kind of inference R_{l+1} is:

1. R_{l+1} is a $\forall^{sk}\colon r$ inference. Permuting down a $\forall^{sk}\colon r$ inference cannot create any blocking inferences and does not change the number of homomorphic inferences in the parallel trees, so the invariant holds. For example, we permute R_{l+1} below a $\forall^{sk}\colon l$ inference:

$$\dfrac{\dfrac{\begin{array}{c}(\psi)\\ \langle \mathbf{GT}\rangle^{\ell_1,\mathbf{T}}, \Gamma \vdash \Delta, \langle \mathbf{FS}\rangle^{\ell_2}\end{array}}{\langle \mathbf{GT}\rangle^{\ell_1,\mathbf{T}}, \Gamma \vdash \Delta, \langle \forall \mathbf{F}\rangle^{\ell_2}}\,R_{l+1}}{\langle \forall \mathbf{G}\rangle^{\ell_1}, \Gamma \vdash \Delta, \langle \forall \mathbf{F}\rangle^{\ell_2}}\,\forall^{sk}\colon l$$

is transformed into

$$\dfrac{\dfrac{\begin{array}{c}(\psi)\\ \langle \mathbf{GT}\rangle^{\ell_1,\mathbf{T}}, \Gamma \vdash \Delta, \langle \mathbf{FS}\rangle^{\ell_2}\end{array}}{\langle \forall \mathbf{G}\rangle^{\ell_1}, \Gamma \vdash \Delta, \langle \mathbf{FS}\rangle^{\ell_2}}\,\forall^{sk}\colon l}{\langle \forall \mathbf{G}\rangle^{\ell_1}, \Gamma \vdash \Delta, \langle \forall \mathbf{F}\rangle^{\ell_2}}\,R_{l+1}$$

2. R_{l+1} is a $\forall^{sk}\colon l$ inference with substitution term $\mathbf{T}$. As R_{l+1} operates on a descendant of ρ, by Proposition 4.1.5, $\mathbf{S}_k > \mathbf{T}$. Therefore $\mathbf{S}_k$ properly contains any Skolem term $\mathbf{R}$ contained in $\mathbf{T}$, so $\mathbf{R} = \mathbf{S}_j$ for some $j > k$. Therefore $\mathbf{BLOCK}_{\pi_l}(\mathbf{S}_p) \geq \mathbf{BLOCK}_{\pi_{l+1}}(\mathbf{S}_p)$ for all $p \leq k$. The parallel trees are untouched, so the invariant holds.

3. R_{l+1} is a unary propositional inference. The invariant trivially holds.

4. R_{l+1} is an $\vee\colon l$ inference. To verify the induction invariant, we perform a case distinction depending on the inference below R_{l+1}. We only consider the interesting cases:

   (a) R_{l+1} is permuted over a $\forall^{sk}\colon l$ inference ξ. At most one copy ξ' of ξ is created in π_{l+1}, and there is no branch containing both ξ and ξ'. So for all $\forall^{sk}\colon r$ inferences above R_{l+1}, there is still at most one of ξ, ξ' below them, so $\mathbf{BLOCK}_{\pi_{l+1}}(\mathbf{S}_i) \leq \mathbf{BLOCK}_{\pi_l}(\mathbf{S}_i)$ for all $i \in \{1, \ldots, k\}$.
   For example, consider the case

$$\dfrac{\dfrac{\dfrac{\overset{(\psi)}{\mathbf{F}_1, \Pi, \langle \mathbf{GT} \rangle^{\ell,\mathbf{T}}, \Gamma_1 \vdash \Delta_1, \Lambda} \qquad \overset{(\psi')}{\mathbf{F}_2, \Pi, \langle \mathbf{GT} \rangle^{\ell,\mathbf{T}}, \Gamma_2 \vdash \Delta_2, \Lambda}}{\mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \langle \mathbf{GT} \rangle^{\ell,\mathbf{T}}, \Pi, \langle \mathbf{GT} \rangle^{\ell,\mathbf{T}}, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \Lambda, \Lambda}\, R_{l+1}}{\langle \mathbf{GT} \rangle^{\ell,\mathbf{T}}, \mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \Lambda}\, \text{contr}\colon *}{\langle \forall \mathbf{G} \rangle^{\ell}, \mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \Lambda}\, \xi$$

   which is transformed to

$$\dfrac{\dfrac{\dfrac{\overset{(\psi)}{\langle \mathbf{GT} \rangle^{\ell,\mathbf{T}}, \mathbf{F}_1, \Pi, \Gamma_1 \vdash \Delta_1, \Lambda}}{\langle \forall \mathbf{G} \rangle^{\ell}, \mathbf{F}_1, \Pi, \Gamma_1 \vdash \Delta_1, \Lambda}\, \xi \qquad \dfrac{\overset{(\psi')}{\langle \mathbf{GT} \rangle^{\ell,\mathbf{T}}, \mathbf{F}_2, \Pi, \Gamma_2 \vdash \Delta_2, \Lambda}}{\langle \forall \mathbf{G} \rangle^{\ell}, \mathbf{F}_2, \Pi, \Gamma_2 \vdash \Delta_2, \Lambda}\, \xi'}{\mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \langle \forall \mathbf{G} \rangle^{\ell}, \Pi, \langle \forall \mathbf{G} \rangle^{\ell}, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \Lambda, \Lambda}\, R_{l+1}}{\langle \forall \mathbf{G} \rangle^{\ell}, \mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \Lambda}\, \text{contr}\colon *$$

   So for all $\forall^{sk}\colon r$ inferences in ψ, ψ' there is still only one copy of ξ below them, and hence $\mathbf{BLOCK}_{\pi_{l+1}}(\mathbf{S}_i) \leq \mathbf{BLOCK}_{\pi_l}(\mathbf{S}_i)$.

   (b) R_{l+1} is permuted over a $\forall^{sk}\colon r$ inference ξ with Skolem term $\mathbf{S}_p$. If $p < k$, then $\mathbf{BLOCK}_{\pi_l}(\mathbf{S}_p) = 0$ and therefore duplicating ξ still gives $\mathbf{BLOCK}_{\pi_{l+1}}(\mathbf{S}_p) = 0$. $p = k$ does not hold, as we chose a lowermost blocked $\forall^{sk}\colon r$ inference ρ.

(c) R_{l+1} is permuted over a binary inference ξ such that one of the auxiliary formulas of ξ is contracted; then the situation in π_l is

$$\dfrac{\dfrac{\dfrac{\mathbf{F}_1, \Pi, \mathbf{G}_1, \Gamma_1 \vdash \Delta_1, \Lambda \qquad \mathbf{F}_2, \Pi, \mathbf{G}_1, \Gamma_2 \vdash \Delta_2, \Lambda}{\mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \mathbf{G}_1, \Pi, \mathbf{G}_1, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \Lambda, \Lambda}\, R_{l+1}}{\mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \mathbf{G}_1, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \Lambda}\,\mathrm{contr}\colon * \qquad \begin{matrix}(\varphi)\\ \mathbf{G}_2, \Sigma \vdash \Theta\end{matrix}}{\mathbf{G}_1 \vee \mathbf{G}_2, \mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \Gamma_1, \Gamma_2, \Sigma \vdash \Theta, \Delta_1, \Delta_2, \Lambda}\,\xi$$

which, in π_{l+1}, is transformed to

$$\dfrac{\dfrac{\dfrac{\mathbf{G}_1, \mathbf{F}_1, \Pi, \Gamma_1 \vdash \Delta_1, \Lambda \qquad \begin{matrix}(\varphi)\\ \mathbf{G}_2, \Sigma \vdash \Theta\end{matrix}}{\mathbf{G}_1 \vee \mathbf{G}_2, \mathbf{F}_1, \Pi, \Gamma_1, \Sigma \vdash \Theta, \Delta_1, \Lambda}\,\xi \qquad (\psi)}{\mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \mathbf{G}_1 \vee \mathbf{G}_2, \Pi, \mathbf{G}_1 \vee \mathbf{G}_2, \Gamma_1, \Gamma_2, \Sigma, \Sigma \vdash \Theta, \Theta, \Delta_1, \Delta_2, \Lambda, \Lambda}\, R_{l+1}}{\mathbf{F}_1 \vee \mathbf{F}_2, \mathbf{G}_1 \vee \mathbf{G}_2, \Pi, \Gamma_1, \Gamma_2, \Sigma \vdash \Theta, \Delta_1, \Delta_2, \Lambda}\,\mathrm{contr}\colon *$$

where ψ is

$$\dfrac{\mathbf{G}_1, \mathbf{F}_2, \Pi, \Gamma_2 \vdash \Delta_2, \Lambda \qquad \begin{matrix}(\varphi)\\ \mathbf{G}_2, \Sigma \vdash \Theta\end{matrix}}{\mathbf{G}_1 \vee \mathbf{G}_2, \mathbf{F}_2, \Pi, \Gamma_2, \Sigma \vdash \Theta, \Delta_2, \Lambda}\,\xi$$

As $\mathbf{BLOCK}_{\pi_l}(\mathbf{S}_p) = 0$ for $p < k$, $\mathbf{BLOCK}_{\pi_{l+1}}(\mathbf{S}_p) = 0$ even when duplicating a subtree. Hence we only have to consider $\mathbf{S}_k$. Assume $\mathbf{BLOCK}_{\pi_{l+1}}(\mathbf{S}_k) > \mathbf{BLOCK}_{\pi_l}(\mathbf{S}_k)$, then there exists a $\forall^{sk}\colon r$ inference ρ' in the duplicated tree φ with Skolem term $\mathbf{S}_k$. As ρ' was created by copying a inference ρ^* that was, by weak regularity, homomorphic to ρ, also ρ' will be homomorphic to ρ due to the applications of contractions contr: $*$ on $\Sigma, \Theta, \mathbf{G}_1 \vee \mathbf{G}_2$. Therefore the inferences blocking ρ' in the copy of φ are counted in $\mathrm{RR}(\pi_{l+1}, \rho, \sigma)$. Let z be the number of inferences blocking inferences ρ' copied in this way, then $\mathrm{RR}(\pi_{l+1}, \rho, \sigma) = \mathrm{RR}(\pi_l, \rho, \sigma) + z$ and $\mathbf{BLOCK}_{\pi_{l+1}}(\mathbf{S}_p) = \mathbf{BLOCK}_{\pi_l}(\mathbf{S}_p) + z$ and hence $\mathrm{BR}(\pi_{l+1}, \rho, \sigma) \leq \mathrm{BR}(\pi_l, \rho, \sigma)$.

This completes the case distinction. Let ω be the inference directly above ρ, then $\mathrm{RR}(\pi_m, \rho, \sigma) = \mathrm{RR}(\pi_m, \omega, \sigma)$. Permute ρ down over σ in the same way as above and apply Lemma 4.1.27 to the resulting proof. This yields a proof π'_m such that $\mathrm{RR}(\pi'_m, \omega, \sigma) = 0$ and, because ρ is now below σ, $\mathbf{BLOCK}_{\pi'_m}(\mathbf{S}_k) < \mathbf{BLOCK}_{\pi}(\mathbf{S}_k)$. □

Example 4.1.39. In this and the following example, we apply the above Lemma. Note that for reasons of clarity, in this example the relevant instances of $\mathrm{RR}(\pi, \xi, \sigma)$ are always 0. The example after this one will exhibit an $\mathbf{LK}_{\mathrm{sk}}$-proof where this is not the case.

So consider the following $\mathbf{LK}_{\mathrm{sk}}$-proof π:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(s)\vdash P(s)}{(\exists x)P(x)\vdash P(s)}\,\exists^{sk}\!:l\ \rho_1 \qquad \dfrac{P(s)\vdash P(s)}{(\forall x)P(x)\vdash P(s)}\,\forall^{sk}\!:l\ \sigma_1}{(\exists x)P(x)\lor(\forall x)P(x)\vdash P(s),P(s)}\,\lor\!:l\ \xi_1}{(\exists x)P(x)\lor(\forall x)P(x)\vdash P(s)}\,\mathrm{contr}\!:r \qquad (\psi)}{(\exists x)P(x)\lor(\forall x)P(x), P(s)\to(\exists^{s}x)Q(x)\vdash Q(f(s))}\,\to\!:l\ \xi_2}{(\exists x)P(x)\lor(\forall x)P(x), (\forall x)(P(x)\to(\exists x)Q(x))\vdash Q(f(s))}\,\forall^{sk}\!:l\ \sigma_2}{(\exists x)P(x)\lor(\forall x)P(x), (\forall x)(P(x)\to(\exists x)Q(x))\vdash(\exists x)Q(x)}\,\exists^{sk}\!:r\ \sigma_3$$

where ψ is

$$\dfrac{Q(f(s))\vdash Q(f(s))}{(\exists^{s}x)Q(x)\vdash Q(f(s))}\,\exists^{sk}\!:l\ \rho_2$$

where the inference labels indicate the denotations of the inferences. The proof contains no homomorphic pairs. σ_3 blocks ρ_1 and ρ_2, and σ_2 blocks ρ_1, hence

$$\begin{aligned}&\mathrm{SK}_\pi = \{s, f(s)\},\\ &\mathbf{S}_1 = f(s), \mathbf{S}_2 = s,\end{aligned}$$

and

$$\begin{aligned}&Q_{\rho_1} = 2,\ Q_{\rho_2} = 1,\\ &\mathbf{BLOCK}_\pi(s) = 2,\ \mathbf{BLOCK}_\pi(f(s)) = 1,\\ &\alpha_\pi = \langle 2, 1\rangle\,,\\ &\rho = \rho_2, \sigma = \sigma_3,\\ &l = 2, R_1 = \xi_2, R_2 = \sigma_2\end{aligned}$$

We start by permuting R_2 below σ, obtaining π_2:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(s)\vdash P(s)}{(\exists x)P(x)\vdash P(s)}\,\exists^{sk}\!:l\ \rho_1 \qquad \dfrac{P(s)\vdash P(s)}{(\forall x)P(x)\vdash P(s)}\,\forall^{sk}\!:l\ \sigma_1}{(\exists x)P(x)\lor(\forall x)P(x)\vdash P(s),P(s)}\,\lor\!:l\ \xi_1}{(\exists x)P(x)\lor(\forall x)P(x)\vdash P(s)}\,\mathrm{contr}\!:r \qquad (\psi)}{(\exists x)P(x)\lor(\forall x)P(x), P(s)\to(\exists^{s}x)Q(x)\vdash Q(f(s))}\,\to\!:l\ \xi_2}{(\exists x)P(x)\lor(\forall x)P(x), P(s)\to(\exists x)Q(x)\vdash(\exists x)Q(x)}\,\exists^{sk}\!:r\ \sigma_3}{(\exists x)P(x)\lor(\forall x)P(x), (\forall x)(P(x)\to(\exists x)Q(x))\vdash(\exists x)Q(x)}\,\forall^{sk}\!:l\ \sigma_2$$

yielding

$$
\begin{aligned}
&Q_{\rho_1} = 2,\ Q_{\rho_2} = 1,\\
&\mathbf{BLOCK}_{\pi}(s) = 2,\ \mathbf{BLOCK}_{\pi}(f(s)) = 1,\\
&\alpha_{\pi} = \langle 2, 1\rangle,\\
&\rho = \rho_2, \sigma = \sigma_3,\\
&l = 1, R_1 = \xi_2
\end{aligned}
$$

Now we permute R_1 below σ, obtaining π_3:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(s) \vdash P(s)}{(\exists x)P(x) \vdash P(s)}\ \exists^{sk}\colon l\ \rho_1 \quad \dfrac{P(s) \vdash P(s)}{(\forall x)P(x) \vdash P(s)}\ \forall^{sk}\colon l\ \sigma_1}{(\exists x)P(x) \lor (\forall x)P(x) \vdash P(s), P(s)}\ \lor\colon l\ \xi_1}{(\exists x)P(x) \lor (\forall x)P(x) \vdash P(s)}\ \mathrm{contr}\colon r \quad (\psi_3)}{(\exists x)P(x) \lor (\forall x)P(x), P(s) \to (\exists^s x)Q(x) \vdash (\exists x)Q(x)}\ \to\colon l\ \xi_2}{(\exists x)P(x) \lor (\forall x)P(x), (\forall x)(P(x) \to (\exists x)Q(x)) \vdash (\exists x)Q(x)}\ \forall^{sk}\colon l\ \sigma_2
$$

where ψ_3 is

$$
\dfrac{\dfrac{Q(f(s)) \vdash Q(f(s))}{(\exists^s x)Q(x) \vdash Q(f(s))}\ \exists^{sk}\colon l\ \rho_2}{(\exists^s x)Q(x) \vdash (\exists x)Q(x)}\ \exists^{sk}\colon r\ \sigma_3
$$

and

$$
\begin{aligned}
&Q_{\rho_1} = Q_{\rho_2} = 1,\\
&\mathbf{BLOCK}_{\pi_3}(s) = \mathbf{BLOCK}_{\pi_2}(f(s)) = 1,\\
&\alpha_{\pi_3} = \langle 1, 1\rangle,\\
&\rho = \rho_2, \sigma = \sigma_3,\\
&l = 0
\end{aligned}
$$

Finally, we permute ρ below σ, obtaining π_4, which is π_3 where ψ_3 is replaced by

$$
\dfrac{\dfrac{Q(f(s)) \vdash Q(f(s))}{Q(f(s)) \vdash (\exists x)Q(x)}\ \exists^{sk}\colon r\ \sigma_3}{(\exists^s x)Q(x) \vdash (\exists x)Q(x)}\ \exists^{sk}\colon l\ \rho_2
$$

We have

$$
\begin{aligned}
&Q_{\rho_1} = 1,\\
&Q_{\rho_2} = 0,\\
&\mathbf{BLOCK}_{\pi_4}(s) = 1,\\
&\mathbf{BLOCK}_{\pi_4}(f(s)) = 0,\\
&\alpha_{\pi_4} = \langle 0, 1\rangle,
\end{aligned}
$$

We move on to $\mathbf{S}_2 = s$. Now we have

$$
\begin{array}{l}
\rho = \rho_1, \sigma = \sigma_2 \\
l = 1, R_1 = \xi_1,
\end{array}
$$

and we start by permuting down ξ_1 over ξ_2. This yields π_5:

$$
\dfrac{\dfrac{\dfrac{\dfrac{P(s) \vdash P(s)}{(\exists x)P(x) \vdash P(s)}\ \exists^{sk}\colon l\ \rho_1 \qquad \dfrac{\dfrac{Q(f(s)) \vdash Q(f(s))}{Q(f(s)) \vdash (\exists x)Q(x)}\ \exists^{sk}\colon r\ \sigma_3}{(\exists^s x)Q(x) \vdash (\exists x)Q(x)}\ \exists^{sk}\colon l\ \rho_2}{(\exists x)P(x), P(s) \to (\exists^s x)Q(x) \vdash (\exists x)Q(x)}\ \to\colon l\ \xi_2 \qquad (\psi_5)}{(\exists x)P(x) \vee (\forall x)P(x), F, F \vdash (\exists x)Q(x), (\exists x)Q(x)}\ \vee\colon l\ \xi_1}{(\exists x)P(x) \vee (\forall x)P(x), P(s) \to (\exists^s x)Q(x) \vdash (\exists x)Q(x)}\ \text{contr}\colon *
$$

$$
\dfrac{(\exists x)P(x) \vee (\forall x)P(x), P(s) \to (\exists^s x)Q(x) \vdash (\exists x)Q(x)}{(\exists x)P(x) \vee (\forall x)P(x), (\forall x)(P(x) \to (\exists x)Q(x)) \vdash (\exists x)Q(x)}\ \forall^{sk}\colon l\ \sigma_2
$$

where $F = P(s) \to (\exists^s x)Q(x)$ and ψ_5 is

$$
\dfrac{\dfrac{P(s) \vdash P(s)}{(\forall x)P(x) \vdash P(s)}\ \forall^{sk}\colon l\ \sigma_1 \qquad \dfrac{\dfrac{Q(f(s)) \vdash Q(f(s))}{Q(f(s)) \vdash (\exists x)Q(x)}\ \exists^{sk}\colon r\ \sigma_3'}{(\exists^s x)Q(x) \vdash (\exists x)Q(x)}\ \exists^{sk}\colon l\ \rho_2'}{(\forall x)P(x), P(s) \to (\exists^s x)Q(x) \vdash (\exists x)Q(x)}\ \to\colon l\ \xi_2'
$$

Note that $\{\sigma_3, \sigma_3'\}$, $\{\rho_2, \rho_2'\}$, and $\{\xi_2, \xi_2'\}$ are all homomorphic pairs. We have

$$
\begin{array}{l}
Q_{\rho_1} = 1, \\
Q_{\rho_2} = Q_{\rho_2'} = 0, \\
\mathbf{BLOCK}_{\pi_5}(s) = 1, \\
\mathbf{BLOCK}_{\pi_5}(f(s)) = 0, \\
\alpha_{\pi_5} = \langle 0, 1 \rangle, \\
l = 1
\end{array}
$$

We have to permute ξ_1 further down, below σ, yielding π_6:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(s) \vdash P(s)}{(\exists x)P(x) \vdash P(s)}\ \exists^{sk}\colon l\ \rho_1 \qquad \dfrac{\dfrac{Q(f(s)) \vdash Q(f(s))}{Q(f(s)) \vdash (\exists x)Q(x)}\ \exists^{sk}\colon r\ \sigma_3}{(\exists^s x)Q(x) \vdash (\exists x)Q(x)}\ \exists^{sk}\colon l\ \rho_2}{(\exists x)P(x), P(s) \to (\exists^s x)Q(x) \vdash (\exists x)Q(x)}\ \to\colon l\ \xi_2}{(\exists x)P(x), (\forall x)(P(x) \to (\exists x)Q(x)) \vdash (\exists x)Q(x)}\ \forall^{sk}\colon l\ \sigma_2 \qquad (\psi_6)}{(\exists x)P(x) \vee (\forall x)P(x), G, G \vdash (\exists x)Q(x), (\exists x)Q(x)}\ \vee\colon l\ \xi_1}{(\exists x)P(x) \vee (\forall x)P(x), (\forall x)(P(x) \to (\exists x)Q(x)) \vdash (\exists x)Q(x)}\ \text{contr}\colon *
$$

where $G = (\forall x)(P(x) \to (\exists x)Q(x))$ and ψ_6 is

$$
\dfrac{\dfrac{\dfrac{P(s) \vdash P(s)}{(\forall x)P(x) \vdash P(s)}\ \forall^{sk}\colon l\ \sigma_1 \qquad \dfrac{\dfrac{Q(f(s)) \vdash Q(f(s))}{Q(f(s)) \vdash (\exists x)Q(x)}\ \exists^{sk}\colon r\ \sigma'_3}{(\exists^s x)Q(x) \vdash (\exists x)Q(x)}\ \exists^{sk}\colon l\ \rho'_2}{(\forall x)P(x), P(s) \to (\exists^s x)Q(x) \vdash (\exists x)Q(x)}\ \to\colon l\ \xi'_2}{(\forall x)P(x), (\forall x)(P(x) \to (\exists x)Q(x)) \vdash (\exists x)Q(x)}\ \forall^{sk}\colon l\ \sigma'_2
$$

with

$$
\begin{aligned}
&Q_{\rho_1} = 1,\\
&Q_{\rho_2} = Q_{\rho'_2} = 0,\\
&\mathbf{BLOCK}_{\pi_6}(s) = 1,\\
&\mathbf{BLOCK}_{\pi_6}(f(s)) = 0,\\
&\alpha_{\pi_6} = \langle 0, 1 \rangle,\\
&l = 0
\end{aligned}
$$

Finally we permute ρ_1 below σ_2 to obtain π_7:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(s) \vdash P(s) \qquad \dfrac{\dfrac{Q(f(s)) \vdash Q(f(s))}{Q(f(s)) \vdash (\exists x)Q(x)}\ \exists^{sk}\colon r\ \sigma_3}{(\exists^s x)Q(x) \vdash (\exists x)Q(x)}\ \exists^{sk}\colon l\ \rho_2}{P(s), P(s) \to (\exists^s x)Q(x) \vdash (\exists x)Q(x)}\ \to\colon l\ \xi_2}{P(s), (\forall x)(P(x) \to (\exists x)Q(x)) \vdash (\exists x)Q(x)}\ \forall^{sk}\colon l\ \sigma_2}{(\exists x)P(x), (\forall x)(P(x) \to (\exists x)Q(x)) \vdash (\exists x)Q(x)}\ \exists^{sk}\colon l\ \rho_1 \qquad (\psi_6)}{(\exists x)P(x) \lor (\forall x)P(x), G, G \vdash (\exists x)Q(x), (\exists x)Q(x)}\ \lor\colon l\ \xi_1}{(\exists x)P(x) \lor (\forall x)P(x), (\forall x)(P(x) \to (\exists x)Q(x)) \vdash (\exists x)Q(x)}\ \text{contr}\colon *
$$

such that

$$
\begin{aligned}
&Q_{\rho_1} = 0,\\
&Q_{\rho_2} = Q_{\rho'_2} = 0,\\
&\mathbf{BLOCK}_{\pi_6}(s) = 0,\\
&\mathbf{BLOCK}_{\pi_6}(f(s)) = 0,\\
&\alpha_{\pi_6} = \langle 0, 0 \rangle,
\end{aligned}
$$

and hence all inferences are correctly placed.

Example 4.1.40. The following example will highlight the need for sequential pruning. Consider the $\mathbf{LK}_{\mathrm{sk}}$-proof π:

$$
\dfrac{\dfrac{\dfrac{\dfrac{(\psi) \qquad \dfrac{\dfrac{P(s) \vdash P(s)}{(\exists x)P(x) \vdash P(s)}\,\exists^{sk}\colon l \qquad A \vdash A}{(\exists x)P(x) \lor A \vdash P(s), A}\,\lor\colon l}{(\exists x)P(x) \lor A, P(s) \to A, (\exists x)P(x) \lor A \vdash A, A \land P(s)}\,\land\colon r}{(\exists x)P(x) \lor A, P(s) \to A \vdash A, A \land P(s)}\,\text{contr}\colon l}{\neg(A \land P(s)), (\exists x)P(x) \lor A, P(s) \to A \vdash A}\,\neg\colon l}{\dfrac{(\exists x)P(x) \lor A, (P(s) \to A) \land \neg(A \land P(s)) \vdash A}{(\exists x)P(x) \lor A, (\forall x)((P(x) \to A) \land \neg(A \land P(x))) \vdash A}\,\forall^{sk}\colon l}\,\land\colon l
$$

where ψ is

$$
\dfrac{\dfrac{\dfrac{\dfrac{P(s) \vdash P(s)}{(\exists x)P(x) \vdash P(s)}\,\exists^{sk}\colon l \qquad A \vdash A}{(\exists x)P(x) \lor A \vdash P(s), A}\,\lor\colon l \qquad A \vdash A}{(\exists x)P(x) \lor A, P(s) \to A \vdash A, A}\,\to\colon l}{(\exists x)P(x) \lor A, P(s) \to A \vdash A}\,\text{contr}\colon r
$$

Here the $\forall^{sk}\colon l$ application blocks both $\exists^{sk}\colon l$ applications, hence we have $\mathbf{BLOCK}_\pi(s) = 2$. We may choose which $\exists^{sk}\colon l$ application we want to permute down first, let us take the one in ψ and denote it ρ. Denote the $\forall^{sk}\colon l$ application by σ. Note that now it is indeed the case that $\mathrm{RR}(\pi, \rho, \sigma) = 1$ (and hence $\mathrm{BR}(\pi, \rho, \sigma) = 1$) due to the second $\exists^{sk}\colon l$ application. $l = 1$ and R_1 is the $\lor\colon l$ application which we permute down and obtain π_1, which is π where ψ is replaced by

$$
\dfrac{\dfrac{\dfrac{\dfrac{P(s) \vdash P(s)}{(\exists x)P(x) \vdash P(s)}\,\exists^{sk}\colon l \qquad A \vdash A}{(\exists x)P(x), P(s) \to A \vdash A}\,\to\colon l \qquad A \vdash A}{(\exists x)P(x) \lor A, P(s) \to A \vdash A, A}\,\lor\colon l}{(\exists x)P(x) \lor A, P(s) \to A \vdash A}\,\text{contr}\colon r
$$

We permute the $\lor\colon l$ application further down together with the contr: r application — now we have to duplicate a subproof containing an $\exists^{sk}\colon l$ application that is homomorphic to ρ, obtaining π_2:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(\psi_1) \qquad (\psi_2)}{(\exists x)P(x), P(s) \to A, (\exists x)P(x) \lor A \vdash A \land P(s), A}\,\land\colon r \qquad (\psi_3)}{F, P(s) \to A, F, F \vdash A \land P(s), A, A \land P(s), A}\,\lor\colon l}{(\exists x)P(x) \lor A, P(s) \to A \vdash A \land P(s), A}\,\text{contr}\colon *}{\neg(A \land P(s)), (\exists x)P(x) \lor A, P(s) \to A \vdash A}\,\neg\colon l}{(P(s) \to A) \land \neg(A \land P(s)), (\exists x)P(x) \lor A \vdash A}\,\land\colon l}{(\forall x)((P(x) \to A) \land \neg(A \land P(x))), (\exists x)P(x) \lor A \vdash A}\,\forall^{sk}\colon l
$$

where $F = (\exists x)P(x) \lor A$ and ψ_1 is

$$
\dfrac{\dfrac{P(s) \vdash P(s)}{(\exists x)P(x) \vdash P(s)}\,\exists^{sk}\colon l \qquad A \vdash A}{(\exists x)P(x), P(s) \to A \vdash A}\,\to\colon l
$$

and ψ_2 is

$$
\dfrac{\dfrac{P(s) \vdash P(s)}{(\exists x)P(x) \vdash P(s)}\,\exists^{sk}\colon l \qquad A \vdash A}{(\exists x)P(x) \lor A \vdash P(s), A}\,\lor\colon l
$$

and ψ_3 is

$$
\dfrac{A \vdash A \qquad \dfrac{\dfrac{P(s) \vdash P(s)}{(\exists x)P(x) \vdash P(s)}\,\exists^{sk}\colon l \qquad A \vdash A}{(\exists x)P(x) \lor A \vdash P(s), A}\,\lor\colon l}{A, (\exists x)P(x) \lor A \vdash A, A \land P(s)}\,\land\colon r
$$

Now we indeed have

$$
\begin{aligned}
&\mathrm{RR}(\pi_2, \rho, \sigma) = 2,\\
&\mathbf{BLOCK}_{\pi_2}(s) = 3,\\
&\mathrm{BR}(\pi_2, \rho, \sigma) = 1
\end{aligned}
$$

We finish by permuting the $\lor\colon l$ application below σ, obtaining π_3:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(\psi_1) \qquad (\psi_2)}{(\exists x)P(x), P(s) \to A, F \vdash A \land P(s), A}\,\land\colon r}{\neg(A \land P(s)), (\exists x)P(x), P(s) \to A, F \vdash A}\,\neg\colon l}{\neg(A \land P(s)) \land (P(s) \to A), (\exists x)P(x), F \vdash A}\,\land\colon l}{G, (\exists x)P(x), F \vdash A}\,\forall^{sk}\colon l \qquad (\psi_4)}{F, G, G, F, F \vdash A, A}\,\lor\colon l}{(\exists x)P(x) \lor A, (\forall x)(\neg(A \land P(x)) \land (P(x) \to A)) \vdash A}\,\text{contr}\colon *
$$

where $G = (\forall x)(\neg(A \wedge P(x)) \wedge (P(x) \to A))$ and ψ_4 is

$$
\dfrac{\dfrac{\dfrac{A \vdash A \qquad \dfrac{\dfrac{P(s) \vdash P(s)}{(\exists x)P(x) \vdash P(s)}\,\exists^{sk}\colon l \qquad A \vdash A}{(\exists x)P(x) \vee A \vdash P(s), A}\,\vee\colon l}{\dfrac{A, (\exists x)P(x) \vee A \vdash A, A \wedge P(s)}{\neg(A \wedge P(s)), A, (\exists x)P(x) \vee A \vdash A}\,\neg\colon l}\,\wedge\colon r}{\dfrac{P(s) \to A, \neg(A \wedge P(s)), A, (\exists x)P(x) \vee A \vdash A}{\neg(A \wedge P(s)) \wedge P(s) \to A, A, (\exists x)P(x) \vee A \vdash A}\,\wedge\colon l}\,\text{weak}\colon l}{(\forall x)(\neg(A \wedge P(x)) \wedge P(x) \to A), A, (\exists x)P(x) \vee A \vdash A}\,\forall^{sk}\colon l
$$

Now, we can finally permute ρ below σ, obtaining π_4:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(s) \vdash P(s) \qquad A \vdash A}{P(s), P(s) \to A \vdash A}\,\to\colon l \qquad (\psi_2)}{P(s), P(s) \to A, F \vdash A \wedge P(s), A}\,\wedge\colon r}{\neg(A \wedge P(s)), P(s), P(s) \to A, F \vdash A}\,\neg\colon l}{\neg(A \wedge P(s)) \wedge (P(s) \to A), P(s), F \vdash A}\,\wedge\colon l}{G, P(s), F \vdash A}\,\forall^{sk}\colon l}{(\exists x)P(x), G, F \vdash A}\,\exists^{sk}\colon l \qquad (\psi_4)}{F, G, G, F, F \vdash A, A}\,\vee\colon l}{(\exists x)P(x) \vee A, (\forall x)(\neg(A \wedge P(x)) \wedge (P(x) \to A)) \vdash A}\,\text{contr}\colon *
$$

Note that $\mathrm{SHP}(\pi_4) \neq \emptyset$. Applying sequential pruning yields π_5:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(s) \vdash P(s) \qquad A \vdash A}{P(s), P(s) \to A \vdash A}\,\to\colon l \qquad \dfrac{P(s) \vdash P(s)}{P(s) \vdash P(s), A}\,\text{weak}\colon r}{P(s), P(s), P(s) \to A \vdash A \wedge P(s), A}\,\wedge\colon r}{\neg(A \wedge P(s)), P(s), P(s), P(s) \to A \vdash A}\,\neg\colon l}{\neg(A \wedge P(s)) \wedge (P(s) \to A), P(s), P(s) \vdash A}\,\wedge\colon l}{G, P(s), P(s) \vdash A}\,\forall^{sk}\colon l}{G, P(s) \vdash A}\,\text{contr}\colon l}{(\exists x)P(x), G \vdash A}\,\exists^{sk}\colon l \qquad (\psi_4)}{F, G, G, F \vdash A, A}\,\vee\colon l}{(\exists x)P(x) \vee A, (\forall x)(\neg(A \wedge P(x)) \wedge (P(x) \to A)) \vdash A}\,\text{contr}\colon *
$$

where indeed $\mathbf{BLOCK}_{\pi_5}(s) = 1 < 2 = \mathbf{BLOCK}_{\pi}(s)$ (the remaining blocking is in ψ_2).

Theorem 4.1.41 (Soundness). *Let π be a $\mathbf{LK}_{\mathrm{sk}}$-proof of S. Then there exists a cut-free* **LK***-proof of S.*

Proof. We apply Lemma 4.1.27 and Lemma 4.1.38 to obtain a sequentially pruned $\mathbf{LK}_{\mathrm{sk}}$-proof π' of S where all inferences are correctly placed.

For the rest of this proof, we allow $\forall\colon r$ inferences in $\mathbf{LK}_{\mathrm{sk}}$-proofs (with the usual eigenvariable condition). By induction on the number of $\forall^{sk}\colon r$ inferences in π', we construct sequentially pruned $\mathbf{LK}_{\mathrm{sk}}$-proofs π'' where all inferences are correctly placed, containing strictly less $\forall^{sk}\colon r$ inferences than π'.

Let ρ

$$\frac{\begin{array}{c}(\psi)\\ \Gamma \vdash \Delta, \langle \overline{\mathbf{FS}} \rangle^{\ell}\end{array}}{\Gamma \vdash \Delta, \langle \forall_{\alpha} \mathbf{F} \rangle^{\ell}}\ \forall^{sk}\colon r$$

be a $\forall^{sk}\colon r$ inference in π' such that $\mathbf{S}$ is a $>$-maximal Skolem term in π'.

Assume that $\mathbf{S}$ occurs in $\Gamma \cup \Delta \cup \ell$. As π' is an $\mathbf{LK}_{\mathrm{sk}}$-proof, S does not contain Skolem symbols and so a descendant of $\mathbf{S}$ must be eliminated by a $\forall^{sk}$ inference σ below ρ. Distinguish:

1. σ is a $\forall^{sk}\colon r$ inference. As π' is sequentially pruned and weakly regular, the Skolem term $\mathbf{T}$ of σ fulfills $\mathbf{S} \neq \mathbf{T}$. Therefore $\mathbf{S} < \mathbf{T}$, which contradicts the assumption of $>$-maximality of $\mathbf{S}$!

2. σ is a $\forall^{sk}\colon l$ inference. Then ρ is not correctly placed!

Hence $\mathbf{S}$ does not occur in $\Gamma \cup \Delta \cup \ell$. Applying Lemma 4.1.19, we obtain $\psi\,[\mathbf{S} \leftarrow \mathbf{Y}]$. We replace ρ in π' by

$$\frac{\begin{array}{c}(\psi\,[\mathbf{S} \leftarrow \mathbf{Y}])\\ \Gamma \vdash \Delta, \langle \overline{\mathbf{FY}} \rangle^{\ell}\end{array}}{\Gamma \vdash \Delta, \langle \forall_{\alpha} \mathbf{F} \rangle^{\ell}}\ \forall\colon r$$

We perform this procedure on all source inferences of $\mathbf{S}$ at once. As π' is sequentially pruned, all such inferences are parallel and the substitutions do not interfere with each other. As $\mathbf{Y}$ is new, it does not cause eigenvariable violations in $\psi\,[\mathbf{S} \leftarrow \mathbf{Y}]$. As we apply the same replacement on the homomorphic paths, weak regularity is preserved.

Finally, we obtain a tree consisting of $\mathbf{LK}_{\mathrm{sk}}$ inferences, except $\forall^{sk}\colon r$, and $\forall\colon r$ inferences obeying the eigenvariable condition. We replace the $\mathbf{LK}_{\mathrm{sk}}$ inferences by the respective **LK** inferences to obtain the desired **LK**-proof. □

Example 4.1.42. Recall the $\mathbf{LK}_{\mathrm{sk}}$-proof π_7 from Example 4.1.39. In it, there are indeed no eigenterm violations. Hence we may apply the replacements $[f(s) \leftarrow \alpha]$ and $[s \leftarrow \beta]$ (α, β being fresh variables of type ι) to obtain the $\mathbf{LK}$-proof π:

$$\dfrac{\dfrac{\dfrac{\dfrac{P(\beta) \vdash P(\beta) \qquad \dfrac{\dfrac{Q(\alpha) \vdash Q(\alpha)}{Q(\alpha) \vdash (\exists x)Q(x)}\,\exists_\iota\colon r}{(\exists x)Q(x) \vdash (\exists x)Q(x)}\,\exists_\iota\colon l}{P(\beta), P(\beta) \to (\exists x)Q(x) \vdash (\exists x)Q(x)}\,\to\colon l}{P(\beta), (\forall x)(P(x) \to (\exists x)Q(x)) \vdash (\exists x)Q(x)}\,\forall_\iota\colon l}{(\exists x)P(x), (\forall x)(P(x) \to (\exists x)Q(x)) \vdash (\exists x)Q(x)}\,\exists_\iota\colon l \qquad (\psi)}{\dfrac{(\exists x)P(x) \lor (\forall x)P(x), G, G \vdash (\exists x)Q(x), (\exists x)Q(x)}{(\exists x)P(x) \lor (\forall x)P(x), (\forall x)(P(x) \to (\exists x)Q(x)) \vdash (\exists x)Q(x)}\,\mathrm{contr}\colon *}\,\lor\colon l$$

where $G = (\forall x)(P(x) \to (\exists x)Q(x))$ and ψ is

$$\dfrac{\dfrac{\dfrac{P(\beta) \vdash P(\beta)}{(\forall x)P(x) \vdash P(\beta)}\,\forall_\iota\colon l \qquad \dfrac{\dfrac{Q(\alpha) \vdash Q(\alpha)}{Q(\alpha) \vdash (\exists x)Q(x)}\,\exists_\iota\colon r}{(\exists x)Q(x) \vdash (\exists x)Q(x)}\,\exists_\iota\colon l}{(\forall x)P(x), P(\beta) \to (\exists x)Q(x) \vdash (\exists x)Q(x)}\,\to\colon l}{(\forall x)P(x), (\forall x)(P(x) \to (\exists x)Q(x)) \vdash (\exists x)Q(x)}\,\forall_\iota\colon l$$

4.1.3 Another notion of regularity

In this section, we study an even more liberal soundness condition for $\mathbf{LK}_{\mathrm{sk}}$-trees. The material presented here is not needed for the CERES$^\omega$ method, so the reader only interested in this part of the present thesis may skip this subsection.

First, we endow the β-reduction relation with some additional structure:

Definition 4.1.43 (β-successors)**.** Consider an occurrence of $\mathbf{T} = (\lambda \mathbf{X}.\mathbf{R})\mathbf{S}$. Then $\mathbf{T} \to_\beta \mathbf{R}\,[\mathbf{X} \leftarrow \mathbf{S}]$. Let ω be an occurrence of an expression $\mathbf{U}$ in $\mathbf{R}$. Then the corresponding occurrence of $\mathbf{U}\,[\mathbf{X} \leftarrow \mathbf{S}]$ in $\mathbf{R}$ is called a *β-successor* of ω. Let ω be an occurrence of an expression $\mathbf{U}$ in $\mathbf{S}$. Then the corresponding occurrences of $\mathbf{U}$ in $\mathbf{R}\,[\mathbf{X} \leftarrow \mathbf{S}]$ are β-successors of ω. The notion of β-successor is extended in the obvious way to the $\twoheadrightarrow_\beta$ relation.

Example 4.1.44. Let $t = (\lambda x.f(x,x))y$ be an expression. Let ω_1 be the occurrence of $f(x,x)$, and let ω_2 be the occurrence of y. Then $t \twoheadrightarrow_\beta f(y,y)$ and the occurrence of $f(y,y)$ is a β-successor of ω_1, and both occurrences of y are β-successors of ω_2.

In the last subsection, we were able to handle different $\forall^{sk}\colon r$ applications with the same Skolem term — provided they were homomorphic, expressed through the notion of weak regularity. In this section, we relax the notion of weak regularity a bit and show that $\mathbf{LK}_{\mathrm{sk}}$-trees that are regular in this relaxed sense can be transformed into $\mathbf{LK}_{\mathrm{sk}}$-proofs. In contrast to homomorphic inferences (which are intuitively "very similar"), we introduce *disconnected* inferences, which are intuitively "unrelated". Towards this end, we introduce a relation that connects occurrences of expressions in $\mathbf{LK}_{\mathrm{sk}}$-trees. This is similar to the logical flow graphs of [15, 17]. The difference is that logical flow graphs consider the *logical* structure of a proof, while we are interested also in the "term" level. Furthermore, logical flow graphs are directed, while the relation we are interested in here is symmetric.

For a fixed tree π, we denote the set of all expression occurrences in π by $\mathcal{O}$.

Definition 4.1.45. Let π be an $\mathbf{LK}_{\mathrm{sk}}$-tree, let $\eta_1, \eta_2, \eta_3 \in \mathcal{O}$. We define the relation $\eta_1 \leftrightsquigarrow_1 \eta_2$:

1. η_1, η_2 occur in an axiom $\mathbf{F}[\mathbf{t}]_p \vdash \mathbf{F}[\mathbf{t}]_p$. If η_1 is the indicated occurrence of $\mathbf{t}$ at p in the antecedent, and η_2 is the indicated occurrence of $\mathbf{t}$ at p in the consequent, then $\eta_1 \leftrightsquigarrow_1 \eta_2$.

2. η_1 occurs in an active formula of a $\forall^{sk}\colon l$ inference.

$$\frac{(\overline{\mathbf{F}\mathbf{T}[\mathbf{t}]_p})^{l,\mathbf{T}[\mathbf{t}]_p}\Gamma \vdash \Delta}{(\forall_\alpha \mathbf{F})^l, \Gamma \vdash \Delta}\ \forall^{sk}\colon l$$

   If η_1, η_2 are distinct β-successors of the indicated occurrence of $\mathbf{t}$ in $\mathbf{T}$ in the auxiliary formula, then $\eta_1 \leftrightsquigarrow_1 \eta_2$. If η_1 is a β-successor of the indicated occurrence of $\mathbf{t}$ in $\mathbf{T}$ in the auxiliary formula, and η_2 is the indicated occurrence of $\mathbf{t}$ in $\mathbf{T}$ in the label, then $\eta_1 \leftrightsquigarrow_1 \eta_2$.

$$\frac{(\overline{\mathbf{F}[\mathbf{t}]_p\mathbf{T}})^{l,\mathbf{T}}, \Gamma \vdash \Delta}{(\forall_\alpha \mathbf{F}[\mathbf{t}]_p)^l, \Gamma \vdash \Delta}\ \forall^{sk}\colon l$$

   If η_1 is the indicated occurrence of $\mathbf{t}$ in the lower sequent, and η_2 is a β-successor of the indicated occurrence of $\mathbf{t}$ in the upper sequent, then $\eta_1 \leftrightsquigarrow_1 \eta_2$.

3. η_1, η_2 occur in the active formulas of a $\forall^{sk}\colon r$ inference.

$$\frac{\Gamma \vdash \Delta, (\overline{\mathbf{F}\mathbf{S}[\mathbf{t}]_p})^{\mathbf{T}_1,\dots,\mathbf{T}_n}}{\Gamma \vdash \Delta, (\forall_\alpha \mathbf{F})^{\mathbf{T}_1,\dots,\mathbf{T}_n}} \forall^{sk}\colon r$$

where $\mathbf{S} = \mathbf{f}\mathbf{S}_1 \dots \mathbf{S}_m$. If η_1, η_2 are distinct β-successors of the indicated occurrence of $\mathbf{t}$ in $\mathbf{S}$, then $\eta_1 \leftrightsquigarrow_1 \eta_2$. For all $i \leq n$, $j \leq m$, if $\mathbf{T}_i = \mathbf{S}_j$ and in

$$\frac{\Gamma \vdash \Delta, (\overline{\mathbf{F}(\mathbf{f}\mathbf{S}_1 \dots \mathbf{S}_j[\mathbf{t}]_p \dots \mathbf{S}_m)})^{\mathbf{T}_1,\dots,\mathbf{T}_i[\mathbf{t}]_p,\dots \mathbf{T}_n}}{\Gamma \vdash \Delta, (\forall_\alpha \mathbf{F})^{\mathbf{T}_1,\dots,\mathbf{T}_i[\mathbf{t}]_p,\dots \mathbf{T}_n}} \forall^{sk}\colon r$$

η_1 is a β-successor of the indicated occurrence of $\mathbf{t}$ in the auxiliary formula and η_2 is the indicated occurrence of $\mathbf{t}$ in the upper label, and η_3 is the indicated occurrence of $\mathbf{t}$ in the lower label, then $\eta_1 \leftrightsquigarrow_1 \eta_2$ and $\eta_1 \leftrightsquigarrow_1 \eta_3$.

4. η_1, η_2 occur in the auxiliary and main formula, respectively, of a $\vee\colon r^1$ inference

$$\frac{\Gamma \vdash \Delta, \mathbf{F}[\mathbf{t}]_p}{\Gamma \vdash \Delta, \mathbf{F}[\mathbf{t}]_p \vee \mathbf{G}} \vee\colon r^1$$

Let η_1 be the indicated occurrence of $\mathbf{t}$ at p in the auxiliary formula, and let η_2 be the indicated occurrence of $\mathbf{t}$ at p in the main formula. Then $\eta_1 \leftrightsquigarrow_1 \eta_2$.

5. η_1, η_2 occur in the active formulas of other propositional inferences: defined analogously to the previous case.

6. η_1, η_2 occur in the auxiliary formulas, and η_3 in the main formula of a contr: r inference

$$\frac{\Gamma \vdash \Delta, \langle \mathbf{F}[\mathbf{t}]_p \rangle^l, \langle \mathbf{F}[\mathbf{t}]_p \rangle^l}{\Gamma \vdash \Delta, \langle \mathbf{F}[\mathbf{t}]_p \rangle^l} \text{contr}\colon r$$

Let η_1 be the indicated occurrence of $\mathbf{t}$ at p in the left auxiliary formula, let η_2 be the indicated occurrence of $\mathbf{t}$ at p in the right auxiliary formula, and let η_3 be the indicated occurrence of $\mathbf{t}$ at p in the main formula. Then $\eta_1 \leftrightsquigarrow_1 \eta_3$, $\eta_2 \leftrightsquigarrow_1 \eta_3$.

$$\frac{\Gamma \vdash \Delta, \langle \mathbf{F} \rangle^{l[\mathbf{t}]_p}, \langle \mathbf{F} \rangle^{l[\mathbf{t}]_p}}{\Gamma \vdash \Delta, \langle \mathbf{F} \rangle^{l[\mathbf{t}]_p}} \text{contr}\colon r$$

Let η_1, η_2 be the indicated occurrences in the upper labels and let η_3 be the indicated occurrence in the lower label. Then $\eta_1 \leftrightsquigarrow_1 \eta_3$ and $\eta_2 \leftrightsquigarrow_1 \eta_3$.

7. η_1, η_2 occur in the auxiliary formulas, and η_3 in the main formula of a contr: l inference: defined analogously to the previous case.

8. η_1, η_2 occur in context formulas of a binary inference:

$$\frac{\Gamma, \mathbf{F}[\mathbf{t}]_p \vdash \Delta \qquad \Pi \vdash \Lambda}{\Gamma', \mathbf{F}[\mathbf{t}]_p \vdash \Delta'}$$

   Let η_1 be the indicated occurrence of $\mathbf{t}$ at p in the premise, and let η_2 be the indicated occurrence of $\mathbf{t}$ at p in the conclusion. Then $\eta_1 \leftrightsquigarrow_1 \eta_2$.

   The definition is made analogously for η_1, η_2 occuring in the consequent, and for η_1 occuring in the left premise.

9. η_1, η_2 occur in the context formulas of a unary inference: defined analogously to the previous case.

10. η_1, η_2 occur in the context labels of a inference: defined analogously to the previous case.

The relation $\leftrightsquigarrow$ is then defined as the symmetric closure of the $\leftrightsquigarrow_1$ relation. By $\leftrightsquigarrow^*$ we denote the reflexive and transitive closure of $\leftrightsquigarrow$. For $\eta \in \mathcal{O}$ we denote the equivalence class of η modulo $\leftrightsquigarrow^*$ by $[\eta]_c$.

Let $\eta \in \mathcal{O}$, then by $t(\eta)$ we denote the expression at η. By definition we have

Lemma 4.1.46. *For all $\eta_1, \eta_2 \in \mathcal{O}$, if $\eta_1 \leftrightsquigarrow \eta_2$ then $t(\eta_1) = t(\eta_2)$.*

Definition 4.1.47. Let $\eta_1, \ldots, \eta_n \in \mathcal{O}$ be pairwise different. If for all $1 \leq i < n$, $\eta_i \leftrightsquigarrow \eta_{i+1}$, then $(\eta_1, \ldots, \eta_n)$ is called a *dependency path (from η_1 to η_n)*.

Lemma 4.1.48. *There exists a dependency path from η_1 to η_n iff $\eta_1 \leftrightsquigarrow^* \eta_n$.*

Proof. One direction is trivial. The other direction follows from the fact that we may remove loops from a witness of $\eta_1 \leftrightsquigarrow^* \eta_n$. □

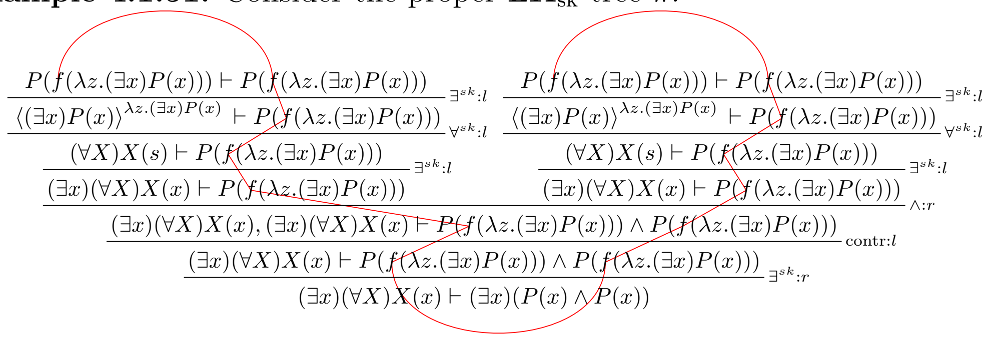
Definition 4.1.49. Let ρ_1, ρ_2 be $\forall^{sk}$: r applications with the same Skolem symbol. Let η_1 be a β-successor of the Skolem symbol occurrence of ρ_1, and let η_2 be a β-successor of the Skolem symbol occurrence of ρ_2. Then a dependency path from η_1 to η_2 is called a *connection from ρ_1 to ρ_2*.

Proposition 4.1.50. *Let ρ_1, ρ_2 be $\forall^{sk}\!:r$ applications with the same Skolem symbol. Then there exists a connection from ρ_1 to ρ_2 iff there exists a connection from ρ_2 to ρ_1.*

Proof. By the symmetry of $\leftrightsquigarrow^*$ and Lemma 4.1.48. □

If a connection from ρ_1 to ρ_2 exists, we say that ρ_1 and ρ_2 are *connected*. If no connection exists, then we say that they are *disconnected*.

Example 4.1.51. Consider the proper $\mathbf{LK}_{\mathrm{sk}}$-tree π:



For reasons of readability, some labels and their $\leftrightsquigarrow$-relations have been omitted from the figure. The connection between the two upper $\exists^{sk}\!:l$ inferences has been indicated. Observe that the two $\exists^{sk}\!:l$ inferences with Skolem term s are disconnected.

We can now state our relaxed regularity condition:

Definition 4.1.52. An $\mathbf{LK}_{\mathrm{sk}}$-tree π is called *weakly+ regular* if for every two distinct $\forall^{sk}\!:r$ applications ρ_1, ρ_2 in π

1. if ρ_1, ρ_2 have identical Skolem terms then either
   (a) ρ_1, ρ_2 are homomorphic, or
   (b) ρ_1, ρ_2 are disconnected,
2. and if ρ_2 operates on a descendent of the main formula of ρ_1, then ρ_1 and ρ_2 have different Skolem symbols.

Observe that due to the second condition, weak regularity does not imply weak+ regularity.

Example 4.1.53. Let π be as in Example 4.1.51. Let ρ_1, ρ_2 be the $\exists^{sk}\!:l$ inferences with Skolem term $f(\lambda z.(\exists x)P(x))$, and let σ_1, σ_2 be the $\exists^{sk}\!:l$ inferences with Skolem term s. Then ρ_1, ρ_2 are disconnected, and σ_1, σ_2 are homomorphic, so π is weakly+ regular.

Lemma 4.1.54. *Let $\omega_1, \omega_2 \in \mathcal{O}$ such that ω_2 is a descendent of ω_1. Then $\omega_1 \leftrightsquigarrow^* \omega_2$.*

Proof. By induction on the number of inferences between ω_1 and ω_2. □

Now follows the main lemma of this section. It shows that we can separate disconnected $\forall^{sk}\colon r$ inferences modulo homomorphism equivalence classes. This is the main technical tool to convert weakly+ regular, proper $\mathbf{LK}_{\text{sk}}$-trees to $\mathbf{LK}_{\text{sk}}$-proofs. In the light of Proposition 4.1.15 we make the following definition:

Definition 4.1.55. Let ρ be an inference in an $\mathbf{LK}_{\text{sk}}$-tree. By $[\rho]_h$ we denote the equivalence class of ρ induced by inference homomorphisms.

Lemma 4.1.56. *Let π be a proper $\mathbf{LK}_{\text{sk}}$-tree of E. Let ρ be a $\forall^{sk}\colon r$ application with Skolem symbol $\mathbf{f}$ in π. Let $\mathbf{g}$ be a Skolem symbol not occuring in π such that $\mathbf{f}$ and $\mathbf{g}$ have the same signature. Let S be the set of the β-successors of the Skolem symbol occurrences of the inferences in $[\rho]_h$. Let $[S]_c = \bigcup_{\mu \in S} [\mu]_c$. Then the result π' of simultaneously replacing in π, for every $\omega \in [S]_c$, $\mathbf{f}$ at ω by $\mathbf{g}$, is an $\mathbf{LK}_{\text{sk}}$-tree of S. Furthermore, if π is weakly+ regular, then so is π'.*

Proof. First, we show that substituting $\mathbf{f}$ for $\mathbf{g}$ at $\omega \in [S]_c$ really produces an $\mathbf{LK}_{\text{sk}}$-tree of S. By Lemma 4.1.46 indeed $t(\omega) = \mathbf{f}$. We proceed by case distinction on the definition of $\leftrightsquigarrow$:

1. ω occurs in an axiom $\mathbf{F}[\mathbf{f}]_p \vdash \mathbf{F}[\mathbf{f}]_p$. After substitution, it is still an axiom $\mathbf{F}[\mathbf{g}]_p \vdash \mathbf{F}[\mathbf{g}]_p$.

2. ω occurs in active formulas of a $\forall^{sk}\colon l$ inference. If it occurs in the β-successors of its substitution term $\mathbf{T}$

$$\frac{(\overline{\mathbf{FT}[\mathbf{f}]_p})^{l,\mathbf{T}[\mathbf{f}]_p}\Gamma \vdash \Delta}{(\forall_\alpha \mathbf{F})^l, \Gamma \vdash \Delta}\ \forall^{sk}\colon l$$

in which case the inference becomes

$$\frac{(\overline{\mathbf{FT}[\mathbf{g}]_p})^{l,\mathbf{T}[\mathbf{g}]_p}\Gamma \vdash \Delta}{(\forall_\alpha \mathbf{F})^l, \Gamma \vdash \Delta}\ \forall^{sk}\colon l$$

as for all β-successors λ of the indicated occurrence of $\mathbf{f}$ in the auxiliary formula, $\omega \leftrightsquigarrow \lambda$, and also for the indicated occurrence σ in the label, $\omega \leftrightsquigarrow \sigma$.

Otherwise if ω occurs as a β successor of a term in $\mathbf{F}$

$$\frac{(\overline{\mathbf{F}[\mathbf{f}]_p\mathbf{T}})^{l,\mathbf{T}}, \Gamma \vdash \Delta}{(\forall_\alpha \mathbf{F}[\mathbf{f}]_p)^l, \Gamma \vdash \Delta} \forall^{sk}\colon l$$

the inference becomes

$$\frac{(\overline{\mathbf{F}[\mathbf{g}]_p\mathbf{T}})^{l,\mathbf{T}}, \Gamma \vdash \Delta}{(\forall_\alpha \mathbf{F}[\mathbf{g}]_p)^l, \Gamma \vdash \Delta} \forall^{sk}\colon l$$

3. ω occurs in active formulas of a $\forall^{sk}\colon r$ inference. If it occurs in the β-successor of its Skolem term $\mathbf{S} = \mathbf{h}\mathbf{S}_1 \dots \mathbf{S}_m$,

$$\frac{\Gamma \vdash \Delta, (\overline{\mathbf{F}\mathbf{S}[\mathbf{f}]_p})^{\mathbf{T}_1,\dots,\mathbf{T}_n}}{\Gamma \vdash \Delta, (\forall_\alpha \mathbf{F})^{\mathbf{T}_1,\dots,\mathbf{T}_n}} \forall^{sk}\colon r$$

distinguish:

   (a) ω is a β-successor of the Skolem symbol $\mathbf{h}$ of the inference. Then the inference becomes

$$\frac{\Gamma \vdash \Delta, (\overline{\mathbf{F}(\mathbf{g}\mathbf{S}_1 \dots \mathbf{S}_m)})^{\mathbf{T}_1,\dots,\mathbf{T}_n}}{\Gamma \vdash \Delta, (\forall_\alpha \mathbf{F})^{\mathbf{T}_1,\dots,\mathbf{T}_n}} \forall^{sk}\colon r$$

   which is correct as $\mathbf{g}$ is a Skolem symbol.

   (b) ω is not a β-successor of the Skolem symbol. Then it is a β-successor of an occurrence in some $\mathbf{S}_i$,

$$\frac{\Gamma \vdash \Delta, (\overline{\mathbf{F}(\mathbf{h}\mathbf{S}_1 \dots \mathbf{S}_i[\mathbf{f}]_p \dots \mathbf{S}_m)})^{\mathbf{T}_1,\dots,\mathbf{T}_n}}{\Gamma \vdash \Delta, (\forall_\alpha \mathbf{F})^{\mathbf{T}_1,\dots,\mathbf{T}_n}} \forall^{sk}\colon r$$

   and hence the inference becomes

$$\frac{\Gamma \vdash \Delta, (\overline{\mathbf{F}(\mathbf{h}\mathbf{S}_1 \dots \mathbf{S}_i[\mathbf{g}]_p \dots \mathbf{S}_m)})^{\mathbf{T}'_1,\dots,\mathbf{T}'_n}}{\Gamma \vdash \Delta, (\forall_\alpha \mathbf{F})^{\mathbf{T}'_1,\dots,\mathbf{T}'_n}} \forall^{sk}\colon r$$

   where $\mathbf{T}'_j = \mathbf{T}_j$ if $\mathbf{S}_i \neq \mathbf{T}_j$ and $\mathbf{T}'_j = \mathbf{S}_i[\mathbf{g}]_p$ otherwise. Clearly $\{\mathbf{T}'_1, \dots, \mathbf{T}'_n\} \subseteq \{\mathbf{S}_1, \dots, \mathbf{S}_i[\mathbf{g}]_p, \dots, \mathbf{S}_m\}$.

4. ω occurs in the auxiliary or main formula of a $\vee\colon r^1$ inference

$$\frac{\Gamma \vdash \Delta, \mathbf{F}[\mathbf{f}]_p}{\Gamma \vdash \Delta, \mathbf{F}[\mathbf{f}]_p \vee \mathbf{G}} \vee\colon r^1$$

This inference becomes

$$\frac{\Gamma \vdash \Delta, \mathbf{F}[\mathbf{g}]_p}{\Gamma \vdash \Delta, \mathbf{F}[\mathbf{g}]_p \vee \mathbf{G}} \vee\colon r^1$$

5. ω occurs in the active formulas of other propositional inferences: analogous to the previous case. For example, consider

$$\frac{\mathbf{F}[\mathbf{f}]_p, \Gamma \vdash \Delta \qquad \mathbf{G}, \Pi \vdash \Lambda}{\mathbf{F}[\mathbf{f}]_p \vee \mathbf{G}, \Gamma, \Pi \vdash \Delta, \Lambda}$$

This inference becomes

$$\frac{\mathbf{F}[\mathbf{g}]_p, \Gamma \vdash \Delta \qquad \mathbf{G}, \Pi \vdash \Lambda}{\mathbf{F}[\mathbf{g}]_p \vee \mathbf{G}, \Gamma, \Pi \vdash \Delta, \Lambda}$$

6. ω occurs in the active formulas of a contr: r inference

$$\frac{\Gamma \vdash \Delta, \mathbf{F}[\mathbf{f}]_p, \mathbf{F}[\mathbf{f}]_p}{\Gamma \vdash \Delta, \mathbf{F}[\mathbf{f}]_p} \text{contr}\colon r$$

This inference becomes

$$\frac{\Gamma \vdash \Delta, \mathbf{F}[\mathbf{g}]_p, \mathbf{F}[\mathbf{g}]_p}{\Gamma \vdash \Delta, \mathbf{F}[\mathbf{g}]_p} \text{contr}\colon r$$

7. ω occurs in the active formulas of a contr: l inference: analogous to the previous case.

8. ω occurs in context formulas of a binary inference:

$$\frac{\Gamma, \mathbf{F}[\mathbf{f}]_p \vdash \Delta \qquad \Pi \vdash \Lambda}{\Gamma', \mathbf{F}[\mathbf{f}]_p \vdash \Delta'}$$

This inference becomes

$$\frac{\Gamma, \mathbf{F}[\mathbf{g}]_p \vdash \Delta \qquad \Pi \vdash \Lambda}{\Gamma', \mathbf{F}[\mathbf{g}]_p \vdash \Delta'}$$

The arguments for ω occuring at other positions is analogous.

9. ω occurs in the context formulas of a unary inference: analogous to the previous case.

As E does not contain Skolem symbols, π' is a $\mathbf{LK}_{\mathrm{sk}}$-tree of E. This completes the first part of the proof.

Now we show that if π is weakly+ regular, then so is π'. Consider two $\forall^{sk}\colon r$ applications ρ, σ in π, and their copies ρ', σ' in π'. It is clear that substituting a new Skolem symbol for another one cannot make occurrences of different Skolem symbols in π become occurrences of the same Skolem symbol in π', so the second condition is fulfilled.

For the first condition, we show that

(1) If ρ and σ are homomorphic and ρ', σ' have the same Skolem term, then ρ' and σ' are homomorphic, and

(2) If ρ and σ are disconnected then so are ρ' and σ'.

(2) is trivial, as a connection in π' gives rise to a connection in π. It remains to show (1).

For contradiction, assume that ρ' and σ' are not homomorphic. As ρ and σ are homomorphic, there exist homomorphic paths μ_ρ and μ_σ in π such that $F(\mu_\rho) = F(\mu_\sigma)$. Consider their copies μ'_ρ and μ'_σ in π'. As ρ' and σ' are not homomorphic, there must be some occurrence ω_ρ of $\mathbf{f}$ on μ_ρ such that $\omega_\rho \in [S]_c$, but for the corresponding occurrence ω_σ on μ_σ, $\omega_\sigma \notin [S]_c$. As E does not contain Skolem symbols, there exists a $\forall$ inference ξ that eliminates a descendent ω^* of ω_ρ. Distinguish:

1. ξ is below the uniting contraction of ρ, σ. But then a descendent of ω_ρ is present in the uniting contraction

$$\frac{\Gamma \vdash \Delta, \mathbf{F}[\mathbf{f}], \mathbf{F}[\mathbf{f}]}{\Gamma \vdash \Delta, \mathbf{F}[\mathbf{f}]}\ \mathrm{contr}\colon r$$

and is therefore $\leftrightsquigarrow^*$-connected to a descendent of ω_σ. Lemma 4.1.54 therefore yields $\omega_\rho \leftrightsquigarrow^* \omega_\sigma$ and hence $\omega_\sigma \in [S]_c$, a contradiction.

2. ξ is above the uniting contraction of ρ, σ. Further distinguish:

(a) ξ is a $\forall^{sk}\colon l$ application

$$\frac{(\mathbf{F}\mathbf{T}[\mathbf{f}]_p)^{l,\mathbf{T}[\mathbf{f}]_\mathbf{P}}, \Gamma \vdash \Delta}{(\forall_\alpha \mathbf{F})^l, \Gamma \vdash \Delta}\ \forall^{sk}\colon l$$

where ω^* is the indicated occurrence of $\mathbf{f}$ in the auxiliary formula. Then by Lemma 4.1.54, ω_ρ is $\leftrightsquigarrow^*$-connected to the indicated occurrence of $\mathbf{f}$ in the label.

This occurrence, in turn, has an ancestor in (and is therefore $\leftrightsquigarrow^*$-connected to) the label of the main formula of ρ, and hence is $\leftrightsquigarrow^*$-connected to an occurrence in a β-successor of the Skolem term of ρ. Hence if the Skolem term of ρ is $\mathbf{t}[\mathbf{f}]_q$, the Skolem term of ρ' is $\mathbf{t}'[\mathbf{g}]_q$. We claim that $\mathbf{t}'[\mathbf{g}]_q$ is not the Skolem term of σ': Assume it is. Let λ be any occurrence of a β-successor of $\mathbf{f}$ at q in the Skolem term $\mathbf{t}[\mathbf{f}]_q$ of σ, then $\lambda \in [S]_c$.

Now consider ω_σ. As ρ and σ are homomorphic, there exists a ξ' with the same auxiliary formula as ξ which removes the descendent of ω_σ and therefore, just like above, ω_σ is $\leftrightsquigarrow^*$-connected to the label of the main formula of σ, and hence to an occurrence in a β-successor of $\mathbf{f}$ at q in the Skolem term $\mathbf{t}[\mathbf{f}]_q$ of σ. But then, $\omega_\sigma \in [S]_c$, a contradiction! Hence ρ' and σ' have different Skolem terms, and we have shown (1).

(b) ξ is a $\forall^{sk}\colon r$ application

$$\frac{\Gamma \vdash \Delta, (\overline{\mathbf{F}\mathbf{S}[\mathbf{f}]_p})^l}{\Gamma \vdash \Delta, (\forall_\alpha \mathbf{F})^l}\ \forall^{sk}\colon r$$

If the indicated occurrence of $\mathbf{f}$ occurs in a proper subterm of $\mathbf{S}$, then it occurs in l and, because π is proper, there exists a $\forall^{sk}\colon l$ application removing a descendent of the label occurrence, and we may apply the argument from the previous case. So we consider the case that p is the head position, and ξ is

$$\frac{\Gamma \vdash \Delta, (\overline{\mathbf{F}(\mathbf{f}\mathbf{S}_1 \ldots \mathbf{S}_m)})^l}{\Gamma \vdash \Delta, (\forall_\alpha \mathbf{F})^l}\ \forall^{sk}\colon r$$

But ξ operates on a descendent of the main formula of ρ and has the same Skolem symbol, which implies that π is not weakly+ regular, and hence contradiction!

□

By applying the previous lemma repeatedly, we can rename Skolem terms of disconnected inference in such a way that weak regularity is attained:

Theorem 4.1.57. *Let π be a weakly+ regular, proper $\mathbf{LK}_{\mathrm{sk}}$-tree of S. Then there exists an $\mathbf{LK}_{\mathrm{sk}}$-proof of S.*

Proof. Let $r(\pi)$ be the number of pairs of $\forall^{sk}\colon r$ inferences ρ, σ with identical Skolem terms in π such that ρ and σ are not homomorphic. We proceed by induction: If $r(\pi) = 0$, then all such inferences are homomorphic and π is weakly regular. Otherwise, we construct a weakly+ regular, proper $\mathbf{LK}_{\mathrm{sk}}$-tree π' of S such that $r(\pi') < r(\pi)$: Let ρ, σ be $\forall^{sk}\colon r$ inferences in π with identical Skolem terms which are not homomorphic. We apply Lemma 4.1.56 to ρ to replace its Skolem symbol $\mathbf{f}$ by a new one $\mathbf{g}$ to obtain π'. We claim that the Skolem symbol of σ is unchanged: assume it is not, then σ is connected to some $\nu \in [\rho]_h$. As σ and ν have the same Skolem term, ν must be homomorphic to σ, but then ρ is homomorphic to σ, a contradiction. So the copies σ', ρ' of σ, ρ in π' now have different Skolem terms. To show that $r(\pi') < r(\pi)$, it now suffices to show for pairs (ξ', λ') of $\forall^{sk}\colon r$ inferences in π' and their copies ξ, λ in π that if ξ', λ' have the same Skolem term and are not homomorphic, then ξ, λ have the same Skolem term, and are not homomorphic. As $\mathbf{g}$ is a new Skolem term, ξ and λ have the same Skolem term if ξ', λ' do. The second part follows directly from (1) in the proof of Lemma 4.1.56. $\square$

Example 4.1.58. Consider the following proper $\mathbf{LK}_{\mathrm{sk}}$-tree π:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{R(t,s)\vdash R(t,s)}{(\exists x)R(x,s)\vdash R(t,s)}\,{\scriptstyle \exists^{sk}:l}}{(\forall X)X(s)\vdash R(t,s)}\,{\scriptstyle \forall^{sk}:l} \quad \dfrac{P(s)\vdash P(s)}{P(s)\vdash (\forall x)P(x)}\,{\scriptstyle \forall^{sk}:r}}{(\forall X)X(s)\vee P(s)\vdash R(t,s),(\forall x)P(x)}\,{\scriptstyle \vee:l}}{(\forall x)((\forall X)X(x)\vee P(x))\vdash R(t,s),(\forall x)P(x)}\,{\scriptstyle \forall^{sk}:l} \quad \dfrac{\dfrac{\dfrac{\dfrac{R(t,s)\vdash R(t,s)}{(\exists x)R(x,s)\vdash R(t,s)}\,{\scriptstyle \exists^{sk}:l}}{(\forall X)X(s)\vdash R(t,s)}\,{\scriptstyle \forall^{sk}:l} \quad \dfrac{P(s)\vdash P(s)}{P(s)\vdash (\forall x)P(x)}\,{\scriptstyle \forall^{sk}:r}}{(\forall X)X(s)\vee P(s)\vdash R(t,s),(\forall x)P(x)}\,{\scriptstyle \vee:l}}{(\forall x)((\forall X)X(x)\vee P(x))\vdash R(t,s),(\forall x)P(x)}\,{\scriptstyle \forall^{sk}:l}}{(\forall x)((\forall X)X(x)\vee P(x)),(\forall x)((\forall X)X(x)\vee P(x))\vdash R(t,s)\wedge R(t,s),(\forall x)P(x),(\forall x)P(x)}\,{\scriptstyle \wedge:r}}{(\forall x)((\forall X)X(x)\vee P(x))\vdash R(t,s)\wedge R(t,s),(\forall x)P(x),(\forall x)P(x)}\,{\scriptstyle \mathrm{contr}:l}}{(\forall x)((\forall X)X(x)\vee P(x))\vdash (\exists u)(\exists v)(\exists w)(\exists x)(R(u,v)\wedge R(w,x)),(\forall x)P(x),(\forall x)P(x)}\,{\scriptstyle \exists^{sk}:r\times 4}$$

where $T = \lambda z.(\exists x)R(x,z)$, $t = f(T,s)$, and $s \in \mathcal{K}_\iota$, $f \in \mathcal{K}_{\iota\to o,\iota,\iota}$, and sequent labels have been left out for readability. The two $\exists^{sk}\colon l$ inferences are homomorphic, and the two $\forall^{sk}\colon r$ inferences are disconnected, so π is weakly+ regular. Hence we may apply Theorem 4.1.57. The two $\forall^{sk}\colon r$ inferences are the only pair of inferences with identical Skolem term which are not homomorphic, so we replace s by a new $r \in \mathcal{K}_\iota$ according to Lemma 4.1.56. One of the two relevant paths is indicated in the figure; the other goes through the sequent labels into an occurrence of t. The result of the replacement is

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{R(t',r)\vdash R(t',r)}{(\exists x)R(x,r)\vdash R(t',r)}\,{\scriptstyle \exists^{sk}_\iota:l}}{(\forall X)X(r)\vdash R(t',r)}\,{\scriptstyle \forall^{sk}_{\iota\to o}:l} \quad \dfrac{P(r)\vdash P(r)}{P(r)\vdash (\forall x)P(x)}\,{\scriptstyle \forall^{sk}_\iota:r}}{(\forall X)X(r)\vee P(r)\vdash R(t',r),(\forall x)P(x)}\,{\scriptstyle \vee:l}}{(\forall x)((\forall X)X(x)\vee P(x))\vdash R(t',r),(\forall x)P(x)}\,{\scriptstyle \forall^{sk}_\iota:l} \quad \dfrac{\dfrac{\dfrac{\dfrac{R(t,s)\vdash R(t,s)}{(\exists x)R(x,s)\vdash R(t,s)}\,{\scriptstyle \exists^{sk}_\iota:l}}{(\forall X)X(s)\vdash R(t,s)}\,{\scriptstyle \forall^{sk}_{\iota\to o}:l} \quad \dfrac{P(s)\vdash P(s)}{P(s)\vdash (\forall x)P(x)}\,{\scriptstyle \forall^{sk}_\iota:r}}{(\forall X)X(s)\vee P(s)\vdash R(t,s),(\forall x)P(x)}\,{\scriptstyle \vee:l}}{(\forall x)((\forall X)X(x)\vee P(x))\vdash R(t,s),(\forall x)P(x)}\,{\scriptstyle \forall^{sk}_\iota:l}}{(\forall x)((\forall X)X(x)\vee P(x)),(\forall x)((\forall X)X(x)\vee P(x))\vdash R(t',r)\wedge R(t,s),(\forall x)P(x),(\forall x)P(x)}\,{\scriptstyle \wedge:r}}{(\forall x)((\forall X)X(x)\vee P(x))\vdash R(t',r)\wedge R(t,s),(\forall x)P(x),(\forall x)P(x)}\,{\scriptstyle \mathrm{contr}:l}}{(\forall x)((\forall X)X(x)\vee P(x))\vdash (\exists u)(\exists v)(\exists w)(\exists x)(R(u,v)\wedge R(w,x)),(\forall x)P(x),(\forall x)P(x)}\,{\scriptstyle \exists^{sk}_\iota:r\times 4}$$

where $t' = f(T,r)$. Now all $\forall^{sk}\colon r$ and $\exists^{sk}\colon l$ inferences have pairwise different Skolem terms, and hence this is an $\mathbf{LK}_{\mathrm{sk}}$-proof.

Theorem 4.1.59 (Soundness)**.** *Let π be a weakly+ regular, proper $\mathbf{LK}_{\mathrm{sk}}$-tree of S. Then there exists a cut-free* **LK***-proof of S.*

Proof. By Theorems 4.1.57 and 4.1.41. □

4.2 The resolution calculus $\mathcal{R}_{al}$

In this section, we introduce the resolution calculus $\mathcal{R}_{al}$ we will use to define the CERES$^\omega$ method in the next section. As in $\mathbf{LK}_{\mathrm{sk}}$, we deal with labelled sequents. Note that $\mathcal{R}$ will include rules for CNF transformation: this is standard in higher-order resolution, as the notion of clause is not closed under substitution. It is also done in the ENAR calculus from [21] for a similar reason.

Definition 4.2.1 ($\mathcal{R}_{al}$ rules, deductions and refutations)**.**

$$\frac{\Gamma \vdash \Delta, \langle \neg \mathbf{A} \rangle^\ell}{\langle \mathbf{A} \rangle^\ell, \Gamma \vdash \Delta}\ \neg^T \qquad \frac{\langle \neg \mathbf{A} \rangle^\ell, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \langle \mathbf{A} \rangle^\ell}\ \neg^F$$

$$\frac{\Gamma \vdash \Delta, \langle \mathbf{A} \vee \mathbf{B} \rangle^\ell}{\Gamma \vdash \Delta, \langle \mathbf{A} \rangle^\ell, \langle \mathbf{B} \rangle^\ell}\ \vee^T \qquad \frac{\langle \mathbf{A} \vee \mathbf{B} \rangle^\ell, \Gamma \vdash \Delta}{\langle \mathbf{A} \rangle^\ell, \Gamma \vdash \Delta}\ \vee^F_l \qquad \frac{\langle \mathbf{A} \vee \mathbf{B} \rangle^\ell, \Gamma \vdash \Delta}{\langle \mathbf{B} \rangle^\ell, \Gamma \vdash \Delta}\ \vee^F_r$$

$$\frac{\Gamma \vdash \Delta, \langle \forall_\alpha \mathbf{A} \rangle^\ell}{\Gamma \vdash \Delta, \langle \mathbf{A}\mathbf{X} \rangle^{\ell,\mathbf{X}}}\ \forall^T \qquad \frac{\langle \forall_\alpha \mathbf{A} \rangle^\ell, \Gamma \vdash \Delta}{\langle \mathbf{A}(\mathbf{f}\mathbf{S}_1 \dots \mathbf{S}_n) \rangle^\ell, \Gamma \vdash \Delta}\ \forall^F \qquad \frac{S}{S\,[\mathbf{X} \leftarrow \mathbf{T}]}\ \mathrm{Sub}$$

$$\frac{\Gamma \vdash \Delta, \langle \mathbf{A} \rangle^{\ell_1}, \dots, \langle \mathbf{A} \rangle^{\ell_n} \quad \langle \mathbf{A} \rangle^{\ell_{n+1}}, \dots, \langle \mathbf{A} \rangle^{\ell_m}, \Pi \vdash \Lambda}{\Gamma, \Pi \vdash \Delta, \Lambda}\ \mathrm{Cut}$$

In Cut, $\mathbf{A}$ is atomic. In $\forall^T$, $\mathbf{X}$ is a variable of appropriate type which does not occur in $\Gamma, \Delta, \mathbf{A}$. In $\forall^F$, $\ell = \mathbf{S}_1, \dots, \mathbf{S}_n$ and if $\tau(\mathbf{S}_i) = \alpha_i$ for $1 \leq i \leq n$ then $\mathbf{f} \in \mathcal{K}_{\alpha_1,\dots,\alpha_n,\alpha}$ is a Skolem symbol. An application of this rule is called *source inference* of $\mathbf{f}\mathbf{S}_1 \dots \mathbf{S}_m$, and $\mathbf{f}\mathbf{S}_1 \dots \mathbf{S}_m$ is called the *Skolem term* of this inference.

Let $\mathcal{C}$ be a set of sequents. A sequence of sequents $S_1, \dots, S_n$ is an $\mathcal{R}_{al}$-*deduction of S_n from $\mathcal{C}$* if for all $1 \leq i \leq n$ either

1. $S_i \in \mathcal{C}$ or
2. S_i is derived from S_j (and S_k) by an $\mathcal{R}_{al}$ rule, where $j, k < i$.

In addition, we require that all $\forall^F$ inferences used have pairwise distinct Skolem symbols. An $\mathcal{R}_{al}$-deduction of the empty sequent from $\mathcal{C}$ is called an $\mathcal{R}_{al}$*-refutation of* $\mathcal{C}$.

The calculus $\mathcal{R}_{al}$ is quite close to Andrews' resolution calculus $\mathcal{R}$ presented in Section 2.3. Just like in $\mathcal{R}$, $\mathcal{R}_{al}$-deductions are defined in a linear fashion (in contrast to **LK**-proofs and $\mathbf{LK}_{\mathrm{sk}}$-trees). The two main differences to $\mathcal{R}$ are (1) the use of labels to control the arguments of the Skolem terms introduced by the $\forall^F$ rule, and (2) the incorporation of Andrews' rules of Simplification and Cut into the Cut rule of $\mathcal{R}_{al}$. Regarding the latter, note that this restriction is not as serious as it may appear at first glance: For example, the sentence $F = \forall x P(x) \to (P(a) \wedge P(b))$ cannot be proved in **LK**, restricted to atomic cut, without using non-atomic contraction. Still, $\neg F$ can be refuted in $\mathcal{R}_{al}$. Relative completeness of $\mathcal{R}_{al}$ is still an open problem:

Conjecture 4.2.2. *Let $\mathcal{S}$ be a set of labelled sequents. If there exists an $\mathcal{R}$-refutation of the reduct of $\mathcal{S}$, then there exists an $\mathcal{R}_{al}$-refutation of $\mathcal{S}$.*

This conjecture will imply completeness of the CERES$^\omega$ method, in conjunction with the following result from [1] (which still holds in the presence of Miller's restriction):

Theorem 4.2.3. *Let $\mathcal{S}$ be a set of sentences. If there exists a $\mathcal{T}$-refutation of $\mathcal{S}$ then there exists an $\mathcal{R}$-refutation of $\mathcal{S}$.*

Note that the above formulation of the conjecture is not the only way to attain this goal: completeness with respect to an appropriate intensional model class (see [10, 45]) for higher-order logic would also suffice (together with a soundness theorem for that class for **LK**). The formulation above has the advantage that an effective proof of it would give an algorithm to transform $\mathcal{R}$-refutations into $\mathcal{R}_{al}$-refutations, allowing proof search to be done in practice in the more convenient $\mathcal{R}$ calculus.

The following subsection will present results which indicate that the conjecture can indeed be resolved positively by studying whether the $\mathcal{R}$ calculus can be sufficiently restricted.

4.2.1 Restricting $\mathcal{R}$ (towards $\mathcal{R}_{al}$)

In this section, we will consider the following calculus:

Definition 4.2.4 (Resolution calculus $\mathcal{R}_a$)**.** We define the calculus $\mathcal{R}_a$ analogously to the calculus $\mathcal{R}$ (see Definition 2.3.1); it consists of the rules:

$$\frac{\Gamma \vdash \Delta, \neg\mathbf{A}}{\mathbf{A}, \Gamma \vdash \Delta}\ \neg^T \qquad \frac{\neg\mathbf{A}, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \mathbf{A}}\ \neg^F \qquad \frac{\Gamma \vdash \Delta, \mathbf{A} \vee \mathbf{B}}{\Gamma \vdash \Delta, \mathbf{A}, \mathbf{B}}\ \vee^T$$

$$\frac{\mathbf{A} \vee \mathbf{B}, \Gamma \vdash \Delta}{\mathbf{A}, \Gamma \vdash \Delta}\ \vee^F_l \qquad \frac{\mathbf{A} \vee \mathbf{B}, \Gamma \vdash \Delta}{\mathbf{B}, \Gamma \vdash \Delta}\ \vee^F_r$$

$$\frac{\Gamma \vdash \Delta, \forall \mathbf{A}}{\Gamma \vdash \Delta, \mathbf{AX}} \forall^T \qquad \frac{\forall \mathbf{A}, \Gamma \vdash \Delta}{\mathbf{A}(\mathbf{fX}_1 \ldots \mathbf{X}_n), \Gamma \vdash \Delta} \forall^F \qquad \frac{S}{S\,[\mathbf{X} \leftarrow \mathbf{T}]} \text{Sub}$$

$$\frac{\Gamma \vdash \Delta, \mathbf{A}, \ldots, \mathbf{A} \quad \mathbf{A}, \ldots, \mathbf{A}, \Pi \vdash \Lambda}{\Gamma, \Pi \vdash \Delta, \Lambda} \text{mCut}$$

where in $\forall^F$, $\mathbf{X}_1, \ldots, \mathbf{X}_n$ are all the free variables occuring in $\mathbf{A}$, and if $\tau(\mathbf{X}_i) = t_i$ for $1 \leq i \leq n$ and $\tau(\mathbf{A}) = t \to o$, then $\mathbf{f} \in \mathcal{K}_{t_1,\ldots,t_n,t}$. In mCut, $\mathbf{A}$ is atomic.

Note that $\mathcal{R}_a$ is "in-between" $\mathcal{R}$ and $\mathcal{R}_{al}$: it does not have the Sim^T, Sim^F rules of $\mathcal{R}$, but the $\forall^F$ and $\forall^T$ rules work as they do in $\mathcal{R}$. In this section, we are interested in the question whether $\mathcal{R}_a$ is still complete (with respect to $\mathcal{R}$). The answer will be positive for a fragment of $\mathcal{R}$:

Definition 4.2.5. Let γ be an $\mathcal{R}$-deduction such that all Skolem terms of $\forall^F$ inferences in γ are constants. Then γ is called an *$\mathcal{R}_c$-deduction.*

Let γ be an $\mathcal{R}$-deduction, and ρ_1, ρ_2 inferences in γ. Then we say that ρ_1 is *direct ancestor* of ρ_2 if the conclusion of ρ_1 is a premise of ρ_2. ρ_2 is a *direct descendant* of ρ_1 if ρ_1 is a direct ancestor of ρ_2. Similarily, if S_1, S_2 are sequent occurrences in γ then S_1 is a direct ancestor of S_2 if there exists an inference with premise S_1 and conclusion S_2 in γ, and then S_2 is a direct descendant of S_1. The *proper ancestor (descendant)* relations are the transitive closures of the direct ancestor (direct descendant) relations. The *ancestor (descendant)* relations are the reflexive closures of the proper ancestor (descendant) relations. If S_1 is a descendant of S_2 then we also say that S_1 *depends on* S_2. Furthermore, we say that an inference ρ operates on a formula occurrence ω if ω is an auxiliary or main formula of ρ (note that the Sub rule does not operate on any formula occurrences).

For notational convenience we will refer to Sim^T and Sim^F inferences simply as Sim inferences.

Definition 4.2.6. We say that a Sim inference ρ is *locked* if all the direct descendants of ρ operate on the main formula of ρ. Let ω be a formula occurrence in γ. Then a sequence of sequents $S_1, \ldots, S_n$ is a *path starting at* ω if S_1 contains ω and for all $1 \leq i < n$, S_i is a direct ancestor of S_{i+1}. A path p starting at ω is called *uninterrupted* if no inference on p operates on a descendant of ω.

Proposition 4.2.7. *Let ω be the occurrence of $\mathbf{F}$ in the sequent $\Gamma \vdash \Delta, \mathbf{F}$ $(\mathbf{F}, \Gamma \vdash \Delta)$ in an $\mathcal{R}$-deduction γ, and let p be an uninterrupted path starting at ω. Then all sequents in p are of the form $\Pi \vdash \Lambda, \mathbf{F}\sigma$ $(\mathbf{F}\sigma, \Pi \vdash \Lambda)$ for some Π, Λ and substitution σ.*

Proof. By induction on the length of p. σ is determined by the Sub inferences on p. □

Proposition 4.2.8. *Let γ be an $\mathcal{R}$-refutation of $\mathcal{C}$. Then there exists an $\mathcal{R}$-refutation ψ of $\mathcal{C}$ such that all* Sim *inferences in ψ are locked and such that the Skolem terms occuring in γ are exactly those occuring in ψ.*

Proof. We may assume that there exists a Sim inference ρ in γ that is not locked. W.l.o.g. assume that ρ is a Sim^T inference. We construct an $\mathcal{R}$-refutation γ' of $\mathcal{C}$ such that γ' contains strictly less non-locked Sim inferences than γ, and conclude by induction.

Let $\gamma = S_1, \ldots, S_k$. As γ is an $\mathcal{R}$-refutation, S_k does not contain formula occurrences and hence (1) every formula occurrence ω has a descendant which is an auxiliary formula. Let ω be the main formula of ρ, let $S_i = \Gamma \vdash \Delta, \mathbf{A}, \mathbf{A}$ be the premise of ρ (where the $\mathbf{A}$'s are the auxiliary formulas of ρ), and let $S_j = \Gamma \vdash \Delta, \mathbf{A}$ be the conclusion of ρ. As ρ is not locked and by (1), there exist non-trivial uninterrupted paths $p_1, \ldots, p_n$ from ω to some auxiliary formulas occurring in sequents T_i ($1 \leq i \leq n$). Define $\psi = \Sigma_1, \ldots, \Sigma_{j-1}, \Sigma_{j+1}, \Sigma_k$ where

(1) if S_l occurs on some p_i then by Proposition 4.2.7, S_l is of the form $\Pi \vdash \Lambda, \mathbf{A}\sigma$ and we define $\Sigma_l = \Pi \vdash \Lambda, \mathbf{A}\sigma, \mathbf{A}\sigma$,

(2) if S_l is inferred from some T_j then $\Sigma_l = T_j, S_l$,

(3) otherwise $\Sigma_l = S_l$.

ψ is an $\mathcal{R}$-refutation of $\mathcal{C}$: W.l.o.g. we treat the case of S_l being inferred in ψ by a unary inference. In case (1) if S_l is inferred from S_j in γ then we can infer Σ_l from $\Sigma_i = S_i$ in ψ. Otherwise it is inferred from some S_m for which also case (1) holds, and we can infer Σ_l from Σ_m. In case (2), we can infer T_j from Σ_j by Sim^T and S_l from T_j as in γ. In case (3) if S_l was inferred from S_m in γ then Σ_m ends in S_m and we can infer S_l from Σ_m just as S_l was inferred from S_m in γ.

Note that we have only introduced locked Sim inferences, and have removed one non-locked Sim inference. Hence ψ contains strictly less non-locked Sim inferences than γ, which concludes the proof. □

Example 4.2.9. Consider the $\mathcal{R}$-deduction γ:

1	$Px \vee Qx, Px \vee Qx \vdash \forall y Ry$	
2	$Px \vee Qx \vdash \forall y Ry$	$\mathrm{Sim}^F : 1$
3	$Px \vee Qx \vdash Rz$	$\forall^F : 2$
4	$Pz \vee Qz \vdash Rz$	$\mathrm{Sub} : 3$
5	$Pz \vdash Rz$	$\vee_l^F : 4$
6	$Pc \vee Qc \vdash Rc$	$\mathrm{Sub} : 4$
7	$Qc \vdash Rc$	$\vee_r^F : 6$

Applying Proposition 4.2.8 to γ yields the $\mathcal{R}$-deduction

1	$Px \vee Qx, Px \vee Qx \vdash \forall y Ry$	
2	$Px \vee Qx, Px \vee Qx \vdash Rz$	$\forall^F : 1$
3	$Pz \vee Qz, Pz \vee Qz \vdash Rz$	$\mathrm{Sub} : 2$
4	$Pz \vee Qz \vdash Rz$	$\mathrm{Sim}^F : 3$
5	$Pz \vdash Rz$	$\vee_l^F : 4$
6	$Pc \vee Qc, Pc \vee Qc \vdash Rc$	$\mathrm{Sub} : 3$
7	$Pc \vee Qc \vdash Rc$	$\mathrm{Sim}^F : 6$
8	$Qc \vdash Rc$	$\vee_r^F : 7$

Hence from now on we will focus on the following set of rules:

Definition 4.2.10 (Rules for $\mathcal{R}'_a$)**.**

$$\frac{\Gamma \vdash \Delta, \neg\mathbf{A}, \ldots, \neg\mathbf{A}}{\mathbf{A}, \Gamma \vdash \Delta}\ \neg^T \qquad \frac{\neg\mathbf{A}, \ldots, \neg\mathbf{A}, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \mathbf{A}}\ \neg^F$$

$$\frac{\mathbf{A} \vee \mathbf{B}, \ldots, \mathbf{A} \vee \mathbf{B}, \Gamma \vdash \Delta}{\mathbf{A}, \Gamma \vdash \Delta}\ \vee_l^F \qquad \frac{\mathbf{A} \vee \mathbf{B}, \ldots, \mathbf{A} \vee \mathbf{B}, \Gamma \vdash \Delta}{\mathbf{B}, \Gamma \vdash \Delta}\ \vee_r^F$$

$$\frac{\Gamma \vdash \Delta, \mathbf{A} \vee \mathbf{B}, \ldots, \mathbf{A} \vee \mathbf{B}}{\Gamma \vdash \Delta, \mathbf{A}, \mathbf{B}}\ \vee^T \qquad \frac{\Gamma \vdash \Delta}{(\Gamma \vdash \Delta)\,[\mathbf{X} \leftarrow \mathbf{T}]}\ \mathrm{Sub}$$

$$\frac{\Gamma \vdash \Delta, \forall\mathbf{A}, \ldots, \forall\mathbf{A}}{\Gamma \vdash \Delta, \mathbf{A}\mathbf{X}}\ \forall^T \qquad \frac{\forall\mathbf{A}, \ldots, \forall\mathbf{A}, \Gamma \vdash \Delta}{\mathbf{A}(\mathbf{f}\mathbf{X}_1 \ldots \mathbf{X}_n), \Gamma \vdash \Delta}\ \forall^F$$

$$\frac{\Gamma \vdash \Delta, \mathbf{A}, \ldots, \mathbf{A} \quad \mathbf{A}, \ldots, \mathbf{A}, \Pi \vdash \Lambda}{\Gamma, \Pi \vdash \Delta, \Lambda}\ \mathrm{mCut}$$

with conditions on mCut, $\forall^F$ as in Definition 4.2.4 (rules of $\mathcal{R}_a$). An inference is called *restricted* if it has at most one auxiliary formula.

Hence the following follows immediately from Proposition 4.2.8:

Proposition 4.2.11. *Let γ be an $\mathcal{R}$-refutation of $\mathcal{C}$. Then there exists an $\mathcal{R}'_a$-refutation ψ of $\mathcal{C}$ such that the Skolem terms occuring in γ are exactly those occuring in ψ.*

Note that an $\mathcal{R}'_a$-deduction γ is an $\mathcal{R}_a$-deduction iff all inferences in γ except mCut are restricted. We introduce some notions regarding the status of inferences in $\mathcal{R}'_a$ deductions:

Definition 4.2.12. An inference is called *relevant* if it is not an mCut or $\forall^F$ inference. Let ρ be an $\forall^F$ inference. ρ is called *unfinished* if there exists an inference μ operating on a proper ancestor of an auxiliary formula of ρ such that μ is not restricted. Otherwise ρ is called *prefinished.* ρ is called *finished* if it is prefinished and restricted.

Example 4.2.13. Consider the $\mathcal{R}'_a$-deduction

1	$A \vee \forall x Px, A \vee \forall x Px \vdash$	
2	$\forall x Px \vdash$	$\vee^F_r : 1$
3	$Ps \vdash$	$\forall^F : 2$

Then inference 3 is unfinished since inference 2 operates on a proper ancestor of the auxiliary formula of 3, and 2 is not restricted. Now consider

1	$A \vee \forall x Px, A \vee \forall x Px \vdash$	
2	$A \vee \forall x Px, \forall x Px \vdash$	$\vee^F_r : 1$
3	$\forall x Px, \forall x Px \vdash$	$\vee^F_r : 2$
4	$Ps \vdash$	$\forall^F : 3$

Here, inference 4 is prefinished but not finished since it is not restricted.

Definition 4.2.14. Let $S = \mathbf{F}_1, \ldots, \mathbf{F}_n \vdash \mathbf{G}_1, \ldots, \mathbf{G}_m$ be a sequent. If there exist $k_1, \ldots, k_n, \ell_1, \ldots, \ell_m \in \mathbb{N}$ such that

$$S' = k_1 \times \mathbf{F}_1, \ldots, k_n \times \mathbf{F}_n \vdash \ell_1 \times \mathbf{G}_1, \ldots, \ell_m \times \mathbf{G}_m,$$

then S' is a *multiple of* S, where the notation $k_i \times \mathbf{F}_i$ means "k_i occurrences of $\mathbf{F}_i$". Abusing notation, we write $\mathbf{F}_1, \ldots, \mathbf{F}_n \vdash_m \mathbf{G}_1, \ldots, \mathbf{G}_m$ for S' if S' is a multiple of S.

If all relevant inferences in an $\mathcal{R}'_a$-deduction γ are restricted, then we say that γ is *restricted.* We define $\mathrm{NF}(\gamma)$ to be the number of $\forall^F$ inferences in γ which are not finished (i.e. unfinished or not restricted).

Proposition 4.2.15. *Let γ be an $\mathcal{R}'_a$-deduction of $\vdash \Gamma$ from $\mathcal{C}$. Then there exists an $\mathcal{R}'_a$-deduction ψ of $\vdash_m \Gamma$ from $\mathcal{C}$ such that ψ is restricted.*

Furthermore, the Skolem terms occuring in ψ are the same as those occuring in γ, and $\mathrm{NF}(\gamma) = \mathrm{NF}(\psi)$.

Proof. Assume γ is not restricted. Let $\gamma = S_1, \ldots, S_n$, and let i be the least such that S_i is inferred by a relevant inference ρ such that ρ is not restricted. We will construct an $\mathcal{R}'_a$-deduction $\psi = S_1, \ldots, S_{i-1}, \Sigma, S'_{i+1}, \ldots, S'_n$ from $\mathcal{C}$ such that (1) if μ is an inference in ψ with conclusion in $S_1, \ldots, S_{i-1}, \Sigma$, then μ is restricted and furthermore, (2) a sequent in ψ is inferred by an $\forall^F$ inference μ iff its corresponding sequent in γ is inferred by an $\forall^F$ inference μ', and μ is not finished iff μ' is. We may then conclude by induction on $n - i$, where i is defined as above.

$S_1, \ldots, S_{i-1}$ are inferred in ψ as they were in γ. By assumption, all these inferences are restricted if they are relevant. Σ is defined as follows: We treat the case of ρ being an $\vee^T$ inference. The other cases are analogous. Let $\Gamma \vdash \Delta, \mathbf{A} \vee \mathbf{B}, \ldots, \mathbf{A} \vee \mathbf{B}$ be the premise of ρ, and let $\Gamma \vdash \Delta, \mathbf{A}, \mathbf{B}$ be the conclusion. Then Σ is the sequence of sequents starting with $\Gamma \vdash \Delta, \mathbf{A} \vee \mathbf{B}, \ldots, \mathbf{A} \vee \mathbf{B}, \mathbf{A}, \mathbf{B}$ and ending with $\Gamma \vdash \Delta, \mathbf{A}, \mathbf{B}, \ldots, \mathbf{A}, \mathbf{B}$, such that every sequent in Σ is inferred from the previous one by the restricted version of ρ. The first sequent in Σ can be inferred from the same S_j, $j < i$, as it was in γ, using the restricted version of ρ. By construction, (1) holds. For (2), note that by assumption ρ cannot be $\forall^F$, as ρ is relevant. All other inferences are as they were in γ, so (2) holds for this part of ψ.

Now, define S'_j for $i < j \leq n$. Let ω be the main formula of ρ, and let $S_j = \Gamma, \Delta$ where Δ are all the descendants of ω in S_j in γ. Define $S'_j = \Gamma, \Delta, \ldots, \Delta$ if there exists an uninterrupted path starting at ω and ending at S_j in γ (for some suitable number of copies of Δ), and $S'_j = S_j$ otherwise. S'_j can be derived in ψ:

1. If S_j was derived in γ from S_k with $k < i$, then Δ is empty and we can derive $S'_j = S_j$ from S_k.

2. If S_j was derived from S_i in γ, we can derive S'_j from the last element of Σ.

3. If S_j was derived from S_k, with $k > i$, in γ then again we can derive S'_j from S'_k in ψ. If the inference with conclusion S_j is the first inference operating on a descendant of ω in γ, we have to increase the number of auxiliary formulas to derive the correct sequent in ψ. For example, if $S_k = \Gamma \vdash \Delta, \mathbf{A} \vee \mathbf{B}$ and $S_j = \Gamma \vdash \Delta, \mathbf{A}, \mathbf{B}$ is derived by $\vee^T$, then $S'_k = \Gamma \vdash \Delta, \mathbf{A} \vee \mathbf{B}, \ldots, \mathbf{A} \vee \mathbf{B}$ and we derive $S'_j = S_j$ from S'_k by $\vee^T$ in ψ.

For (2), it is clear by construction that S'_j is inferred by $\forall^F$ iff S_j is. Note that inferences from γ are changed iff they operate on descendants of ω, in which case they are unfinished if they are instances of $\forall^F$ in both γ and ψ. $\square$

The second $\mathcal{R}'_a$-deduction in Example 4.2.13 is obtained from the first by applying Proposition 4.2.15.

Proposition 4.2.16. *Let ρ_1, ρ_2 be $\forall^F$ inferences in an $\mathcal{R}'_a$-deduction such that ρ_1 operates on an ancestor of the main formula of ρ_2. Then if ρ_1 is not finished, ρ_2 is not finished.*

Proof. As ρ_1 is not finished, an inference operating on an ancestor of the main formula ω of ρ_1 is not restricted. By assumption ω is an ancestor of the main formula of ρ_2, so ρ_2 is unfinished and hence not finished. □

For the final results, we will allow the rule of weakening in $\mathcal{R}'_a$-deductions to ease the presentation of the proofs:

$$\frac{\Gamma \vdash \Delta}{\Gamma, \Pi \vdash \Delta, \Lambda}\ weak$$

Proposition 4.2.17. *Let γ be an $\mathcal{R}'_a$-refutation of $\mathcal{C}$ using weakening. Then there exists an $\mathcal{R}'_a$-refutation ψ of $\mathcal{C}$ without weakening such that* $\mathrm{NF}(\psi) \leq \mathrm{NF}(\gamma)$.

Proof. By deleting formula occurrences, sequents and inferences. □

Proposition 4.2.18. *Let γ be an $\mathcal{R}'_a$-refutation of $\mathcal{C}$ such that all Skolem terms of $\forall^F$ inferences in γ are constants. Then there exists an $\mathcal{R}_a$-refutation of $\mathcal{C}$.*

Proof. Note that if γ is restricted and $\mathrm{NF}(\gamma) = 0$, γ is the desired $\mathcal{R}$-refutation.

By Proposition 4.2.15, we may assume that γ is restricted. We proceed by induction on $\mathrm{NF}(\gamma)$, showing that if γ is a restricted $\mathcal{R}'_a$-deduction of S from $\mathcal{C}$, then there exists a restricted $\mathcal{R}'_a$-deduction ψ of S from $\mathcal{C}$ with $\mathrm{NF}(\psi) = 0$.

If $\mathrm{NF}(\gamma) = 0$, we may take $\psi = \gamma$. Hence assume as inductive hypothesis that for all $\mathcal{R}'_a$-deductions λ of S from $\mathcal{C}$ with $\mathrm{NF}(\lambda) < \mathrm{NF}(\gamma)$, there exists an $\mathcal{R}'_a$-deduction λ' of S from $\mathcal{C}$ with $\mathrm{NF}(\lambda') = 0$.

We say that an $\forall^F$ inference ρ is *uppermost* if no unfinished $\forall^F$ inference operates on a proper ancestor of the auxiliary formula of ρ. By assumption, there exists an $\forall^F$ inference in γ that is not finished. Then there exists an uppermost $\forall^F$ inference ρ in γ that is not finished. Observe that ρ is prefinished and not restricted, as it is uppermost and all relevant inferences are restricted.

Let $\gamma = S_1, \ldots, S_n$, and let the premise of ρ be $S_i = \forall \mathbf{A}, \ldots, \forall \mathbf{A}, \Gamma \vdash \Delta$ (containing $k+1 \geq 2$ auxiliary formulas), the conclusion be $S_j = \mathbf{Ac}, \Gamma \vdash \Delta$, and denote the main formula of ρ by ω. Note that S_n is the empty sequent since γ is an $\mathcal{R}'_a$-refutation. If S_n does not depend on S_j, then clearly we can simply remove S_j and the sequents that depend on it from γ to obtain a restricted $\mathcal{R}'_a$-deduction of S_n from $\mathcal{C}$ containing strictly less $\forall^F$ inferences which are not finished, and we may conclude by the inductive hypothesis. Hence assume S_n depends on S_j. Note that $\mathbf{A}$ does not contain free variables since $\mathbf{c}$ is a constant. Let $\mathbf{c}_1, \dots, \mathbf{c}_k$ be fresh Skolem constants.

For $1 \leq q \leq k$, we will construct restricted $\mathcal{R}'_a$-deductions

1. ψ_0 of $(\Gamma \vdash \Delta) \circ (\mathbf{Ac}_1, \dots, \mathbf{Ac}_k \vdash_m)$ from $\mathcal{C}$, and

2. ψ_q of $(\Gamma \vdash \Delta) \circ (\mathbf{Ac}_{q+1}, \dots, \mathbf{Ac}_k \vdash_m)$ from

$$\mathcal{C} \cup \{(\Gamma \vdash \Delta) \circ (\mathbf{Ac}_q, \dots, \mathbf{Ac}_k \vdash_m)\}.$$

such that for $0 \leq p \leq k$, $\mathrm{NF}(\psi_p) < \mathrm{NF}(\gamma)$. We may then apply the inductive hypothesis to ψ_p to obtain restricted $\mathcal{R}'_a$-deductions ψ'_p with $\mathrm{NF}(\psi'_p) = 0$. Hence all inferences except mCut are restricted in ψ'_p. We may then rename the Skolem symbols of the ψ'_p such that their sets of Skolem symbols are pairwise disjoint. Then clearly $\psi = \psi'_0, \dots, \psi'_k$ has $\mathrm{NF}(\psi) = 0$ and is therefore the desired $\mathcal{R}'_a$-refutation.

We start by defining ψ_0. For $j+1 \leq r \leq n$, if S_r does not depend on S_j then $S'_r = S_r$, and otherwise $S'_r = S_r \circ (\mathbf{Ac}_1, \dots, \mathbf{Ac}_k \vdash_m)$. Note that $S'_n = \mathbf{Ac}_1, \dots, \mathbf{Ac}_k \vdash_m$. So let

$$\psi_0 = S_1, \dots, S_{j-1}, \Sigma, S'_{j+1}, \dots, S'_n, (\Gamma \vdash \Delta) \circ (\mathbf{Ac}_1, \dots, \mathbf{Ac}_k \vdash_m),$$

where Σ is a sequence of sequents deriving $\mathbf{Ac}, \mathbf{Ac}_1, \dots, \mathbf{Ac}_k, \Gamma \vdash \Delta$ from S_i using only restricted $\forall^F$. Clearly $S_1, \dots, S_{j-1}$ can be derived from $\mathcal{C}$ as they were in γ. Since ρ is prefinished, all the $\forall^F$ inferences introduced in deriving Σ are finished. Letting S'_j be the last sequent in Σ, we show that S'_r can be derived in ψ_0 for $j < r \leq n$. Distinguish:

1. If S_r does not depend on S_j, then neither do its premise(s) S_p (S_q). Hence $S'_r = S_r$ and $S'_p = S_p$ (and $S'_q = S_q$) and S'_r can be inferred from S'_p (S'_q) just as it was in γ.

2. If S_r depends on S_j and was inferred by a unary inference μ from S_p, then $p \geq j$ and hence we can infer S'_r from S'_p by the same unary inference. If μ is Sub, remember that $\mathbf{A}$ is closed and hence not affected by the substitution.

3. If S_r depends on S_j and was inferred by mCut from S_p and S_t, then at least one of the premises depends on S_j. Hence we may infer S'_r from S'_p and S'_t by mCut. Note that if both premises depend on S_j, the multiplicities of the $\mathbf{Ac}_q$ increase.

Note that $S'_n = (\mathbf{Ac}_1, \ldots, \mathbf{Ac}_k \vdash_m)$, so the last sequent of ψ_0 can be derived from S'_n by weakening. By construction, for every $\forall^F$ inference in ψ_0 that is not finished there exists a unique $\forall^F$ inference in γ that is not finished, hence $\mathrm{NF}(\psi_0) < \mathrm{NF}(\gamma)$ (because ρ induces only finished inferences in ψ_0). Since all relevant inferences in γ are restricted, this is also the case for ψ_0. Hence ψ_0 is as desired.

We turn to the construction of ψ_q for $1 \leq q \leq k$. Let

$$\psi'_q = (\Gamma \vdash \Delta) \circ (\mathbf{Ac}_q, \ldots, \mathbf{Ac}_k \vdash_m), S_{1,q}, \ldots, S_{j-1,q}, S_{j+1,q}, \ldots, S_{n,q}$$

where $S_{r,q}$ is defined in the following way:

1. If S_r does not depend on S_j, then $S_{r,q} = S_r\,[\mathbf{c} \leftarrow \mathbf{c}_q]$.

2. If S_r depends on S_j, denote the inference whose conclusion S_r is by ρ. Distinguish:

   (a) If no inference in γ on the path from ω to S_r operates on a descendant of ω, then S_r is of the form $\mathbf{Ac}, \Pi \vdash \Lambda$. Then let $S_{r,q} = (\Pi \vdash \Lambda) \circ (\mathbf{Ac}_q, \ldots, \mathbf{Ac}_k \vdash_m)$.

   (b) ρ is the first inference operating on a descendant of ω. We treat the case where ρ is $\vee^T$, the other cases are similar. So if $S_r = \Pi \vdash \Lambda, \mathbf{B}, \mathbf{C}$ is inferred from $S_\ell = \Pi \vdash \Lambda, \mathbf{B} \vee \mathbf{C}$ then $S_{\ell,q} = (\Pi \vdash \Lambda, \mathbf{B} \vee \mathbf{C}, \ldots, \mathbf{B} \vee \mathbf{C}) \circ (\mathbf{Ac}_{q+1}, \ldots, \mathbf{Ac}_k \vdash_m)$ by the previous case (note that by assumption $\mathbf{Ac}_q = \mathbf{B} \vee \mathbf{C}$). Then let $S_{r,q} = (\Pi \vdash \Lambda, \mathbf{B}, \mathbf{C}) \circ (\mathbf{Ac}_{q+1}, \ldots, \mathbf{Ac}_k \vdash_m)$.

   (c) Otherwise, $S_{r,q} = S_r \circ (\mathbf{Ac}_{q+1}, \ldots, \mathbf{Ac}_k \vdash_m)$.

For $r \in \{1, \ldots, j-1, j+1, \ldots, n\}$, we show that $S_{r,q}$ can be derived in ψ'_q by distinguishing how S_r is derived in γ:

1. $S_r \in \mathcal{C}$. Then S_r does not contain $\mathbf{c}$ and does not depend on S_j, hence $S_{r,q} \in \mathcal{C}$.

2. If S_r is inferred by Sub with $[\mathbf{X} \leftarrow \mathbf{T}]$ from S_p, then we may use Sub with $[\mathbf{X} \leftarrow \mathbf{T}\,[\mathbf{c} \leftarrow \mathbf{c}_q]]$ to derive $S_{r,q}$ from $S_{p,q}$, again noting that $\mathbf{A}$ is closed.

3. S_r is derived from S_p by a CNF inference. We may use the same inference to infer $S_{r,q}$ from $S_{p,q}$ (In case $S_{r,q}$ is constructed in case 2(b) above, the number of auxiliary formulas of the inference increases).

4. S_r is derived from S_p and S_t by an mCut. We may derive $S_{r,q}$ from $S_{p,q}$ and $S_{t,q}$ using mCut. Again if $S_{r,q}$ is constructed in case 2(b) above, the number of auxiliary formulas of the inference increases. Also, note again that if both premises depend on S_j, then the multiplicities of the $\mathbf{Ac}_\ell$ increase.

By construction, for every $\forall^F$ inference in ψ'_q that is not finished there exists a unique $\forall^F$ inference in γ that is not finished, hence $\mathrm{NF}(\psi'_q) < \mathrm{NF}(\gamma)$ (because ρ does not induce an $\forall^F$ inference in ψ'_q). Note that due to 2(b), also the $\forall^F$ inferences operating on descendants of $\mathbf{Ac}_q$ are not finished, but their corresponding inferences in γ operate on descendants of ω and are hence not finished, too.

Set $\psi''_q = \psi'_q, (\Gamma \vdash \Delta) \circ (\mathbf{Ac}_{q+1}, \ldots, \mathbf{Ac}_k \vdash_m)$ Note that the last sequent of ψ'_q is $S_{n,q} = \mathbf{Ac}_{q+1}, \ldots, \mathbf{Ac}_k \vdash_m$, hence the last sequent of ψ''_q can again be derived by weakening. Finally, we may apply Proposition 4.2.15 to ψ''_q to obtain a restricted ψ_q such that $\mathrm{NF}(\psi_q) = \mathrm{NF}(\psi''_q) = \mathrm{NF}(\psi'_q) < \mathrm{NF}(\gamma)$. Hence ψ_q is as desired. Finally, we apply Proposition 4.2.17 to ψ, which completes the proof. □

Example 4.2.19. Consider the $\mathcal{R}'_a$-refutation of $\{\forall x(Px \vee \neg Px), \forall x(Px \vee \neg Px) \vdash\}$:

1	$\forall x(Px \vee \neg Px), \forall x(Px \vee \neg Px) \vdash$	
2	$Ps \vee \neg Ps \vdash$	$\forall^F : 1$
3	$Ps \vdash$	$\vee_l^F : 2$
4	$\neg Ps \vdash$	$\vee_r^F : 2$
5	$\vdash Ps$	$\neg^F : 4$
6	$\vdash$	$\mathrm{mCut} : 5, 3$

In the proof of Proposition 4.2.18 we obtain ψ_0

1	$\forall x(Px \vee \neg Px), \forall x(Px \vee \neg Px) \vdash$	
2	$\forall x(Px \vee \neg Px), Ps \vee \neg Ps \vdash$	$\forall^F : 1$
3	$Ps_1 \vee \neg Ps_1, Ps \vee \neg Ps \vdash$	$\forall^F : 2$
4	$Ps_1 \vee \neg Ps_1, Ps \vdash$	$\vee_l^F : 3$
5	$Ps_1 \vee \neg Ps_1, \neg Ps \vdash$	$\vee_r^F : 3$
6	$Ps_1 \vee \neg Ps_1, \vdash Ps$	$\neg^F : 5$
7	$Ps_1 \vee \neg Ps_1, Ps_1 \vee \neg Ps_1 \vdash$	$\mathrm{mCut} : 6, 4$

and ψ'_1

8	$Ps_1 \vee \neg Ps_1, Ps_1 \vee \neg Ps_1 \vdash$	
9	$Ps_1 \vdash$	$\vee^F_l : 8$
10	$\neg Ps_1 \vdash$	$\vee^F_r : 8$
11	$\vdash Ps_1$	$\neg^F : 10$
12	$\vdash$	$\mathrm{mCut} : 9, 11$

ψ'_1 is not restricted, but after application of Proposition 4.2.15 we obtain the restricted ψ_1

8	$Ps_1 \vee \neg Ps_1, Ps_1 \vee \neg Ps_1 \vdash$	
9	$Ps_1 \vee \neg Ps_1, Ps_1 \vdash$	$\vee^F_l : 8$
10	$Ps_1, Ps_1 \vdash$	$\vee^F_l : 9$
11	$Ps_1 \vee \neg Ps_1, \neg Ps_1 \vdash$	$\vee^F_r : 8$
12	$\neg Ps_1, \neg Ps_1 \vdash$	$\vee^F_r : 11$
13	$\neg Ps_1 \vdash Ps_1$	$\neg^F : 12$
14	$\vdash Ps_1, Ps_1$	$\neg^F : 13$
15	$\vdash$	$\mathrm{mCut} : 10, 14$

Clearly $\psi = \psi_0, \psi_1$ is the desired $\mathcal{R}_a$-refutation of $\{\forall x(Px \vee \neg Px), \forall x(Px \vee \neg Px) \vdash\}$.

We can now state the main result of this section, showing that a restricted class of $\mathcal{R}$-refutations can indeed be translated into $\mathcal{R}_a$:

Theorem 4.2.20. *Let γ be an $\mathcal{R}_c$-refutation of $\mathcal{C}$. Then there exists an $\mathcal{R}_a$-refutation of $\mathcal{C}$.*

Proof. By Propositions 4.2.11 and 4.2.18. □

4.3 Cut-elimination by resolution

So far, we have investigated the cut-free fragment of $\mathbf{LK}_{\mathrm{sk}}$. To deal with the problem of cut-elimination, we first connect ordinary **LK** with the rules of $\mathbf{LK}_{\mathrm{sk}}$. The following definition will provide the analogue to the normal form provided by proof Skolemization in first-order logic, but in higher-order logic:

Definition 4.3.1 ($\mathbf{LK}_{\mathrm{skc}}$-trees)**.** An $\mathbf{LK}_{\mathrm{skc}}$-*tree* is a tree formed according to the rules of $\mathbf{LK}_{\mathrm{sk}}$ and **LK** such that

1. rules of **LK** operate only on cut-ancestors, and
2. rules of $\mathbf{LK}_{\mathrm{sk}}$ operate only on end-sequent ancestors.

We extend the notions of paths, homomorphic inferences, and weak regularity to $\mathbf{LK}_{\text{skc}}$-trees. Let π be an $\mathbf{LK}_{\text{skc}}$-tree with end-sequent S. We say that π is an $\mathbf{LK}_{\text{skc}}$*-proof* if it is weakly regular and proper.

Definition 4.3.2. Let π be an $\mathbf{LK}_{\text{skc}}$-tree. π is called *regular* if

1. each $\forall^{sk}\colon r$ inference has a unique Skolem symbol and
2. the eigenvariable of each $\forall\colon r$ inference ρ only occurs above ρ in π.

Proposition 4.3.3. *Let π be an $\mathbf{LK}_{\text{skc}}$-tree. If π is regular, then π is weakly regular.*

Lemma 4.3.4 (Skolemization)**.** *Let π be a regular* $\mathbf{LK}$*-proof of S. Then there exists a regular* $\mathbf{LK}_{\text{skc}}$*-proof ψ of S.*

Proof. Let ρ be a inference in π with conclusion $\mathbf{F}_1, \ldots, \mathbf{F}_n \vdash \mathbf{F}_{n+1}, \ldots, \mathbf{F}_m$. By induction on the height of ρ, we define a regular $\mathbf{LK}_{\text{skc}}$-tree π_ρ with conclusion $\langle \mathbf{F}_1 \rangle^{\ell_1}, \ldots, \langle \mathbf{F}_n \rangle^{\ell_n} \vdash \langle \mathbf{F}_{n+1} \rangle^{\ell_{n+1}}, \ldots, \langle \mathbf{F}_m \rangle^{\ell_m}$ such that for all $1 \leq i \leq m$, ℓ_i is the sequence of substitution terms of $\forall\colon l$ inferences operating on descendants of $\mathbf{F}_i$ in π, and such that π_ρ fulfills an *eigenterm* condition, i.e. every Skolem symbol occurs only above its source inference.[1]

1. ρ is an axiom $\mathbf{A} \vdash \mathbf{A}$. Let ℓ_1 be the sequence of substitution terms of the $\forall\colon l$ inferences operating on the descendants of the left occurrence of $\mathbf{A}$, and let ℓ_2 be the sequence of substitution terms of the $\forall\colon l$ inferences operating on descendants of the right occurrence of $\mathbf{A}$. Then take as π_ρ the axiom $\langle \mathbf{A} \rangle^{\ell_1} \vdash \langle \mathbf{A} \rangle^{\ell_2}$.

2. ρ is a $\forall\colon l$ inference operating on an end-sequent ancestor:

$$\dfrac{\begin{matrix}(\varphi)\\ \overline{\mathbf{F}\mathbf{T}}, \Gamma \vdash \Delta\end{matrix}}{\forall_\alpha \mathbf{F}, \Gamma \vdash \Delta}\ \forall\colon l$$

   By (IH) we obtain a regular $\mathbf{LK}_{\text{skc}}$-tree φ' of $\left\langle \overline{\mathbf{F}\mathbf{T}} \right\rangle^{\ell, \mathbf{T}}, \Gamma' \vdash \Delta'$ where Γ', Δ' are Γ, Δ with the respective labels. We take for π_ρ

$$\dfrac{\begin{matrix}(\varphi')\\ \left\langle \overline{\mathbf{F}\mathbf{T}} \right\rangle^{l, \mathbf{T}}, \Gamma' \vdash \Delta'\end{matrix}}{\langle \forall_\alpha \mathbf{F} \rangle^{l}, \Gamma' \vdash \Delta'}\ \forall^{sk}\colon l$$

[1] It is possible to assign arbitrary labels to cut-ancestors in $\mathbf{LK}_{\text{skc}}$-trees. To avoid a case distinction, cut-ancestors are assigned labels in the same way as end-sequent ancestors in this proof.

3. ρ is a $\forall\colon l$ inference operating on a cut-ancestor. Then we simply take the regular $\mathbf{LK}_{\mathrm{skc}}$-tree obtained by (IH) and apply ρ to it.

4. ρ is a $\forall\colon r$ inference operating on an end-sequent ancestor:

$$\dfrac{\begin{matrix}(\varphi)\\ \Gamma \vdash \Delta, \overline{\mathbf{F}\mathbf{X}}\end{matrix}}{\Gamma \vdash \Delta, \forall_\alpha \mathbf{F}}\ \forall\colon r$$

By (IH) we obtain a regular $\mathbf{LK}_{\mathrm{skc}}$-tree φ' of $\Gamma' \vdash \Delta', \langle\overline{\mathbf{F}\mathbf{X}}\rangle^{\mathbf{T}_1,\ldots,\mathbf{T}_n}$, with Γ', Δ' as above. Let $\mathbf{f} \in \mathcal{K}_{\alpha_1,\ldots,\alpha_n,\alpha}$, where for $1 \leq i \leq n$ $\tau(\mathbf{T}_i) = \alpha_i$, be a new Skolem symbol, and let $\mathbf{S} = \mathbf{f}(\mathbf{T_1} \ldots \mathbf{T}_n)$. Let σ be the substitution $[\mathbf{X} \leftarrow \mathbf{S}]$. By regularity, $\mathbf{X}$ is not an eigenvariable in φ', and does not occur in $\mathbf{T}_1, \ldots, \mathbf{T}_n$. Hence $\varphi'\sigma$ is a regular $\mathbf{LK}_{\mathrm{skc}}$-tree of $\Gamma' \vdash \Delta', \langle\overline{\mathbf{F}\mathbf{S}}\rangle^{\mathbf{T}_1,\ldots,\mathbf{T}_n}$. Take for π_ρ

$$\dfrac{\begin{matrix}(\varphi'\sigma)\\ \Gamma' \vdash \Delta', \langle\overline{\mathbf{F}\mathbf{S}}\rangle^{\mathbf{T}_1,\ldots,\mathbf{T}_n}\end{matrix}}{\Gamma' \vdash \Delta', \langle\forall_\alpha \mathbf{F}\rangle^{\mathbf{T}_1,\ldots,\mathbf{T}_n}}\ \forall^{sk}\colon r$$

5. ρ is a $\forall\colon r$ inference operating on a cut ancestor. Again we take the regular $\mathbf{LK}_{\mathrm{skc}}$-tree obtained by (IH) and apply ρ to it.

6. ρ is a cut inference

$$\dfrac{\begin{matrix}(\varphi)\\ \Gamma \vdash \Delta, \mathbf{F}\end{matrix} \qquad \begin{matrix}(\lambda)\\ \mathbf{F}, \Pi \vdash \Lambda\end{matrix}}{\Gamma, \Pi \vdash \Delta, \Lambda}\ \mathrm{cut}$$

By (IH) we obtain regular $\mathbf{LK}_{\mathrm{skc}}$-trees φ', λ' of $\Gamma' \vdash \Delta', \langle\mathbf{F}\rangle^{\ell_1}$ and $\langle\mathbf{F}\rangle^{\ell_2}, \Pi' \vdash \Lambda'$, respectively. If the intersection of the Skolem symbols of φ', λ' is non-empty, by the eigenterm condition we can rename Skolem symbols to achieve this. Hence the $\mathbf{LK}_{\mathrm{skc}}$-tree π_ρ

$$\dfrac{\begin{matrix}(\varphi')\\ \Gamma' \vdash \Delta', \langle\mathbf{F}\rangle^{\ell_1}\end{matrix} \qquad \begin{matrix}(\lambda')\\ \langle\mathbf{F}\rangle^{\ell_2}, \Pi' \vdash \Lambda'\end{matrix}}{\Gamma', \Pi' \vdash \Delta', \Lambda'}\ \mathrm{cut}$$

is regular.

7. ρ is a contr: r inference

$$\cfrac{\begin{array}{c}(\varphi)\\ \Gamma \vdash \Delta, \mathbf{F}, \mathbf{F}\end{array}}{\Gamma \vdash \Delta, \mathbf{F}}\ \text{contr}: r$$

   By (IH) we obtain a regular $\mathbf{LK}_{\text{skc}}$-tree φ' of $\Gamma' \vdash \Delta', \langle \mathbf{F} \rangle^{\ell_1}, \langle \mathbf{F} \rangle^{\ell_2}$. Note that the inferences operating on descendants of the occurrences of $\mathbf{F}$ coincide, so $\ell_1 = \ell_2$ and we may take for π_ρ

$$\cfrac{\begin{array}{c}(\varphi')\\ \Gamma' \vdash \Delta', \langle \mathbf{F} \rangle^{\ell_1}, \langle \mathbf{F} \rangle^{\ell_1}\end{array}}{\Gamma' \vdash \Delta', \langle \mathbf{F} \rangle^{\ell_1}}\ \text{contr}: r$$

8. ρ is a contr: l inference: symmetric.

9. ρ is a propositional inference or weakening: analogous to the previous cases.

Let ρ be the last inference in π, then $\psi = \pi_\rho$ is the desired regular $\mathbf{LK}_{\text{skc}}$-proof. □

We will now set up some notation for the main definitions of CERES$^\omega$. Let π be an $\mathbf{LK}_{\text{skc}}$-tree, and let S be a sequent in π. Then by cutanc(S) we denote the sub-sequent of S consisting of the cut-ancestors of S, and by esanc(S) we denote the sub-sequent of S consisting of the end-sequent ancestors of S. Note that for any sequent $S = \text{cutanc}(S) \circ \text{esanc}(S)$. Let ρ be a unary inference, σ a binary inference, ψ, χ $\mathbf{LK}_{\text{sk}}$-trees, then $\rho(\psi)$ is the $\mathbf{LK}_{\text{sk}}$-tree obtained by applying ρ to the end-sequent of ψ, and $\sigma(\psi, \chi)$ is the $\mathbf{LK}_{\text{sk}}$-tree obtained from the $\mathbf{LK}_{\text{sk}}$-trees ψ and χ by applying σ. Note that while this notation is ambigous, it will always be clear from the context what the auxiliary formulas of the $\rho(\psi)$ and $\sigma(\psi, \chi)$ are. Let P, Q be sets of $\mathbf{LK}_{\text{sk}}$-trees. Then $P^{\Gamma \vdash \Delta} = \{\psi^{\Gamma \vdash \Delta} \mid \psi \in P\}$, where $\psi^{\Gamma \vdash \Delta}$ is ψ followed by weakenings adding $\Gamma \vdash \Delta$, and $P \times_\sigma Q = \{\sigma(\psi, \chi) \mid \psi \in P, \chi \in Q\}$.

Definition 4.3.5 (Characteristic sequent set and proof projections)**.** Let π be a regular $\mathbf{LK}_{\text{skc}}$-proof. For each inference ρ in π, we define a set of $\mathbf{LK}_{\text{sk}}$-trees, the set of *projections* $\mathcal{P}_\rho(\pi)$, and a set of labelled sequents, the *characteristic sequent set* $\text{CS}_\rho(\pi)$.

- If ρ is an axiom with conclusion $S = \langle \mathbf{A} \rangle^{\ell_1} \vdash \langle \mathbf{A} \rangle^{\ell_2}$, distinguish:
  - $\mathrm{cutanc}(S) = S$. Then $\mathrm{CS}_\rho(\pi) = \mathcal{P}_\rho(\pi) = \emptyset$.
  - $\mathrm{cutanc}(S) \neq S$. Distinguish:
    (a) If $\mathrm{cutanc}(S) = \vdash \langle \mathbf{A} \rangle^{\ell_2}$ then $\mathrm{CS}_\rho(\pi) = \{\vdash \langle \mathbf{A} \rangle^{\ell_1}\}$ and $\mathcal{P}_\rho(\pi) = \{\langle \mathbf{A} \rangle^{\ell_1} \vdash \langle \mathbf{A} \rangle^{\ell_1}\}$,
    (b) if $\mathrm{cutanc}(S) = \langle \mathbf{A} \rangle^{\ell_1} \vdash$ then $\mathrm{CS}_\rho(\pi) = \{\langle \mathbf{A} \rangle^{\ell_2} \vdash\}$ and $\mathcal{P}_\rho(\pi) = \{\langle \mathbf{A} \rangle^{\ell_2} \vdash \langle \mathbf{A} \rangle^{\ell_2}\}$,
    (c) if $\mathrm{cutanc}(S) = \vdash$ then $\mathrm{CS}_\rho(\pi) = \{\vdash\}$ and $\mathcal{P}_\rho(\pi) = \{S\}$.
- If ρ is a unary inference with immediate predecessor ρ' with $\mathcal{P}_{\rho'}(\pi) = \{\psi_1, \ldots, \psi_n\}$, distinguish:

  (a) ρ operates on ancestors of cut formulas. Then

  $$\mathcal{P}_\rho(\pi) = \mathcal{P}_{\rho'}(\pi)$$

  (b) ρ operates on ancestors of the end-sequent. Then

  $$\mathcal{P}_\rho(\pi) = \{\rho(\psi_1), \ldots, \rho(\psi_n)\}$$

  In any case, $\mathrm{CS}_\rho(\pi) = \mathrm{CS}_{\rho'}(\pi)$.
- Let ρ be a binary inference with immediate predecessors ρ_1 and ρ_2.

  (a) If ρ operates on ancestors of cut-formulas, let $\Gamma_i \vdash \Delta_i$ be the ancestors of the end-sequent in the conclusion sequent of ρ_i and define

  $$\mathcal{P}_\rho(\pi) = \mathcal{P}_{\rho_1}(\pi)^{\Gamma_2 \vdash \Delta_2} \cup \mathcal{P}_{\rho_2}(\pi)^{\Gamma_1 \vdash \Delta_1}$$

  For the characteristic sequent set, define

  $$\mathrm{CS}_\rho(\pi) = \mathrm{CS}_{\rho_1}(\pi) \cup \mathrm{CS}_{\rho_2}(\pi)$$

  (b) If ρ operates on ancestors of the end-sequent, then

  $$\mathcal{P}_\rho(\pi) = \mathcal{P}_{\rho_1}(\pi) \times_\rho \mathcal{P}_{\rho_2}(\pi).$$

  For the characteristic sequent set, define

  $$\mathrm{CS}_\rho(\pi) = \mathrm{CS}_{\rho_1}(\pi) \times \mathrm{CS}_{\rho_2}(\pi)$$

The *set of projections* of π, $\mathcal{P}(\pi)$ is defined as $\mathcal{P}_{\rho_0}(\pi)$, and the *characteristic sequent set* of π, $\mathrm{CS}(\pi)$ is defined as $\mathrm{CS}_{\rho_0}(\pi)$, where ρ_0 is the last inference of π.

Note that for $\mathbf{LK}_{\text{skc}}$-proofs π containing only atomic axioms, $\text{CS}(\pi)$ consists of sequents containing only atomic formulas. This is not required, though.

Proposition 4.3.6. *Let π be a regular $\mathbf{LK}_{\text{skc}}$-proof. Then there exists an $\mathbf{LK}$-refutation of the reduct of $\text{CS}(\pi)$.*

Proof. We inductively define, for each inference ρ with conclusion S in π, an $\mathbf{LK}$-tree γ_ρ of the reduct of $\text{cutanc}(S)$ from the reduct of $\text{CS}_\rho(\pi)$.

- If ρ is an axiom $\langle \mathbf{A} \rangle^{\ell_1} \vdash \langle \mathbf{A} \rangle^{\ell_2}$, distinguish:
  - $\text{cutanc}(S) = S$. Take the axiom ρ for γ_ρ.
  - $\text{cutanc}(S) \neq S$. Then $\text{CS}_\rho(\pi) = \{S'\}$ and we may take the reduct of S'.
- If ρ is a unary inference with immediate predecessor ρ', let S' be the conclusion of ρ' and distinguish:
  - ρ operates on ancestors of cut formulas. By (IH) we have an $\mathbf{LK}$-tree $\gamma_{\rho'}$ of $\text{cutanc}(S')$ from $\text{CS}_{\rho'}(\pi)$. Apply ρ to $\gamma_{\rho'}$ to obtain γ_ρ. Note that as $\text{cutanc}(S')$ is a sub-sequent of S', if ρ' is a strong quantifier inference, its eigenvariable condition is fulfilled. As $\text{CS}_\rho(\pi) = \text{CS}_{\rho'}(\pi)$ by definition, γ_ρ is the desired $\mathbf{LK}$-tree of $\text{cutanc}(S)$.
  - ρ operates on ancestors of the end-sequent. Then $\text{cutanc}(S) = \text{cutanc}(S')$ and $\text{CS}_\rho(\pi) = \text{CS}_{\rho'}(\pi)$ and hence we may take for γ_ρ the $\mathbf{LK}$-tree obtained by (IH).
- If ρ is a binary inference with immediate predecessors ρ_1, ρ_2, let $\gamma_{\rho_1}, \gamma_{\rho_2}$ be the $\mathbf{LK}$-trees obtained by (IH) and distinguish:
  - ρ operates on ancestors of cut-formulas. Then obtain γ_ρ by applying ρ to $\gamma_{\rho_1}, \gamma_{\rho_2}$: As $\text{CS}_\rho(\pi) = \text{CS}_{\rho_1}(\pi) \cup \text{CS}_{\rho_1}(\pi)$ it is the desired $\mathbf{LK}$-tree.
  - ρ operates on ancestors of the end-sequent. Then $\text{CS}_\rho(\pi) = \text{CS}_{\rho_1}(\pi) \times \text{CS}_{\rho_2}(\pi)$. We may assume that the eigenvariables of γ_{ρ_1} are distinct from the variables occurring in γ_{ρ_2} and vice-versa, otherwise we perform renamings. Let S_1, S_2 be the conclusions of ρ_1, ρ_2 respectively. For every $C \in \text{CS}_{\rho_1}(\pi)$, construct an $\mathbf{LK}$-tree γ_C of $\text{cutanc}(S_2) \circ C$ from $\text{CS}_{\rho_2}(\pi) \times \{C\}$ by taking γ_{ρ_2} and adding C to every sequent, and appending contractions on C at

the end. As the eigenvariables of γ_{ρ_2} are distinct from the variables of C by the consideration above, γ_C is really an **LK**-tree. Now, construct γ_ρ by taking γ_{ρ_1} and appending, at every leaf of the form $C \in \mathrm{CS}_{\rho_1}(\pi)$, the **LK**-tree γ_C, and adding contractions on $\mathrm{cutanc}(S_2)$ at the end. Again, no eigenvariable conditions are violated by the above consideration and γ_C is an **LK**-tree of $\mathrm{cutanc}(S_1) \circ \mathrm{cutanc}(S_2)$ from $\mathrm{CS}_\rho(\pi)$, as required.

Let ρ be the last inference in π, then γ_ρ is the desired **LK**-refutation. □

We will now address a central problem of CERES$^\omega$: how to combine an $\mathcal{R}_{al}$-refutation of $\mathrm{CS}(\pi)$ with the $\mathbf{LK}_{\mathrm{sk}}$-trees from $\mathcal{P}(\pi)$ into an $\mathbf{LK}_{\mathrm{sk}}$-proof of the end-sequent of π. The following definitions set up the main properties of the $\mathbf{LK}_{\mathrm{sk}}$-trees in $\mathcal{P}(\pi)$:

Definition 4.3.7 (Restrictedness)**.** Let $\mathcal{S}$ be a set of formula occurrences in an $\mathbf{LK}_{\mathrm{skc}}$-tree π. We say that π *is $\mathcal{S}$-linear* if no inferences operate on ancestors of occurrences in $\mathcal{S}$. We say that π *is $\mathcal{S}$-restricted* if no inferences except contraction operate on ancestors of occurrences in $\mathcal{S}$.

If $\mathcal{S}$ is the set of occurrences of cut-formulas of π and π is $\mathcal{S}$-restricted, we say that π is restricted.

Example 4.3.8. Consider the $\mathbf{LK}_{\mathrm{skc}}$-tree π

$$\dfrac{\dfrac{P(a) \vdash P(a) \quad Y(b) \vdash Y(b)}{P(a) \vee Y(b) \vdash P(a), Y(b)}\ \vee{:}\,l \qquad \dfrac{\dfrac{\dfrac{Y(b) \vdash \langle Y(b)\rangle^T \quad (b) \vdash \langle Y(b)\rangle^T}{Y(b), Y(b) \vdash \langle Y(b) \wedge Y(b)\rangle^T}\ \wedge{:}\,r}{Y(b) \vdash \langle Y(b) \wedge Y(b)\rangle^T}\ \mathrm{contr}{:}\,l}{Y(b) \vdash (\exists X)X(b)}\ \exists^{sk}{:}\,r}{P(a) \vee Y(b) \vdash (\exists X)X(b), P(a)}\ \mathrm{cut}$$

where $T = \lambda x.Y(x) \wedge Y(x)$. Let $\mathcal{S}$ be the ancestors of $P(a)$ in the end-sequent, and let C be the ancestors of cut-formulas in π. Then π is $\mathcal{S}$-linear and C-restricted, and thus restricted.

In principle, labels of linear occurrences in $\mathbf{LK}_{\mathrm{skc}}$-trees may be deleted:

Proposition 4.3.9. *Let π be an $\mathbf{LK}_{\mathrm{skc}}$-tree, and $\mathcal{S}$ a set of formula occurrences in π that is closed under descendants, and let π be $\mathcal{S}$-linear. If π' is obtained from π by replacing all labels of ancestors of occurrences in $\mathcal{S}$ by the empty label, then π' is an $\mathbf{LK}_{\mathrm{skc}}$-tree.*

Proof. As π is S-linear, no inferences operate on the respective occurrences. As no inference has restrictions on labels of context formulas (except that direct descendants have the same labels as their direct ancestors), and also axioms pose no restrictions on labels, the proposition holds. □

Definition 4.3.10 (Skolem parallel)**.** Let ρ_1, ρ_2 be a $\forall^{sk}\colon r$ inferences in $\mathbf{LK}_{\mathrm{skc}}$-trees π_1, π_2 with Skolem terms $\mathbf{S}_1$, $\mathbf{S}_2$ respectively. ρ_1, ρ_2 are called *Skolem parallel* if for all substitutions σ_1, σ_2, if $\mathbf{S}_1\sigma_1 = \mathbf{S}_2\sigma_2$ then $\mu_1\sigma_1, \mu_2\sigma_2$ are homomorphic, where μ_1, μ_2 are the maximal downwards paths starting at $\mathbf{S}_1, \mathbf{S}_2$ respectively. π_1, π_2 are called Skolem parallel if for all $\forall^{sk}\colon r$ inferences ρ_1, ρ_2 in π_1, π_2 respectively, ρ_1, ρ_2 are Skolem parallel.

Example 4.3.11. Consider the $\mathbf{LK}_{\mathrm{skc}}$-trees π

$$\dfrac{\dfrac{\Theta(f(\Theta)) \vdash \langle\Theta(f(\Theta))\rangle^{\Theta}}{\Theta(f(\Theta)) \vdash \langle(\forall y)\Theta(y)\rangle^{\Theta}}\ \forall^{sk}\colon r}{\Theta(f(\Theta)) \vdash (\exists X)(\forall y)X(y)}\ \exists^{sk}\colon r$$

and ψ

$$\dfrac{\dfrac{\dfrac{\dfrac{P(f(T)) \vdash \langle P(f(T))\rangle^{T} \quad Q(\alpha) \vdash \langle Q(\alpha)\rangle^{T}}{P(f(T)) \lor Q(\alpha) \vdash \langle P(f(T))\rangle^{T}, \langle Q(\alpha)\rangle^{T}}\ \lor\colon l}{P(f(T)) \lor Q(\alpha) \vdash \langle P(f(T)) \lor Q(\alpha)\rangle^{T}}\ \lor\colon r}{P(f(T)) \lor Q(\alpha) \vdash \langle(\forall y)(P(y) \lor Q(\alpha))\rangle^{T}}\ \forall^{sk}\colon r}{P(f(T)) \lor Q(\alpha) \vdash (\exists X)(\forall y)X(y)}\ \exists^{sk}\colon r$$

where $T = \lambda x.P(x) \lor Q(\alpha)$ and $f \in \mathcal{K}_{\iota\to o,\iota}$. Then π and ψ are Skolem parallel.

Proposition 4.3.12. *Let π_1, π_2 be $\mathbf{LK}_{\mathrm{skc}}$-trees and σ a substitution. If π_1, π_2 are Skolem parallel, then $\pi_1\sigma, \pi_2$ are.*

Proof. Consider Skolem terms $\mathbf{S}_1, \mathbf{S}_2$ occurring as auxiliary formulas of $\forall^{sk}\colon r$ inferences ρ_1, ρ_2 in $\pi_1\sigma, \pi_2$ respectively. Then by construction of $\pi_1\sigma$, $\mathbf{S}_1 = \mathbf{S}'_1\sigma$ for some Skolem term $\mathbf{S}'_1$ occurring as auxiliary formula of a $\forall^{sk}\colon r$ inference ρ'_1 in π_1. Let μ'_1 be the maximal downwards path starting at $\mathbf{S}'_1$, and μ_2 the maximal downwards path in π_2 starting at $\mathbf{S}_2$. Let σ_1, σ_2 be substitutions such that $\mathbf{S}_2\sigma_2 = \mathbf{S}_1\sigma_1 = \mathbf{S}'_1\sigma\sigma_1$. As ρ'_1, ρ_2 are Skolem parallel, $F(\mu'_1\sigma\sigma_1) = F(\mu_2\sigma_2)$. But by construction of $\pi_1\sigma$, $\mu'_1\sigma$ is the maximal downwards path starting at $\mathbf{S}_1$ in $\pi_1\sigma$, so ρ_1, ρ_2 are Skolem parallel. □

Definition 4.3.13 (Axiom labels)**.** Let π be an $\mathbf{LK}_{\mathrm{skc}}$-tree, let ω be a formula occurrence in π, and let μ be an ancestor of ω that occurs in an axiom A. Then A is called a *source axiom* for ω. Let $\mathcal{S}$ be a set of formula occurrences in π. We say that π *has suitable axiom labels with respect to* $\mathcal{S}$ if for all formula occurrences ω in $\mathcal{S}$, the source axioms of ω are of the form $\langle\mathbf{F}\rangle^{\ell} \vdash \langle\mathbf{F}\rangle^{\ell}$.

Example 4.3.14. Consider the $\mathbf{LK}_{\text{skc}}$-tree π

$$\dfrac{\dfrac{\langle Y(b)\rangle^T \vdash \langle Y(b)\rangle^T \quad Y(b) \vdash \langle Y(b)\rangle^T}{\langle Y(b)\rangle^T, Y(b) \vdash \langle Y(b) \wedge Y(b)\rangle^T}\ \wedge: r}{\langle Y(b)\rangle^T, Y(b) \vdash (\exists X)X(b)}\ \exists^{sk}: r$$

where $T = \lambda x.Y(x) \wedge Y(x)$. Let ω be the occurrence of $\langle Y(b)\rangle^T$ in the end-sequent. Then π has suitable axiom labels with respect to $\{\omega\}$. Note that π does not have suitable axiom labels with respect to the occurrence of $Y(b)$ in the end-sequent.

Definition 4.3.15 (Balancedness)**.** Let π be an $\mathbf{LK}_{\text{skc}}$-tree, and let $\mathcal{S}$ be a set of formula occurrences in π. We call π $\mathcal{S}$*-balanced* if for every axiom $\langle \mathbf{F}\rangle^{\ell_1} \vdash \langle \mathbf{F}\rangle^{\ell_2}$ in π, at least one occurrence of $\mathbf{F}$ is an ancestor of a formula occurrence in $\mathcal{S}$.

Example 4.3.16. Consider the $\mathbf{LK}_{\text{skc}}$-tree π from Example 4.3.8. Let ω_1 be the occurrence of $P(a) \vee \Theta(b)$ in the end-sequent of π, and let ω_2 be the occurrence of $(\exists X)X(b)$ in the end-sequent of π. Then π is neither $\{\omega_1\}$-balanced nor $\{\omega_2\}$-balanced, but π is $\{\omega_1, \omega_2\}$-balanced.

Definition 4.3.17 (CERES-projections)**.** Let S be a proper sequent, and C be a sequent. Then an $\mathbf{LK}_{\text{skc}}$-tree π is called a CERES-*projection* for (S, C) if the end-sequent of π is $S \circ C$ and π is weakly regular, $\mathcal{C}$-linear, $\mathcal{S}$-balanced, restricted, and has suitable axiom labels with respect to $\mathcal{C}$, where $\mathcal{S}$ resp. $\mathcal{C}$ is the set of formula occurrences of S resp. C in the end-sequent of π.

Let $\mathcal{C}$ be a set of sequents. A set of $\mathbf{LK}_{\text{skc}}$-trees $\mathcal{P}$ is called a *set of* CERES-*projections for* $(S, \mathcal{C})$ if for all $C \in \mathcal{C}$ there exists a $\pi(C) \in \mathcal{P}$ such that $\pi(C)$ is a CERES-projection for (S, C) and moreover, for all $\pi_1, \pi_2 \in \mathcal{P}$, π_1 and π_2 are Skolem parallel.

Lemma 4.3.18. *Let π be a regular $\mathbf{LK}_{\text{skc}}$-proof of S. Then $\mathcal{P}(\pi)$ is a set of* CERES-*projections for $(S, \mathrm{CS}(\pi))$. Furthermore, for all $\psi \in \mathcal{P}(\pi)$, $|\psi| \leq |\pi|$.*

Proof. By inspecting Definition 4.3.5. Let ρ be an inference in π with conclusion R. By induction on height(ρ), it is easy to see that for every $C \in \mathrm{CS}_\rho(\pi)$, $\mathcal{P}_\rho(\pi)$ contains an $\mathbf{LK}_{\text{sk}}$-tree of esanc$(R) \circ C$. Hence $\mathcal{P}(\pi)$ contains an $\mathbf{LK}_{\text{sk}}$-tree $\pi(C)$ of $S \circ C$ for every $C \in \mathrm{CS}(\pi)$. It remains to verify that (1) $\pi(C)$ is a CERES-projection for (S, C) and (2) every $\pi(C_1), \pi(C_2) \in \mathcal{P}(\pi)$ are Skolem parallel.

Regarding (1): $\pi(C)$ is regular, which follows from the fact that π is regular, and that in constructing $\pi(C)$ from π, every inference in π induces at most one copy of it in $\pi(C)$. Hence $\pi(C)$ is also weakly regular. S-balancedness, C-linearity and suitable axiom labels follow immediately from the definition. As $\pi(C)$ is cut-free, it is trivially restricted.

Regarding (2): Consider $\mu_1, \mu_2, \mathbf{S}_1, \mathbf{S}_2, \sigma_1, \sigma_2$ as in Definition 4.3.10. By construction, if an inference ρ of π is applied in both $\pi(C_1)$ and $\pi(C_2)$, also all inferences operating on descendants of the main formula of ρ are applied in both $\pi(C_1)$ and $\pi(C_2)$. Therefore by regularity of π, $\mu_1 = \mu_2$. $\mu_1 = \mu_2$ implies $\mathbf{S}_1 = \mathbf{S}_2$, hence $\mathbf{S}_1\sigma_1 = \mathbf{S}_1\sigma_2$ and therefore $\sigma_1 \restriction \mathrm{FV}(\mathbf{S}_1) = \sigma_2 \restriction \mathrm{FV}(\mathbf{S}_2)$. Therefore $\mu_1\sigma_1 = \mu_2\sigma_2$ by Proposition 4.1.9. □

Lemma 4.3.19. *Let S be a proper sequent. Let $\mathcal{C}$ be a set of sequents, and $\mathcal{P}$ a set of* CERES*-projections for $(S, \mathcal{C})$. Then, if there exists an $\mathcal{R}$-refutation of $\mathcal{C}$, there exists a restricted, weakly regular, balanced $\mathbf{LK}_{\mathrm{skc}}$-tree of S.*

Proof. Let $\gamma : S_1, \ldots, S_n$ be an $\mathcal{R}$-refutation of $\mathcal{C}$ (hence $S_n = \vdash$). Let $S = \Gamma \vdash \Delta$. By induction on $0 \leq i \leq n$, we construct sets of $\mathbf{LK}_{\mathrm{skc}}$-trees $\mathcal{P}_i \supseteq \mathcal{P}$ such that $\mathcal{P}_i$ is a set of CERES-projections for $(S, \mathcal{C} \cup \{S_1, \ldots, S_i\})$ and such that $\mathcal{P}_i$ contains only Skolem symbols from $\mathcal{P}$ and $S_1, \ldots, S_i$. Then $\mathcal{P}_n$ contains a CERES-projection for $(S, \vdash)$ which is the desired $\mathbf{LK}_{\mathrm{skc}}$-tree of S. We set $\mathcal{P}_0 = \mathcal{P}$.

For $i > 0$, distinguish how S_i is inferred in γ:

1. $S_i \in \mathcal{C}$. Then we may take $\mathcal{P}_i = \mathcal{P}_{i-1}$ by $\mathcal{P} \subseteq \mathcal{P}_{i-1}$ and (IH).

2. S_i is derived from S_j (and S_k). Then by (IH) we obtain a set of CERES-projections $\mathcal{P}_{i-1}$ for $(S, \mathcal{C} \cup \{S_1, \ldots, S_{i-1})$. By definition there exist CERES-projections $\pi_j \in \mathcal{P}_{i-1}$ for (S, S_j) (and $\pi_k \in \mathcal{P}_{i-1}$ for (S, S_k)). We set $\mathcal{P}_i = \mathcal{P}_{i-1} \cup \{\pi_i\}$, where π_i is an $\mathbf{LK}_{\mathrm{skc}}$-tree defined by distinguishing how S_i is inferred in γ:

   (a) $S_i = \langle \mathbf{A} \rangle^\ell, \Pi \vdash \Lambda$ is derived from $S_j = \Pi \vdash \Lambda, \langle \neg\mathbf{A} \rangle^\ell$ by $\neg^T$. Then the end-sequent of π_j is $S \circ S_j = \Gamma, \Pi \vdash \Lambda, \Delta, \langle \neg\mathbf{A} \rangle^\ell$. By S_j-linearity of π_j, the maximal upwards path μ starting at $\langle \neg\mathbf{A} \rangle^\ell$ is unique. Let μ end in $\langle \neg\mathbf{A} \rangle^\ell \vdash \langle \neg\mathbf{A} \rangle^\ell$ (the labels are identical because π_j has suitable axiom labels with respect to S_j). By S-balancedness, we may replace this axiom in π_j by

   $$\frac{\langle \mathbf{A} \rangle^\ell \vdash \langle \mathbf{A} \rangle^\ell}{\langle \mathbf{A} \rangle^\ell, \langle \neg\mathbf{A} \rangle^\ell \vdash} \neg\colon l$$

   to obtain π_i of $\langle \mathbf{A} \rangle^\ell, \Gamma, \Pi \vdash \Lambda, \Delta = S \circ S_i$. The desired properties of π_i and $\mathcal{P}_i$ follow trivially from the fact that they hold for π_j and $\mathcal{P}_{i-1}$ respectively.

(b) S_i is derived from S_j by either $\neg^F$, $\vee^T$, $\vee^F_l$, $\vee^F_r$: analogously to the previous case, there exists a unique axiom introducing the auxiliary formula of the inference in π_j. Depending on the rule applied, we perform one of the following replacements to obtain π_i:

$$\neg^F : \langle\neg\mathbf{A}\rangle^\ell \vdash \langle\neg\mathbf{A}\rangle^\ell \quad \rightsquigarrow \quad \frac{\langle\mathbf{A}\rangle^\ell \vdash \langle\mathbf{A}\rangle^\ell}{\vdash \langle\neg\mathbf{A}\rangle^\ell, \langle\mathbf{A}\rangle^\ell}\ \neg\colon r$$

$$\vee^T : \langle\mathbf{A}\vee\mathbf{B}\rangle^\ell \vdash \langle\mathbf{A}\vee\mathbf{B}\rangle^\ell \quad \rightsquigarrow \quad \frac{\langle\mathbf{A}\rangle^\ell \vdash \langle\mathbf{A}\rangle^\ell \quad \langle\mathbf{B}\rangle^\ell \vdash \langle\mathbf{B}\rangle^\ell}{\langle\mathbf{A}\vee\mathbf{B}\rangle^\ell \vdash \langle\mathbf{A}\rangle^\ell, \langle\mathbf{B}\rangle^\ell}\ \vee\colon l$$

$$\vee^F_l : \langle\mathbf{A}\vee\mathbf{B}\rangle^\ell \vdash \langle\mathbf{A}\vee\mathbf{B}\rangle^\ell \quad \rightsquigarrow \quad \frac{\langle\mathbf{A}\rangle^\ell \vdash \langle\mathbf{A}\rangle^\ell}{\langle\mathbf{A}\rangle^\ell \vdash \langle\mathbf{A}\vee\mathbf{B}\rangle^\ell}\ \vee\colon r^1$$

$$\vee^F_r : \langle\mathbf{A}\vee\mathbf{B}\rangle^\ell \vdash \langle\mathbf{A}\vee\mathbf{B}\rangle^\ell \quad \rightsquigarrow \quad \frac{\langle\mathbf{B}\rangle^\ell \vdash \langle\mathbf{B}\rangle^\ell}{\langle\mathbf{B}\rangle^\ell \vdash \langle\mathbf{A}\vee\mathbf{B}\rangle^\ell}\ \vee\colon r^2$$

As in the previous case, the desired properties of π_i and $\mathcal{P}_i$ follow from those of π_j and $\mathcal{P}_{i-1}$.

(c) $S_i = \langle\mathbf{AS}\rangle^\ell, \Pi \vdash \Lambda$ is derived from $S_j = \langle\forall\mathbf{A}\rangle^\ell, \Pi \vdash \Lambda$ by $\forall^F$. Then the end-sequent of π_j is $\langle\forall\mathbf{A}\rangle^\ell, \Pi, \Gamma \vdash \Delta, \Lambda$. By S_j-linearity and suitable axiom labels there exists a unique axiom $\langle\forall\mathbf{A}\rangle^\ell \vdash \langle\forall\mathbf{A}\rangle^\ell$ introducing the ancestor of $\langle\forall\mathbf{A}\rangle^\ell$. By S-balancedness, we may replace it by

$$\frac{\langle\mathbf{AS}\rangle^\ell \vdash \langle\mathbf{AS}\rangle^\ell}{\langle\mathbf{AS}\rangle^\ell \vdash \langle\forall\mathbf{A}\rangle^\ell}\ \forall^{sk}\colon r$$

to obtain π_i of $\langle\mathbf{AS}\rangle^\ell, \Pi, \Gamma \vdash \Delta, \Lambda$. As π_j is weakly regular, so is π_i (note that the Skolem symbol of this inference does not occur in π_j by assumption and the fact that it is fresh in γ). As π_j is Skolem parallel to the $\mathbf{LK}_{\mathrm{skc}}$-trees in $\mathcal{P}_{i-1}$, so is π_i as the downwards paths of auxiliary formulas of $\forall^{sk}\colon r$ inferences are unchanged, except for the new inference which has a fresh symbol. Restrictedness, S-balancedness and suitable axiom labels carry over from π_j.

(d) $S_i = \Pi \vdash \Lambda, \langle\mathbf{AX}\rangle^{\ell,\mathbf{X}}$ is derived from $S_j = \Pi \vdash \Lambda, \langle\forall\mathbf{A}\rangle^\ell$ by $\forall^T$. By (IH) we have an $\mathbf{LK}_{\mathrm{skc}}$-tree π_j of $\Pi, \Gamma \vdash \Delta, \Lambda, \langle\forall\mathbf{A}\rangle^\ell$. By S_j-linearity there exists a unique axiom $\langle\forall\mathbf{A}\rangle^\ell \vdash \langle\forall\mathbf{A}\rangle^\ell$ introducing the ancestor of $\langle \forall \mathbf{A} \rangle^{\ell}$. By S-balancedness, we may replace it by

$$\frac{\langle \mathbf{AX} \rangle^{\ell,\mathbf{X}} \vdash \langle \mathbf{AX} \rangle^{\ell,\mathbf{X}}}{\langle \forall \mathbf{A} \rangle^{\ell} \vdash \langle \mathbf{AX} \rangle^{\ell,\mathbf{X}}} \; \forall^{sk}\colon l$$

to obtain π_i of $\Pi, \Gamma \vdash \Delta, \Lambda, \langle \mathbf{AX} \rangle^{\ell,\mathbf{X}}$. Again the desired properties carry over from π_j.

(e) S_i is inferred from S_j by Sub with substitution σ. As S is proper, $\pi_i = \pi_j \sigma$ is an $\mathbf{LK}_{\mathrm{skc}}$-tree of $S_j \sigma \circ S$ which is restricted, S-balanced, weakly regular, and Skolem parallel to the $\mathbf{LK}_{\mathrm{skc}}$-trees in $\mathcal{P}_{i-1}$ by Proposition 4.3.12 and (IH).

(f) $S_i = \Gamma_j, \Gamma_k \vdash \Delta_j, \Delta_k$ is derived from $S_j = \Gamma_j \vdash \Delta_j, \langle \mathbf{A} \rangle^{\ell_1}, \ldots, \langle \mathbf{A} \rangle^{\ell_n}$ and $S_k = \langle \mathbf{A} \rangle^{\ell_{n+1}}, \ldots, \langle \mathbf{A} \rangle^{\ell_m}, \Gamma_k \vdash \Delta_k$ by Cut. By Proposition 4.3.9, we may delete labels from the ancestors of occurrences of $\mathbf{A}$ from π_j, π_k respectively, denote these trees by π'_j, π'_k. Take for π_i

$$\dfrac{\dfrac{\dfrac{(\pi'_j)}{\Gamma, \Gamma_j \vdash \Delta, \Delta_j, \mathbf{A}, \ldots, \mathbf{A}}}{\Gamma, \Gamma_j \vdash \Delta, \Delta_j, \mathbf{A}} \text{contr}\colon r \quad \dfrac{\dfrac{(\pi'_k)}{\mathbf{A}, \ldots, \mathbf{A}, \Gamma_k, \Gamma \vdash \Delta_k, \Delta}}{\mathbf{A}, \Gamma_k, \Gamma \vdash \Delta_k, \Delta} \text{contr}\colon l}{\dfrac{\Gamma, \Gamma, \Gamma_j, \Gamma_k \vdash \Delta, \Delta, \Delta_j, \Delta_k}{\Gamma, \Gamma_j, \Gamma_k \vdash \Delta, \Delta_j, \Delta_k} \text{contr}\colon *} \text{cut}$$

As π_j, π_k are Skolem parallel and weakly regular, and we contract on Γ, Δ, π_i is weakly regular. As the downwards paths of ancestors of S only change by some repetitions, π_i and the $\mathbf{LK}_{\mathrm{skc}}$-trees in $\mathcal{P}_{i-1}$ are Skolem parallel. π_i is restricted because π_j, π_k are S_j-linear and S_k-linear, respectively. S_i-linearity follows from S_j-linearity and S_k-linearity. As π_j, π_k are S-balanced, also π_i is. As π_j, π_k have suitable axiom labels, also π_i has: going from π_j to π'_j, we only delete labels of occurrences that are cut-ancestors in π_i (analogously for π_k). The suitable axiom labels hence remain by S-balancedness.

□

Lemma 4.3.20. *Let* π *be a restricted* $\mathbf{LK}_{\mathrm{skc}}$*-proof of* S*. Then there exists a* $\mathbf{LK}_{\mathrm{sk}}$*-proof of* S*.*

Proof. We proceed by induction on the number of Cut inferences in π. Consider a subtree φ of π that ends in an uppermost Cut ρ. Let the end-sequent of φ be $S_1 \circ S_2$, where S_1 are the end-sequent ancestors and S_2 are the cut-ancestors (in π)). We will transform φ into an $\mathbf{LK}_{\mathrm{sk}}$-tree φ' such that replacing φ by φ' in π results in a restricted $\mathbf{LK}_{\mathrm{skc}}$-proof of S (in particular φ' will be S_2-restricted). We proceed by induction on the height of ρ.

1. ρ occurs directly below axioms. Then ρ is

$$\frac{\langle \mathbf{A} \rangle^{\ell_1} \vdash \langle \mathbf{A} \rangle^{\ell_2} \quad \langle \mathbf{A} \rangle^{\ell_3} \vdash \langle \mathbf{A} \rangle^{\ell_4}}{\langle \mathbf{A} \rangle^{\ell_1} \vdash \langle \mathbf{A} \rangle^{\ell_4}} \text{ Cut}$$

   and we replace it by $\langle \mathbf{A} \rangle^{\ell_1} \vdash \langle \mathbf{A} \rangle^{\ell_4}$.

2. ρ does not occur directly below axioms. Then we permute ρ up. The only interesting case is permuting ρ over a contraction — here, the Cut is duplicated and the context contracted. By this contraction, weak regularity is preserved. Since the heights of both cuts is decreased, we may apply the induction hypothesis twice to obtain the desired $\mathbf{LK}_{\mathrm{skc}}$-proof.

□

Theorem 4.3.21. *Let π be a regular, proper* $\mathbf{LK}_{\mathrm{skc}}$*-proof of S such that there exists an $\mathcal{R}_{al}$-refutation of* $\mathrm{CS}(\pi)$*. Then there exists a cut-free* $\mathbf{LK}$*-proof of S.*

Proof. By Lemma 4.3.18 and Lemma 4.3.19, there exists a restricted $\mathbf{LK}_{\mathrm{skc}}$-proof of S. By Lemma 4.3.20, there exists an $\mathbf{LK}_{\mathrm{sk}}$-proof of S. By Theorem 4.1.41, there exists a cut-free $\mathbf{LK}$-proof of S. □

Conjecture 4.2.2 implies completeness of the cut-elimination method:

Theorem 4.3.22. *Assume Conjecture 4.2.2. Let π be an* $\mathbf{LK}$*-proof of a proper sequent S. Then there exists a cut-free* $\mathbf{LK}$*-proof of S.*

Proof. π can be transformed into a regular $\mathbf{LK}$-proof of S. By Lemma 4.3.4, there exists a regular $\mathbf{LK}_{\mathrm{skc}}$-proof of S. Let $\mathrm{CS}_R(\pi)$ be the reduct of $\mathrm{CS}(\pi)$. By Proposition 4.3.6, Proposition 2.2.3, and Theorem 4.2.3, there exists an $\mathcal{R}'_a$-refutation γ of $F(\mathrm{CS}_R(\pi))$. By deleting some $\vee^T$ and $\neg^T$ inferences from γ, we obtain an $\mathcal{R}'_a$-refutation of $\mathrm{CS}_R(\pi)$. By Conjecture 4.2.2, we may apply Theorem 4.3.21. □

Of course, cut-elimination implies consistency. Hence by the seminal result of Gödel from [28] and by the fact that through relativization (see [59]) it can be shown that cut-elimination for $\mathbf{LK}$ implies the consistency of higher-order arithmetic, at some point in the proof of the theorem above we must use assumptions which can not be proven in higher-order arithmetic. This strength is to be found in the proof of Theorem 4.2.3.

4.3.1 A simple example

In this section, we apply the CERES$^\omega$ method to a simple example for illustration. Let X, Y, X_0, Y_0 be variables of type $\iota \to o$, Z a variable of type $(\iota \to o) \to o$, x, x_0 variables of type ι, and f a constant of type $\iota \to \iota$. Since all our symbols are unary, we will not write superflous parentheses.

Consider the regular **LK**-proof π

$$\dfrac{\dfrac{\dfrac{\dfrac{(\pi_1) \qquad (\pi_2)}{\forall Z(ZX_0 \to ZY_0), (\forall x(X_0x \to X_0fx) \vdash \forall x(Y_0x \to Y_0ffx)}\ cut}{\vdash \forall Z(ZX_0 \to ZY_0) \to (\forall x(X_0x \to X_0fx) \to \forall x(Y_0x \to Y_0ffx))}\ 2\times \to: r}{\vdash \forall X(\forall Z(ZX \to ZY_0) \to (\forall x(Xx \to Xfx) \to \forall x(Y_0x \to Y_0ffx)))}\ \forall: r}{\vdash \exists Y \forall X(\forall Z(ZX \to ZY) \to (\forall x(Xx \to Xfx) \to \forall x(Yx \to Yffx)))}\ \exists: r$$

where π_1 is

$$\dfrac{\dfrac{F \vdash F \quad G \vdash G}{\forall x(X_0x \to X_0fx) \to \forall x(Y_0x \to Y_0fx), F \vdash G}\ \to: l}{\forall Z(ZX_0 \to ZY_0), F \vdash G}\ \forall: l$$

where the substitution term of the $\forall: l$ inference is $\lambda X_0.\forall x(X_0x \to X_0fx)$, $F = (\forall x(X_0x \to X_0fx)$, $G = \forall x(Y_0x \to Y_0fx)$, and π_2 is

$$\dfrac{\dfrac{\dfrac{\dfrac{Y_0x_0 \vdash Y_0x_0 \quad Y_0fx_0 \vdash Y_0fx_0 \quad Y_0ffx_0 \vdash Y_0ffx_0}{Y_0x_0, Y_0x_0 \to Y_0fx_0, Y_0fx_0 \to Y_0ffx_0 \vdash Y_0ffx_0}\ 2\times \to: l}{Y_0x_0 \to Y_0fx_0, Y_0fx_0 \to Y_0ffx_0 \vdash Y_0x_0 \to Y_0ffx_0}\ \to: r}{\forall x(Y_0x \to Y_0fx) \vdash Y_0x_0 \to Y_0ffx_0}\ \text{contr}: l, 2 \times \forall: l}{\forall x(Y_0x \to Y_0fx) \vdash \forall x(Y_0x \to Y_0ffx)}\ \forall: r$$

Let $h \in \mathcal{K}_{\iota\to o,\iota\to o}$, and $g \in \mathcal{K}_{\iota\to o,\iota}$. Then application of Lemma 4.3.4 yields the regular $\mathbf{LK}_{\text{skc}}$-proof φ:

$$\dfrac{\dfrac{\dfrac{\dfrac{(\psi_1) \qquad (\psi_2)}{\langle\forall Z(ZhY_0 \to ZY_0)\rangle^{Y_0}, \langle(\forall x(hY_0x \to hY_0fx)\rangle^{Y_0} \vdash \langle\forall x(Y_0x \to Y_0ffx)\rangle^{Y_0}}\ cut}{\vdash \langle\forall Z(ZhY_0 \to ZY_0) \to (\forall x(hY_0x \to hY_0fx) \to \forall x(Y_0x \to Y_0ffx))\rangle^{Y_0}}\ 2\times \to: r}{\vdash \langle\forall X(\forall Z(ZX \to ZY_0) \to (\forall x(Xx \to Xfx) \to \forall x(Y_0x \to Y_0ffx)))\rangle^{Y_0}}\ \forall^{sk}: r}{\vdash \exists Y \forall X(\forall Z(ZX \to ZY) \to (\forall x(Xx \to Xfx) \to \forall x(Yx \to Yffx)))}\ \exists^{sk}: r$$

where ψ_1 is

$$\dfrac{\dfrac{\langle F'\rangle^{Y_0} \vdash \langle F'\rangle^{Y_0,T} \quad \langle G\rangle^{Y_0,T} \vdash G}{\langle\forall x(hY_0x \to hY_0fx) \to \forall x(Y_0x \to Y_0fx)\rangle^{Y_0,T}, \langle F'\rangle^{Y_0} \vdash G}\ \to: l}{\langle\forall Z(ZhY_0 \to ZY_0)\rangle^{Y_0}, \langle F'\rangle^{Y_0} \vdash G}\ \forall^{sk}: l$$

where $T = \lambda X_0.\forall x(X_0 x \to X_0 f x)$ and $F' = (\forall x(hY_0 x \to hY_0 f x)$, and ψ_2 is

$$\dfrac{\dfrac{\dfrac{\dfrac{\langle Y_0gY_0\rangle^{Y_0} \vdash Y_0gY_0 \quad Y_0fgY_0 \vdash Y_0fgY_0 \quad Y_0ffgY_0 \vdash \langle Y_0ffgY_0\rangle^{Y_0}}{\langle Y_0gY_0\rangle^{Y_0}, Y_0gY_0 \to Y_0fgY_0, Y_0fgY_0 \to Y_0ffgY_0 \vdash \langle Y_0ffgY_0\rangle^{Y_0}}\ 2\times \to: l}{Y_0gY_0 \to Y_0fgY_0, Y_0fgY_0 \to Y_0ffgY_0 \vdash \langle Y_0gY_0 \to Y_0ffgY_0\rangle^{Y_0}}\ \to: r}{\forall x(Y_0x \to Y_0fx) \vdash \langle Y_0gY_0 \to Y_0ffgY_0\rangle^{Y_0}}\ \mathrm{contr}: l, 2\times\forall: l}{\forall x(Y_0x \to Y_0fx) \vdash \langle \forall x(Y_0x \to Y_0ffx)\rangle^{Y_0}}\ \forall^{sk}: r$$

Then

$$\begin{aligned}\mathrm{CS}(\varphi) &= \{\vdash \langle G\rangle^{Y_0,T}\} \cup \{\vdash \langle Y_0gY_0\rangle^{Y_0}\} \cup \emptyset \cup \{\langle Y_0ffgY_0\rangle^{Y_0} \vdash\} \\ &= \left\{\begin{array}{l}\vdash \langle \forall x(Y_0x \to Y_0fx)\rangle^{Y_0,\lambda X_0.\forall x(X_0x \to X_0fx)};\\ \vdash \langle Y_0gY_0\rangle^{Y_0};\ \langle Y_0ffgY_0\rangle^{Y_0} \vdash\end{array}\right\}.\end{aligned}$$

Proposition 4.3.6 yields the following **LK**-refutation of the reduct of $\mathrm{CS}(\varphi)$:

$$\dfrac{\vdash \forall x(Y_0x \to Y_0fx) \qquad \dfrac{\dfrac{\vdash Y_0gY_0 \quad Y_0fgY_0 \vdash Y_0fgY_0 \quad Y_0ffgY_0 \vdash}{Y_0gY_0 \to Y_0fgY_0, Y_0fgY_0 \to Y_0ffgY_0 \vdash}\ 2\times \to: l}{\forall x(Y_0x \to Y_0fx) \vdash}\ \mathrm{contr}: l, 2\times\forall: l}{\vdash}\ cut$$

Let S be the end-sequent of φ. Then $\mathcal{P}(\varphi)$ consists of the following $\mathbf{LK}_{\mathrm{skc}}$-trees: ψ_1 is

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\langle F'\rangle^{Y_0} \vdash \langle F'\rangle^{Y_0,T} \quad \langle G\rangle^{Y_0,T} \vdash \langle G\rangle^{Y_0,T}}{\langle F' \to G\rangle^{Y_0,T}, \langle F'\rangle^{Y_0} \vdash \langle G\rangle^{Y_0,T}}\ \to: l}{\langle \forall Z(ZhY_0 \to ZY_0)\rangle^{Y_0}, \langle F'\rangle^{Y_0} \vdash \langle G\rangle^{Y_0,T}}\ \forall^{sk}: l}{\langle \forall Z(ZhY_0 \to ZY_0)\rangle^{Y_0}, \langle F'\rangle^{Y_0} \vdash \langle G\rangle^{Y_0,T}, \langle \forall x(Y_0x \to Y_0ffx)\rangle^{Y_0}}\ \mathrm{weak}: r}{\vdash \langle G\rangle^{Y_0,T}, \langle \forall Z(ZhY_0 \to ZY_0) \to (F' \to \forall x(Y_0x \to Y_0ffx))\rangle^{Y_0}}\ 2\times \to: r}{(\vdash \langle G\rangle^{Y_0,T}) \circ S}\ \forall^{sk}: r, \exists^{sk}: r$$

ψ_2 is

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\langle Y_0gY_0\rangle^{Y_0} \vdash \langle Y_0gY_0\rangle^{Y_0}}{\langle Y_0gY_0\rangle^{Y_0} \vdash \langle Y_0gY_0\rangle^{Y_0}, \langle Y_0ffgY_0\rangle^{Y_0}}\ \mathrm{weak}: r}{\vdash \langle Y_0gY_0\rangle^{Y_0}, \langle Y_0gY_0 \to Y_0ffgY_0\rangle^{Y_0}}\ \to: r}{\vdash \langle Y_0gY_0\rangle^{Y_0}, \langle \forall x(Y_0x \to Y_0ffx)\rangle^{Y_0}}\ \forall^{sk}: r}{\dfrac{\langle \forall Z(ZhY_0 \to ZY_0)\rangle^{Y_0}, \langle F'\rangle^{Y_0} \vdash \langle Y_0gY_0\rangle^{Y_0}, \langle \forall x(Y_0x \to Y_0ffx)\rangle^{Y_0}}{\vdash \langle Y_0gY_0\rangle^{Y_0}, \langle \forall Z(ZhY_0 \to ZY_0) \to (F' \to \forall x(Y_0x \to Y_0ffx))\rangle^{Y_0}}\ 2\times \to: r}\ 2\times\mathrm{weak}: l}{(\vdash \langle Y_0gY_0\rangle^{Y_0}) \circ S}\ \forall^{sk}: r, \exists^{sk}: r$$

and ψ_3 is

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\langle Y_0ffgY_0\rangle^{Y_0} \vdash \langle Y_0ffgY_0\rangle^{Y_0}}{\langle Y_0gY_0\rangle^{Y_0}, \langle Y_0ffgY_0\rangle^{Y_0} \vdash \langle Y_0ffgY_0\rangle^{Y_0}}\ \text{weak:}\,l}{\langle Y_0ffgY_0\rangle^{Y_0} \vdash \langle Y_0gY_0 \to Y_0ffgY_0\rangle^{Y_0}}\ \to\colon r}{\langle Y_0ffgY_0\rangle^{Y_0} \vdash \langle \forall x(Y_0x \to Y_0ffx)\rangle^{Y_0}}\ \forall^{sk}\colon r}{\langle \forall Z(ZhY_0 \to ZY_0)\rangle^{Y_0}, \langle F'\rangle^{Y_0}, \langle Y_0ffgY_0\rangle^{Y_0} \vdash \langle \forall x(Y_0x \to Y_0ffx)\rangle^{Y_0}}\ 2\times\text{weak:}\,l}{\langle Y_0ffgY_0\rangle^{Y_0} \vdash \langle \forall Z(ZhY_0 \to ZY_0) \to (F' \to \forall x(Y_0x \to Y_0ffx))\rangle^{Y_0}}\ 2\times \to\colon r}{(\langle Y_0ffgY_0\rangle^{Y_0} \vdash) \circ S}\ \forall^{sk}\colon r, \exists^{sk}\colon r$$

Consider the following $\mathcal{R}$-refutation of CS(φ):

1	$\vdash \langle \forall x(Y_0x \to Y_0fx)\rangle^{Y_0,T}$	CS(φ)
2	$\vdash \langle Y_0x_0 \to Y_0fx_0\rangle^{Y_0,T,x_0}$	$\forall^T : 1$
3	$\langle Y_0x_0\rangle^{Y_0,T,x_0} \vdash \langle Y_0fx_0\rangle^{Y_0,T,x_0}$	$\to^T : 2$
4	$\langle Y_0gY_0\rangle^{Y_0,T,gY_0} \vdash \langle Y_0fgY_0\rangle^{Y_0,T,gY_0}$	Sub : $3\,[x_0 \leftarrow gY_0]$
5	$\vdash \langle Y_0gY_0\rangle^{Y_0}$	CS(φ)
6	$\vdash \langle Y_0fgY_0\rangle^{Y_0,T,gY_0}$	Cut : $4, 5$
7	$\langle Y_0fgY_0\rangle^{Y_0,T,fgY_0} \vdash \langle Y_0ffgY_0\rangle^{Y_0,T,fgY_0}$	Sub : $3\,[x_0 \leftarrow fgY_0]$
8	$\vdash \langle Y_0ffgY_0\rangle^{Y_0,T,fgY_0}$	Cut : $6, 7$
9	$\langle Y_0ffgY_0\rangle^{Y_0} \vdash$	CS(φ)
10	$\vdash$	Cut : $8, 9$

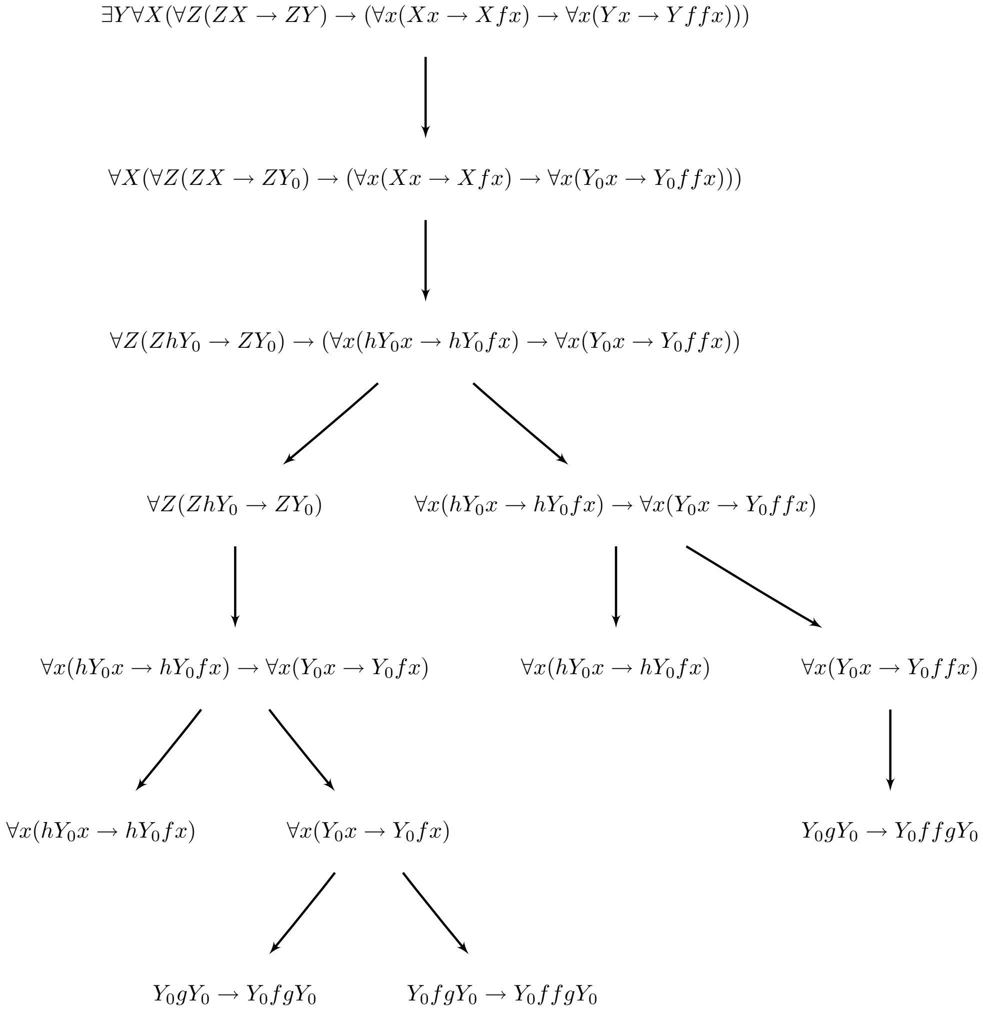
We sketch the application of Lemma 4.3.19: To improve readability, we suppress the labels in the following figures. The $\forall^T, \to^T$ inferences (steps 2–3) cause ψ_1 to become ψ_1':

$$\dfrac{\dfrac{\dfrac{\dfrac{F' \vdash F' \qquad \dfrac{\dfrac{Y_0x_0 \vdash Y_0x_0 \quad Y_0fx_0 \vdash Y_0fx_0}{Y_0x_0, Y_0x_0 \to Y_0fx_0 \vdash Y_0fx_0}\ \to\colon l}{Y_0x_0, G \vdash Y_0fx_0}\ \forall^{sk}\colon l}{Y_0x_0, F' \to G, F' \vdash Y_0fx_0}\ \to\colon l}{Y_0x_0, \forall Z(ZhY_0 \to ZY_0), F' \vdash Y_0fx_0}\ \forall^{sk}\colon l}{\dfrac{Y_0x_0, \forall Z(ZhY_0 \to ZY_0), F' \vdash Y_0fx_0, \forall x(Y_0x \to Y_0ffx)}{\dfrac{Y_0x_0 \vdash Y_0fx_0, \forall Z(ZhY_0 \to ZY_0) \to (F' \to \forall x(Y_0x \to Y_0ffx))}{(Y_0x_0 \vdash Y_0fx_0) \circ S}\ \forall^{sk}\colon r, \exists^{sk}\colon r}\ 2\times \to\colon r}\ \text{weak:}\,r$$

Eventually we obtain the following $\mathbf{LK}_{\mathrm{skc}}$-proof ψ:

$$\dfrac{\dfrac{\dfrac{\dfrac{\psi_2 \quad \psi_1' [x_0 \leftarrow gY_0]}{S \circ S \circ (\vdash Y_0 f g Y_0)}\,\mathrm{cut} \quad \psi_1' [x_0 \leftarrow fgY_0]}{S \circ S \circ S \circ (\vdash Y_0 f f g Y_0)}\,\mathrm{cut} \quad \psi_3}{S \circ S \circ S \circ S}\,\mathrm{cut}}{S}\,\mathrm{contr}{:}\ast$$

Applying Lemma 4.3.20 and Lemma 4.1.27 yields a sequentially pruned $\mathbf{LK}_{\mathrm{sk}}$-proof of S. We summarize the instantiation information of this proof in the following figure (based on the Skolem expansion tree formalism of [44]).



Observe that this basically corresponds to the proof obtained by applying reductive cut-elimination (i.e. the reduction rules extracted from Gentzen's cut-elimination proof from [23]) to φ.

Now, let $R = \lambda x.X_0x \wedge \neg X_0x$ and consider a second $\mathcal{R}$-refutation of $\mathrm{CS}(\varphi)$:

1	$\vdash \langle Y_0gY_0\rangle^{Y_0}$	$\mathrm{CS}(\varphi)$
2	$\vdash \langle X_0gR \wedge \neg X_0gR\rangle^R$	Sub : 1 $[Y_0 \leftarrow R]$
3	$\vdash \langle X_0gR\rangle^R$	$\wedge_l^T : 2$
4	$\vdash \langle \neg X_0gR\rangle^R$	$\wedge_r^T : 2$
5	$\langle X_0gR\rangle^R \vdash$	$\neg^T : 4$
6	$\vdash$	Cut : 3, 5

Steps 1–3 transform (via Lemma 4.3.19) ψ_2 into ψ_2':

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\langle X_0gR\rangle^R \vdash \langle X_0gR\rangle^R}{\langle RgR\rangle^R \vdash \langle X_0gR\rangle^R}\ \wedge\colon l_1}{\langle RgR\rangle^R \vdash \langle X_0gR\rangle^R, \langle RffgR\rangle^R}\ \text{weak}\colon r}{\vdash \langle X_0gR\rangle^R, \langle RgR \to RffgR\rangle^R}\ \to\colon r}{\vdash \langle X_0gR\rangle^R, \langle \forall x(Rx \to Rffx)\rangle^R}\ \forall^{sk}\colon r}{\langle \forall Z(ZhR \to ZR)\rangle^R, \langle F'\rangle^R \vdash \langle X_0gR\rangle^R, \langle \forall x(Rx \to Rffx)\rangle^R}\ 2\times\text{weak}\colon l}{\vdash \langle X_0gR\rangle^R, \langle \forall Z(ZhR \to ZR) \to (F' \to \forall x(Rx \to Rffx))\rangle^R}\ 2\times \to\colon r}{(\vdash \langle X_0gR\rangle^R) \circ S}\ \forall^{sk}\colon r, \exists^{sk}\colon r
$$

and steps 1,2,4,5 yield ψ_2'':

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\langle X_0gR\rangle^R \vdash \langle X_0gR\rangle^R}{\langle \neg X_0gR\rangle^R, \langle X_0gR\rangle^R \vdash}\ \neg\colon l}{\langle RgR\rangle^R, \langle X_0gR\rangle^R \vdash}\ \wedge\colon l_2}{\langle RgR\rangle^R, \langle X_0gR\rangle^R \vdash \langle RffgR\rangle^R}\ \text{weak}\colon r}{\langle X_0gR\rangle^R \vdash \langle RgR \to RffgR\rangle^R}\ \to\colon r}{\langle X_0gR\rangle^R \vdash \langle \forall x(Rx \to Rffx)\rangle^R}\ \forall^{sk}\colon r}{\langle \forall Z(ZhR \to ZR)\rangle^R, \langle F'\rangle^R, \langle X_0gR\rangle^R \vdash \langle \forall x(Rx \to Rffx)\rangle^R}\ 2\times\text{weak}\colon l}{\langle X_0gR\rangle^R \vdash \langle \forall Z(ZhR \to ZR) \to (F' \to \forall x(Rx \to Rffx))\rangle^R}\ 2\times \to\colon r}{(\langle X_0gR\rangle^R \vdash) \circ S}\ \forall^{sk}\colon r, \exists^{sk}\colon r
$$

Finally we obtain the $\mathbf{LK}_{\mathrm{skc}}$-proof ψ':

$$
\dfrac{\dfrac{\psi_2' \qquad \psi_2''}{S \circ S}\ \text{cut}}{S}\ \text{contr}\colon *
$$

Application of Lemma 4.3.20 yields the $\mathbf{LK}_{\mathrm{sk}}$-proof ψ'':

$$
\begin{array}{c}
\langle X_0gR\rangle^R \vdash \langle X_0gR\rangle^R \\ \hline
\langle \neg X_0gR\rangle^R, \langle X_0gR\rangle^R \vdash \quad \neg\colon l \\ \hline
\langle RgR\rangle^R, \langle X_0gR\rangle^R \vdash \quad \wedge\colon l_2 \\ \hline
\langle RgR\rangle^R, \langle X_0gR\rangle^R \vdash \langle RffgR\rangle^R \quad \text{weak}\colon r \\ \hline
\langle X_0gR\rangle^R \vdash \langle RgR \to RffgR\rangle^R \quad \to\colon r \\ \hline
\langle X_0gR\rangle^R \vdash \langle \forall x(Rx \to Rffx)\rangle^R \quad \forall^{sk}\colon r \\ \hline
\langle \forall Z(ZhR \to ZR)\rangle^R, \langle F'\rangle^R, \langle X_0gR\rangle^R \vdash \langle \forall x(Rx \to Rffx)\rangle^R \quad 2\times\text{weak}\colon l \\ \hline
\langle X_0gR\rangle^R \vdash \langle \forall Z(ZhR \to ZR) \to (F' \to \forall x(Rx \to Rffx))\rangle^R \quad 2\times \to\colon r \\ \hline
(\langle X_0gR\rangle^R \vdash) \circ S \quad \forall^{sk}\colon r, \exists^{sk}\colon r \\ \hline
(\langle RgR\rangle^R \vdash) \circ S \quad \wedge\colon l_1 \\ \hline
(\langle RgR\rangle^R \vdash \langle RffgR\rangle^R) \circ S \quad \text{weak}\colon r \\ \hline
(\vdash \langle RgR \to RffgR\rangle^R) \circ S \quad \to\colon r \\ \hline
(\vdash \langle \forall x(Rx \to Rffx)\rangle^R) \circ S \quad \forall^{sk}\colon r \\ \hline
(\langle \forall Z(ZhR \to ZR)\rangle^R, \langle F'\rangle^R, \vdash \langle \forall x(Rx \to Rffx)\rangle^R) \circ S \quad 2\times\text{weak}\colon l \\ \hline
(\vdash \langle \forall Z(ZhR \to ZR) \to (F' \to \forall x(Rx \to Rffx))\rangle^R) \circ S \quad 2\times \to\colon r \\ \hline
S \circ S \quad \forall^{sk}\colon r, \exists^{sk}\colon r \\ \hline
S \quad \text{contr}\colon *
\end{array}
$$

Application of Lemma 4.1.27 yields the sequentially pruned $\mathbf{LK}_{\mathrm{sk}}$-proof ψ''':

$$
\begin{array}{c}
\langle X_0gR\rangle^R \vdash \langle X_0gR\rangle^R \\ \hline
\langle X_0gR\rangle^R, \langle \neg X_0gR\rangle^R \vdash \quad \neg\colon l \\ \hline
\langle X_0gR\rangle^R, \langle RgR\rangle^R \vdash \quad \wedge\colon l_2 \\ \hline
\langle RgR\rangle^R, \langle RgR\rangle^R \vdash \quad \wedge\colon l_1 \\ \hline
\langle RgR\rangle^R \vdash \quad \text{contr}\colon l \\ \hline
\langle RgR\rangle^R \vdash \langle RffgR\rangle^R \quad \text{weak}\colon r \\ \hline
\vdash \langle RgR \to RffgR\rangle^R \quad \to\colon r \\ \hline
\vdash \langle \forall x(Rx \to Rffx)\rangle^R \quad \forall^{sk}\colon r \\ \hline
\langle \forall Z(ZhR \to ZR)\rangle^R, \langle F'\rangle^R, \vdash \langle \forall x(Rx \to Rffx)\rangle^R \quad 2\times\text{weak}\colon l \\ \hline
\vdash \langle \forall Z(ZhR \to ZR) \to (F' \to \forall x(Rx \to Rffx))\rangle^R \quad 2\times \to\colon r \\ \hline
\vdash \exists Y \forall X(\forall Z(ZX \to ZY) \to (\forall x(Xx \to Xfx) \to \forall x(Yx \to Yffx))) \quad \forall^{sk}\colon r, \exists^{sk}\colon r
\end{array}
$$

Note that in ψ''', all $\forall^{sk}\colon r$ inferences are already correctly placed. Hence Theorem 4.1.41 yields the following $\mathbf{LK}$-proof, where Θ is a variable of type

$\iota \to o$ and α is a variable of type ι:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\langle X_0\alpha\rangle^R \vdash \langle X_0\alpha\rangle^R}{\langle X_0\alpha\rangle^R, \langle\neg X_0\alpha\rangle^R \vdash}\ \neg\colon l}{\langle X_0\alpha\rangle^R, \langle R\alpha\rangle^R \vdash}\ \wedge\colon l_2}{\langle R\alpha\rangle^R, \langle R\alpha\rangle^R \vdash}\ \wedge\colon l_1}{\langle R\alpha\rangle^R \vdash}\ \text{contr}\colon l}{\langle R\alpha\rangle^R \vdash \langle Rff\alpha\rangle^R}\ \text{weak}\colon r}{\vdash \langle R\alpha \to Rff\alpha\rangle^R}\ \to\colon r}{\vdash \langle \forall x(Rx \to Rffx)\rangle^R}\ \forall^{sk}\colon r}{\langle \forall Z(Z\Theta \to ZR)\rangle^R, \langle \forall x(\Theta x \to \Theta fx)\rangle^R, \vdash \langle \forall x(Rx \to Rffx)\rangle^R}\ 2\times\text{weak}\colon l}{\vdash \langle \forall Z(Z\Theta \to ZR) \to (\forall x(\Theta x \to \Theta fx) \to \forall x(Rx \to Rffx))\rangle^R}\ 2\times \to\colon r}{\vdash \exists Y \forall X(\forall Z(ZX \to ZY) \to (\forall x(Xx \to Xfx) \to \forall x(Yx \to Yffx)))}\ \forall\colon r, \exists\colon r
$$

Intuitively, this proof shows that one can prove the end-sequent of π by instantiating Y not by an arbitrary set Y_0, but by the empty set (represented here by the term R). Even though this observation is trivial in itself, observe that the proof ψ''' (or any proof using the empty set as witness for Y) *cannot* be obtained by applying reductive cut-elimination to π! This shows that the CERES$^\omega$ method can really produce more cut-free proofs than the reductive methods. Hence, when applied to a more sophisticated proof, the CERES$^\omega$ method may produce interesting cut-free proofs not obtainable by reductive methods.

4.3.2 An application of non-standard projections

In Section 4.3, the notion of CERES-projection was defined and the standard set of CERES-projections $\mathcal{P}(\pi)$ was constructed for use in the CERES$^\omega$ method. Using the machinery we have developed, together with non-standard sets of CERES-projections, we can give an easy constructive proof of the fact that **LK** is relatively complete to $\mathcal{R}_{al}$:

Proposition 4.3.23. *Let* $\mathbf{F}$ *be a closed formula such that an* $\mathcal{R}_{al}$*-refutation of* $\{\mathbf{F} \vdash\}$ *exists. Then there exists an* **LK***-proof of* $\vdash \mathbf{F}$.

Proof. Let $S = \vdash \mathbf{F}$, $\mathcal{C} = \{\mathbf{F} \vdash\}$ and let π be the $\mathbf{LK}_{\text{sk}}$-proof $\mathbf{F} \vdash \mathbf{F}$. It is easy to check that $\{\pi\}$ is a set of CERES-projections for $(S, \mathcal{C})$. The result then follows from Lemmas 4.3.19 and 4.3.20, and Theorem 4.1.41. □

Chapter 5

Conclusion

In this thesis, we have worked towards a generalization of the cut-elimination method CERES to higher-order logic. In Chapter 3, we have given a preliminary formulation of CERES for the class of **QFC**-proofs, which is roughly the class of proofs in second-order logic using only quantifier-free comprehension. We have shown that in this setting, one of the main ingredients of the first-order CERES method, proof Skolemization, can still be applied successfully.

In Chapter 4, we formulate the CERES$^\omega$ method, which generalizes the CERES method to full higher-order logic. To this end, we introduced the cut-free calculus $\mathbf{LK}_{\mathrm{sk}}$ in Section 4.1. The point of $\mathbf{LK}_{\mathrm{sk}}$ was to transfer the essential properties of proof Skolemization from first-order to the higher-order logic setting. This was achieved by using Skolem terms in place of the eigenvariables of **LK**, and abandoning the eigenvariable condition.

To achieve soundness for $\mathbf{LK}_{\mathrm{sk}}$, two notions of regularity (weak regularity and weak+ regularity) where introduced. Soundness was shown by giving an algorithm transforming weakly regular $\mathbf{LK}_{\mathrm{sk}}$-trees to **LK**-proofs, and by reducing soundness of weak+ regularity to soundness of weak regularity.

In Section 4.2, we introduced the resolution calculus $\mathcal{R}_{al}$. In order to relate this non-standard calculus to the standard resolution calculus $\mathcal{R}$, we introduced an intermediary calculus $\mathcal{R}_a$ and showed a restricted relative completeness result. We conjectured that $\mathcal{R}_{al}$ is indeed relatively complete to $\mathcal{R}$ and left the proof of this for future work.

Finally, in Section 4.3, we put together the results of this chapter to define the CERES$^\omega$ method. We first define a calculus with cut, $\mathbf{LK}_{\mathrm{skc}}$, by combining the rules of $\mathbf{LK}_{\mathrm{sk}}$ and **LK**. We give a translation from **LK**-proofs to $\mathbf{LK}_{\mathrm{skc}}$-proofs and state the central definitions of the method, culminating in the notions of CERES-projection and characteristic sequent sets. Using these notions we show that we can eliminate the cuts from an $\mathbf{LK}_{\mathrm{skc}}$-proof by modifying its set of CERES-projections according to a $\mathcal{R}_{al}$-refutation of its characteristic sequent set. Our conjecture implies that such a $\mathcal{R}_{al}$-refutation always exists, which in turn implies completeness of the method. We conclude by illustrating the CERES$^\omega$ method by applying it to a simple example, observing that the method may produce cut-free proofs that cannot be obtained by traditional reductive cut-elimination methods.

The main motivation of this work was to define a cut-elimination method that can be successfully used in practice; i.e. to apply cut-elimination to concrete mathematical proofs (formalized in **LK**). This thesis therefore provides a theoretical basis for such applications, but there is still much to be done: As mentioned, completeness of the method relies on a conjecture which, although likely to have a positive resolution, still needs to be proven. Also, the calculus $\mathcal{R}$ we consider here is quite basic and does not have some of the features that modern higher-order resolution calculi have (e.g. integration of unification, treatment of extensionality). Such calculi should be integrated into the method in the future. In general, the behaviour of extensionality with respect to CERES$^\omega$ remains to be investigated — this is such a natural feature that practical proof analysis will greatly benefit from its incorporation. Furthermore, since sequent calculus proofs are hard to read, and since there exist more convenient formats like the Skolem expansion trees of [44], for practical purposes it will be useful to develop an algorithm transforming $\mathbf{LK}_{\mathrm{sk}}$-proofs to Skolem expansion trees (without the detour via **LK**).

It remains to note that some parts of this thesis have been implemented in the context of the GAPT framework[1]. In particular, the calculi **LK**, $\mathbf{LK}_{\mathrm{sk}}$ and $\mathbf{LK}_{\mathrm{skc}}$ have been implemented, along with the translation from **LK** to $\mathbf{LK}_{\mathrm{skc}}$, and the extraction of the characteristic sequent set. It is therefore already now possible to semi-automatically analyse higher-order proofs using the CERES$^\omega$ method.

[1]See `http://www.logic.at/ceres` and `http://code.google.com/p/gapt/`

Bibliography

[1] Peter B. Andrews. Resolution in Type Theory. *Journal of Symbolic Logic*, 36(3):414–432, September 1971.

[2] Jeremy Avigad. Algebraic proofs of cut elimination. *Journal of Logic and Algebraic Programming*, 49(1–2):15–30, 2001.

[3] M. Baaz and A. Leitsch. Towards a clausal analysis of cut-elimination. *Journal of Symbolic Computation*, 41(3-4):381–410, 2006.

[4] Matthias Baaz, Stefan Hetzl, Alexander Leitsch, Clemens Richter, and Hendrik Spohr. Cut-elimination: Experiments with ceres. In Franz Baader and Andrei Voronkov, editors, *Logic for Programming, Artificial Intelligence, and Reasoning (LPAR) 2004*, volume 3452 of *Lecture Notes in Computer Science*, pages 481–495. Springer, 2005.

[5] Matthias Baaz, Stefan Hetzl, Alexander Leitsch, Clemens Richter, and Hendrik Spohr. Proof transformation by ceres. In Jonathan M. Borwein and William M. Farmer, editors, *Mathematical Knowledge Management (MKM) 2006*, volume 4108 of *Lecture Notes in Artificial Intelligence*, pages 82–93. Springer, 2006.

[6] Matthias Baaz, Stefan Hetzl, Alexander Leitsch, Clemens Richter, and Hendrik Spohr. Ceres: An analysis of fürstenberg's proof of the infinity of primes. *Theoretical Computer Science*, 403:160–175, 2008.

[7] Matthias Baaz and Alexander Leitsch. Skolemization and proof complexity. *Fundamenta Informaticae*, 20(4):353–379, 1994.

[8] Matthias Baaz and Alexander Leitsch. Cut normal forms and proof complexity. *Annals of Pure and Applied Logic*, 97:127–177, 1999.

[9] Matthias Baaz and Alexander Leitsch. Cut-elimination and redundancy-elimination by resolution. *Journal of Symbolic Computation*, 29(2):149–176, 2000.

[10] C. Benzmüller, C. E. Brown, and M. Kohlhase. Higher-order semantics and extensionality. *Journal of Symbolic Logic*, 69(4):1027–1088, 2004.

[11] Christoph Benzmüller. Extensional higher-order paramodulation and rue-resolution. In *Automated Deduction (CADE-16)*, volume 1632 of *Lecture Notes in Computer Science*, pages 399–413. Springer Berlin, 1999.

[12] Christoph Benzmüller. Comparing approaches to resolution based higher-order theorem proving. *Synthese*, 133(1–2):203–235, October 2002.

[13] Christoph Benzmüller and Michael Kohlhase. Extensional higher-order resolution. In *Automated Deduction (CADE-15)*, pages 56–71, London, UK, 1998. Springer-Verlag.

[14] Wilfried Buchholz. Ein ausgezeichnetes modell für die intuitionistische typenlogik. *Archive for Mathematical Logic*, 17, 1975.

[15] Samuel R. Buss. The undecidability of k-provability. *Annals of Pure and Applied Logic*, 53:75–102, 1991.

[16] Domenico Cantone and Marianna Nicolosi Asmundo. A further and effective liberalization of the δ-rule in free variable semantic tableaux. In *Automated Deduction in Classical and Non-Classical Logics*, volume 1761, 2000.

[17] Alessandra Carbone. The cost of a cycle is a square. *Journal of Symbolic Logic*, 67(1):35–60, mar 2002.

[18] Alonzo Church. A formulation of the simple theory of types. *JSL*, 5(2):56–68, June 1940.

[19] Vincent Danos, Jean-Baptiste Joinet, and Harold Schellinx. A New Deconstructive Logic: Linear Logic. *Journal of Symbolic Logic*, 62(3):755–807, 1997.

[20] Gilles Dowek. Higher-order unification and matching. pages 1009–1062, 2001.

[21] Gilles Dowek, Thérèse Hardin, and Claude Kirchner. Theorem proving modulo. *Journal of Automated Reasoning*, 31(1):33–72, September 2003.

[22] Dov M. Gabbay. *Labelled Deductive Systems*, volume I of *Oxford Logic Guides*. Oxford Science Publications, 1996.

[23] Gerhard Gentzen. Untersuchungen über das logische Schließen I. *Mathematische Zeitschrift*, 39(1):176–210, dec 1935.

[24] Gerhard Gentzen. Untersuchungen über das logische Schließen II. *Mathematische Zeitschrift*, 39(1):405–431, dec 1935.

[25] Gerhard Gentzen. Die Widerspruchsfreiheit der reinen Zahlentheorie. *Mathematische Annalen*, 112:493–565, 1936.

[26] Jean-Yves Girard. *Interprétation fonctionnelle et élimination des coupures de l'arithmétique d'ordre supérieure*. PhD thesis, Université Paris 7, 1972.

[27] Jean-Yves Girard. Linear logic. *Theoretical Computer Science*, 50:1–102, 1987.

[28] Kurt Gödel. Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I. *Monatshefte für Mathematik und Physik*, 38(1):173–198, 1931.

[29] Kurt Gödel. Über eine bisher noch nicht benutzte Erweiterung des finiten Standpunktes. *Dialectica*, 12:280–287, 1958.

[30] Reiner Hähnle and Peter H. Schmitt. The liberalized δ-rule in free variable semantic tableaux. *Journal of Automated Reasoning*, 13(2):211–221, jun 1994.

[31] Stefan Hetzl. Notes on $\mathbf{LK}^{\restriction}$. Unpublished manuscript, 2007.

[32] Stefan Hetzl, Alexander Leitsch, Daniel Weller, and Bruno Woltzenlogel Paleo. CERES in second-order logic. Technical report, Vienna University of Technology, 2008.

[33] Stefan Hetzl, Alexander Leitsch, Daniel Weller, and Bruno Woltzenlogel Paleo. Herbrand sequent extraction. In Serge Autexier, John Campbell, Julio Rubio, Volker Sorge, Masakazu Suzuki, and Freek Wiedijk, editors, *Intelligent Computer Mathematics*, volume 5144 of *Lecture Notes in Computer Science*, pages 462–477. Springer Berlin, 2008.

[34] Stefan Hetzl, Alexander Leitsch, Daniel Weller, and Bruno Woltzenlogel Paleo. A clausal approach to proof analysis in second-order logic. In Sergei Artemov and Anil Nerode, editors, *Logical Foundations of Computer Science*, volume 5407 of *Lecture Notes in Computer Science*, pages 214–229. Springer Berlin, 2009.

[35] Gérard Huet. A mechanization of type theory. In *Proceedings of the Third International Joint Conference on Artificial Intelligence*, pages 139–146, 1973.

[36] Gérard Huet. A unification algorithm for typed λ-calculus. *Theoretical Computer Science*, 1(1):27–57, 1975.

[37] Gérard Huet. Higher order unification 30 years later. In *Theorem Proving in Higher Order Logics (TPHOLs) 2002*, volume 2410 of *Lecture Notes in Computer Science*, pages 241–258. Springer Berlin, 2002.

[38] D.C. Jensen and T. Pietrzykowski. Mechanizing ω-order type theory through unification. *Theoretical Computer Science*, 3(2):123–171, November 1976.

[39] Ulrich Kohlenbach. *Applied Proof Theory: Proof Interpretations and their Use in Mathematics*. Springer Berlin, 2008.

[40] M. Kohlhase. *A Mechanization of Sorted Higher-Order Logic Based on the Resolution Principle*. PhD thesis, Universität des Saarlandes, 1994.

[41] H. Luckhardt. Herbrand-Analysen zweier Beweise des Satzes von Roth: Polynomiale Anzahlschranken. *Journal of Symbolic Logic*, 54(1):234–263, March 1989.

[42] Per Martin-Löf. An intuitionistic theory of types: Predicative part. In H. E. Rose and J.C. Shepherdson, editors, *Logic Colloquium '73*, volume 80 of *Studies in Logic and the Foundations of Mathematics*, pages 73–118, 1975.

[43] Dale Miller. *Proofs in Higher-Order Logic*. PhD thesis, Carnegie Mellon University, Department of Mathematics, 1983.

[44] Dale Miller. A compact representation of proofs. *Studia Logica*, 46(4):347–370, 1987.

[45] Reinhard Muskens. Intensional models for the theory of types. *Journal of Symbolic Logic*, 72(1):98–118, 2007.

[46] Mitsuhiro Okada. A uniform semantic proof for cut-elimination and completeness of various first and higher order logics. *Theoretical Computer Science*, 281:471–498, 2002.

[47] Michel Parigot. Proofs of strong normalisation for second order classical natural deduction. *Journal of Symbolic Logic*, 62(4):1461–1479, 1997.

[48] Tomasz Pietrzykowski. A complete mechanization of second-order type theory. *Journal of the ACM*, 20(2):333–364, 1973.

[49] Dag Prawitz. Hauptsatz for higher order logic. *Journal of Symbolic Logic*, 33(3):452–457, sep 1968.

[50] Diana Ratiu and Trifon Trifonov. Exploring the computational content of the infinite pigeonhole principle. *accepted by the Journal of Logic and Computation*, 2010.

[51] J. A. Robinson. A machine-oriented logic based on the resolution principle. *Journal of the ACM*, 12(1):23–41, 1965.

[52] Kurt Schütte. Syntactical and semantical properties of simple type theory. *Journal of Symbolic Logic*, 25(4):305–326, dec 1960.

[53] Helmut Schwichtenberg. Content in proofs of list reversal. In *Marktoberdorf*, 2007.

[54] Helmut Schwichtenberg. Dialectica interpretation of well-founded induction. *Mathematical Logic Quarterly*, 54(3):229–239, 2007.

[55] Stephen G. Simpson. *Subsystems of Second-Order Arithmetic*. Springer Berlin, 1999.

[56] W. Snyder and Jean Gallier. Higher-order unification revisited: complete sets of transformations. *J. Symb. Comput.*, 8(1-2):101–140, 1989.

[57] W. W. Tait. Intensional interpretations of functionals of finite type i. *Journal of Symbolic Logic*, 32(2):198–212, jun 1967.

[58] M. Takahashi. A proof of cut-elimination theorem in simple type theory. *Journal of the Mathematical Society of Japan*, 19:399–410, 1967.

[59] Gaisi Takeuti. *Proof Theory*. North-Holland/American Elsevier, 1975.

[60] A. S. Troelstra and H. Schwichtenberg. *Basic Proof Theory*. Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, second edition, 2000.

Index

Dipl.-Ing. (FH) Daniel Weller
Schrekergasse 16/2
1160 Vienna
Austria
tel.: +43 650 2569477
mail: weller@logic.at

Curriculum Vitae Daniel Weller

Affiliation

Project Assistant at the University of Technology Vienna,
Theory and Logic Group (E185-2),
Project “Proof Theoretic Applications of the CERES Method”,
supported by the Austrian Science Fund

Education

2006-	PhD studies at the University of Technology Vienna
2006/06/20	Graduated with honors, title: Dipl.-Ing. (FH)
2002-2006	Undergraduate studies “Telecommunications and System Engineering” at the University of Applied Sciences Salzburg
1993-2001	High School Sacré Cœur Pressbaum

Diploma Thesis

Title:	Automated Formal Verification of a Time-Stamping Protocol
Awards:	Received the “Würdigungspreis” of the Austrian ministry of education, science and culture Nominated for the “best of” award of the chamber of commerce Salzburg

Publications

D. Weller. *On Quantifier-Free Cuts in First-Order Logic*, submitted for publication, 2010

T. Dunchev, A. Leitsch, T. Libal, D. Weller, B. Woltzenlogel Paleo. *System Description: The Proof Transformation System CERES*, Int. Joint Conference on Automated Reasoning 2010, R. Goebel et al. (eds.), Springer LNCS 6173, 2010

S. Hetzl, A. Leitsch, T. Libal, D. Weller, B. Woltzenlogel Paleo. *Resolution Refinements for Cut-Elimination based on Reductive Methods*, ESSLLI Workshop on Structures and Deduction, 2009

S. Hetzl, A. Leitsch, D. Weller, B. Woltzenlogel Paleo. *A clausal approach to proof analysis in second-order logic*, Logical Foundations of Computer Science 2009, Sergei Artemov and Anil Nerode (eds.), Springer LNCS 5407, 2009

S. Hetzl, A. Leitsch, D. Weller, B. Woltzenlogel Paleo. *Transforming and Analyzing Proofs in the* CERES-*system*, Proc. of the LPAR 2008 Workshops, Piotr Rudnicki et al. (eds.), ISSN 1613-0073, 2008

S. Hetzl, A. Leitsch, D. Weller, B. Woltzenlogel Paleo. *Herbrand Sequent Extraction*, Intelligent Computer Mathematics, S. Autexier et al. (eds.), Springer LNAI 5144, 2008

S. Hetzl, A. Leitsch, D. Weller, B. Woltzenlogel Paleo. *Proof Analysis with* HLK, CERES *and* ProofTool*: Current Status and Future Directions*, Proc. of CICM Workshop ESARM'08, G. Sutcliffe et al. (eds.), ISSN 1613-0073, 2008

S. Hetzl, A. Leitsch, D. Weller, B. Woltzenlogel Paleo. *CERES in Second-Order Logic*, Technical Report, Vienna University of Technology, 2008

Scientific Talks

Research Talks

Towards CERES in Higher-Order Logic, Talk at the Laboratoire d'Informatique de l'Ecole Polytechnique, France, 2010

Proof Skolemization and De-Skolemization, Workshop on Logic and Computation, Vienna, Austria, 2009

CERES: a program for cut-elimination, Kurt Gödel Research Center, Vienna, Austria, 2009

Transforming and Analyzing Proofs in the CERES-*system*, KEAPPA 2008, Doha, Qatar, 2008

Cut-Elimination by Resolution and Skolemization in Second-Order Logic, APS 2008, Doha, Qatar, 2008

Skolemization of Sequent Calculus Proofs in Higher-Order Logic (Work in Progress), Structural Proof Theory 2008, Paris, France, 2008

Implementing CERES: tools for proof analysis, Collegium Logicum 2007, Vienna, Austria, 2007

Summer School Lectures

From Cut-elimination by Reduction to Cut-elimination by Resolution. Fifth International Tbilisi Summer School in Logic and Language, Tbilisi, Georgia, 2009

Proof Analysis. Sixth International Tbilisi Summer School in Logic and Language, Svaneti, Georgia, 2010

Participation in Scientific Organisations

2010-	Deputy Publicity Chair of the Kurt Gödel Society

Industry Projects

2006	Design and implementation of software for the creation and administration of cryptographic certificates
2006	Design and implementation of an LDAP directory for X.509 certificates
2005	Design and implementation of a time-stamping service

Personal information

Nationality:	Austria
Date of birth:	1983/06/12
Language skills:	English, German
Hobbies:	Music, Basketball, Software development