CERES in Higher-Order Logic

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Abstract

We define a generalization CERES$^\omega$ of the first-order cut-elimination method CERES to higher-order logic. At the core of CERES$^\omega$ lies the computation of an (unsatisfiable) set of sequents $\mathrm{CS}(\pi)$ (the *characteristic sequent set*) from a proof π of a sequent S. A refutation of $\mathrm{CS}(\pi)$ in a higher-order resolution calculus can be used to transform cut-free parts of π (the *proof projections*) into a cut-free proof of S. An example illustrates the method and shows that CERES$^\omega$ can produce meaningful cut-free proofs in mathematics that traditional cut-elimination methods cannot reach.

1. Introduction

Proof analysis is a central mathematical activity which proved crucial to the development of mathematics. Indeed many mathematical concepts such as the notion of group or the notion of probability were introduced by analyzing existing arguments. In some sense the analysis and synthesis of proofs form the very core of mathematical progress [19].

Cut-elimination introduced by Gentzen [11] is the most prominent form of proof transformation in logic and plays a key role in automating the analysis of mathematical proofs. The removal of cuts corresponds to the elimination of intermediate statements (lemmas) from proofs, resulting in a purely combinatorial proof.

In a formal sense Girard's analysis of van der Waerden's theorem [13] is the application of cut-elimination to the (topological) proof of Fürstenberg/Weiss with the "perspective" of obtaining van der Waerden's (combinatorial) proof. Naturally, an application of a complex proof transformation like cut-elimination by humans requires a goal oriented strategy.

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The development of the method CERES (cut-elimination by resolution) was inspired by the idea to fully automate cut-elimination on real mathematical proofs, with the aim of obtaining new interesting elementary proofs. While a fully automated treatment proved successful for mathematical proofs of moderate complexity (e.g. the tape proof [3] and the lattice proof [15]), more complex mathematical proofs required an interactive use of CERES; this way we successfully analyzed Füstenberg's proof of the infinitude of primes (see [4] and [1]) and obtained Euclid's argument of prime construction. Even in its interactive use CERES proved to be superior to reductive cut-elimination due to additional structural information given by the *characteristic clause set* (see below).

So far the CERES-method was defined within first-order logic. This made the analysis of Fürstenberg's proof of the infinitude of primes rather problematic. In fact the problem could not be formalized as a single proof but only as an infinite schema of proofs. On the other hand it is shown in [4] that the proof can be formalized in second-order arithmetic in a simple and natural way. As higher-order logic is quite close to mathematical practice, the extension of CERES to higher-order logic became a matter of major importance. An extension to a (relatively small) subclass of second-order logic was given in [16].

In this paper we define an extension of CERES to higher-order logic. Some features of the CERES-method like proof Skolemization do not carry over to higher-order, while others (like proof projection) become much more complicated. To overcome the Skolemization problem we define a calculus $\mathbf{LK}_{\mathrm{sk}}$, where eigenvariables are replaced by Skolem terms (this technique can be also found in [17]). The proof projections become proofs which may be locally unsound (due to violations of eigenvariable conditions), but fulfill some global soundness properties. It is shown that, by the global soundness property, a transformation into an ordinary **LK**-proof is possible. The underlying resolution calculus is a restricted variant of Andrews' higher-order resolution calculus (see [2]), where only atomic simplification is admitted. Despite the complicated behavior of CERES in higher-order logic, the characteristic sequent set $\mathrm{CS}(\pi)$ for a proof π of a sequent S remains the major advantage of the method. Roughly speaking, the problem of finding a cut-free proof of S is reduced to finding a resolution refutation of $\mathrm{CS}(\pi)$. In general, it is easier to refute $\mathrm{CS}(\pi)$ than to prove S directly in a cut-free way. Hence CERES can be seen as a "semi-semantic" method of cut-elimination. Furthermore, CERES can find more cut-free proofs of S than can be found by application of Gentzen-style proof reduction rules.

The method is demonstrated by transforming a proof in second-order arithmetic using order induction into another one using the least number principle. The proof transformation is achieved by cut-elimination on the second-order induction axiom. The analysis by CERES^{ω} also shows that a solution can be found which cannot be obtained by the reductive Gentzen method.

2. Preliminaries

We work in a version of Church's simple theory of types [8], using the base types ι, o of individuals and booleans, respectively. The only binding operator in our language is λ, and we assume logical constants $\vee_{o\to o\to o}$, $\wedge_{o\to o\to o}$, $\to_{o\to o\to o}$, $\neg_{o\to o}$, $\forall_{(\alpha\to o)\to o}$, and $\exists_{(\alpha\to o)\to o}$ for all types α. As metavariables for terms we use $\mathbf{T},\mathbf{S},\mathbf{R},\ldots$, for variables we use $\mathbf{X},\mathbf{Y},\mathbf{Z},\ldots$, for formulas we use $\mathbf{F},\mathbf{G},\mathbf{H},\ldots$, and for lists of formulas we use $\Gamma,\Delta,\Lambda,\Pi,\ldots$ (all possibly with subscripts). We will not provide type information if it can be inferred from the context. Terms of type o are called *formulas*. If the uppermost symbol of a formula $\mathbf{F}$ is not one of the logical constants, then $\mathbf{F}$ is called *atomic*. We consider terms only modulo α-equality, i.e. modulo renaming of bound variables. If $\mathbf{T},\mathbf{S}$ are terms, then we write $\mathbf{T} > \mathbf{S}$ if $\mathbf{S}$ is a proper subterm of $\mathbf{T}$ (i.e. $\mathbf{S}$ is a subterm of $\mathbf{T}$ and $\mathbf{T} \neq \mathbf{S}$).

Our terms will contain Skolem symbols (i.e. function symbols to be introduced by Skolemization). To obtain sound proof systems, we will need to restrict the terms that can be used: we follow the approach of Miller [17], who provides a precise definition of such a restriction.

2.1. Sequent calculus

A *sequent* is a pair of lists of formulas, written $\Gamma \vdash \Delta$. While we define sequents as lists to be able to define occurrences in sequents and proofs, we will treat them as multisets most of the time. Hence we do not explicitly include exchange or permutation rules in our calculi. For simplicity, we restrict ourselves to prooftrees in which all formulas are in β-normal form. Hence we note that the quantifier rules below include an implicit β-reduction.

Definition 1 (LK rules and proofs). The following figures are the rules of **LK**:

Propositional rules:

$$\frac{\Gamma \vdash \Delta, \mathbf{F}}{\neg\mathbf{F}, \Gamma \vdash \Delta}\ \neg\colon l \qquad \frac{\mathbf{F}, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg\mathbf{F}}\ \neg\colon r$$

$$\frac{\mathbf{F}, \Gamma \vdash \Delta \quad \mathbf{G}, \Pi \vdash \Lambda}{\mathbf{F}\vee\mathbf{G}, \Gamma, \Pi \vdash \Delta, \Lambda}\ \vee\colon l \qquad \frac{\Gamma \vdash \Delta, \mathbf{F}}{\Gamma \vdash \Delta, \mathbf{F}\vee\mathbf{G}}\ \vee\colon r^1 \qquad \frac{\Gamma \vdash \Delta, \mathbf{G}}{\Gamma \vdash \Delta, \mathbf{F}\vee\mathbf{G}}\ \vee\colon r^2$$

$$\frac{\Gamma \vdash \Delta, \mathbf{F} \quad \Pi \vdash \Lambda, \mathbf{G}}{\Gamma, \Pi \vdash \Delta, \Lambda, \mathbf{F}\wedge\mathbf{G}}\ \wedge\colon r \qquad \frac{\mathbf{F}, \Gamma \vdash \Delta}{\mathbf{F}\wedge\mathbf{G}, \Gamma \vdash \Delta}\ \wedge\colon l^1 \qquad \frac{\mathbf{G}, \Gamma \vdash \Delta}{\mathbf{F}\wedge\mathbf{G}, \Gamma \vdash \Delta}\ \wedge\colon l^2$$

$$\frac{\Gamma \vdash \Delta, \mathbf{F} \quad \mathbf{G}, \Pi \vdash \Lambda}{\mathbf{F}\to\mathbf{G}, \Gamma, \Pi \vdash \Delta, \Lambda}\ \to\colon l \qquad \frac{\mathbf{F}, \Gamma \vdash \Delta, \mathbf{G}}{\Gamma \vdash \Delta, \mathbf{F}\to\mathbf{G}}\ \to\colon r$$

Structural rules:

$$\frac{\Gamma \vdash \Delta, \mathbf{F}, \mathbf{F}}{\Gamma \vdash \Delta, \mathbf{F}}\ \mathrm{contr}\colon r \qquad \frac{\mathbf{F}, \mathbf{F}, \Gamma \vdash \Delta}{\mathbf{F}, \Gamma \vdash \Delta}\ \mathrm{contr}\colon l$$

$$\frac{\Gamma \vdash \Delta}{\mathbf{F}, \Gamma \vdash \Delta}\ \mathrm{weak}\colon l \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \mathbf{F}}\ \mathrm{weak}\colon r \qquad \frac{\Gamma \vdash \Delta, \mathbf{F} \quad \mathbf{F}, \Pi \vdash \Lambda}{\Gamma, \Pi \vdash \Delta, \Lambda}\ \mathrm{cut}$$

Quantifier rules:

$$\frac{\mathbf{RT}, \Gamma \vdash \Delta}{\forall\mathbf{R}, \Gamma \vdash \Delta}\ \forall\colon l \qquad \frac{\Gamma \vdash \Delta, \mathbf{RX}}{\Gamma \vdash \Delta, \forall\mathbf{R}}\ \forall\colon r$$

$$\frac{\Gamma \vdash \Delta, \mathbf{RT}}{\Gamma \vdash \Delta, \exists \mathbf{R}}\ \exists\colon r \qquad \frac{\mathbf{RX}, \Gamma \vdash \Delta}{\exists \mathbf{R}, \Gamma \vdash \Delta}\ \exists\colon l$$

The $\forall\colon r$ and $\exists\colon l$ are called *strong quantifier rules*. In the strong quantifier rules, $\mathbf{X}$ must not occur free in $\Gamma, \Delta, \mathbf{R}$. $\mathbf{X}$ is called the *eigenvariable* of this rule. In $\forall\colon l$ and $\exists\colon r$, $\mathbf{T}$ is called the *substitution term* of the rule.

An **LK**-*proof* is a tree formed according to the rules of **LK** such that all leaves are of the form $\mathbf{F} \vdash \mathbf{F}$. The formulas in $\Gamma, \Delta, \Pi, \Lambda$ are called *context formulas*. The formulas in the upper sequents which are not context formulas are called *auxiliary formulas*, those in the lower sequents are called *main formulas*. The auxiliary formulas of a cut rule are also called *cut formulas*. If π is an **LK**-proof, by $|\pi|$ we denote the number of sequent occurrences in π.

If $\mathcal{S}$ is a set of sequents, then an **LK**-*refutation of* $\mathcal{S}$ is an **LK**-tree π where the end-sequent of π is the empty sequent, and the leaves of π are either axioms $\mathbf{F} \vdash \mathbf{F}$ or sequents in $\mathcal{S}$.

A formula *occurrence* in a sequent or prooftree is a formula together with its position in the sequent or prooftree. Formula occurrences in prooftrees come equipped with an *ancestor* and *descendent* relation which is defined in the usual way (see [6]). An *end-sequent ancestor (cut ancestor)* is an ancestor of a formula in the end-sequent (a cut formula). An inference ρ in a prooftree is said to *operate* on an occurrence ω if ω is an auxiliary or main formula of ρ. An **LK**-proof π is called *regular* if for all $\forall\colon r$ inferences ρ with eigenvariable $\mathbf{X}$ in π, $\mathbf{X}$ only occurs in the subproof ending in ρ. It is well-known that every **LK**-proof of a closed sequent S can be transformed into a regular **LK**-proof of S by renaming eigenvariables.

Recall the system $\mathcal{T}$ introduced in [8] and used in [2]. $\mathcal{T}$ is a Hilbert-type system for higher-order logic. Using the well-known transformations from sequent calculi to Hilbert-type systems (see [12, 20]), one can prove a relative soundness result. If $S = \Gamma \vdash \Delta$ is a sequent, then $F(S) = \bigwedge \Gamma \to \bigvee \Delta$. If $\mathcal{S}$ is a set of sequents, then $F(\mathcal{S}) = \{F(S) \mid S \in \mathcal{S}\}$.

Proposition 1. *If there exists an* **LK**-*refutation of* $\mathcal{S}$*, then there exists a* $\mathcal{T}$-*refutation of* $F(\mathcal{S})$.

3. CERES

The CERES method in first-order logic is defined via two crucial structures: the *characteristic clause set* $\mathrm{CL}(\pi)$, and the *proof projections* $\mathcal{P}(\pi)$ of some proof π of S with arbitrary cuts. The proof projections are cut-free parts of π. One can show that $\mathrm{CL}(\pi)$ is always unsatisfiable. The main transformation of CERES is to combine a resolution refutation of $\mathrm{CL}(\pi)$ and the cut-free proofs from $\mathcal{P}(\pi)$ into a proof of S with at most atomic cuts.

In first-order logic, CERES is restricted to proofs of Skolemized sequents, i.e. sequents not containing $\forall$ in a positive or $\exists$ in a negative context. This is justified by the following well-known proposition:

Proposition 2. *For every first-order sequent S there exists a Skolemized sequent S' such that S is provable iff S' is.*

Furthermore, constructive proofs of this proposition are known (see e.g. [5]). The fact about proofs of Skolemized sequents most important to the CERES method is that inferences with eigenvariable conditions only operate on cut-ancestors:

Proposition 3. *Let π be a first-order* **LK***-proof of a Skolemized sequents S. Then there does not exist a strong quantifier inference in π that operates on an end-sequent ancestor.*

In higher-order logic, this does not hold anymore. Furthermore, it seems that proof Skolemization used in Proposition 2 cannot be generalized to yield **LK**-proofs fulfilling Proposition 3, see [14]. For example, the following **LK**-proof proves a sequent that does not contain strong quantifiers, but the proof contains a strong quantifier inference:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(\beta,a)\vdash P(\beta,a)}{(\forall x)P(x,a)\vdash P(\beta,a)}\,\forall\colon l}{(\forall x)P(x,a)\vdash(\forall z)P(z,a)}\,\forall\colon r \qquad \dfrac{P(c,b)\vdash P(c,b)}{(\forall z)P(z,b)\vdash P(c,b)}\,\forall\colon l}{(\forall x)P(x,a),(\forall z)P(z,a)\to(\forall z)P(z,b)\vdash P(c,b)}\,\to\colon l}{(\forall x)P(x,a),(\forall X)(X(a)\to X(b))\vdash P(c,b)}\,\forall\colon l\ \lambda x.(\forall z)P(z,x)}{(\forall X)(X(a)\to X(b))\vdash(\forall x)P(x,a)\to P(c,b)}\,\to\colon r$$

Note that the auxiliary formula of the lowermost $\forall\colon l$ inference can not be Skolemized. For this reason, we now introduce a sequent calculus without eigenvariable conditions.

4. The calculus $\mathbf{LK_{sk}}$

Definition 2 (Labelled sequents). A *label* is a finite multiset of terms. A *labelled sequent* is a sequent $\mathbf{F}_1,\ldots,\mathbf{F}_n\vdash\mathbf{F}_{n+1},\ldots,\mathbf{F}_m$ together with labels ℓ_i for $1\le i\le m$; we write $\langle\mathbf{F}_1\rangle^{\ell_1},\ldots,\langle\mathbf{F}_n\rangle^{\ell_n}\vdash\langle\mathbf{F}_{n+1}\rangle^{\ell_{n+1}},\ldots,\langle\mathbf{F}_m\rangle^{\ell_m}$. We identify labelled formulas with empty labels with the respective unlabelled formulas. If S is a labelled sequent, then the *reduct* of S is S where all labels are empty. If $\mathcal{C}$ is a set of labelled sequents, then the reduct of $\mathcal{C}$ is $\{S\mid S$ a reduct of some $S'\in\mathcal{C}\}$.

We extend substitutions to labelled sequents: Let σ be a substitution and $S=\langle\mathbf{F}_1\rangle^{\ell_1},\ldots,\langle\mathbf{F}_n\rangle^{\ell_n}\vdash\langle\mathbf{F}_{n+1}\rangle^{\ell_{n+1}},\ldots,\langle\mathbf{F}_m\rangle^{\ell_m}$, then

$$S\sigma=\langle\mathbf{F}_1\sigma\rangle^{\ell_1\sigma},\ldots,\langle\mathbf{F}_n\sigma\rangle^{\ell_n\sigma}\vdash\langle\mathbf{F}_{n+1}\sigma\rangle^{\ell_{n+1}\sigma},\ldots,\langle\mathbf{F}_m\sigma\rangle^{\ell_m\sigma}.$$

Labels such as ours are often used to add (syntactic) information to formulas, see [10]. They have been used in a setting very similar to ours in [9].

The purpose of the labels will be twofold: first, they will track quantifier instantiation information throughout prooftrees (as expressed in Proposition 4).

Second, they will enable us to combine resolution refutations and sequent calculus proofs in a certain way — this will be one of the main constructions of the CERES$^\omega$ method; see Lemma 3.

From now on, we will only consider labelled sequents, and therefore we will call them only sequents. Analogously, we will refer to labelled formula occurrences as formula occurrences. We will denote the union of labels ℓ_1 and ℓ_2 by ℓ_1, ℓ_2. Let $\mathbf{T}$ be a term and ℓ a label, then we denote by $\ell, \mathbf{T}$ the union $\ell \cup \{\mathbf{T}\}$.

Definition 3 ($\mathbf{LK}_{\mathrm{sk}}$ rules). The following figures are the rules of $\mathbf{LK}_{\mathrm{sk}}$:
Labelled quantifier rules:

$$\frac{\Gamma \vdash \Delta, \langle \mathbf{F}(\mathbf{f}\mathbf{S}_1 \ldots \mathbf{S}_n)\rangle^\ell}{\Gamma \vdash \Delta, \langle \forall_\alpha \mathbf{F}\rangle^\ell} \; \forall^{sk}\colon r$$

where $\ell = \mathbf{S}_1, \ldots, \mathbf{S}_n$ and, if $\tau(\mathbf{S}_i) = \alpha_i$ for $1 \leq i \leq n$, then $\mathbf{f} \in \mathcal{K}_{\alpha_1,\ldots,\alpha_n,\alpha}$ is a Skolem symbol. An application of this rule is called *source inference* of $\mathbf{f}\mathbf{S}_1 \ldots \mathbf{S}_m$, and $\mathbf{f}\mathbf{S}_1 \ldots \mathbf{S}_m$ is called the *Skolem term* of this inference. Note that we do *not* impose an eigenvariable or eigenterm restriction on this rule.

$$\frac{\langle \mathbf{F}\mathbf{T}\rangle^{\ell,\mathbf{T}}, \Gamma \vdash \Delta}{\langle \forall_\alpha \mathbf{F}\rangle^\ell, \Gamma \vdash \Delta} \; \forall^{sk}\colon l$$

$\mathbf{T}$ is called the *substitution term* of this inference. The $\exists^{sk}\colon l$ and $\exists^{sk}\colon r$ rules are defined analogously. The $\forall^{sk}\colon r$ and $\exists^{sk}\colon l$ rules will be called *strong labelled quantifier rules*, and the $\forall^{sk}\colon l$ and $\exists^{sk}\colon r$ will be called *weak labelled quantifier rules.* The other rules of $\mathbf{LK}$ are transferred directly to $\mathbf{LK}_{\mathrm{sk}}$:
Propositional rules:

$$\frac{\langle \mathbf{F}\rangle^\ell, \Gamma \vdash \Delta \quad \langle \mathbf{G}\rangle^\ell, \Pi \vdash \Lambda}{\langle \mathbf{F} \vee \mathbf{G}\rangle^\ell, \Gamma, \Pi \vdash \Delta, \Lambda} \; \vee\colon l \qquad \frac{\Gamma \vdash \Delta, \langle \mathbf{F}\rangle^\ell}{\Gamma \vdash \Delta, \langle \mathbf{F} \vee \mathbf{G}\rangle^\ell} \; \vee\colon r^1$$

The rest of the propositional rules of $\mathbf{LK}$ are adapted analogously.
Structural rules:

$$\frac{\Gamma \vdash \Delta, \langle \mathbf{F}\rangle^\ell, \langle \mathbf{F}\rangle^\ell}{\Gamma \vdash \Delta, \langle \mathbf{F}\rangle^\ell} \; \mathrm{contr}\colon r \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \langle \mathbf{F}\rangle^\ell} \; \mathrm{weak}\colon r$$

and analogously for contr: l and weak: l. An $\mathbf{LK}_{\mathrm{sk}}$*-tree* is a tree formed according to the rules of $\mathbf{LK}_{\mathrm{sk}}$, such that all leaves are of the form $\langle \mathbf{F}\rangle^{\ell_1} \vdash \langle \mathbf{F}\rangle^{\ell_2}$ for some formula $\mathbf{F}$ and some labels ℓ_1, ℓ_2. The *axiom partner* of $\langle \mathbf{F}\rangle^{\ell_1}$ is defined to be $\langle \mathbf{F}\rangle^{\ell_2}$, and vice-versa. Let π be an $\mathbf{LK}_{\mathrm{sk}}$-tree with end-sequent S. If S does not contain Skolem terms or free variables, and all labels in S are empty, then S is called *proper.* If the end-sequent of π is proper, we say that π is proper.

Note that $\mathbf{LK}_{\mathrm{sk}}$ is a cut-free calculus.

Example 1. The following figure shows a proper $\mathbf{LK}_{\mathrm{sk}}$-tree of a valid sequent:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\langle S(f(\lambda x.\neg S(x)))\rangle^{\lambda x.\neg S(x)} \vdash \langle S(f(\lambda x.\neg S(x)))\rangle^{\lambda x.\neg S(x)}}{\langle \neg S(f(\lambda x.\neg S(x)))\rangle^{\lambda x.\neg S(x)}, \langle S(f(\lambda x.\neg S(x)))\rangle^{\lambda x.\neg S(x)} \vdash}\ \neg\colon l}{\langle S(f(\lambda x.\neg S(x)))\rangle^{\lambda x.\neg S(x)} \vdash \langle \neg\neg S(f(\lambda x.\neg S(x)))\rangle^{\lambda x.\neg S(x)}}\ \neg\colon r}{\vdash \langle S(f(\lambda x.\neg S(x))) \to \neg\neg S(f(\lambda x.\neg S(x)))\rangle^{\lambda x.\neg S(x)}}\ \to\colon r}{\vdash \langle (\forall z)(S(z) \to \neg\neg S(z))\rangle^{\lambda x.\neg S(x)}}\ \forall^{sk}\colon r}{\vdash \langle (\exists Y)(\forall z)(S(z) \to \neg Y(z))\rangle}\ \exists^{sk}\colon r}{\vdash \langle (\forall X)(\exists Y)(\forall z)(X(z) \to \neg Y(z))\rangle}\ \forall^{sk}\colon r
$$

where $S \in \mathcal{K}_{\iota\to o}$, $f \in \mathcal{K}_{\iota\to o,\iota}$, and the substitution term of the $\exists^{sk}\colon r$ is $\lambda x.\neg S(x)$. Note that although the labels in the axiom coincide, this is not required in general.

So far, we have not called the trees built up using the rules of $\mathbf{LK}_{\mathrm{sk}}$ *proofs*. The reason is that without further restrictions, $\mathbf{LK}_{\mathrm{sk}}$-trees are unsound:

Example 2. Consider the following $\mathbf{LK}_{\mathrm{sk}}$-tree of $(\exists x)P(x) \vdash (\forall x)P(x)$:

$$
\dfrac{\dfrac{P(s) \vdash P(s)}{(\exists x)P(x) \vdash P(s)}\ \exists^{sk}\colon l}{(\exists x)P(x) \vdash (\forall x)P(x)}\ \forall^{sk}\colon r
$$

where $s \in \mathcal{K}_\iota$. The source of unsoundness in this example stems from the fact that in $\mathbf{LK}_{\mathrm{sk}}$-trees, it is possible to use the same Skolem term for distinct and "unrelated" strong quantifier inferences.

Towards introducing our global soundness condition, which will be more general than the eigenvariable condition of $\mathbf{LK}$, we introduce some definitions and facts about occurrences in $\mathbf{LK}_{\mathrm{sk}}$-trees.

Proposition 4. *Let ω be a formula occurrence in a proper $\mathbf{LK}_{\mathrm{sk}}$-tree π with label $\{\mathbf{T}_1, \ldots, \mathbf{T}_n\}$. Then $\mathbf{T}_1, \ldots, \mathbf{T}_n$ are exactly the substitution terms of the weak labelled quantifier inferences operating on descendents of ω.*

Proof. By induction on the number of sequents between ω and the end-sequent of π. If ω occurs in the end-sequent, then it has no descendents and, as π is proper, ω has the empty label.

Assume ω occurs in the premise of an inference. Denote the direct descendent of ω by ω'. If ω occurs in the context, then ω has the same label as ω', the weak labelled quantifier inferences operating on descendents of ω are the same as those operating on descendents of ω', so we conclude with the induction hypothesis. If ω is the auxiliary formula of a propositional inference, a contraction inference, or a strong labelled quantifier inference, the argument is analogous. Finally, assume ω is the auxiliary formula of a weak labelled quantifier inference ρ with substitution term $\mathbf{T}$, and that the label of ω is $\mathbf{T}_1, \ldots, \mathbf{T}_n, \mathbf{T}$. Then the label of ω' is $\mathbf{T}_1, \ldots, \mathbf{T}_n$, and by (IH) these are exactly the substitution terms of the weak labelled quantifier inferences $\rho_1, \ldots, \rho_n$ operating on descendents of ω'. Then the weak labelled quantifier inferences operating on descendents of ω are $\rho_1, \ldots, \rho_n, \rho$, and hence the label of ω is as desired. $\square$

Definition 4 (Paths). Let μ be a sequence of formula occurrences $\mu_1, \ldots, \mu_n$ in an $\mathbf{LK}_{\mathrm{sk}}$-tree. If for all $1 \leq i < n$, μ_i is an immediate ancestor (immediate descendent) of μ_{i+1}, then μ is called a *downwards (upwards) path*. If μ is a downwards (upwards) path ending in an occurrence in the end-sequent (a leaf), then μ is called *maximal*.

Definition 5 (Homomorphic paths). If ω is a formula occurrence, then denote by $F(\omega)$ the formula at ω. If μ is a sequence of formula occurrences, we define $F(\mu)$ as μ where every formula occurrence ω is replaced by $F(\omega)$, and repetitions are ommited. Two sequences of formula occurrences μ, ν are called *homomorphic* if $F(\mu) = F(\nu)$.

Example 3. Consider the $\mathbf{LK}_{\mathrm{sk}}$-tree π:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\langle R(a,f(a))\rangle^a \vdash \langle R(a,f(a))\rangle^a}{\vdash \langle R(a,f(a))\rangle^a, \langle \neg R(a,f(a))\rangle^a}\ \neg\colon r}{\vdash \langle R(a,f(a))\rangle^a, \langle R(a,f(a)) \lor \neg R(a,f(a))\rangle^a}\ \lor\colon r^2}{\vdash \langle R(a,f(a)) \lor \neg R(a,f(a))\rangle^a, \langle R(a,f(a)) \lor \neg R(a,f(a))\rangle^a}\ \lor\colon r^1}{\vdash \langle R(a,f(a)) \lor \neg R(a,f(a))\rangle^a}\ \mathrm{contr}\colon r}{\vdash \langle (\forall y)(R(a,y) \lor \neg R(a,y))\rangle^a}\ \forall^{sk}\colon r}{\vdash (\exists x)(\forall y)(R(x,y) \lor \neg R(x,y))}\ \exists^{sk}\colon r}{}$$

π contains the following maximal downwards paths μ_1, μ_2:

$$\begin{aligned}
\mu_1 = \quad & \langle R(a,f(a))\rangle^a, \langle \neg R(a,f(a))\rangle^a, \langle R(a,f(a)) \lor \neg R(a,f(a))\rangle^a, \\
& \langle R(a,f(a)) \lor \neg R(a,f(a))\rangle^a, \langle R(a,f(a)) \lor \neg R(a,f(a))\rangle^a, \\
& \langle (\forall y)(R(a,y) \lor \neg R(a,y))\rangle^a, (\exists x)(\forall y)(R(x,y) \lor \neg R(x,y)) \\[1ex]
\mu_2 = \quad & \langle R(a,f(a))\rangle^a, \langle R(a,f(a))\rangle^a, \langle R(a,f(a))\rangle^a, \\
& \langle R(a,f(a)) \lor \neg R(a,f(a))\rangle^a, \langle R(a,f(a)) \lor \neg R(a,f(a))\rangle^a, \\
& \langle (\forall y)(R(a,y) \lor \neg R(a,y))\rangle^a, (\exists x)(\forall y)(R(x,y) \lor \neg R(x,y)) \\[1ex]
F(\mu_1) = \quad & \langle R(a,f(a))\rangle^a, \langle \neg R(a,f(a))\rangle^a, \langle R(a,f(a)) \lor \neg R(a,f(a))\rangle^a, \\
& \langle (\forall y)(R(a,y) \lor \neg R(a,y))\rangle^a, (\exists x)(\forall y)(R(x,y) \lor \neg R(x,y)) \\[1ex]
F(\mu_2) = \quad & \langle R(a,f(a))\rangle^a, \langle R(a,f(a)) \lor \neg R(a,f(a))\rangle^a, \\
& \langle (\forall y)(R(a,y) \lor \neg R(a,y))\rangle^a, (\exists x)(\forall y)(R(x,y) \lor \neg R(x,y))
\end{aligned}$$

Proposition 5. *Let π be a proper $\mathbf{LK}_{\mathrm{sk}}$-tree, let ρ be a strong labelled quantifier inference in π with Skolem term $\mathbf{S}$ and auxiliary formula α, and let μ be a maximal downwards path starting at α. Then* $\mathrm{FV}(\mathbf{S}) = \mathrm{FV}(\mu)$.

Proof. As π is proper, its end-sequent does not contain free variables. Hence all free variables in μ are contained in substitution terms of weak labelled quantifier inferences, and they are exactly the free variables of $\mathbf{S}$ by Proposition 4. $\square$

Proposition 6. *Let α_1, α_2 be formula occurrences. If there exists a downwards path from α_1 to α_2, then it is unique.*

Proof. Every formula occurrence has at most one direct descendent. $\square$

Corollary 1. *If α is a formula occurrence, then there exists a unique maximal downwards path starting at α.*

Our investigation of paths allows us to define a relation between inferences in a tree that, through paths, are connected in a strong sense.

Definition 6 (Homomorphic inferences). Let α_1, α_2 be formula occurrences in an $\mathbf{LK}_{\mathrm{sk}}$-tree π. Let c be a contraction inference below both α_1, α_2 with auxiliary occurrences γ_1, γ_2. Then α_1, α_2 are *homomorphic in* c if the downwards paths $\alpha_1, \ldots, \gamma_1$ and $\alpha_2, \ldots, \gamma_2$ exist and are homomorphic. α_1, α_2 are called *homomorphic* if there exists a c such that they are homomorphic in c.

Let ρ_1, ρ_2 be inferences of the same type with auxiliary formula occurrences α_1^1 (α_1^2) and α_2^1 (α_2^2). ρ_1, ρ_2 are called *homomorphic* if there exists a contraction inference c such that α_1^1 and α_2^1 are homomorphic in c and α_1^2 and α_2^2 are homomorphic in c. Call this contraction inference the *uniting contraction* of ρ_1, ρ_2.

Example 4. Consider the following $\mathbf{LK}_{\mathrm{sk}}$-tree π:

$$
\dfrac{
\dfrac{\dfrac{\langle P(s)\rangle^{s} \vdash P(s)}{(\forall x)P(x) \vdash P(s)}\ \forall^{sk}\colon l}{(\forall x)P(x) \vdash (\forall x)P(x)}\ \forall^{sk}\colon r\ (1)
\qquad
\dfrac{\dfrac{\langle P(s)\rangle^{s} \vdash P(s)}{\langle P(s)\rangle^{s} \vdash (\forall x)P(x)}\ \forall^{sk}\colon r\ (3)}{(\forall x)P(x) \vdash (\forall x)P(x)}\ \forall^{sk}\colon l
}{
\dfrac{(\forall x)P(x) \vee (\forall x)P(x) \vdash (\forall x)P(x), (\forall x)P(x)}{(\forall x)P(x) \vee (\forall x)P(x) \vdash (\forall x)P(x)}\ \mathrm{contr}\colon r\ (2)
}\ \vee\colon l
$$

The inferences $(1), (3)$ in π are homomorphic, and (2) is their uniting contraction. More concretely, let μ be the path from the auxiliary formula of (1) to the auxiliary formula of (2). Let ν be the path from the auxiliary formula of (3) to the auxiliary formula of (2). Then $F(\mu) = P(s), (\forall x)P(x) = F(\nu)$.

On the other hand, consider π':

$$
\dfrac{
\dfrac{\dfrac{\langle P(s_1)\rangle^{s_1} \vdash P(s_1)}{(\forall x)P(x) \vdash P(s_1)}\ \forall^{sk}\colon l}{(\forall x)P(x) \vdash (\forall x)P(x)}\ \forall^{sk}\colon r\ (1)
\qquad
\dfrac{\dfrac{\langle P(s_2)\rangle^{s_2} \vdash P(s_2)}{\langle P(s_2)\rangle^{s_2} \vdash (\forall x)P(x)}\ \forall^{sk}\colon r\ (3)}{(\forall x)P(x) \vdash (\forall x)P(x)}\ \forall^{sk}\colon l
}{
\dfrac{(\forall x)P(x) \vee (\forall x)P(x) \vdash (\forall x)P(x), (\forall x)P(x)}{(\forall x)P(x) \vee (\forall x)P(x) \vdash (\forall x)P(x)}\ \mathrm{contr}\colon r\ (2)
}\ \vee\colon l
$$

In π', there are no homomorphic inferences because the auxiliary formulas of the $\forall^{sk}: r$ applications differ: Define μ, ν as above, then $F(\mu) = P(s_1), (\forall x)P(x) \neq P(s_2), (\forall x)P(x) = F(\nu)$.

The previous example motivates the following statement about homomorphic quantifier inferences.

Proposition 7. *If two strong labelled quantifier inferences are homomorphic, they have identical Skolem terms.*

Proof. Denote the two strong labelled quantifier inferences applications by ρ_1, ρ_2. Then there exist homomorphic paths p_1, p_2 starting at the auxiliary formulas of ρ_1, ρ_2 respectively. The second elements of p_1, p_2 are the main formula occurrences of ρ_1, ρ_2 respectively. As p_1, p_2 are homomorphic the formula lists induced by them are equal, therefore ρ_1, ρ_2 have the same auxiliary and main formulas and therefore their Skolem terms are identical. □

Proposition 8. *The homomorphism relation on inferences is a partial equivalence relation.*

Proof. The homomorphism relation on inferences is symmetric because the homomorphism relation on sequences of formula occurrences is. It is transitive: Assume ρ_1, ρ_2 are homomorphic, and ρ_2, ρ_3 are homomorphic. We assume that ρ_1, ρ_2, ρ_3 are unary inferences, the binary case is analogous. Designate the respective auxiliary formulas by $\alpha_1, \alpha_2, \alpha_3$. Then there is a contraction c on formula occurrences γ_1, γ_2 s.t. the downwards paths $\alpha_1, \ldots, \gamma_1$ and $\alpha_2, \ldots, \gamma_2$ exist and are homomorphic, and there is a contraction c' on formula occurrences γ_2', γ_3 s.t. the paths $\alpha_2, \ldots, \gamma_2'$ and $\alpha_3, \ldots, \gamma_3$ exist and are homomorphic. From the existence of these paths, it follows that c, c' cannot be parallel. W.l.o.g. assume that c is above c', then

$$\alpha_2, \ldots, \gamma_2' = \alpha_2, \ldots, \gamma_2, \gamma_2^*, \ldots, \gamma_2'$$

by Proposition 6, and there exists a path

$$\alpha_1, \ldots, \gamma_1, \gamma_2^*, \ldots, \gamma_2'.$$

For $i \in \{1, 2\}$, let ω_i be the first formula occurrence from the right in $\alpha_i, \ldots, \gamma_i$ such that $F(\omega_i) \neq F(\gamma_i)$, ρ_1, ρ_3 are homomorphic by the following chain of equalities:

$$\begin{array}{l} F(\alpha_1, \ldots, \gamma_1, \gamma_2^*, \ldots, \gamma_2') = \\ F(\alpha_1, \ldots, \omega_1), F(\gamma_2^*, \ldots, \gamma_2') = \\ F(\alpha_2, \ldots, \omega_2), F(\gamma_2^*, \ldots, \gamma_2') = \\ F(\alpha_2, \ldots, \gamma_2, \ldots, \gamma_2') = \\ F(\alpha_3, \ldots, \gamma_3) \end{array}$$

□

We can now define the notion of an $\mathbf{LK}_{\mathrm{sk}}$-proof, for which we will require the converse of the Proposition 7 to hold.

Definition 7 (Weak regularity and $\mathbf{LK}_{\mathrm{sk}}$-proofs). Let π be an $\mathbf{LK}_{\mathrm{sk}}$-tree with end-sequent S. π is *weakly regular* if for all distinct strong labelled quantifier inferences ρ_1, ρ_2 in π: If ρ_1, ρ_2 have identical Skolem terms, then ρ_1, ρ_2 are homomorphic. We say that π is an $\mathbf{LK}_{\mathrm{sk}}$*-proof* if it is weakly regular and proper.

In ordinary **LK**, it follows directly from the definition of regularity that all strong quantifier inferences in a regular **LK**-tree π fulfill the eigenvariable condition, and thus are **LK**-proofs. Hence the name "weak regularity": inferences are allowed to use the same eigenterm, provided they are homomorphic.

Example 5. The $\mathbf{LK}_{\mathrm{sk}}$-tree from Example 1 is (trivially) an $\mathbf{LK}_{\mathrm{sk}}$-proof. Also the first $\mathbf{LK}_{\mathrm{sk}}$-tree from Example 4 is an $\mathbf{LK}_{\mathrm{sk}}$-proof: the only two strong labelled quantifier applications in the tree are homomorphic.

Finally, consider the following example:

$$\dfrac{\langle R(s,f(s))\rangle^{s} \vdash \langle R(s,f(s))\rangle^{f(s)}}{\langle(\exists y)R(s,y)\rangle^{s} \vdash \langle R(s,f(s))\rangle^{f(s)}}\ \exists^{sk}: l \qquad \dfrac{\dfrac{\langle R(s,f(s))\rangle^{s} \vdash \langle R(s,f(s))\rangle^{f(s)}}{\langle(\exists y)R(s,y)\rangle^{s} \vdash \langle R(s,f(s))\rangle^{f(s)}}\ \exists^{sk}: l}{\langle(\exists y)R(s,y)\rangle^{s}, \langle\neg R(s,f(s))\rangle^{f(s)} \vdash}\ \neg: l$$

$$\dfrac{\cdots \qquad \cdots}{\langle(\exists y)R(s,y)\rangle^{s}, \langle(\exists y)R(s,y)\rangle^{s}, \langle R(s,f(s)) \to \neg R(s,f(s))\rangle^{f(s)} \vdash}\ \to: l$$

$$\dfrac{\langle(\exists y)R(s,y)\rangle^{s}, \langle(\exists y)R(s,y)\rangle^{s}, \langle R(s,f(s)) \to \neg R(s,f(s))\rangle^{f(s)} \vdash}{(\forall x)(\exists y)R(x,y), \langle(\exists y)R(s,y)\rangle^{s}, \langle R(s,f(s)) \to \neg R(s,f(s))\rangle^{f(s)} \vdash}\ \forall^{sk}: l$$

$$\dfrac{(\forall x)(\exists y)R(x,y), \langle(\exists y)R(s,y)\rangle^{s}, \langle R(s,f(s)) \to \neg R(s,f(s))\rangle^{f(s)} \vdash}{(\forall x)(\exists y)R(x,y), (\forall x)(\exists y)R(x,y), \langle R(s,f(s)) \to \neg R(s,f(s))\rangle^{f(s)} \vdash}\ \forall^{sk}: l$$

$$\dfrac{\dfrac{\dfrac{\dfrac{(\forall x)(\exists y)R(x,y), (\forall x)(\exists y)R(x,y), \langle R(s,f(s)) \to \neg R(s,f(s))\rangle^{f(s)} \vdash}{(\forall x)(\exists y)R(x,y), \langle R(s,f(s)) \to \neg R(s,f(s))\rangle^{f(s)} \vdash}\ \mathrm{contr}: l}{(\forall x)(\exists y)R(x,y), (\forall y)(R(s,y) \to \neg R(s,y)) \vdash}\ \forall^{sk}: l}{(\forall x)(\exists y)R(x,y), (\exists x)(\forall y)(R(x,y) \to \neg R(x,y)) \vdash}\ \exists^{sk}: l}{(\forall x)(\exists y)R(x,y), \vdash \neg(\exists x)(\forall y)(R(x,y) \to \neg R(x,y))}\ \neg: r$$

where $f \in \mathcal{K}_{\iota,\iota}$ and $s \in \mathcal{K}_{\iota}$.

Denote the upper-left $\exists^{sk}: l$ application by ρ_1, the upper-right $\exists^{sk}: l$ application by ρ_2, and the bottommost $\exists^{sk}: l$ application by ρ_3. ρ_3 is the only $\exists^{sk}: l$ application with Skolem term s, so there is nothing to check. On the other hand, ρ_1 and ρ_2 have the same Skolem term $f(s)$. They are indeed homomorphic: the contr: l application is their uniting contraction, and the homomorphic paths are

$$\begin{aligned}\mu(\rho_1) = \ & \langle R(s,f(s))\rangle^{s}, \langle(\exists y)R(s,y)\rangle^{s}, \\ & \langle(\exists y)R(s,y)\rangle^{s}, (\forall x)(\exists y)R(x,y), \\ & (\forall x)(\exists y)R(x,y)\end{aligned}$$

$$\begin{aligned}\mu(\rho_2) = \ & \langle R(s,f(s))\rangle^{s}, \langle(\exists y)R(s,y)\rangle^{s}, \\ & \langle(\exists y)R(s,y)\rangle^{s}, \langle(\exists y)R(s,y)\rangle^{s}, \\ & \langle(\exists y)R(s,y)\rangle^{s}, (\forall x)(\exists y)R(x,y)\end{aligned}$$

because $F(\mu(\rho_1)) = F(\mu(\rho_2)) = \langle R(s,f(s))\rangle^{s}, \langle(\exists y)R(s,y)\rangle^{s}, (\forall x)(\exists y)R(x,y)$.

We postpone the proof of soundness of $\mathbf{LK}_{\mathrm{sk}}$ to Section 7 and instead consider the problem of cut-elimination. Since $\mathbf{LK}_{\mathrm{sk}}$ is cut-free, we first connect ordinary **LK** with the rules of $\mathbf{LK}_{\mathrm{sk}}$. The following definition will provide an analogue to Proposition 3, but in higher-order logic:

Definition 8 ($\mathbf{LK}_{\mathrm{skc}}$-trees). An $\mathbf{LK}_{\mathrm{skc}}$*-tree* is a tree formed according to the rules of $\mathbf{LK}_{\mathrm{sk}}$ and **LK** such that

1. rules of **LK** operate only on cut-ancestors, and
2. rules of $\mathbf{LK}_{\mathrm{sk}}$ operate only on end-sequent ancestors.

Hence the cut-ancestors in an $\mathbf{LK}_{\mathrm{skc}}$-tree have empty labels.

The method for showing cut-elimination for $\mathbf{LK}_{\mathrm{skc}}$ will be cut-elimination by resolution. Hence we will now introduce our resolution calculus.

5. The resolution calculus $\mathcal{R}_{al}$

In this section, we introduce the resolution calculus $\mathcal{R}_{al}$ we will use to define the CERES$^{\omega}$ method in the next section. As in $\mathbf{LK}_{\mathrm{sk}}$, we deal with labelled sequents. Note that $\mathcal{R}_{al}$ will include rules for CNF transformation: this is standard in higher-order resolution, as the notion of clause is not closed under substitution. It is also done in the ENAR calculus from [9] for a similar reason.

Definition 9 ($\mathcal{R}_{al}$ rules, deductions and refutations).

$$\frac{\Gamma \vdash \Delta, \langle \neg \mathbf{A} \rangle^{\ell}}{\langle \mathbf{A} \rangle^{\ell}, \Gamma \vdash \Delta}\ \neg^{T} \qquad \frac{\langle \neg \mathbf{A} \rangle^{\ell}, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \langle \mathbf{A} \rangle^{\ell}}\ \neg^{F}$$

$$\frac{\Gamma \vdash \Delta, \langle \mathbf{A} \vee \mathbf{B} \rangle^{\ell}}{\Gamma \vdash \Delta, \langle \mathbf{A} \rangle^{\ell}, \langle \mathbf{B} \rangle^{\ell}}\ \vee^{T} \qquad \frac{\langle \mathbf{A} \vee \mathbf{B} \rangle^{\ell}, \Gamma \vdash \Delta}{\langle \mathbf{A} \rangle^{\ell}, \Gamma \vdash \Delta}\ \vee^{F}_{l} \qquad \frac{\langle \mathbf{A} \vee \mathbf{B} \rangle^{\ell}, \Gamma \vdash \Delta}{\langle \mathbf{B} \rangle^{\ell}, \Gamma \vdash \Delta}\ \vee^{F}_{r}$$

$$\frac{\langle \mathbf{A} \wedge \mathbf{B} \rangle^{\ell}, \Gamma \vdash \Delta}{\langle \mathbf{A} \rangle^{\ell}, \langle \mathbf{B} \rangle^{\ell}, \Gamma \vdash \Delta}\ \wedge^{F} \qquad \frac{\Gamma \vdash \Delta, \langle \mathbf{A} \wedge \mathbf{B} \rangle^{\ell}}{\Gamma \vdash \Delta, \langle \mathbf{A} \rangle^{\ell}}\ \wedge^{T}_{l} \qquad \frac{\Gamma \vdash \Delta, \langle \mathbf{A} \wedge \mathbf{B} \rangle^{\ell}}{\Gamma \vdash \Delta, \langle \mathbf{B} \rangle^{\ell}}\ \wedge^{T}_{r}$$

$$\frac{\Gamma \vdash \Delta, \langle \mathbf{A} \rightarrow \mathbf{B} \rangle^{\ell}}{\langle \mathbf{A} \rangle^{\ell}, \Gamma \vdash \Delta, \langle \mathbf{B} \rangle^{\ell}}\ \rightarrow^{T} \qquad \frac{\langle \mathbf{A} \rightarrow \mathbf{B} \rangle^{\ell}, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \langle \mathbf{A} \rangle^{\ell}}\ \rightarrow^{F}_{l} \qquad \frac{\langle \mathbf{A} \rightarrow \mathbf{B} \rangle^{\ell}, \Gamma \vdash \Delta}{\langle \mathbf{B} \rangle^{\ell}, \Gamma \vdash \Delta}\ \rightarrow^{F}_{r}$$

$$\frac{\Gamma \vdash \Delta, \langle \forall_{\alpha} \mathbf{A} \rangle^{\ell}}{\Gamma \vdash \Delta, \langle \mathbf{A}\mathbf{X} \rangle^{\ell, \mathbf{X}}}\ \forall^{T} \qquad \frac{\langle \forall_{\alpha} \mathbf{A} \rangle^{\ell}, \Gamma \vdash \Delta}{\langle \mathbf{A}(\mathbf{f}\mathbf{S}_1 \dots \mathbf{S}_n) \rangle^{\ell}, \Gamma \vdash \Delta}\ \forall^{F}$$

$$\frac{\langle \exists_{\alpha} \mathbf{A} \rangle^{\ell}, \Gamma \vdash \Delta}{\langle \mathbf{A}\mathbf{X} \rangle^{\ell, \mathbf{X}}, \Gamma \vdash \Delta}\ \exists^{F} \qquad \frac{\Gamma \vdash \Delta, \langle \exists_{\alpha} \mathbf{A} \rangle^{\ell}}{\Gamma \vdash \Delta, \langle \mathbf{A}(\mathbf{f}\mathbf{S}_1 \dots \mathbf{S}_n) \rangle^{\ell}}\ \exists^{T}$$

$$\frac{S}{S\,[\mathbf{X} \leftarrow \mathbf{T}]}\ \mathrm{Sub}$$

$$\frac{\Gamma \vdash \Delta, \langle \mathbf{A} \rangle^{\ell_1}, \dots, \langle \mathbf{A} \rangle^{\ell_n} \quad \langle \mathbf{A} \rangle^{\ell_{n+1}}, \dots, \langle \mathbf{A} \rangle^{\ell_m}, \Pi \vdash \Lambda}{\Gamma, \Pi \vdash \Delta, \Lambda}\ \mathrm{Cut}$$

In Cut, $\mathbf{A}$ is atomic. In $\forall^T$ and $\exists^F$, $\mathbf{X}$ is a variable of appropriate type which does not occur in $\Gamma, \Delta, \mathbf{A}$. In $\forall^F$ and $\exists^T$, $\ell = \mathbf{S}_1, \ldots, \mathbf{S}_n$ and if $\tau(\mathbf{S}_i) = \alpha_i$ for $1 \leq i \leq n$ then $\mathbf{f} \in \mathcal{K}_{\alpha_1,\ldots,\alpha_n,\alpha}$ is a Skolem symbol. An application of this rule is called *source inference* of $\mathbf{fS}_1 \ldots \mathbf{S}_m$, and $\mathbf{fS}_1 \ldots \mathbf{S}_m$ is called the *Skolem term* of this inference.

Let $\mathcal{C}$ be a set of sequents. A sequence of sequents $S_1, \ldots, S_n$ is an $\mathcal{R}_{al}$-*deduction of* S_n *from* $\mathcal{C}$ if for all $1 \leq i \leq n$ either

1. $S_i \in \mathcal{C}$ or
2. S_i is derived from S_j (and S_k) by an $\mathcal{R}_{al}$ rule, where $j, k < i$.

In addition, we require that all $\forall^F$ and $\exists^T$ inferences used have pairwise distinct Skolem symbols. An $\mathcal{R}_{al}$-deduction of the empty sequent from $\mathcal{C}$ is called an $\mathcal{R}_{al}$-*refutation of* $\mathcal{C}$.

The calculus $\mathcal{R}_{al}$ is quite close to Andrews' resolution calculus $\mathcal{R}$ from [2]. Just like in $\mathcal{R}$, $\mathcal{R}_{al}$-deductions are defined in a linear fashion (in contrast to **LK**-proofs and $\mathbf{LK}_{\text{sk}}$-trees). The two main differences to $\mathcal{R}$ are (1) the use of labels to control the arguments of the Skolem terms introduced by the $\forall^F$ rule, and (2) the incorporation of Andrews' rules of Simplification and Cut into the Cut rule of $\mathcal{R}_{al}$. Regarding the latter, note that this restriction is not as serious as it may appear at first glance: For example, the sentence $F = \forall x P(x) \rightarrow (P(a) \land P(b))$ cannot be proved in **LK**, restricted to atomic cut, without using non-atomic contraction. Still, $\neg F$ can be refuted in $\mathcal{R}_{al}$. We state the relative completeness problem of $\mathcal{R}_{al}$:

Relative Completeness of $\mathcal{R}_{al}$. *Let $\mathcal{S}$ be a set of labelled sequents. $\mathcal{R}_{al}$ is relatively complete if the following holds: If there exists an $\mathcal{R}$-refutation of the reduct of $\mathcal{S}$, then there exists an $\mathcal{R}_{al}$-refutation of $\mathcal{S}$.*

Relative completeness will imply completeness of the CERES$^\omega$ method, in conjunction with the following result from [2] (which still holds in the presence of Miller's restriction):

Theorem 1. *Let $\mathcal{S}$ be a set of sentences. If there exists a $\mathcal{T}$-refutation of $\mathcal{S}$ then there exists an $\mathcal{R}$-refutation of $\mathcal{S}$.*

Note that the above formulation of relative completeness is not the only way to attain this goal: completeness with respect to an appropriate intensional model class (see [7, 18]) for higher-order logic would also suffice (together with a soundness theorem for that class for **LK**). The formulation above has the advantage that an effective proof of it would give an algorithm to transform $\mathcal{R}$-refutations into $\mathcal{R}_{al}$-refutations, allowing proof search to be done in practice in the more convenient $\mathcal{R}$ calculus.

6. CERES$^\omega$

In this section, we will show cut-elimination for $\mathbf{LK}_{\text{skc}}$. To connect this result to **LK**, our first task is to show that **LK**-proofs can be translated to $\mathbf{LK}_{\text{skc}}$-proofs.

We extend the notions of paths, homomorphic inferences, and weak regularity to $\mathbf{LK}_{\text{skc}}$-trees. Let π be an $\mathbf{LK}_{\text{skc}}$-tree with end-sequent S. We say that π is an $\mathbf{LK}_{\text{skc}}$*-proof* if it is weakly regular and proper.

Definition 10. Let π be an $\mathbf{LK}_{\text{skc}}$-tree. π is called *regular* if

1. each strong labelled quantifier inference has a unique Skolem symbol and
2. the eigenvariable of each strong quantifier inference ρ only occurs above ρ in π.

Proposition 9. *Let π be an $\mathbf{LK}_{\text{skc}}$-tree. If π is regular, then π is weakly regular.*

The following lemma provides an analogue to the $\Rightarrow$-direction of Proposition 2.

Lemma 1 (Skolemization). *Let π be a regular $\mathbf{LK}$-proof of S. Then there exists a regular $\mathbf{LK}_{\text{skc}}$-proof ψ of S.*

Proof. Let ρ be an inference in π with conclusion $\mathbf{F}_1, \ldots, \mathbf{F}_n \vdash \mathbf{F}_{n+1}, \ldots, \mathbf{F}_m$. By induction on the height of ρ, we define a regular $\mathbf{LK}_{\text{skc}}$-tree π_ρ with conclusion $\langle \mathbf{F}_1 \rangle^{\ell_1}, \ldots, \langle \mathbf{F}_n \rangle^{\ell_n} \vdash \langle \mathbf{F}_{n+1} \rangle^{\ell_{n+1}}, \ldots, \langle \mathbf{F}_m \rangle^{\ell_m}$ such that for all $1 \leq i \leq m$, ℓ_i is the sequence of substitution terms of $\forall\colon l$ inferences operating on descendents of $\mathbf{F}_i$ in π, and such that π_ρ fulfills an *eigenterm* condition, i.e. every Skolem symbol occurs only above its source inference.[2]

1. ρ is an axiom $\mathbf{A} \vdash \mathbf{A}$. Let ℓ_1 be the sequence of substitution terms of the weak quantifier inferences operating on the descendents of the left occurrence of $\mathbf{A}$, and let ℓ_2 be the sequence of substitution terms of the weak quantifier inferences operating on descendents of the right occurrence of $\mathbf{A}$. Then take as π_ρ the axiom $\langle \mathbf{A} \rangle^{\ell_1} \vdash \langle \mathbf{A} \rangle^{\ell_2}$.
2. ρ is a $\forall\colon l$ inference operating on an end-sequent ancestor:

$$\cfrac{\begin{matrix}(\varphi)\\ \overline{\mathbf{FT}}, \Gamma \vdash \Delta\end{matrix}}{\forall_\alpha \mathbf{F}, \Gamma \vdash \Delta}\ \forall\colon l$$

By (IH) we obtain a regular $\mathbf{LK}_{\text{skc}}$-tree φ' of $\left\langle \overline{\mathbf{FT}} \right\rangle^{\ell, \mathbf{T}}, \Gamma' \vdash \Delta'$ where Γ', Δ' are Γ, Δ with the respective labels. We take for π_ρ

$$\cfrac{\begin{matrix}(\varphi')\\ \left\langle \overline{\mathbf{FT}} \right\rangle^{l, \mathbf{T}}, \Gamma' \vdash \Delta'\end{matrix}}{\langle \forall_\alpha \mathbf{F} \rangle^{l}, \Gamma' \vdash \Delta'}\ \forall^{sk}\colon l$$

[2]It is possible to assign arbitrary labels to cut-ancestors in $\mathbf{LK}_{\text{skc}}$-trees. To avoid a case distinction, cut-ancestors are assigned labels in the same way as end-sequent ancestors in this proof.

3. ρ is a $\forall\colon l$ inference operating on a cut-ancestor. Then we simply take the regular $\mathbf{LK}_{\mathrm{skc}}$-tree obtained by (IH) and apply ρ to it.
4. ρ is a $\forall\colon r$ inference operating on an end-sequent ancestor:

$$\cfrac{\begin{matrix}(\varphi)\\ \Gamma \vdash \Delta, \overline{\mathbf{FX}}\end{matrix}}{\Gamma \vdash \Delta, \forall_\alpha \mathbf{F}}\ \forall\colon r$$

By (IH) we obtain a regular $\mathbf{LK}_{\mathrm{skc}}$-tree φ' of $\Gamma' \vdash \Delta', \left\langle \overline{\mathbf{FX}} \right\rangle^{\mathbf{T}_1,\ldots,\mathbf{T}_n}$, with Γ', Δ' as above. Let $\mathbf{f} \in \mathcal{K}_{\alpha_1,\ldots,\alpha_n,\alpha}$, where for $1 \leq i \leq n$ $\tau(\mathbf{T}_i) = \alpha_i$, be a new Skolem symbol, and let $\mathbf{S} = \mathbf{f}(\mathbf{T_1} \ldots \mathbf{T}_n)$. Let σ be the substitution $[\mathbf{X} \leftarrow \mathbf{S}]$. By regularity, $\mathbf{X}$ is not an eigenvariable in φ', and does not occur in $\mathbf{T}_1, \ldots, \mathbf{T}_n$. Hence $\varphi'\sigma$ is a regular $\mathbf{LK}_{\mathrm{skc}}$-tree of $\Gamma' \vdash \Delta', \left\langle \overline{\mathbf{FS}} \right\rangle^{\mathbf{T}_1,\ldots,\mathbf{T}_n}$. Take for π_ρ

$$\cfrac{\begin{matrix}(\varphi'\sigma)\\ \Gamma' \vdash \Delta', \left\langle \overline{\mathbf{FS}} \right\rangle^{\mathbf{T}_1,\ldots,\mathbf{T}_n}\end{matrix}}{\Gamma' \vdash \Delta', \left\langle \forall_\alpha \mathbf{F} \right\rangle^{\mathbf{T}_1,\ldots,\mathbf{T}_n}}\ \forall^{sk}\colon r$$

5. ρ is a $\forall\colon r$ inference operating on a cut ancestor. Again we take the regular $\mathbf{LK}_{\mathrm{skc}}$-tree obtained by (IH) and apply ρ to it.
6. ρ is a cut inference

$$\cfrac{\begin{matrix}(\varphi)\\ \Gamma \vdash \Delta, \mathbf{F}\end{matrix} \qquad \begin{matrix}(\lambda)\\ \mathbf{F}, \Pi \vdash \Lambda\end{matrix}}{\Gamma, \Pi \vdash \Delta, \Lambda}\ \mathrm{cut}$$

By (IH) we obtain regular $\mathbf{LK}_{\mathrm{skc}}$-trees φ', λ' of $\Gamma' \vdash \Delta', \langle \mathbf{F} \rangle^{\ell_1}$ and $\langle \mathbf{F} \rangle^{\ell_2}, \Pi' \vdash \Lambda'$, respectively. If the intersection of the Skolem symbols of φ', λ' is non-empty, by the eigenterm condition we can rename Skolem symbols to achieve this. Hence the $\mathbf{LK}_{\mathrm{skc}}$-tree π_ρ

$$\cfrac{\begin{matrix}(\varphi')\\ \Gamma' \vdash \Delta', \langle \mathbf{F} \rangle^{\ell_1}\end{matrix} \qquad \begin{matrix}(\lambda')\\ \langle \mathbf{F} \rangle^{\ell_2}, \Pi' \vdash \Lambda'\end{matrix}}{\Gamma', \Pi' \vdash \Delta', \Lambda'}\ \mathrm{cut}$$

is regular.
7. ρ is a $\mathrm{contr}\colon r$ inference

$$\cfrac{\begin{matrix}(\varphi)\\ \Gamma \vdash \Delta, \mathbf{F}, \mathbf{F}\end{matrix}}{\Gamma \vdash \Delta, \mathbf{F}}\ \mathrm{contr}\colon r$$

By (IH) we obtain a regular $\mathbf{LK}_{\mathrm{skc}}$-tree φ' of $\Gamma' \vdash \Delta', \langle \mathbf{F} \rangle^{\ell_1}, \langle \mathbf{F} \rangle^{\ell_2}$. Note that the inferences operating on descendents of the occurrences of $\mathbf{F}$ coincide, so $\ell_1 = \ell_2$ and we may take for π_ρ

$$\frac{\begin{array}{c}(\varphi')\\ \Gamma' \vdash \Delta', \langle \mathbf{F} \rangle^{\ell_1}, \langle \mathbf{F} \rangle^{\ell_1}\end{array}}{\Gamma' \vdash \Delta', \langle \mathbf{F} \rangle^{\ell_1}}\ \mathrm{contr}\colon r$$

8. ρ is another type of inference: analogous to the previous cases.

Let ρ be the last inference in π, then $\psi = \pi_\rho$ is the desired regular $\mathbf{LK}_{\mathrm{skc}}$-proof. $\square$

We will now set up some notation for the main definitions of CERES$^\omega$. Let π be an $\mathbf{LK}_{\mathrm{skc}}$-tree, and let S be a sequent in π. Then by $\mathrm{cutanc}(S)$ we denote the sub-sequent of S consisting of the cut-ancestors of S, and by $\mathrm{esanc}(S)$ we denote the sub-sequent of S consisting of the end-sequent ancestors of S. Note that for any sequent $S = \mathrm{cutanc}(S) \circ \mathrm{esanc}(S)$. Let ρ be a unary inference, σ a binary inference, ψ, χ $\mathbf{LK}_{\mathrm{sk}}$-trees, then $\rho(\psi)$ is the $\mathbf{LK}_{\mathrm{sk}}$-tree obtained by applying ρ to the end-sequent of ψ, and $\sigma(\psi, \chi)$ is the $\mathbf{LK}_{\mathrm{sk}}$-tree obtained from the $\mathbf{LK}_{\mathrm{sk}}$-trees ψ and χ by applying σ. Note that while this notation is ambigous, it will always be clear from the context what the auxiliary formulas of the $\rho(\psi)$ and $\sigma(\psi, \chi)$ are. Let P, Q be sets of $\mathbf{LK}_{\mathrm{sk}}$-trees. Then $P^{\Gamma \vdash \Delta} = \{\psi^{\Gamma \vdash \Delta} \mid \psi \in P\}$, where $\psi^{\Gamma \vdash \Delta}$ is ψ followed by weakenings adding $\Gamma \vdash \Delta$, and $P \times_\sigma Q = \{\sigma(\psi, \chi) \mid \psi \in P, \chi \in Q\}$.

Definition 11 (Characteristic sequent set and proof projections). Let π be a regular $\mathbf{LK}_{\mathrm{skc}}$-proof. For each inference ρ in π, we define a set of $\mathbf{LK}_{\mathrm{sk}}$-trees, the set of *projections* $\mathcal{P}_\rho(\pi)$, and a set of labelled sequents, the *characteristic sequent set* $\mathrm{CS}_\rho(\pi)$.

- If ρ is an axiom with conclusion $S = \langle \mathbf{A} \rangle^{\ell_1} \vdash \langle \mathbf{A} \rangle^{\ell_2}$, distinguish:
  - $\mathrm{cutanc}(S) = S$. Then $\mathrm{CS}_\rho(\pi) = \mathcal{P}_\rho(\pi) = \emptyset$.
  - $\mathrm{cutanc}(S) \neq S$. Distinguish:
    (a) If $\mathrm{cutanc}(S) = \vdash \langle \mathbf{A} \rangle^{\ell_2}$ then $\mathrm{CS}_\rho(\pi) = \{\vdash \langle \mathbf{A} \rangle^{\ell_1}\}$ and $\mathcal{P}_\rho(\pi) = \{\langle \mathbf{A} \rangle^{\ell_1} \vdash \langle \mathbf{A} \rangle^{\ell_1}\}$,
    (b) if $\mathrm{cutanc}(S) = \langle \mathbf{A} \rangle^{\ell_1} \vdash$ then $\mathrm{CS}_\rho(\pi) = \{\langle \mathbf{A} \rangle^{\ell_2} \vdash\}$ and $\mathcal{P}_\rho(\pi) = \{\langle \mathbf{A} \rangle^{\ell_2} \vdash \langle \mathbf{A} \rangle^{\ell_2}\}$,
    (c) if $\mathrm{cutanc}(S) = \vdash$ then $\mathrm{CS}_\rho(\pi) = \{\vdash\}$ and $\mathcal{P}_\rho(\pi) = \{S\}$.

- If ρ is a unary inference with immediate predecessor ρ' with $\mathcal{P}_{\rho'}(\pi) = \{\psi_1, \ldots, \psi_n\}$, distinguish:

  (a) ρ operates on ancestors of cut formulas. Then

$$\mathcal{P}_\rho(\pi) = \mathcal{P}_{\rho'}(\pi)$$

  (b) ρ operates on ancestors of the end-sequent. Then

$$\mathcal{P}_\rho(\pi) = \{\rho(\psi_1), \ldots, \rho(\psi_n)\}$$

In any case, $\mathrm{CS}_\rho(\pi) = \mathrm{CS}_{\rho'}(\pi)$.

- Let ρ be a binary inference with immediate predecessors ρ_1 and ρ_2.
  (a) If ρ operates on ancestors of cut-formulas, let $\Gamma_i \vdash \Delta_i$ be the ancestors of the end-sequent in the conclusion sequent of ρ_i and define

  $$\mathcal{P}_\rho(\pi) = \mathcal{P}_{\rho_1}(\pi)^{\Gamma_2 \vdash \Delta_2} \cup \mathcal{P}_{\rho_2}(\pi)^{\Gamma_1 \vdash \Delta_1}$$

  For the characteristic sequent set, define

  $$\mathrm{CS}_\rho(\pi) = \mathrm{CS}_{\rho_1}(\pi) \cup \mathrm{CS}_{\rho_2}(\pi)$$

  (b) If ρ operates on ancestors of the end-sequent, then

  $$\mathcal{P}_\rho(\pi) = \mathcal{P}_{\rho_1}(\pi) \times_\rho \mathcal{P}_{\rho_2}(\pi).$$

  For the characteristic sequent set, define

  $$\mathrm{CS}_\rho(\pi) = \mathrm{CS}_{\rho_1}(\pi) \times \mathrm{CS}_{\rho_2}(\pi)$$

The *set of projections* of π, $\mathcal{P}(\pi)$ is defined as $\mathcal{P}_{\rho_0}(\pi)$, and the *characteristic sequent set* of π, $\mathrm{CS}(\pi)$ is defined as $\mathrm{CS}_{\rho_0}(\pi)$, where ρ_0 is the last inference of π.

Note that for $\mathbf{LK}_{\mathrm{skc}}$-proofs π containing only atomic axioms, $\mathrm{CS}(\pi)$ consists of sequents containing only atomic formulas. This is not required, though.

Proposition 10. *Let π be a regular $\mathbf{LK}_{\mathrm{skc}}$-proof. Then there exists an $\mathbf{LK}$-refutation of the reduct of* $\mathrm{CS}(\pi)$.

Proof. We inductively define, for each inference ρ with conclusion S in π, an $\mathbf{LK}$-tree γ_ρ of the reduct of $\mathrm{cutanc}(S)$ from the reduct of $\mathrm{CS}_\rho(\pi)$.

- If ρ is an axiom $\langle \mathbf{A} \rangle^{\ell_1} \vdash \langle \mathbf{A} \rangle^{\ell_2}$, distinguish:
  - $\mathrm{cutanc}(S) = S$. Take the axiom ρ for γ_ρ.
  - $\mathrm{cutanc}(S) \neq S$. Then $\mathrm{CS}_\rho(\pi) = \{S'\}$ and we may take the reduct of S'.
- If ρ is a unary inference with immediate predecessor ρ', let S' be the conclusion of ρ' and distinguish:
  - ρ operates on ancestors of cut formulas. By (IH) we have an $\mathbf{LK}$-tree $\gamma_{\rho'}$ of $\mathrm{cutanc}(S')$ from $\mathrm{CS}_{\rho'}(\pi)$. Apply ρ to $\gamma_{\rho'}$ to obtain γ_ρ. Note that as $\mathrm{cutanc}(S')$ is a sub-sequent of S', if ρ' is a strong quantifier inference, its eigenvariable condition is fulfilled. As $\mathrm{CS}_\rho(\pi) = \mathrm{CS}_{\rho'}(\pi)$ by definition, γ_ρ is the desired $\mathbf{LK}$-tree of $\mathrm{cutanc}(S)$.
  - ρ operates on ancestors of the end-sequent. Then $\mathrm{cutanc}(S) = \mathrm{cutanc}(S')$ and $\mathrm{CS}_\rho(\pi) = \mathrm{CS}_{\rho'}(\pi)$ and hence we may take for γ_ρ the $\mathbf{LK}$-tree obtained by (IH).

- If ρ is a binary inference with immediate predecessors ρ_1, ρ_2, let $\gamma_{\rho_1}, \gamma_{\rho_2}$ be the **LK**-trees obtained by (IH) and distinguish:
  - ρ operates on ancestors of cut-formulas. Then obtain γ_ρ by applying ρ to $\gamma_{\rho_1}, \gamma_{\rho_2}$: As $\mathrm{CS}_\rho(\pi) = \mathrm{CS}_{\rho_1}(\pi) \cup \mathrm{CS}_{\rho_1}(\pi)$ it is the desired **LK**-tree.
  - ρ operates on ancestors of the end-sequent. Then $\mathrm{CS}_\rho(\pi) = \mathrm{CS}_{\rho_1}(\pi) \times \mathrm{CS}_{\rho_2}(\pi)$. We may assume that the eigenvariables of γ_{ρ_1} are distinct from the variables occurring in γ_{ρ_2} and vice-versa, otherwise we perform renamings. Let S_1, S_2 be the conclusions of ρ_1, ρ_2 respectively. For every $C \in \mathrm{CS}_{\rho_1}(\pi)$, construct an **LK**-tree γ_C of $\mathrm{cutanc}(S_2) \circ C$ from $\mathrm{CS}_{\rho_2}(\pi) \times \{C\}$ by taking γ_{ρ_2} and adding C to every sequent, and appending contractions on C at the end. As the eigenvariables of γ_{ρ_2} are distinct from the variables of C by the consideration above, γ_C is really an **LK**-tree. Now, construct γ_ρ by taking γ_{ρ_1} and appending, at every leaf of the form $C \in \mathrm{CS}_{\rho_1}(\pi)$, the **LK**-tree γ_C, and adding contractions on $\mathrm{cutanc}(S_2)$ at the end. Again, no eigenvariable conditions are violated by the above consideration and γ_C is an **LK**-tree of $\mathrm{cutanc}(S_1) \circ \mathrm{cutanc}(S_2)$ from $\mathrm{CS}_\rho(\pi)$, as required.

Let ρ be the last inference in π, then γ_ρ is the desired **LK**-refutation. □

We will now address a central problem of CERES$^\omega$: how to combine an $\mathcal{R}_{al}$-refutation of $\mathrm{CS}(\pi)$ with the $\mathbf{LK}_{\mathrm{sk}}$-trees from $\mathcal{P}(\pi)$ into an $\mathbf{LK}_{\mathrm{sk}}$-proof of the end-sequent of π. The following definitions set up the main properties of the $\mathbf{LK}_{\mathrm{sk}}$-trees in $\mathcal{P}(\pi)$:

Definition 12 (Restrictedness). Let $\mathcal{S}$ be a set of formula occurrences in an $\mathbf{LK}_{\mathrm{skc}}$-tree π. We say that π *is $\mathcal{S}$-linear* if no inferences operate on ancestors of occurrences in $\mathcal{S}$. We say that π *is $\mathcal{S}$-restricted* if no inferences except contraction operate on ancestors of occurrences in $\mathcal{S}$.

If $\mathcal{S}$ is the set of occurrences of cut-formulas of π and π is $\mathcal{S}$-restricted, we say that π is restricted.

Example 6. Consider the $\mathbf{LK}_{\mathrm{skc}}$-tree π

$$\dfrac{\dfrac{P(a) \vdash P(a) \quad Y(b) \vdash Y(b)}{P(a) \lor Y(b) \vdash P(a), Y(b)}\ \lor\colon l \qquad \dfrac{\dfrac{\dfrac{Y(b) \vdash \langle Y(b) \rangle^T \quad Y(b) \vdash \langle Y(b) \rangle^T}{Y(b), Y(b) \vdash \langle Y(b) \land Y(b) \rangle^T}\ \land\colon r}{Y(b) \vdash \langle Y(b) \land Y(b) \rangle^T}\ \mathrm{contr}\colon l}{Y(b) \vdash (\exists X) X(b)}\ \exists^{sk}\colon r}{P(a) \lor Y(b) \vdash (\exists X) X(b), P(a)}\ \mathrm{cut}$$

where $T = \lambda x. Y(x) \land Y(x)$. Let $\mathcal{S}$ be the ancestors of $P(a)$ in the end-sequent, and let C be the ancestors of cut-formulas in π. Then π is $\mathcal{S}$-linear and C-restricted, and thus restricted.

In principle, labels of linear occurrences in $\mathbf{LK}_{\mathrm{skc}}$-trees may be deleted:

Proposition 11. *Let π be an $\mathbf{LK}_{\mathrm{skc}}$-tree, and $\mathcal{S}$ a set of formula occurrences in π that is closed under descendents, and let π be $\mathcal{S}$-linear. If π' is obtained from π by replacing all labels of ancestors of occurrences in $\mathcal{S}$ by the empty label, then π' is an $\mathbf{LK}_{\mathrm{skc}}$-tree.*

Proof. As π is $\mathcal{S}$-linear, no inferences operate on the respective occurrences. As no inference has restrictions on labels of context formulas (except that direct descendents have the same labels as their direct ancestors), and also axioms pose no restrictions on labels, the proposition holds. $\square$

Definition 13 (Skolem parallel). Let ρ_1, ρ_2 be strong labelled quantifier inferences in $\mathbf{LK}_{\mathrm{skc}}$-trees π_1, π_2 with Skolem terms $\mathbf{S}_1$, $\mathbf{S}_2$ respectively. ρ_1, ρ_2 are called *Skolem parallel* if for all substitutions σ_1, σ_2, if $\mathbf{S}_1\sigma_1 = \mathbf{S}_2\sigma_2$ then $\mu_1\sigma_1, \mu_2\sigma_2$ are homomorphic, where μ_1, μ_2 are the maximal downwards paths starting at $\mathbf{S}_1, \mathbf{S}_2$ respectively. π_1, π_2 are called Skolem parallel if for all strong labelled quantifier inferences ρ_1, ρ_2 in π_1, π_2 respectively, ρ_1, ρ_2 are Skolem parallel.

Example 7. Consider the $\mathbf{LK}_{\mathrm{skc}}$-trees π

$$\dfrac{\dfrac{Y(f(Y)) \vdash \langle Y(f(Y))\rangle^{Y}}{Y(f(Y)) \vdash \langle(\forall y)Y(y)\rangle^{Y}}\ \forall^{sk}: r}{Y(f(Y)) \vdash (\exists X)(\forall y)X(y)}\ \exists^{sk}: r$$

and ψ

$$\dfrac{\dfrac{\dfrac{\dfrac{P(f(T)) \vdash \langle P(f(T))\rangle^{T} \quad Q(\alpha) \vdash \langle Q(\alpha)\rangle^{T}}{P(f(T)) \vee Q(\alpha) \vdash \langle P(f(T))\rangle^{T}, \langle Q(\alpha)\rangle^{T}}\ \vee: l}{P(f(T)) \vee Q(\alpha) \vdash \langle P(f(T)) \vee Q(\alpha)\rangle^{T}}\ \vee: r}{P(f(T)) \vee Q(\alpha) \vdash \langle(\forall y)(P(y) \vee Q(\alpha))\rangle^{T}}\ \forall^{sk}: r}{P(f(T)) \vee Q(\alpha) \vdash (\exists X)(\forall y)X(y)}\ \exists^{sk}: r$$

where $T = \lambda x.P(x) \vee Q(\alpha)$ and $f \in \mathcal{K}_{\iota \to o, \iota}$. Then π and ψ are Skolem parallel.

Proposition 12. *Let π_1, π_2 be $\mathbf{LK}_{\mathrm{skc}}$-trees and σ a substitution. If π_1, π_2 are Skolem parallel, then $\pi_1\sigma, \pi_2$ are.*

Proof. Consider Skolem terms $\mathbf{S}_1, \mathbf{S}_2$ occurring in auxiliary formulas of strong labelled quantifier inferences ρ_1, ρ_2 in $\pi_1\sigma, \pi_2$ respectively. Then by construction of $\pi_1\sigma$, $\mathbf{S}_1 = \mathbf{S}_1'\sigma$ for some Skolem term $\mathbf{S}_1'$ occurring in the auxiliary formula of a strong labelled quantifier inference ρ_1' in π_1. Let μ_1' be the maximal downwards path starting at $\mathbf{S}_1'$, and μ_2 the maximal downwards path in π_2 starting at $\mathbf{S}_2$. Let σ_1, σ_2 be substitutions such that $\mathbf{S}_2\sigma_2 = \mathbf{S}_1\sigma_1 = \mathbf{S}_1'\sigma\sigma_1$. As ρ_1', ρ_2 are Skolem parallel, $F(\mu_1'\sigma\sigma_1) = F(\mu_2\sigma_2)$. But by construction of $\pi_1\sigma$, $\mu_1'\sigma$ is the maximal downwards path starting at $\mathbf{S}_1$ in $\pi_1\sigma$, so ρ_1, ρ_2 are Skolem parallel. $\square$

Definition 14 (Axiom labels). Let π be an $\mathbf{LK}_{\text{skc}}$-tree, let ω be a formula occurrence in π, and let μ be an ancestor of ω that occurs in an axiom A. Then A is called a *source axiom* for ω. Let $\mathcal{S}$ be a set of formula occurrences in π. We say that π *has suitable axiom labels with respect to* $\mathcal{S}$ if for all formula occurrences ω in $\mathcal{S}$, the source axioms of ω are of the form $\langle \mathbf{F} \rangle^{\ell} \vdash \langle \mathbf{F} \rangle^{\ell}$.

Example 8. Consider the $\mathbf{LK}_{\text{skc}}$-tree π

$$\dfrac{\dfrac{\langle Y(b) \rangle^{T} \vdash \langle Y(b) \rangle^{T} \quad Y(b) \vdash \langle Y(b) \rangle^{T}}{\langle Y(b) \rangle^{T}, Y(b) \vdash \langle Y(b) \wedge Y(b) \rangle^{T}} \wedge : r}{\langle Y(b) \rangle^{T}, Y(b) \vdash (\exists X) X(b)} \exists^{sk} : r$$

where $T = \lambda x. Y(x) \wedge Y(x)$. Let ω be the occurrence of $\langle Y(b) \rangle^{T}$ in the end-sequent. Then π has suitable axiom labels with respect to $\{\omega\}$. Note that π does not have suitable axiom labels with respect to the occurrence of $Y(b)$ in the end-sequent.

Definition 15 (Balancedness). Let π be an $\mathbf{LK}_{\text{skc}}$-tree, and let $\mathcal{S}$ be a set of formula occurrences in π. We call π $\mathcal{S}$*-balanced* if for every axiom $\langle \mathbf{F} \rangle^{\ell_1} \vdash \langle \mathbf{F} \rangle^{\ell_2}$ in π, at least one occurrence of $\mathbf{F}$ is an ancestor of a formula occurrence in $\mathcal{S}$. We say that π is *balanced* if π is $\mathcal{S}$-balanced, where $\mathcal{S}$ is the set of end-sequent occurrences of π.

Example 9. Consider the $\mathbf{LK}_{\text{skc}}$-tree π from Example 6. Let ω_1 be the occurrence of $P(a) \vee Y(b)$ in the end-sequent of π, and let ω_2 be the occurrence of $(\exists X) X(b)$ in the end-sequent of π. Then π is neither $\{\omega_1\}$-balanced nor $\{\omega_2\}$-balanced, but π is $\{\omega_1, \omega_2\}$-balanced.

Definition 16 (CERES-projections). Let S be a proper sequent, and C be a sequent. Then an $\mathbf{LK}_{\text{skc}}$-tree π is called a CERES-*projection* for (S, C) if the end-sequent of π is $S \circ C$ and π is weakly regular, $\mathcal{O}_C$-linear, $\mathcal{O}_S$-balanced, restricted, and has suitable axiom labels with respect to $\mathcal{O}_C$, where $\mathcal{O}_S$ resp. $\mathcal{O}_C$ is the set of formula occurrences of S resp. C in the end-sequent of π.

Let $\mathcal{C}$ be a set of sequents. A set of $\mathbf{LK}_{\text{skc}}$-trees $\mathcal{P}$ is called a *set of* CERES-*projections for* $(S, \mathcal{C})$ if for all $C \in \mathcal{C}$ there exists a $\pi(C) \in \mathcal{P}$ such that $\pi(C)$ is a CERES-projection for (S, C) and moreover, for all $\pi_1, \pi_2 \in \mathcal{P}$, π_1 and π_2 are Skolem parallel.

Lemma 2. *Let π be a regular $\mathbf{LK}_{\text{skc}}$-proof of S. Then $\mathcal{P}(\pi)$ is a set of* CERES-*projections for $(S, \mathrm{CS}(\pi))$. Furthermore, for all $\psi \in \mathcal{P}(\pi)$, $|\psi| \leq |\pi|$.*

Proof. By inspecting Definition 11. Let ρ be an inference in π with conclusion R. By induction on height(ρ), it is easy to see that for every $C \in \mathrm{CS}_{\rho}(\pi)$, $\mathcal{P}_{\rho}(\pi)$ contains an $\mathbf{LK}_{\text{sk}}$-tree of esanc$(R) \circ C$. Hence $\mathcal{P}(\pi)$ contains an $\mathbf{LK}_{\text{sk}}$-tree $\pi(C)$ of $S \circ C$ for every $C \in \mathrm{CS}(\pi)$. It remains to verify that (1) $\pi(C)$ is a CERES-projection for (S, C) and (2) every $\pi(C_1), \pi(C_2) \in \mathcal{P}(\pi)$ are Skolem parallel.

Regarding (1): $\pi(C)$ is regular, which follows from the fact that π is regular, and that in constructing $\pi(C)$ from π, every inference in π induces at most one copy of it in $\pi(C)$. Hence $\pi(C)$ is also weakly regular. S-balancedness, C-linearity and suitable axiom labels follow immediately from the definition. As $\pi(C)$ is cut-free, it is trivially restricted.

Regarding (2): Consider $\mu_1, \mu_2, \mathbf{S}_1, \mathbf{S}_2, \sigma_1, \sigma_2$ as in Definition 13. By construction, if an inference ρ of π is applied in both $\pi(C_1)$ and $\pi(C_2)$, also all inferences operating on descendents of the main formula of ρ are applied in both $\pi(C_1)$ and $\pi(C_2)$. Therefore by regularity of π, $\mu_1 = \mu_2$. $\mu_1 = \mu_2$ implies $\mathbf{S}_1 = \mathbf{S}_2$, hence $\mathbf{S}_1\sigma_1 = \mathbf{S}_1\sigma_2$ and therefore $\sigma_1 \upharpoonright \mathrm{FV}(\mathbf{S}_1) = \sigma_2 \upharpoonright \mathrm{FV}(\mathbf{S}_2)$. Therefore $\mu_1\sigma_1 = \mu_2\sigma_2$ by Proposition 5. □

Lemma 3. *Let S be a proper sequent. Let $\mathcal{C}$ be a set of sequents, and $\mathcal{P}$ a set of* CERES*-projections for $(S, \mathcal{C})$. Then, if there exists an $\mathcal{R}_{al}$-refutation of $\mathcal{C}$, there exists a restricted, weakly regular, balanced $\mathbf{LK}_{\mathrm{skc}}$-tree of S.*

Proof. Let $\gamma : S_1, \ldots, S_n$ be an $\mathcal{R}_{al}$-refutation of $\mathcal{C}$ (hence $S_n = \vdash$). Let $S = \Gamma \vdash \Delta$. By induction on $0 \leq i \leq n$, we construct sets of $\mathbf{LK}_{\mathrm{skc}}$-trees $\mathcal{P}_i \supseteq \mathcal{P}$ such that $\mathcal{P}_i$ is a set of CERES-projections for $(S, \mathcal{C} \cup \{S_1, \ldots, S_i\})$ and such that $\mathcal{P}_i$ contains only Skolem symbols from $\mathcal{P}$ and $S_1, \ldots, S_i$. Then $\mathcal{P}_n$ contains a CERES-projection for $(S, \vdash)$ which is the desired $\mathbf{LK}_{\mathrm{skc}}$-tree of S. We set $\mathcal{P}_0 = \mathcal{P}$.

For $i > 0$, distinguish how S_i is inferred in γ:

1. $S_i \in \mathcal{C}$. Then we may take $\mathcal{P}_i = \mathcal{P}_{i-1}$ by $\mathcal{P} \subseteq \mathcal{P}_{i-1}$ and (IH).
2. S_i is derived from S_j (and S_k). Then by (IH) we obtain a set of CERES-projections $\mathcal{P}_{i-1}$ for $(S, \mathcal{C} \cup \{S_1, \ldots, S_{i-1}\})$. By definition there exist CERES-projections $\pi_j \in \mathcal{P}_{i-1}$ for (S, S_j) (and $\pi_k \in \mathcal{P}_{i-1}$ for (S, S_k)). We set $\mathcal{P}_i = \mathcal{P}_{i-1} \cup \{\pi_i\}$, where π_i is an $\mathbf{LK}_{\mathrm{skc}}$-tree defined by distinguishing how S_i is inferred in γ:
   (a) $S_i = \langle\mathbf{A}\rangle^\ell, \Pi \vdash \Lambda$ is derived from $S_j = \Pi \vdash \Lambda, \langle\neg\mathbf{A}\rangle^\ell$ by $\neg^T$. Then the end-sequent of π_j is $S \circ S_j = \Gamma, \Pi \vdash \Lambda, \Delta, \langle\neg\mathbf{A}\rangle^\ell$. By S_j-linearity of π_j, the maximal upwards path μ starting at $\langle\neg\mathbf{A}\rangle^\ell$ is unique. Let μ end in $\langle\neg\mathbf{A}\rangle^\ell \vdash \langle\neg\mathbf{A}\rangle^\ell$ (the labels are identical because π_j has suitable axiom labels with respect to S_j). By $\mathcal{S}$-balancedness, we may replace this axiom in π_j by

$$\frac{\langle\mathbf{A}\rangle^\ell \vdash \langle\mathbf{A}\rangle^\ell}{\langle\mathbf{A}\rangle^\ell, \langle\neg\mathbf{A}\rangle^\ell \vdash} \neg : l$$

   to obtain π_i of $\langle\mathbf{A}\rangle^\ell, \Gamma, \Pi \vdash \Lambda, \Delta = S \circ S_i$. The desired properties of π_i and $\mathcal{P}_i$ follow trivially from the fact that they hold for π_j and $\mathcal{P}_{i-1}$ respectively.
   (b) S_i is derived from S_j by some other propositional rule: analogously to the previous case, there exists a unique axiom introducing the auxiliary formula of the inference in π_j. Depending on the rule applied,

we perform one of the following replacements to obtain π_i:

$$\neg^F : \langle\neg\mathbf{A}\rangle^\ell \vdash \langle\neg\mathbf{A}\rangle^\ell \quad\rightsquigarrow\quad \dfrac{\langle\mathbf{A}\rangle^\ell \vdash \langle\mathbf{A}\rangle^\ell}{\vdash \langle\neg\mathbf{A}\rangle^\ell, \langle\mathbf{A}\rangle^\ell}\ \neg\colon r$$

$$\vee^T : \langle\mathbf{A}\vee\mathbf{B}\rangle^\ell \vdash \langle\mathbf{A}\vee\mathbf{B}\rangle^\ell \quad\rightsquigarrow\quad \dfrac{\langle\mathbf{A}\rangle^\ell \vdash \langle\mathbf{A}\rangle^\ell \quad \langle\mathbf{B}\rangle^\ell \vdash \langle\mathbf{B}\rangle^\ell}{\langle\mathbf{A}\vee\mathbf{B}\rangle^\ell \vdash \langle\mathbf{A}\rangle^\ell, \langle\mathbf{B}\rangle^\ell}\ \vee\colon l$$

$$\vee_l^F : \langle\mathbf{A}\vee\mathbf{B}\rangle^\ell \vdash \langle\mathbf{A}\vee\mathbf{B}\rangle^\ell \quad\rightsquigarrow\quad \dfrac{\langle\mathbf{A}\rangle^\ell \vdash \langle\mathbf{A}\rangle^\ell}{\langle\mathbf{A}\rangle^\ell \vdash \langle\mathbf{A}\vee\mathbf{B}\rangle^\ell}\ \vee\colon r^1$$

$$\vee_r^F : \langle\mathbf{A}\vee\mathbf{B}\rangle^\ell \vdash \langle\mathbf{A}\vee\mathbf{B}\rangle^\ell \quad\rightsquigarrow\quad \dfrac{\langle\mathbf{B}\rangle^\ell \vdash \langle\mathbf{B}\rangle^\ell}{\langle\mathbf{B}\rangle^\ell \vdash \langle\mathbf{A}\vee\mathbf{B}\rangle^\ell}\ \vee\colon r^2$$

The replacements for the cases of $\wedge^F, \wedge_l^T, \wedge_r^T, \rightarrow^T, \rightarrow_l^F, \rightarrow_r^F$ are analogous. As in the previous case, the desired properties of π_i and $\mathcal{P}_i$ follow from those of π_j and $\mathcal{P}_{i-1}$.

(c) $S_i = \langle\mathbf{AS}\rangle^\ell, \Pi \vdash \Lambda$ is derived from $S_j = \langle\forall\mathbf{A}\rangle^\ell, \Pi \vdash \Lambda$ by $\forall^F$. Then the end-sequent of π_j is $\langle\forall\mathbf{A}\rangle^\ell, \Pi, \Gamma \vdash \Delta, \Lambda$. By S_j-linearity and suitable axiom labels there exists a unique axiom $\langle\forall\mathbf{A}\rangle^\ell \vdash \langle\forall\mathbf{A}\rangle^\ell$ introducing the ancestor of $\langle\forall\mathbf{A}\rangle^\ell$. By $\mathcal{S}$-balancedness, we may replace it by

$$\dfrac{\langle\mathbf{AS}\rangle^\ell \vdash \langle\mathbf{AS}\rangle^\ell}{\langle\mathbf{AS}\rangle^\ell \vdash \langle\forall\mathbf{A}\rangle^\ell}\ \forall^{sk}\colon r$$

to obtain π_i of $\langle\mathbf{AS}\rangle^\ell, \Pi, \Gamma \vdash \Delta, \Lambda$. As π_j is weakly regular, so is π_i (note that the Skolem symbol of this inference does not occur in π_j by assumption and the fact that it is fresh in γ). As π_j is Skolem parallel to the $\mathbf{LK}_{\mathrm{skc}}$-trees in $\mathcal{P}_{i-1}$, so is π_i as the downwards paths of auxiliary formulas of strong labelled quantifier inferences are unchanged, except for the new inference which has a fresh symbol. Restrictedness, $\mathcal{S}$-balancedness and suitable axiom labels carry over from π_j.

(d) $S_i = \Pi \vdash \Lambda, \langle\mathbf{AX}\rangle^{\ell,\mathbf{X}}$ is derived from $S_j = \Pi \vdash \Lambda, \langle\forall\mathbf{A}\rangle^\ell$ by $\forall^T$. By (IH) we have an $\mathbf{LK}_{\mathrm{skc}}$-tree π_j of $\Pi, \Gamma \vdash \Delta, \Lambda, \langle\forall\mathbf{A}\rangle^\ell$. By S_j-linearity there exists a unique axiom $\langle\forall\mathbf{A}\rangle^\ell \vdash \langle\forall\mathbf{A}\rangle^\ell$ introducing the ancestor of $\langle\forall\mathbf{A}\rangle^\ell$. By $\mathcal{S}$-balancedness, we may replace it by

$$\dfrac{\langle\mathbf{AX}\rangle^{\ell,\mathbf{X}} \vdash \langle\mathbf{AX}\rangle^{\ell,\mathbf{X}}}{\langle\forall\mathbf{A}\rangle^\ell \vdash \langle\mathbf{AX}\rangle^{\ell,\mathbf{X}}}\ \forall^{sk}\colon l$$

to obtain π_i of $\Pi, \Gamma \vdash \Delta, \Lambda, \langle\mathbf{AX}\rangle^{\ell,\mathbf{X}}$. Again the desired properties carry over from π_j.

(e) S_i is inferred from S_j by Sub with substitution σ. As S is proper, $\pi_i = \pi_j\sigma$ is an $\mathbf{LK}_{\mathrm{skc}}$-tree of $S_j\sigma \circ S$ which is restricted, $\mathcal{S}$-balanced, weakly regular, and Skolem parallel to the $\mathbf{LK}_{\mathrm{skc}}$-trees in $\mathcal{P}_{i-1}$ by Proposition 12 and (IH).

(f) $S_i = \Gamma_j, \Gamma_k \vdash \Delta_j, \Delta_k$ is derived from $S_j = \Gamma_j \vdash \Delta_j, \langle\mathbf{A}\rangle^{\ell_1}, \ldots, \langle\mathbf{A}\rangle^{\ell_n}$ and $S_k = \langle\mathbf{A}\rangle^{\ell_{n+1}}, \ldots, \langle\mathbf{A}\rangle^{\ell_m}, \Gamma_k \vdash \Delta_k$ by Cut. By Proposition 11, we may delete labels from the ancestors of occurrences of $\mathbf{A}$ from π_j, π_k respectively, denote these trees by π'_j, π'_k. Take for π_i

$$\dfrac{\dfrac{\dfrac{(\pi'_j)}{\Gamma, \Gamma_j \vdash \Delta, \Delta_j, \mathbf{A}, \ldots, \mathbf{A}}}{\Gamma, \Gamma_j \vdash \Delta, \Delta_j, \mathbf{A}}\ \text{contr}: r \quad \dfrac{\dfrac{(\pi'_k)}{\mathbf{A}, \ldots, \mathbf{A}, \Gamma_k, \Gamma \vdash \Delta_k, \Delta}}{\mathbf{A}, \Gamma_k, \Gamma \vdash \Delta_k, \Delta}\ \text{contr}: l}{\dfrac{\Gamma, \Gamma, \Gamma_j, \Gamma_k \vdash \Delta, \Delta, \Delta_j, \Delta_k}{\Gamma, \Gamma_j, \Gamma_k \vdash \Delta, \Delta_j, \Delta_k}\ \text{contr}: *}\ \text{cut}$$

As π_j, π_k are Skolem parallel and weakly regular, and we contract on Γ, Δ, π_i is weakly regular. As the downwards paths of ancestors of S only change by some repetitions, π_i and the $\mathbf{LK}_{\mathrm{skc}}$-trees in $\mathcal{P}_{i-1}$ are Skolem parallel. π_i is restricted because π_j, π_k are S_j-linear and S_k-linear, respectively. S_i-linearity follows from S_j-linearity and S_k-linearity. As π_j, π_k are $\mathcal{S}$-balanced, also π_i is. As π_j, π_k have suitable axiom labels, also π_i has: going from π_j to π'_j, we only delete labels of occurrences that are cut-ancestors in π_i (analogously for π_k). The suitable axiom labels hence remain by $\mathcal{S}$-balancedness. □

Lemma 4. *Let π be a restricted $\mathbf{LK}_{\mathrm{skc}}$-proof of S. Then there exists a $\mathbf{LK}_{\mathrm{sk}}$-proof of S.*

Proof. We proceed by induction on the number of Cut inferences in π. Consider a subtree φ of π that ends in an uppermost Cut ρ. Let the end-sequent of φ be $S_1 \circ S_2$, where S_1 are the end-sequent ancestors and S_2 are the cut-ancestors (in π)). We will transform φ into an $\mathbf{LK}_{\mathrm{sk}}$-tree φ' such that replacing φ by φ' in π results in a restricted $\mathbf{LK}_{\mathrm{skc}}$-proof of S (in particular φ' will be S_2-restricted). We proceed by induction on the height of ρ.

1. ρ occurs directly below axioms. Then ρ is

$$\dfrac{\langle\mathbf{A}\rangle^{\ell_1} \vdash \langle\mathbf{A}\rangle^{\ell_2} \quad \langle\mathbf{A}\rangle^{\ell_3} \vdash \langle\mathbf{A}\rangle^{\ell_4}}{\langle\mathbf{A}\rangle^{\ell_1} \vdash \langle\mathbf{A}\rangle^{\ell_4}}\ \text{Cut}$$

and we replace it by $\langle\mathbf{A}\rangle^{\ell_1} \vdash \langle\mathbf{A}\rangle^{\ell_4}$.

2. ρ does not occur directly below axioms. Then we permute ρ up. The only interesting case is permuting ρ over a contraction — here, the Cut is duplicated and the context contracted. By this contraction, weak regularity is preserved. Since the heights of both cuts is decreased, we may apply the induction hypothesis twice to obtain the desired $\mathbf{LK}_{\mathrm{skc}}$-proof. □

We may now state the main theorem of this section:

Theorem 2. *Let π be a regular, proper $\mathbf{LK}_{\mathrm{skc}}$-proof of S such that there exists an $\mathcal{R}_{al}$-refutation of* $\mathrm{CS}(\pi)$*. Then there an $\mathbf{LK}_{\mathrm{sk}}$-proof of S.*

Proof. By Lemma 2 and Lemma 3, there exists a restricted $\mathbf{LK}_{\mathrm{skc}}$-proof of S. By Lemma 4, there exists an $\mathbf{LK}_{\mathrm{sk}}$-proof of S. □

To see that CERES$^{\omega}$ is a cut-elimination method for **LK**, we will show in the next section that $\mathbf{LK}_{\mathrm{sk}}$-proofs can be translated to cut-free **LK**-proofs.

7. Soundness of $\mathbf{LK}_{\mathbf{sk}}$

This section will be devoted to proving that weak regularity suffices for soundness of $\mathbf{LK}_{\mathrm{sk}}$-proofs.

Definition 17. Let π be an $\mathbf{LK}_{\mathrm{sk}}$-tree, and ρ an inference in π. Define the *height* of ρ, height(ρ), as the maximal number of sequents between ρ and an axiom in π.

Lemma 5. *Let $\mathbf{T}$ be a Skolem term and π be a $\mathbf{LK}_{\mathrm{sk}}$-tree of S such that π does not contain a source inference of $\mathbf{T}$. Let $\mathbf{X}$ be a variable not occurring in π, then there exists an $\mathbf{LK}_{\mathrm{sk}}$-tree $\pi\,[\mathbf{T} \leftarrow \mathbf{X}]$ of $S\,[\mathbf{T} \leftarrow \mathbf{X}]$. Furthermore, if π is weakly regular (proper) then $\pi\,[\mathbf{T} \leftarrow \mathbf{X}]$ is weakly regular (proper).*

Proof. Let $\sigma = [\mathbf{T} \leftarrow \mathbf{X}]$, and let ρ be an inference in π with conclusion S. By induction on height(ρ), we construct $\mathbf{LK}_{\mathrm{sk}}$-trees π_ρ of $S\sigma$.

1. ρ is an axiom $\langle \mathbf{A} \rangle^{\ell_1} \vdash \langle \mathbf{A} \rangle^{\ell_2}$. Take for π_ρ the axiom $\langle \mathbf{A}\sigma \rangle^{\ell_1\sigma} \vdash \langle \mathbf{A}\sigma \rangle^{\ell_2\sigma}$.
2. ρ is a $\forall^{sk}: r$ inference

$$\frac{\Gamma \vdash \Delta, \langle \mathbf{FR} \rangle^{\ell}}{\Gamma \vdash \Delta, \langle \forall \mathbf{F} \rangle^{\ell}}\ \forall^{sk}: r$$

where $\mathbf{R}$ is the Skolem term of ρ. By (IH) we have a $\mathbf{LK}_{\mathrm{sk}}$-tree ψ of $\Gamma\sigma \vdash \Delta\sigma, \langle \mathbf{FR}\sigma \rangle^{l\sigma}$. Note that $\mathbf{FR}\sigma =_\beta \mathbf{F}\sigma\mathbf{R}\sigma$. Hence we may take for π_ρ

$$\frac{\begin{array}{c}(\psi)\\ \Gamma\sigma \vdash \Delta\sigma, \langle \mathbf{F}\sigma\mathbf{R}\sigma \rangle^{\ell\sigma}\end{array}}{\Gamma\sigma \vdash \Delta\sigma, \langle \forall \mathbf{F}\sigma \rangle^{\ell\sigma}}\ \forall^{sk}: r$$

3. ρ is a $\forall^{sk}: l$ inference

$$\frac{\langle \mathbf{FR} \rangle^{\ell,\mathbf{R}}, \Gamma \vdash \Delta}{\langle \forall \mathbf{F} \rangle^{\ell}, \Gamma \vdash \Delta}\ \forall^{sk}: l$$

By (IH) we have an $\mathbf{LK}_{\mathrm{sk}}$-tree ψ of $\langle \mathbf{FR}\sigma \rangle^{\ell\sigma,\mathbf{R}\sigma}, \Gamma\sigma \vdash \Delta\sigma$. By the soundness assumption for Skolem terms from [17], $\mathbf{T}$ does not contain variables bound in $\mathbf{F}$, hence $\mathbf{FR}\sigma =_\beta \mathbf{F}\sigma\mathbf{R}\sigma$. Therefore we may take as π_ρ:

$$\dfrac{\begin{matrix}(\psi)\\ \langle \mathbf{F}\sigma\mathbf{R}\sigma\rangle^{\ell\sigma,\mathbf{R}\sigma}, \Gamma\sigma \vdash \Delta\sigma\end{matrix}}{\langle \forall \mathbf{F}\sigma\rangle^{\ell\sigma}, \Gamma\sigma \vdash \Delta\sigma}\ \forall^{sk}: l$$

4. ρ is a structural or propositional inference. As in the previous cases, we simply apply the rule to the tree(s) obtained by hypothesis to obtain π_ρ.

Let ρ be the last inference in π; then we set $\pi\sigma = \pi_\rho$. It remains to show that weak regularity is preserved. As we apply σ on the whole tree, every path μ in $\pi\sigma$ induces a path ν in π such that $\mu = \nu\sigma$. Hence homomorphisms of downwards paths are preserved. □

Example 10. Consider the following $\mathbf{LK}_{\text{sk}}$-tree π, where $s \in \mathcal{K}_\iota$ and $f \in \mathcal{K}_{\iota,\iota}$:

$$\dfrac{\dfrac{\dfrac{\langle R(s,f(s),s)\rangle^{f(s)} \vdash \langle R(s,f(s),s)\rangle^{s}}{\langle R(s,f(s),s)\rangle^{f(s)} \vdash \langle (\forall x)R(s,x,s)\rangle^{s}}\ \forall^{sk}: r}{\langle R(s,f(s),s)\rangle^{f(s)} \vdash (\exists y)(\forall x)R(s,x,y)}\ \exists^{sk}: r}{(\forall y)R(s,y,s) \vdash (\exists y)(\forall x)R(s,x,y)}\ \forall^{sk}: l$$

Then $\pi\,[s \leftarrow z]$:

$$\dfrac{\dfrac{\dfrac{\langle R(z,f(z),z)\rangle^{f(z)} \vdash \langle R(s,f(z),z)\rangle^{z}}{\langle R(z,f(z),z)\rangle^{f(z)} \vdash \langle (\forall x)R(z,x,z)\rangle^{z}}\ \forall^{sk}: r}{\langle R(z,f(z),z)\rangle^{f(z)} \vdash (\exists y)(\forall x)R(z,x,y)}\ \exists^{sk}: r}{(\forall y)R(z,y,z) \vdash (\exists y)(\forall x)R(z,x,y)}\ \forall^{sk}: l$$

is an $\mathbf{LK}_{\text{sk}}$-tree.

Lemma 6. *Let ρ, ρ' be homomorphic inferences, and c their uniting contraction. Let $\rho_1, \ldots, \rho_n$ and $\rho'_1, \ldots, \rho'_m$ be the logical inferences operating on descendents of the auxiliary formulas of ρ, ρ' above c. Then $n = m$ and for all $1 \leq i \leq n$, ρ_i and ρ'_i are homomorphic.*

Proof. By induction on n. $n = 0$ is trivial. For the induction step, let μ, μ' be the homomorphic downwards paths from ρ, ρ' respectively to c. Consider ρ_1. As it is a logical inference, its auxiliary formula is different from its main formula. As $F(\mu) = F(\mu')$, there exists the logical inference ρ'_1 of the same type (and even with the same substitution or Skolem term, if applicable), and the downwards paths from ρ_1, ρ'_1 respectively to c exist and are homomorphic. Hence ρ_1, ρ'_1 are homomorphic and we may conclude with the induction hypothesis. □

7.1. Sequential Pruning

To show soundness of $\mathbf{LK}_{\text{sk}}$, we will transform $\mathbf{LK}_{\text{sk}}$-proofs into $\mathbf{LK}$-proofs. Roughly, this will be accomplished by permuting inferences and substituting eigenvariables for Skolem terms. In $\mathbf{LK}_{\text{sk}}$-proofs, a certain kind of redundancy may be present: namely, it may be the case that two strong labelled inferences on a common branch use the same Skolem term. This will prevent an eigenterm condition from holding, and hence in this situation we cannot substitute an eigenvariable for the Skolem term. This subsection is devoted to showing how to eliminate this redundancy.

Definition 18 (Sequential pruning). Let π be an $\mathbf{LK}_{\text{sk}}$-tree and ρ, ρ' inferences in π. Then ρ, ρ' are called *sequential* if they are on a common branch in π. We define the set of *sequential homomorphic pairs* as

$$\mathrm{SHP}(\pi) = \{\langle \rho, \rho' \rangle \mid \rho, \rho' \text{ homomorphic in } \pi \text{ and } \rho, \rho' \text{ sequential}\}.$$

We say that π is *sequentially pruned* if $\mathrm{SHP}(\pi) = \emptyset$.

Towards pruning sequential homomorphic pairs, we analyze the permutation of contraction inferences over independent inferences:

Definition 19. Let ρ be an inference above an inference σ. Then ρ and σ are *independent* if the auxiliary formula of σ is not a descendent of the main formula of ρ.

Definition 20 (The relation $\rhd_c$). We will now define the rewrite relation $\rhd_c$ for $\mathbf{LK}_{\text{sk}}$-trees π, π', where we assume the inferences contr: $*$ and σ to be independent:

1. If π is

$$\dfrac{\dfrac{\Pi, \Pi, \Gamma \vdash \Delta, \Lambda, \Lambda}{\Pi, \Gamma \vdash \Delta, \Lambda}\,\text{contr}\colon *}{\Pi, \Gamma' \vdash \Delta', \Lambda}\,\sigma$$

   and π' is

$$\dfrac{\dfrac{\Pi, \Pi, \Gamma \vdash \Delta, \Lambda, \Lambda}{\Pi, \Pi, \Gamma' \vdash \Delta', \Lambda, \Lambda}\,\sigma}{\Pi, \Gamma' \vdash \Delta', \Lambda}\,\text{contr}\colon *$$

   then $\pi \rhd_c^1 \pi'$.
2. If π is

$$\dfrac{\dfrac{\Pi, \Pi, \Gamma \vdash \Delta, \Lambda, \Lambda}{\Pi, \Gamma \vdash \Delta, \Lambda}\,\text{contr}\colon * \qquad \Sigma \vdash \Theta}{\Pi, \Gamma' \vdash \Delta', \Lambda}\,\sigma$$

   and π' is

$$\dfrac{\dfrac{\Pi, \Pi, \Gamma \vdash \Delta, \Lambda, \Lambda \qquad \Sigma \vdash \Theta}{\Pi, \Pi, \Gamma' \vdash \Delta', \Lambda, \Lambda}\,\sigma}{\Pi, \Gamma' \vdash \Delta', \Lambda}\,\text{contr}\colon *$$

then $\pi \rhd_c^1 \pi'$.

3. If π is

$$\dfrac{\Sigma \vdash \Theta \qquad \dfrac{\Pi, \Pi, \Gamma \vdash \Delta, \Lambda, \Lambda}{\Pi, \Gamma \vdash \Delta, \Lambda}\ \mathrm{contr}\colon *}{\Pi, \Gamma' \vdash \Delta', \Lambda}\ \sigma$$

and π' is

$$\dfrac{\dfrac{\Sigma \vdash \Theta \qquad \Pi, \Pi, \Gamma \vdash \Delta, \Lambda, \Lambda}{\Pi, \Pi, \Gamma' \vdash \Delta', \Lambda, \Lambda}\ \sigma}{\Pi, \Gamma' \vdash \Delta', \Lambda}\ \mathrm{contr}\colon *$$

then $\pi \rhd_c^1 \pi'$.

The $\rhd_c$ relation is then defined as the transitive and reflexive closure of the compatible closure of the $\rhd_c^1$ relation.

Lemma 7. *Let π be a weakly regular $\mathbf{LK}_{\mathrm{sk}}$-tree of S. If $\pi \rhd_c \psi$ then ψ is a weakly regular $\mathbf{LK}_{\mathrm{sk}}$-tree of S.*

Proof. By induction on the length of the $\rhd_c$-rewrite sequence. The case of $\pi = \psi$ is trivial, so assume there exists a subtree φ of π such that $\varphi \rhd_c^1 \varphi'$ and ψ is obtained from π by replacing φ by φ'. Then the end-sequent of ψ is the same as that of π. Also weak regularity is preserved: The paths in ψ and π are the same modulo some repetitions. □

Lemma 8. *Let π be a $\mathbf{LK}_{\mathrm{sk}}$-tree with end-sequent S such that π is not sequentially pruned. Then there exists a $\mathbf{LK}_{\mathrm{sk}}$-tree π' with end-sequent S such that*

$$|\mathrm{SHP}(\pi')| < |\mathrm{SHP}(\pi)|$$

Furthermore, if π is weakly regular, so is π'.

Proof. Consider a sequential homomorphic pair in π with uniting contraction c. By Lemma 6, there exists a sequential homomorphic pair ρ, ρ' with uniting contraction c such that no logical inference operates on descendents of the auxiliary formulas of ρ, ρ' above c (ρ, ρ' are the lowermost ρ_i, ρ'_j of Lemma 6, respectively). W.l.o.g. assume that ρ is above ρ'. As no logical inference operates on descendents ω of the auxiliary formula of ρ on the path to c, we can permute all contraction inferences operating on such ω below ρ' using $\rhd_c$. By Lemma 7 the resulting tree is weakly regular and its end-sequent is S. Clearly the number of sequential homomorphic pairs stays the same.

For example, if there are two such contractions inferences between ρ and ρ', the situation is

$$
\begin{array}{l}
\overline{\qquad\quad}\ \rho \\
\quad\vdots \\
\overline{\qquad\quad}\ \mathrm{contr}\colon l \\
\quad\vdots \\
\overline{\qquad\quad}\ \mathrm{contr}\colon l \\
\quad\vdots \\
\overline{\qquad\quad}\ \rho'
\end{array}
$$

which is transformed to

$$
\begin{array}{l}
\overline{\qquad\quad}\ \rho \\
\quad\vdots \\
\overline{\qquad\quad}\ \rho' \\
\overline{\qquad\quad}\ \mathrm{contr}\colon l \\
\overline{\qquad\quad}\ \mathrm{contr}\colon l
\end{array}
$$

Hence we may assume that no inference operates on descendents of the auxiliary formula of ρ between ρ, ρ'. Now distinguish the cases

1. ρ is a unary inference. W.l.o.g. assume that the auxiliary and main formulas of ρ occur on the right. Then the situation is:

$$
\begin{array}{c}
\dfrac{\Gamma \vdash \Delta, \langle \mathbf{F} \rangle^{\ell_1}}{\Gamma \vdash \Delta, \langle \mathbf{G} \rangle^{\ell_2}}\ \rho \\
\vdots \\
\dfrac{\Gamma' \vdash \Delta', \langle \mathbf{F} \rangle^{\ell_1}, \langle \mathbf{G} \rangle^{\ell_2}}{\Gamma' \vdash \Delta', \langle \mathbf{G} \rangle^{\ell_2}, \langle \mathbf{G} \rangle^{\ell_2}}\ \rho' \\
\vdots \\
\dfrac{\Gamma^* \vdash \Delta^*, \langle \mathbf{G} \rangle^{\ell_2}, \langle \mathbf{G} \rangle^{\ell_2}}{\Gamma^* \vdash \Delta^*, \langle \mathbf{G} \rangle^{\ell_2}}\ c
\end{array}
$$

   We replace this subtree by

$$
\begin{array}{c}
\Gamma \vdash \Delta, \langle \mathbf{F} \rangle^{\ell_1} \\
\vdots \\
\dfrac{\dfrac{\Gamma' \vdash \Delta', \langle \mathbf{F} \rangle^{\ell_1}, \langle \mathbf{F} \rangle^{\ell_1}}{\Gamma' \vdash \Delta', \langle \mathbf{F} \rangle^{\ell_1}}\ c}{\Gamma' \vdash \Delta', \langle \mathbf{G} \rangle^{\ell_2}}\ \rho' \\
\vdots \\
\Gamma^* \vdash \Delta^*, \langle \mathbf{G} \rangle^{\ell_2}
\end{array}
$$

2. ρ is a $\vee\colon l$ inference. W.l.o.g. the situation is

$$\dfrac{\langle\mathbf{F}\rangle^{\ell}, \Gamma \vdash \Delta \qquad \langle\mathbf{G}\rangle^{\ell}, \Pi \vdash \Lambda}{\langle\mathbf{F} \vee \mathbf{G}\rangle^{\ell}, \Gamma, \Pi \vdash \Delta, \Lambda}\ \rho$$

$$\vdots$$

$$\dfrac{\langle\mathbf{F}\rangle^{\ell}, \langle\mathbf{F} \vee \mathbf{G}\rangle^{\ell}, \Gamma^{*} \vdash \Delta^{*} \qquad \langle\mathbf{G}\rangle^{\ell}, \Pi^{*} \vdash \Lambda^{*}}{\langle\mathbf{F} \vee \mathbf{G}\rangle^{\ell}, \langle\mathbf{F} \vee \mathbf{G}\rangle^{\ell}, \Gamma^{*}, \Pi^{*} \vdash \Delta^{*}, \Lambda^{*}}\ \rho'$$

$$\vdots$$

$$\dfrac{\langle\mathbf{F} \vee \mathbf{G}\rangle^{\ell}, \langle\mathbf{F} \vee \mathbf{G}\rangle^{\ell}, \Gamma^{+} \vdash \Delta^{+}}{\langle\mathbf{F} \vee \mathbf{G}\rangle^{\ell}, \Gamma^{+} \vdash \Delta^{+}}\ c$$

This is transformed to

$$\dfrac{\langle\mathbf{F}\rangle^{\ell}, \Gamma \vdash \Delta}{\langle\mathbf{F}\rangle^{\ell}, \Gamma, \Pi \vdash \Delta, \Lambda}\ \text{weak}\colon *$$

$$\vdots$$

$$\dfrac{\dfrac{\langle\mathbf{F}\rangle^{\ell}, \langle\mathbf{F}\rangle^{\ell}, \Gamma^{*} \vdash \Delta^{*}}{\langle\mathbf{F}\rangle^{\ell}, \Gamma^{*} \vdash \Delta^{*}}\ c \qquad \langle\mathbf{G}\rangle^{\ell}, \Pi^{*} \vdash \Lambda^{*}}{\langle\mathbf{F} \vee \mathbf{G}\rangle^{\ell}, \Gamma^{*}, \Pi^{*} \vdash \Delta^{*}, \Lambda^{*}}\ \rho'$$

$$\vdots$$

$$\langle\mathbf{F} \vee \mathbf{G}\rangle^{\ell}, \Gamma^{+} \vdash \Delta^{+}$$

As we only permute contractions and delete inferences, weak regularity is preserved by this transformation. Furthermore, consider a sequential homomorphic pair $\langle\sigma, \sigma'\rangle$ in π' (w.l.o.g. we consider the case that ρ is $\vee\colon l$). Clearly σ, σ' also exist in π and $\langle\sigma, \sigma'\rangle$ is a homomorphic pair in π (if its uniting contraction in π' is c in the second figure, then the c in the first figure is its uniting contraction in π). It is sequential since we have not changed the branching structure of the tree (except for deleting a subtree from π to obtain π').

Hence the number of sequentially homomorphic pairs is reduced, which was to show. □

Lemma 9 (Sequential Pruning). *Let π be a $\mathbf{LK}_{\mathrm{sk}}$-tree of S, then there exists $\mathbf{LK}_{\mathrm{sk}}$-tree π' of S s.t. π' is sequentially pruned. Furthermore, if π is weakly regular, so is π'.*

Proof. Repeated application of Lemma 8 does the job. □

Example 11. Consider the $\mathbf{LK}_{\mathrm{sk}}$-tree π:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(s_1)\vdash P(s_1) \qquad P(s_1)\vdash P(s_1)}{P(s_1)\lor P(s_1)\vdash P(s_1),P(s_1)}\,\lor\colon l}{P(s_1)\lor P(s_1)\vdash P(s_1),(\forall x)P(x)}\,\forall^{sk}\colon r}{P(s_1)\lor P(s_1)\vdash (\forall x)P(x),(\forall x)P(x)}\,\forall^{sk}\colon r \qquad \dfrac{Q(t_1)\vdash Q(t_1)}{(\forall x)Q(x)\vdash Q(t_1)}\,\forall^{sk}\colon l}{P(s_1)\lor P(s_1),(\forall x)P(x)\to(\forall x)Q(x)\vdash(\forall x)P(x),Q(t_1)}\,\to\colon l \qquad \dfrac{Q(t_2)\vdash Q(t_2)}{(\forall x)Q(x)\vdash Q(t_2)}\,\forall^{sk}\colon l}{P(s_1)\lor P(s_1),(\forall x)P(x)\to(\forall x)Q(x),(\forall x)P(x)\to(\forall x)Q(x)\vdash Q(t_1),Q(t_2)}\,\to\colon l}{P(s_1)\lor P(s_1),(\forall x)P(x)\to(\forall x)Q(x)\vdash Q(t_1),Q(t_2)}\,\mathrm{contr}\colon l
$$

where $s_1, s_2 \in \mathcal{K}_\iota$.

Denote the upper-left $\forall^{sk}\colon r$ application by ρ_1, the $\forall^{sk}\colon r$ application directly below ρ_1 by ρ_2, the upper $\to\colon l$ application by η_1 and the lower $\to\colon l$ application by η_2. Then

$$\mathrm{SHP}(\pi) = \{\{\rho_1,\rho_2\},\{\eta_1,\eta_2\}\}$$

and the contr: l application is the uniting contraction of both pairs. We apply Lemma 8, removing $\{\eta_1,\eta_2\}$ and obtaining π':

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(s_1)\vdash P(s_1) \qquad P(s_1)\vdash P(s_1)}{P(s_1)\lor P(s_1)\vdash P(s_1),P(s_1)}\,\lor\colon l}{P(s_1)\lor P(s_1)\vdash P(s_1),(\forall x)P(x)}\,\forall^{sk}\colon r}{P(s_1)\lor P(s_1)\vdash (\forall x)P(x),(\forall x)P(x)}\,\forall^{sk}\colon r}{P(s_1)\lor P(s_1)\vdash (\forall x)P(x),(\forall x)P(x),Q(t_1)}\,\mathrm{weak}\colon r}{P(s_1)\lor P(s_1)\vdash (\forall x)P(x),Q(t_1)}\,\mathrm{contr}\colon r \qquad \dfrac{Q(t_2)\vdash Q(t_2)}{(\forall x)Q(x)\vdash Q(t_2)}\,\forall^{sk}\colon l}{P(s_1)\lor P(s_1),(\forall x)P(x)\to(\forall x)Q(x)\vdash Q(t_1),Q(t_2)}\,\to\colon l
$$

such that

$$\mathrm{SHP}(\pi') = \{\{\rho_1,\rho_2\}\}$$

We apply Lemma 8 again, removing $\{\rho_1,\rho_2\}$ and obtaining the sequentially pruned π'':

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(s_1)\vdash P(s_1) \qquad P(s_1)\vdash P(s_1)}{P(s_1)\lor P(s_1)\vdash P(s_1),P(s_1)}\,\lor\colon l}{P(s_1)\lor P(s_1)\vdash P(s_1)}\,\mathrm{contr}\colon r}{P(s_1)\lor P(s_1)\vdash (\forall x)P(x)}\,\forall^{sk}\colon r}{P(s_1)\lor P(s_1)\vdash (\forall x)P(x),Q(t_1)}\,\mathrm{weak}\colon r \qquad \dfrac{Q(t_2)\vdash Q(t_2)}{(\forall x)Q(x)\vdash Q(t_2)}\,\forall^{sk}\colon l}{P(s_1)\lor P(s_1),(\forall x)P(x)\to(\forall x)Q(x)\vdash Q(t_1),Q(t_2)}\,\to\colon l
$$

7.2. *Translating* $\mathbf{LK}_{\mathrm{sk}}$ *to* $\mathbf{LK}$

The main result of this subsection will be to show that $\mathbf{LK}_{\mathrm{sk}}$-proofs can be translated into **LK**-proofs. The proof will be effective, and will be based on permuting inferences and pruning. To this end, we will analyze the permutation of inferences in $\mathbf{LK}_{\mathrm{sk}}$-trees. Such an analysis is often useful, see for example [20] for the case of a first-order sequent calculus. In $\mathbf{LK}_{\mathrm{sk}}$, we have more freedom in the permutation of inferences since we do not have to consider an eigenvariable condition, although we will want to preserve weak regularity.

To ease the following case distinctions, we introduce the following notation:

$$\begin{aligned} \Gamma, A^1 &= \Gamma, A \\ \Gamma, A^0 &= \Gamma \end{aligned}$$

and let $i, i_1, \ldots, i_4 \in \{0,1\}$, $\bar{x} = |x-1|$. In the following transformations, we do not display the labels of the labelled formula occurrences since we always leave them unchanged (what this means exactly will be clear from the context).

Definition 21 (The relation $\triangleright_u$). This definition shows how to permute down a unary logical inference ρ over an inference σ, assuming that ρ and σ are independent. We do not write down the cases involving $\wedge\colon r, \to\colon l, \to\colon r$ inferences, since they are analogous. In case 1, σ is a unary logical inference, in case 2 σ is a weakening inference, in case 3 σ is a contraction inference, and in cases 4–5 σ is an $\vee\colon l$ inference. We define a relation $\triangleright_u^1$ between $\mathbf{LK}_{\mathrm{sk}}$-trees π and π':

1. If π is

$$\dfrac{\dfrac{\mathbf{F}^{i_1}, \mathbf{G}^{i_2}, \Gamma \vdash \Delta, \mathbf{G}^{\bar{i_2}}, \mathbf{F}^{\bar{i_1}}}{\mathbf{M}^{i_3}, \mathbf{G}^{i_2}, \Gamma \vdash \Delta, \mathbf{G}^{\bar{i_2}}, \mathbf{M}^{\bar{i_3}}}\,\rho}{\mathbf{M}^{i_3}, \mathbf{N}^{i_4}, \Gamma \vdash \Delta, \mathbf{N}^{\bar{i_4}}, \mathbf{M}^{\bar{i_3}}}\,\sigma$$

   and π' is

$$\dfrac{\dfrac{\mathbf{F}^{i_1}, \mathbf{G}^{i_2}, \Gamma \vdash \Delta, \mathbf{G}^{\bar{i_2}}, \mathbf{F}^{\bar{i_1}}}{\mathbf{F}^{i_1}, \mathbf{N}^{i_4}, \Gamma \vdash \Delta, \mathbf{N}^{\bar{i_4}}, \mathbf{F}^{\bar{i_1}}}\,\sigma}{\mathbf{M}^{i_3}, \mathbf{N}^{i_2}, \Gamma \vdash \Delta, \mathbf{N}^{\bar{i_2}}, \mathbf{M}^{\bar{i_3}}}\,\rho$$

   then $\pi \triangleright_u^1 \pi'$.
2. If π is

$$\dfrac{\dfrac{\mathbf{F}^{i_1}, \Gamma \vdash \Delta, \mathbf{F}^{\bar{i_1}}}{\mathbf{M}^{i_2}, \Gamma \vdash \Delta, \mathbf{M}^{\bar{i_2}}}\,\rho}{\mathbf{N}^{i_3}, \mathbf{M}^{i_2}, \Gamma \vdash \Delta, \mathbf{M}^{\bar{i_2}}, \mathbf{N}^{\bar{i_3}}}\,\sigma\ (\mathrm{weak}\colon *)$$

   and π' is

$$\dfrac{\dfrac{\mathbf{F}^{i_1}, \Gamma \vdash \Delta, \mathbf{F}^{\bar{i_1}}}{\mathbf{N}^{i_3}, \mathbf{F}^{i_1}, \Gamma \vdash \Delta, \mathbf{F}^{\bar{i_1}}, \mathbf{N}^{\bar{i_3}}}\,\sigma\ (\mathrm{weak}\colon *)}{\mathbf{N}^{i_3}, \mathbf{M}^{i_2}, \Gamma \vdash \Delta, \mathbf{M}^{\bar{i_2}}, \mathbf{N}^{\bar{i_3}}}\,\rho$$

   then $\pi \triangleright_u^1 \pi'$.
3. If π is

$$\dfrac{\dfrac{\mathbf{F}^{i_1}, \mathbf{G}^{i_2}, \mathbf{G}^{i_2}, \Gamma \vdash \Delta, \mathbf{G}^{\bar{i_2}}, \mathbf{G}^{\bar{i_2}}, \mathbf{F}^{\bar{i_1}}}{\mathbf{M}^{i_3}, \mathbf{G}^{i_2}, \mathbf{G}^{i_2}, \Gamma \vdash \Delta, \mathbf{G}^{\bar{i_2}}, \mathbf{G}^{\bar{i_2}}, \mathbf{M}^{\bar{i_3}}}\,\rho}{\mathbf{M}^{i_3}, \mathbf{G}^{i_2}, \Gamma \vdash \Delta, \mathbf{G}^{\bar{i_2}}, \mathbf{M}^{\bar{i_3}}}\,\sigma\ (\mathrm{contr}\colon *)$$

and π' is

$$\dfrac{\dfrac{\mathbf{F}^{i_1}, \mathbf{G}^{i_2}, \mathbf{G}^{i_2}, \Gamma \vdash \Delta, \mathbf{G}^{\bar{i_2}}, \mathbf{G}^{\bar{i_2}}, \mathbf{F}^{\bar{i_1}}}{\mathbf{F}^{i_1}, \mathbf{G}^{i_2}, \Gamma \vdash \Delta, \mathbf{G}^{\bar{i_2}}, \mathbf{F}^{\bar{i_1}}}\ \sigma\ (\text{contr}: *)}{\mathbf{M}^{i_3}, \mathbf{G}^{i_2}, \Gamma \vdash \Delta, \mathbf{G}^{\bar{i_2}}, \mathbf{M}^{\bar{i_3}}}\ \rho$$

then $\pi \rhd_u^1 \pi'$.

4. If π is

$$\dfrac{\dfrac{\mathbf{F}^{i_1}, \mathbf{G}_1, \Gamma \vdash \Delta, \mathbf{F}^{\bar{i_1}}}{\mathbf{M}^{i_2}, \mathbf{G}_1, \Gamma \vdash \Delta, \mathbf{M}^{\bar{i_2}}}\ \rho \qquad \mathbf{G}_2, \Pi \vdash \Lambda}{\mathbf{G}_1 \vee \mathbf{G}_2, \mathbf{M}^{i_2}, \Gamma, \Pi \vdash \Delta, \Lambda, \mathbf{M}^{\bar{i_2}}}\ \sigma$$

and π' is

$$\dfrac{\dfrac{\mathbf{F}^{i_1}, \mathbf{G}_1, \Gamma \vdash \Delta, \mathbf{F}^{\bar{i_1}} \qquad \mathbf{G}_2, \Pi \vdash \Lambda}{\mathbf{G}_1 \vee \mathbf{G}_2, \mathbf{F}^{i_1}, \Gamma, \Pi \vdash \Delta, \Lambda, \mathbf{F}^{\bar{i_1}}}\ \sigma}{\mathbf{G}_1 \vee \mathbf{G}_2, \mathbf{M}^{i_2}, \Gamma, \Pi \vdash \Delta, \Lambda, \mathbf{M}^{\bar{i_2}}}\ \rho$$

then $\pi \rhd_u^1 \pi'$.

5. If π is

$$\dfrac{\mathbf{G}_1, \Gamma \vdash \Delta \qquad \dfrac{\mathbf{F}^{i_1}, \mathbf{G}_2, \Pi \vdash \Lambda, \mathbf{F}^{\bar{i_1}}}{\mathbf{M}^{i_2}, \mathbf{G}_2, \Pi \vdash \Lambda, \mathbf{M}^{\bar{i_2}}}\ \rho}{\mathbf{G}_1 \vee \mathbf{G}_2, \mathbf{M}^{i_2}, \Gamma, \Pi \vdash \Delta, \Lambda, \mathbf{M}^{\bar{i_2}}}\ \sigma$$

and π' is

$$\dfrac{\dfrac{\mathbf{G}_1, \Gamma \vdash \Delta \qquad \mathbf{F}^{i_1}, \mathbf{G}_2, \Pi \vdash \Lambda, \mathbf{F}^{\bar{i_1}}}{\mathbf{G}_1 \vee \mathbf{G}_2, \mathbf{F}^{i_1}, \Gamma, \Pi \vdash \Delta, \Lambda, \mathbf{F}^{\bar{i_1}}}\ \sigma}{\mathbf{G}_1 \vee \mathbf{G}_2, \mathbf{M}^{i_2}, \Gamma, \Pi \vdash \Delta, \Lambda, \mathbf{M}^{\bar{i_2}}}\ \rho$$

then $\pi \rhd_u^1 \pi'$.

Finally, we define the $\rhd_u$ relation as the transitive and reflexive closure of the compatible closure of the $\rhd_u^1$ relation.

Lemma 10. *Let π be a weakly regular $\mathbf{LK}_{\mathrm{sk}}$-tree of S. If $\pi \rhd_u \psi$ then ψ is a weakly regular $\mathbf{LK}_{\mathrm{sk}}$-tree of S.*

Proof. By induction on the length of the $\rhd_u$-rewrite sequence. The case of $\pi = \psi$ is trivial, so assume there exists a subtree φ of π such that $\varphi \rhd_u^1 \varphi'$ and ψ is obtained from π by replacing φ by φ'. Then the end-sequent of ψ is the same as that of π. Also weak regularity is preserved since the paths in ψ and π are the same modulo some repetitions. $\square$

Definition 22 (The relation $\rhd_b$). This definition shows how to permute down a $\vee\colon l$ inference ρ (the cases for $\wedge\colon r, \rightarrow\colon l$ are analogous), together with some contractions the auxiliary formulas of which come from both permises of ρ. In the prooftrees, the indicated occurrences of $\mathbf{F}_1$ and $\mathbf{F}_2$ will be the auxiliary occurrences of ρ. Again, we leave out the cases involving $\wedge\colon r, \rightarrow\colon l, \rightarrow\colon r$ since they are analogous. We will now define the rewrite relation $\rhd_b$ on $\mathbf{LK}_{\mathrm{sk}}$-trees, where we assume ρ and σ to be independent. Cases 1–3 treat the case of σ being a unary logical inference, in case 4 σ is a weakening inference, in cases 5–6 σ is a contraction inference, and in cases 7–9 σ is $\vee\colon l$.

1. If π is

$$\dfrac{\dfrac{\dfrac{\mathbf{F}_1, \Pi, \Gamma_1, \mathbf{G}^{i_1} \vdash \Delta_1, \mathbf{G}^{\bar{i_1}}, \Lambda \qquad \mathbf{F}_2, \Pi, \Gamma_2 \vdash \Delta_2, \Lambda}{\mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \Pi, \Gamma_1, \Gamma_2, \mathbf{G}^{i_1} \vdash \Delta_1, \mathbf{G}^{\bar{i_1}}, \Delta_2, \Lambda, \Lambda}\,\rho}{\mathbf{G}^{i_1}, \mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \Lambda, \mathbf{G}^{\bar{i_1}}}\,\mathrm{contr}\colon *}{\mathbf{M}^{i_2}, \mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \Lambda, \mathbf{M}^{\bar{i_2}}}\,\sigma$$

and π' is

$$\dfrac{\dfrac{\dfrac{\mathbf{G}^{i_1}, \mathbf{F}_1, \Pi, \Gamma_1 \vdash \Delta_1, \Lambda, \mathbf{G}^{\bar{i_1}}}{\mathbf{M}^{i_2}, \mathbf{F}_1, \Pi, \Gamma_1 \vdash \Delta_1, \Lambda, \mathbf{F}_1, \mathbf{M}^{\bar{i_2}}}\,\sigma \qquad \mathbf{F}_2, \Pi, \Gamma_2 \vdash \Delta_2, \Lambda}{\mathbf{F}_1 \vee \mathbf{F}_2, \mathbf{M}^{i_2}, \Pi, \Pi, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \Lambda, \Lambda, \mathbf{M}^{\bar{i_2}}}\,\rho}{\mathbf{F}_1 \vee \mathbf{F}_2, \mathbf{M}^{i_2}, \Pi, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \Lambda, \mathbf{M}^{\bar{i_2}}}\,\mathrm{contr}\colon *$$

then $\pi \rhd_b^1 \pi'$.

2. If π is

$$\dfrac{\dfrac{\dfrac{\mathbf{F}_1, \Pi, \Gamma_1 \vdash \Delta_1, \Lambda \qquad \mathbf{F}_2, \Pi, \Gamma_2, \mathbf{G}^{i_1} \vdash \Delta_2, \Lambda, \mathbf{G}^{\bar{i_1}}}{\mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \Pi, \Gamma_1, \Gamma_2, \mathbf{G}^{i_1} \vdash \Delta_1, \Delta_2, \Lambda, \Lambda, \mathbf{G}^{\bar{i_1}}}\,\rho}{\mathbf{G}^{i_1}, \mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \Lambda, \mathbf{G}^{\bar{i_1}}}\,\mathrm{contr}\colon *}{\mathbf{M}^{i_2}, \mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \Lambda, \mathbf{M}^{\bar{i_2}}}\,\sigma$$

and π' is

$$\dfrac{\dfrac{\mathbf{F}_1, \Pi, \Gamma_1 \vdash \Delta_1, \Lambda \qquad \dfrac{\mathbf{G}^{i_1}, \mathbf{F}_2, \Pi, \Gamma_2 \vdash \Delta_2, \Lambda, \mathbf{G}^{\bar{i_1}}}{\mathbf{M}^{i_2}, \mathbf{F}_2, \Pi, \Gamma_2 \vdash \Delta_2, \Lambda, \mathbf{M}^{\bar{i_2}}}\,\sigma}{\mathbf{F}_1 \vee \mathbf{F}_2, \mathbf{M}^{i_2}, \Pi, \Pi, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \Lambda, \Lambda, \mathbf{M}^{\bar{i_2}}}\,\rho}{\mathbf{F}_1 \vee \mathbf{F}_2, \mathbf{M}^{i_2}, \Pi, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \Lambda, \mathbf{M}^{\bar{i_2}}}\,\mathrm{contr}\colon *$$

then $\pi \rhd_b^1 \pi'$.

3. If π is

$$\dfrac{\dfrac{\dfrac{\mathbf{F}_1, \Pi, \mathbf{G}^{i_1}, \Gamma_1 \vdash \Delta_1, \Lambda, \mathbf{G}^{\bar{i_1}} \qquad \mathbf{F}_2, \Pi, \mathbf{G}^{i_1}, \Gamma_2 \vdash \Delta_2, \Lambda, \mathbf{G}^{\bar{i_1}}}{\mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \mathbf{G}^{i_1}, \Pi, \mathbf{G}^{i_1}, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \Lambda, \mathbf{G}^{\bar{i_1}}, \Lambda, \mathbf{G}^{\bar{i_1}}}\,\rho}{\mathbf{G}^{i_1}, \mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \Lambda, \mathbf{G}^{\bar{i_1}}}\,\mathrm{contr}\colon *}{\mathbf{M}^{i_2}, \mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \Lambda, \mathbf{M}^{\bar{i_2}}}\,\sigma$$

and π' is

$$\dfrac{\dfrac{\dfrac{\mathbf{G}^{i_1},\mathbf{F}_1,\Pi,\Gamma_1 \vdash \Delta_1,\Lambda,\mathbf{G}^{\bar{i}_1}}{\mathbf{M}^{i_2},\mathbf{F}_1,\Pi,\Gamma_1 \vdash \Delta_1,\Lambda,\mathbf{M}^{\bar{i}_2}}\,\sigma \qquad \dfrac{\mathbf{G}^{i_1},\mathbf{F}_2,\Pi,\Gamma_2 \vdash \Delta_2,\Lambda,\mathbf{G}^{\bar{i}_1}}{\mathbf{M}^{i_2},\mathbf{F}_2,\Pi,\Gamma_2 \vdash \Delta_2,\Lambda,\mathbf{M}^{\bar{i}_2}}\,\sigma}{\mathbf{F}_1 \vee \mathbf{F}_2,\Pi,\mathbf{M}^{i_2},\Pi,\mathbf{M}^{i_2},\Gamma_1,\Gamma_2 \vdash \Delta_1,\Delta_2,\Lambda,\mathbf{M}^{\bar{i}_2},\Lambda,\mathbf{M}^{\bar{i}_2}}\,\rho}{\mathbf{F}_1 \vee \mathbf{F}_2,\Pi,\mathbf{M}^{i_2},\Gamma_1,\Gamma_2 \vdash \Delta_1,\Delta_2,\Lambda,\mathbf{M}^{\bar{i}_2}}\,\text{contr}\colon *$$

then $\pi \rhd_b^1 \pi'$.

4. If π is

$$\dfrac{\dfrac{\dfrac{\mathbf{F}_1,\Pi,\Gamma_1 \vdash \Delta_1,\Lambda \qquad \mathbf{F}_2,\Pi,\Gamma_2 \vdash \Delta_2,\Lambda}{\mathbf{F}_1 \vee \mathbf{F}_2,\Pi,\Pi,\Gamma_1,\Gamma_2 \vdash \Delta_1,\Delta_2,\Lambda,\Lambda}\,\rho}{\mathbf{F}_1 \vee \mathbf{F}_2,\Pi,\Gamma_1,\Gamma_2 \vdash \Delta_1,\Delta_2,\Lambda}\,\text{contr}\colon *}{\mathbf{M}^{i},\mathbf{F}_1 \vee \mathbf{F}_2,\Pi,\Gamma_1,\Gamma_2 \vdash \Delta_1,\Delta_2,\Lambda,\mathbf{M}^{\bar{i}}}\,\sigma\ (\text{weak}\colon *)$$

and π' is

$$\dfrac{\dfrac{\dfrac{\mathbf{F}_1,\Pi,\Gamma_1 \vdash \Delta_1,\Lambda}{\mathbf{M}^{i},\mathbf{F}_1,\Pi,\Gamma_1 \vdash \Delta_1,\Lambda,\mathbf{M}^{\bar{i}}}\,\sigma\ (\text{weak}\colon *) \qquad \mathbf{F}_2,\Pi,\Gamma_2 \vdash \Delta_2,\Lambda}{\mathbf{F}_1 \vee \mathbf{F}_2,\mathbf{M}^{i},\Pi,\Pi,\Gamma_1,\Gamma_2 \vdash \Delta_1,\Delta_2,\Lambda,\Lambda,\mathbf{M}^{\bar{i}}}\,\rho}{\mathbf{F}_1 \vee \mathbf{F}_2,\mathbf{M}^{i},\Pi,\Gamma_1,\Gamma_2 \vdash \Delta_1,\Delta_2,\Lambda,\mathbf{M}^{\bar{i}}}\,\text{contr}\colon *$$

then $\pi \rhd_b^1 \pi'$.

5. If π is

$$\dfrac{\dfrac{\dfrac{\mathbf{F}_1,\Pi,\Gamma_1,\mathbf{G}^{i},\mathbf{G}^{i} \vdash \Delta_1,\Lambda,\mathbf{G}^{\bar{i}},\mathbf{G}^{\bar{i}} \qquad \mathbf{F}_2,\Pi,\Gamma_2 \vdash \Delta_2,\Lambda}{\mathbf{F}_1 \vee \mathbf{F}_2,\Pi,\Pi,\Gamma_1,\Gamma_2,\mathbf{G}^{i},\mathbf{G}^{i} \vdash \Delta_1,\Delta_2,\Lambda,\Lambda,\mathbf{G}^{\bar{i}},\mathbf{G}^{\bar{i}}}\,\rho}{\mathbf{G}^{i},\mathbf{G}^{i},\mathbf{F}_1 \vee \mathbf{F}_2,\Pi,\Gamma_1,\Gamma_2 \vdash \Delta_1,\Delta_2,\Lambda,\mathbf{G}^{\bar{i}},\mathbf{G}^{\bar{i}}}\,\text{contr}\colon *}{\mathbf{G}^{i},\mathbf{F}_1 \vee \mathbf{F}_2,\Pi,\Gamma_1,\Gamma_2 \vdash \Delta_1,\Delta_2,\Lambda,\mathbf{G}^{\bar{i}}}\,\sigma\ (\text{contr}\colon *)$$

and π' is

$$\dfrac{\dfrac{\dfrac{\mathbf{F}_1,\Pi,\Gamma_1,\mathbf{G}^{i},\mathbf{G}^{i} \vdash \Delta_1,\Lambda,\mathbf{G}^{\bar{i}},\mathbf{G}^{\bar{i}}}{\mathbf{F}_1,\Pi,\Gamma_1,\mathbf{G}^{i} \vdash \Delta_1,\Lambda,\mathbf{G}^{\bar{i}}}\,\sigma\ (\text{contr}\colon *) \qquad \mathbf{F}_2,\Pi,\Gamma_2 \vdash \Delta_2,\Lambda}{\mathbf{F}_1 \vee \mathbf{F}_2,\Pi,\Pi,\Gamma_1,\Gamma_2,\mathbf{G}^{i} \vdash \Delta_1,\Delta_2,\Lambda,\Lambda,\mathbf{G}^{\bar{i}}}\,\rho}{\mathbf{G}^{i},\mathbf{F}_1 \vee \mathbf{F}_2,\Pi,\Gamma_1,\Gamma_2 \vdash \Delta_1,\Delta_2,\Lambda,\mathbf{G}^{\bar{i}}}\,\text{contr}\colon *$$

then $\pi \rhd_b^1 \pi'$.

6. If π is

$$\dfrac{\dfrac{\dfrac{\mathbf{F}_1,\Pi,\Gamma_1 \vdash \Delta_1,\Lambda \qquad \mathbf{F}_2,\Pi,\Gamma_2,\mathbf{G}^{i},\mathbf{G}^{i} \vdash \Delta_2,\Lambda,\mathbf{G}^{\bar{i}},\mathbf{G}^{\bar{i}}}{\mathbf{F}_1 \vee \mathbf{F}_2,\Pi,\Pi,\Gamma_1,\Gamma_2,\mathbf{G}^{i},\mathbf{G}^{i} \vdash \Delta_1,\Delta_2,\Lambda,\Lambda,\mathbf{G}^{\bar{i}},\mathbf{G}^{\bar{i}}}\,\rho}{\mathbf{G}^{i},\mathbf{G}^{i},\mathbf{F}_1 \vee \mathbf{F}_2,\Pi,\Gamma_1,\Gamma_2 \vdash \Delta_1,\Delta_2,\Lambda,\mathbf{G}^{\bar{i}},\mathbf{G}^{\bar{i}}}\,\text{contr}\colon *}{\mathbf{G}^{i},\mathbf{F}_1 \vee \mathbf{F}_2,\Pi,\Gamma_1,\Gamma_2 \vdash \Delta_1,\Delta_2,\Lambda,\mathbf{G}^{\bar{i}}}\,\sigma\ (\text{contr}\colon *)$$

and π' is

$$\dfrac{\dfrac{\mathbf{F}_1, \Pi, \Gamma_1 \vdash \Delta_1, \Lambda \qquad \dfrac{\mathbf{F}_2, \Pi, \Gamma_2, \mathbf{G}^i, \mathbf{G}^i \vdash \Delta_2, \Lambda, \mathbf{G}^{\bar{i}}, \mathbf{G}^{\bar{i}}}{\mathbf{F}_2, \Pi, \Gamma_2, \mathbf{G}^i \vdash \Delta_2, \Lambda, \mathbf{G}^{\bar{i}}}\ \sigma\ (\text{contr}: *)}{\mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \Pi, \Gamma_1, \Gamma_2, \mathbf{G}^i \vdash \Delta_1, \Delta_2, \Lambda, \Lambda, \mathbf{G}^{\bar{i}}}\ \rho}{\mathbf{G}^i, \mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \Lambda, \mathbf{G}^{\bar{i}}}\ \text{contr}: *$$

then $\pi \rhd_b^1 \pi'$.

7. If π is

$$\dfrac{\dfrac{\dfrac{\mathbf{F}_1, \Pi, \Gamma_1, \mathbf{G}_1 \vdash \Delta_1, \Lambda \qquad \mathbf{F}_2, \Pi, \Gamma_2 \vdash \Delta_2, \Lambda}{\mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \Pi, \Gamma_1, \mathbf{G}_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \Lambda, \Lambda}\ \rho}{\mathbf{G}_1, \mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \Lambda}\ \text{contr}: * \qquad \mathbf{G}_2, \Sigma \vdash \Theta}{\mathbf{G}_1 \vee \mathbf{G}_2, \mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \Gamma_1, \Gamma_2, \Sigma \vdash \Theta, \Delta_1, \Delta_2, \Lambda}\ \sigma$$

and π' is

$$\dfrac{\dfrac{\dfrac{\mathbf{G}_1, \mathbf{F}_1, \Pi, \Gamma_1 \vdash \Delta_1, \Lambda \qquad \mathbf{G}_2, \Sigma \vdash \Theta}{\mathbf{G}_1 \vee \mathbf{G}_2, \mathbf{F}_1, \Pi, \Gamma_1, \Sigma \vdash \Theta, \Delta_1, \Lambda}\ \sigma \qquad \mathbf{F}_2, \Pi, \Gamma_2 \vdash \Delta_2, \Lambda}{\mathbf{F}_1 \vee \mathbf{F}_2, \mathbf{G}_1 \vee \mathbf{G}_2, \Pi, \Pi, \Gamma_1, \Gamma_2, \Sigma \vdash \Theta, \Delta_1, \Delta_2, \Lambda, \Lambda}\ \rho}{\mathbf{F}_1 \vee \mathbf{F}_2, \mathbf{G}_1 \vee \mathbf{G}_2, \Pi, \Gamma_1, \Gamma_2, \Sigma \vdash \Theta, \Delta_1, \Delta_2, \Lambda}\ \text{contr}: *$$

then $\pi \rhd_b^1 \pi'$.

8. If π is

$$\dfrac{\dfrac{\dfrac{\mathbf{F}_1, \Pi, \Gamma_1 \vdash \Delta_1, \Lambda \qquad \mathbf{F}_2, \Pi, \Gamma_2, \mathbf{G}_1 \vdash \Delta_2, \Lambda}{\mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \Pi, \Gamma_1, \Gamma_2, \mathbf{G}_1 \vdash \Delta_1, \Delta_2, \Lambda, \Lambda}\ \rho}{\mathbf{G}_1, \mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \Lambda}\ \text{contr}: * \qquad \mathbf{G}_2, \Sigma \vdash \Theta}{\mathbf{G}_1 \vee \mathbf{G}_2, \mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \Gamma_1, \Gamma_2, \Sigma \vdash \Theta, \Delta_1, \Delta_2, \Lambda}\ \sigma$$

and π' is

$$\dfrac{\dfrac{\mathbf{F}_1, \Pi, \Gamma_1 \vdash \Delta_1, \Lambda \qquad \dfrac{\mathbf{G}_1, \mathbf{F}_2, \Pi, \Gamma_2 \vdash \Delta_2, \Lambda \qquad \mathbf{G}_2, \Sigma \vdash \Theta}{\mathbf{G}_1 \vee \mathbf{G}_2, \mathbf{F}_2, \Pi, \Gamma_2, \Sigma \vdash \Theta, \Delta_2, \Lambda}\ \sigma}{\mathbf{F}_1 \vee \mathbf{F}_2, \mathbf{G}_1 \vee \mathbf{G}_2, \Pi, \Pi, \Gamma_1, \Gamma_2, \Sigma \vdash \Theta, \Delta_1, \Delta_2, \Lambda, \Lambda}\ \rho}{\mathbf{F}_1 \vee \mathbf{F}_2, \mathbf{G}_1 \vee \mathbf{G}_2, \Pi, \Gamma_1, \Gamma_2, \Sigma \vdash \Theta, \Delta_1, \Delta_2, \Lambda}\ \text{contr}: *$$

then $\pi \rhd_b^1 \pi'$.

9. If π is

$$\dfrac{\dfrac{\dfrac{\mathbf{F}_1, \Pi, \mathbf{G}_1, \Gamma_1 \vdash \Delta_1, \Lambda \qquad \mathbf{F}_2, \Pi, \mathbf{G}_1, \Gamma_2 \vdash \Delta_2, \Lambda}{\mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \mathbf{G}_1, \Pi, \mathbf{G}_1, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \Lambda, \Lambda}\ \rho}{\mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \mathbf{G}_1, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \Lambda}\ \text{contr}: * \qquad \mathbf{G}_2, \Sigma \vdash \Theta}{\mathbf{G}_1 \vee \mathbf{G}_2, \mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \Gamma_1, \Gamma_2, \Sigma \vdash \Theta, \Delta_1, \Delta_2, \Lambda}\ \sigma$$

and π' is

$$\dfrac{\dfrac{\mathbf{G}_1, \mathbf{F}_1, \Pi, \Gamma_1 \vdash \Delta_1, \Lambda \qquad \mathbf{G}_2, \Sigma \vdash \Theta}{\mathbf{G}_1 \vee \mathbf{G}_2, \mathbf{F}_1, \Pi, \Gamma_1, \Sigma \vdash \Theta, \Delta_1, \Lambda}\sigma \qquad \dfrac{\mathbf{G}_1, \mathbf{F}_2, \Pi, \Gamma_2 \vdash \Delta_2, \Lambda \qquad \mathbf{G}_2, \Sigma \vdash \Theta}{\mathbf{G}_1 \vee \mathbf{G}_2, \mathbf{F}_2, \Pi, \Gamma_2, \Sigma \vdash \Theta, \Delta_2, \Lambda}\sigma}{\dfrac{\mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \mathbf{G}_1 \vee \mathbf{G}_2, \Pi, \mathbf{G}_1 \vee \mathbf{G}_2, \Gamma_1, \Gamma_2, \Sigma, \Sigma \vdash \Theta, \Theta, \Delta_1, \Delta_2, \Lambda, \Lambda}{\mathbf{F}_1 \vee \mathbf{F}_2, \mathbf{G}_1 \vee \mathbf{G}_2, \Pi, \Gamma_1, \Gamma_2, \Sigma \vdash \Theta, \Delta_1, \Delta_2, \Lambda}\text{contr}: *}\rho$$

then $\pi \rhd_b^1 \pi'$.

Finally, we define the $\rhd_b$ relation as the transitive and reflexive closure of the compatible closure of the $\rhd_b^1$ relation.

Lemma 11. *Let π be a weakly regular $\mathbf{LK}_{\text{sk}}$-tree of S. If $\pi \rhd_b \psi$ then ψ is a weakly regular $\mathbf{LK}_{\text{sk}}$-tree of S.*

Proof. By induction on the length of the $\rhd_b$-rewrite sequence. The case of $\pi = \psi$ is trivial, so assume there exists a subtree φ of π such that $\varphi \rhd_b^1 \varphi'$ and ψ is obtained from π by replacing φ by φ'. Then the end-sequent of ψ is the same as that of π. Also weak regularity is preserved:

1. In cases 1, 2 and 4–8 of Definition 22, the paths in ψ and π are the same modulo some repetitions.
2. In case 3, the paths in ψ and π are the same modulo some repetitions, but a new copy of σ is introduced. Note that the two copies are homomorphic, so we may conclude by Proposition 8.
3. In case 9, σ is duplicated together with the subtree ending in $\Sigma \vdash \Theta$. Observe that all the descendents of the two copies of $\Sigma \vdash \Theta$ are contracted, and hence all the duplicated inferences are homomorphic. Therefore we may again conclude by Proposition 8. □

Summarizing, we obtain

Lemma 12. *Let π be a weakly regular $\mathbf{LK}_{\text{sk}}$-tree of S. If $\pi \rhd_b \psi$, $\pi \rhd_u \psi$, or $\pi \rhd_c \psi$, then ψ is a weakly regular $\mathbf{LK}_{\text{sk}}$-tree of S.*

Proof. By Lemmas 11, 10, and 7. □

The following definitions will be used in the algorithm translating $\mathbf{LK}_{\text{sk}}$-proofs into such $\mathbf{LK}_{\text{sk}}$-proofs which fulfil an eigenterm condition.

Definition 23. Let π be a $\mathbf{LK}_{\text{sk}}$-tree, and let ξ be a branch in π. Let σ, ρ be inferences on ξ and w.l.o.g. let σ be above ρ. Let $\xi_1, \ldots, \xi_n$ be the binary inferences between σ and ρ. For $1 \leq i \leq n$, let λ_i be the subproofs ending in a premise sequent of ξ_i such that λ_i do not contain σ. Then $\lambda_1, \ldots, \lambda_n$ are called the *parallel trees between σ and ρ*.

Definition 24. Let σ be a strong labelled quantifier inference in π with Skolem term $\mathbf{S}$, and ρ be a weak labelled quantifier inference in π with substitution term $\mathbf{T}$. We say that ρ *blocks* σ if ρ is below σ and $\mathbf{T}$ contains $\mathbf{S}$. We call σ *correctly placed* if no weak labelled quantifier inference in π blocks σ.

Example 12. Consider the $\mathbf{LK}_{\mathrm{sk}}$-proof π:

$$\dfrac{\dfrac{\langle P(c)\rangle^{c} \vdash P(c)}{\langle P(c)\rangle^{c} \vdash (\forall x)P(x)}\ \forall^{sk}: r}{(\forall x)P(x) \vdash (\forall x)P(x)}\ \forall^{sk}: l$$

Here, the $\forall^{sk}: l$ inference blocks the $\forall^{sk}: r$ inference.

As indicated before, we will rearrange the quantifier inferences in an $\mathbf{LK}_{\mathrm{sk}}$-proof π in such a way that there are no eigenterm violations: this will allow us to convert the $\mathbf{LK}_{\mathrm{sk}}$-proof into an $\mathbf{LK}$ proof. During this rearranging, we may have to permute binary inferences, causing duplication of subproofs. This is bad for showing termination of the rearranging algorithm because our termination measure will be based on the number of inferences in π. As Example 11 shows, sequential pruning may severly reduce the number of inferences in an $\mathbf{LK}_{\mathrm{sk}}$-proof (especially when pruning binary inferences). In fact, this pruning will be sufficient to show termination of the rearranging procedure in the subsequent lemma. For the termination argument, we will use the notion of lexicographic order:

Definition 25 (Lexicographic order). Let $X_1, \ldots, X_n$ be sets and for $i \leq n$ let $\leq_i$ be a partial order on X_i. Then the *lexicographic order on* $X_1 \times \ldots \times X_n$*:* $<_{\mathrm{LEX}}$ is defined by

$$(x_1, \ldots, x_n) <_{\mathrm{LEX}} (x'_1, \ldots, x'_n) \iff (\exists m > 0)(\forall i < m)(x_i = x'_i) \wedge (x_m <_m x'_m)$$

Lemma 13. *Let π be a $\mathbf{LK}_{\mathrm{sk}}$-proof of S. Then there exists an $\mathbf{LK}_{\mathrm{sk}}$-proof π' of S such that all strong labelled quantifier inferences in π' are correctly placed.*

Proof. We introduce some notations that will be useful. Let π be an $\mathbf{LK}_{\mathrm{sk}}$-tree, ρ be a strong labelled quantifier inference in π with Skolem term $\mathbf{S}$. Define Q_ρ as the number of inferences blocking ρ. Then define $\mathbf{BLOCK}_\pi(\mathbf{S}) = \sum_\sigma Q_\sigma$ where σ ranges over the strong labelled quantifier inferences in π with Skolem term $\mathbf{S}$. If $\mathbf{S}, \mathbf{T}$ are expressions, define $\mathbf{S} \prec \mathbf{T}$ if $\mathbf{S}$ occurs in $\mathbf{T}$.

Define SK_π as the set of Skolem terms occurring in π. Let $|\mathrm{SK}_\pi| = n$, then denote the elements of SK_π by $\mathbf{S}_1, \ldots, \mathbf{S}_n$ s.t. for all $1 \leq i \leq n$ and all $j < i$: either $\mathbf{S}_i \prec \mathbf{S}_j$ or $\mathbf{S}_j$, $\mathbf{S}_i$ are incomparable w.r.t. $\prec$. Then define the n-tupel $\alpha_\pi = \langle \mathbf{BLOCK}_\pi(\mathbf{S}_1), \ldots, \mathbf{BLOCK}_\pi(\mathbf{S}_n)\rangle$.

We show that there exists an $\mathbf{LK}_{\mathrm{sk}}$-proof π' of S such that $\alpha_{\pi'} = \langle 0, \ldots, 0\rangle$, which implies that there are no blocking inferences in π'.

We may assume that some member of α_π is not 0. We will transform π into an $\mathbf{LK}_{\mathrm{sk}}$-proof π' of S such that $\alpha_{\pi'} <_{\mathrm{LEX}} \alpha_\pi$ — existence of the desired $\mathbf{LK}_{\mathrm{sk}}$-proof then follows by induction. Let k be the least integer such that $\mathbf{BLOCK}_\pi(\mathbf{S}_k) > 0$. Then there exists a lowermost strong labelled quantifier inference ρ with Skolem term $\mathbf{S}_k$ such that there is a weak labelled quantifier inference σ blocking ρ. Observe that σ does not operate on a descendent of the main formula of ρ: Assume it does, then by Proposition 4, $\mathbf{S}_k$ properly contains the substitution term of σ and, by the definition of blocking, therefore properly contains itself!

Let σ, ξ be inferences in π. Then define $\mathrm{RR}(\pi, \xi, \sigma) = \sum_{\mu} Q_{\mu}$ where μ ranges over the inferences homomorphic to ρ in the parallel trees between ξ and σ. Define $\mathrm{BR}(\pi, \xi, \sigma) = \mathbf{BLOCK}_{\pi}(\mathbf{S}_k) - \mathrm{RR}(\pi, \xi, \sigma)$. The intuitive idea is: When we permute down inferences, new subtrees can be created which contain inferences homomorphic to ρ. $\mathrm{RR}(\pi, \xi, \sigma)$ counts the number of "blockings" created by these inferences. The point then is that these inferences will eventually be deleted, and then $\mathrm{BR}(\pi, \xi, \sigma) = \mathbf{BLOCK}_{\pi}(\mathbf{S}_k)$ and therefore $\mathbf{BLOCK}_{\pi}(\mathbf{S}_k)$ will properly decrease by permuting ρ below σ.

Formally, let $R_n, \ldots, R_1$ be the inferences between ρ and σ (exluding ρ and σ) operating on descendents of the main formula of ρ, i.e.:

$$
\cfrac{\vdots}{\Gamma \vdash \Delta}\ \rho
$$

$$
\cfrac{\vdots}{\Gamma_n \vdash \Delta_n}\ R_n
$$

$$
\vdots
$$

$$
\cfrac{\vdots}{\Gamma_1 \vdash \Delta_1}\ R_1
$$

$$
\cfrac{\vdots}{\Pi \vdash \Lambda}\ \sigma
$$

We construct by induction $\mathbf{LK}_{\mathrm{sk}}$-proofs $\pi_1, \ldots, \pi_l$ where one of the inferences is permuted down below σ. The induction invariant is: $\forall j < k(\mathbf{BLOCK}_{\pi_{l+1}}(\mathbf{S}_j) = 0) \wedge \mathrm{BR}(\pi_l, \rho, \sigma) \geq \mathrm{BR}(\pi_{l+1}, \rho, \sigma)$. Assume l inferences have been shifted, that is

$$\frac{\vdots}{\Gamma \vdash \Delta}\ \rho$$

$$\frac{\vdots}{\Gamma_n \vdash \Delta_n}\ R_n$$

$$\vdots$$

$$\frac{\vdots}{\Gamma_{l+1} \vdash \Delta_{l+1}}\ R_{l+1}$$

$$\frac{\vdots}{\Pi' \vdash \Lambda'}\ \sigma$$

$$\frac{\vdots}{\Gamma'_l \vdash \Delta'_l}\ R_l$$

$$\vdots$$

$$\frac{\vdots}{\Pi \vdash \Lambda}\ R_1$$

Depending on whether R_{l+1} is a unary, binary, or contraction inference, we use $\triangleright_u$, $\triangleright_b$, or $\triangleright_c$ respectively to permute it below σ, obtaining π_{l+1}. By Lemma 12, π_{l+1} is an $\mathbf{LK}_{\text{sk}}$-proof of S. We verify the induction invariant by distinguishing what kind of inference R_{l+1} is:

1. R_{l+1} is a $\forall^{sk}\colon r$ inference. Permuting down a $\forall^{sk}\colon r$ inference cannot create any blocking inferences and does not change the number of homomorphic inferences in the parallel trees, so the invariant holds. For example, we permute R_{l+1} below a $\forall^{sk}\colon l$ inference:

$$\dfrac{\dfrac{\begin{array}{c}(\psi)\\ \langle \mathbf{GT}\rangle^{\ell_1,\mathbf{T}}, \Gamma \vdash \Delta, \langle \mathbf{FS}\rangle^{\ell_2}\end{array}}{\langle \mathbf{GT}\rangle^{\ell_1,\mathbf{T}}, \Gamma \vdash \Delta, \langle \forall\mathbf{F}\rangle^{\ell_2}}\ R_{l+1}}{\langle \forall\mathbf{G}\rangle^{\ell_1}, \Gamma \vdash \Delta, \langle \forall\mathbf{F}\rangle^{\ell_2}}\ \forall^{sk}\colon l$$

is transformed into

$$\dfrac{\dfrac{\begin{array}{c}(\psi)\\ \langle \mathbf{GT}\rangle^{\ell_1,\mathbf{T}}, \Gamma \vdash \Delta, \langle \mathbf{FS}\rangle^{\ell_2}\end{array}}{\langle \forall\mathbf{G}\rangle^{\ell_1}, \Gamma \vdash \Delta, \langle \mathbf{FS}\rangle^{\ell_2}}\ \forall^{sk}\colon l}{\langle \forall\mathbf{G}\rangle^{\ell_1}, \Gamma \vdash \Delta, \langle \forall\mathbf{F}\rangle^{\ell_2}}\ R_{l+1}$$

2. R_{l+1} is a $\forall^{sk}\colon l$ inference with substitution term $\mathbf{T}$. As R_{l+1} operates on a descendent of ρ, by Proposition 4, $\mathbf{T} \prec \mathbf{S}_k$. Therefore $\mathbf{S}_k$ properly contains any Skolem term $\mathbf{R}$ contained in $\mathbf{T}$, so $\mathbf{R} = \mathbf{S}_j$ for some $j > k$. Therefore $\mathbf{BLOCK}_{\pi_l}(\mathbf{S}_p) \geq \mathbf{BLOCK}_{\pi_{l+1}}(\mathbf{S}_p)$ for all $p \leq k$. The parallel trees are untouched, so the invariant holds.

3. R_{l+1} is an $\exists^{sk}: l$ or an $\exists^{sk}: r$ inference: analogous to the previous case.
4. R_{l+1} is a unary propositional inference. The invariant trivially holds.
5. R_{l+1} is an $\vee: l$ inference. To verify the induction invariant, we perform a case distinction depending on the inference below R_{l+1}. We only consider the interesting cases:
   (a) R_{l+1} is permuted over a $\forall^{sk}: l$ inference ξ. At most one copy ξ' of ξ is created in π_{l+1}, and there is no branch containing both ξ and ξ'. So for all $\forall^{sk}: r$ inferences above R_{l+1}, there is still at most one of ξ, ξ' below them, so $\mathbf{BLOCK}_{\pi_{l+1}}(\mathbf{S}_i) \leq \mathbf{BLOCK}_{\pi_l}(\mathbf{S}_i)$ for all $i \in \{1, \ldots, k\}$.
   For example, consider the case

$$\dfrac{\dfrac{\dfrac{\begin{matrix}(\psi)\\ \mathbf{F}_1, \Pi, \langle\mathbf{GT}\rangle^{\ell,\mathbf{T}}, \Gamma_1 \vdash \Delta_1, \Lambda\end{matrix} \quad \begin{matrix}(\psi')\\ \mathbf{F}_2, \Pi, \langle\mathbf{GT}\rangle^{\ell,\mathbf{T}}, \Gamma_2 \vdash \Delta_2, \Lambda\end{matrix}}{\mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \langle\mathbf{GT}\rangle^{\ell,\mathbf{T}}, \Pi, \langle\mathbf{GT}\rangle^{\ell,\mathbf{T}}, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \Lambda, \Lambda}\,R_{l+1}}{\langle\mathbf{GT}\rangle^{\ell,\mathbf{T}}, \mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \Lambda}\,\mathrm{contr}\colon *}{\langle\forall\mathbf{G}\rangle^{\ell}, \mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \Lambda}\,\xi$$

   which is transformed to

$$\dfrac{\dfrac{\dfrac{\begin{matrix}(\psi)\\ \langle\mathbf{GT}\rangle^{\ell,\mathbf{T}}, \mathbf{F}_1, \Pi, \Gamma_1 \vdash \Delta_1, \Lambda\end{matrix}}{\langle\forall\mathbf{G}\rangle^{\ell}, \mathbf{F}_1, \Pi, \Gamma_1 \vdash \Delta_1, \Lambda}\,\xi \quad \dfrac{\begin{matrix}(\psi')\\ \langle\mathbf{GT}\rangle^{\ell,\mathbf{T}}, \mathbf{F}_2, \Pi, \Gamma_2 \vdash \Delta_2, \Lambda\end{matrix}}{\langle\forall\mathbf{G}\rangle^{\ell}, \mathbf{F}_2, \Pi, \Gamma_2 \vdash \Delta_2, \Lambda}\,\xi'}{\mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \langle\forall\mathbf{G}\rangle^{\ell}, \Pi, \langle\forall\mathbf{G}\rangle^{\ell}, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \Lambda, \Lambda}\,R_{l+1}}{\langle\forall\mathbf{G}\rangle^{\ell}, \mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \Lambda}\,\mathrm{contr}\colon *$$

   So for all $\forall^{sk}: r$ inferences in ψ, ψ' there is still only one copy of ξ below them, and hence $\mathbf{BLOCK}_{\pi_{l+1}}(\mathbf{S}_i) \leq \mathbf{BLOCK}_{\pi_l}(\mathbf{S}_i)$.
   (b) R_{l+1} is permuted over a $\forall^{sk}: r$ inference ξ with Skolem term $\mathbf{S}_p$. If $p < k$, then $\mathbf{BLOCK}_{\pi_l}(\mathbf{S}_p) = 0$ and therefore duplicating ξ still gives $\mathbf{BLOCK}_{\pi_{l+1}}(\mathbf{S}_p) = 0$. $p = k$ does not hold, as we chose a lowermost blocked $\forall^{sk}: r$ inference ρ.
   (c) R_{l+1} is permuted over a binary inference ξ such that one of the auxiliary formulas of ξ is contracted; then the situation in π_l is

$$\dfrac{\dfrac{\dfrac{\mathbf{F}_1, \Pi, \mathbf{G}_1, \Gamma_1 \vdash \Delta_1, \Lambda \quad \mathbf{F}_2, \Pi, \mathbf{G}_1, \Gamma_2 \vdash \Delta_2, \Lambda}{\mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \mathbf{G}_1, \Pi, \mathbf{G}_1, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \Lambda, \Lambda}\,R_{l+1}}{\mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \mathbf{G}_1, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \Lambda}\,\mathrm{contr}\colon * \quad \begin{matrix}(\varphi)\\ \mathbf{G}_2, \Sigma \vdash \Theta\end{matrix}}{\mathbf{G}_1 \vee \mathbf{G}_2, \mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \Gamma_1, \Gamma_2, \Sigma \vdash \Theta, \Delta_1, \Delta_2, \Lambda}\,\xi$$

   which is transformed to

$$\dfrac{\dfrac{\mathbf{G}_1, \mathbf{F}_1, \Pi, \Gamma_1 \vdash \Delta_1, \Lambda \qquad \overset{(\varphi)}{\mathbf{G}_2, \Sigma \vdash \Theta}}{\mathbf{G}_1 \vee \mathbf{G}_2, \mathbf{F}_1, \Pi, \Gamma_1, \Sigma \vdash \Theta, \Delta_1, \Lambda}\,\xi \qquad \dfrac{\mathbf{G}_1, \mathbf{F}_2, \Pi, \Gamma_2 \vdash \Delta_2, \Lambda \qquad \overset{(\varphi)}{\mathbf{G}_2, \Sigma \vdash \Theta}}{\mathbf{G}_1 \vee \mathbf{G}_2, \mathbf{F}_2, \Pi, \Gamma_2, \Sigma \vdash \Theta, \Delta_2, \Lambda}\,\xi}{\dfrac{\mathbf{F}_1 \vee \mathbf{F}_2, \Pi, \mathbf{G}_1 \vee \mathbf{G}_2, \Pi, \mathbf{G}_1 \vee \mathbf{G}_2, \Gamma_1, \Gamma_2, \Sigma, \Sigma \vdash \Theta, \Theta, \Delta_1, \Delta_2, \Lambda, \Lambda}{\mathbf{F}_1 \vee \mathbf{F}_2, \mathbf{G}_1 \vee \mathbf{G}_2, \Pi, \Gamma_1, \Gamma_2, \Sigma \vdash \Theta, \Delta_1, \Delta_2, \Lambda}\,\text{contr}\colon *}\,R_{l+1}$$

in π_{l+1}.
As $\mathbf{BLOCK}_{\pi_l}(\mathbf{S}_p) = 0$ for $p < k$, $\mathbf{BLOCK}_{\pi_{l+1}}(\mathbf{S}_p) = 0$ even when duplicating a subtree. Hence we only have to consider $\mathbf{S}_k$. Assume $\mathbf{BLOCK}_{\pi_{l+1}}(\mathbf{S}_k) > \mathbf{BLOCK}_{\pi_l}(\mathbf{S}_k)$, then there exists a $\forall^{sk}\colon r$ inference ρ' in the duplicated tree φ with Skolem term $\mathbf{S}_k$. As ρ' was created by copying a inference ρ^* that was, by weak regularity, homomorphic to ρ, also ρ' will be homomorphic to ρ due to the applications of contractions contr: $*$ on $\Sigma, \Theta, \mathbf{G}_1 \vee \mathbf{G}_2$. Therefore the inferences blocking ρ' in the copy of φ are counted in $\mathrm{RR}(\pi_{l+1}, \rho, \sigma)$. Let z be the number of inferences blocking inferences ρ' copied in this way, then $\mathrm{RR}(\pi_{l+1}, \rho, \sigma) = \mathrm{RR}(\pi_l, \rho, \sigma) + z$ and $\mathbf{BLOCK}_{\pi_{l+1}}(\mathbf{S}_p) = \mathbf{BLOCK}_{\pi_l}(\mathbf{S}_p) + z$ and hence $\mathrm{BR}(\pi_{l+1}, \rho, \sigma) \leq \mathrm{BR}(\pi_l, \rho, \sigma)$.

6. R_{l+1} is another binary inference: analogous to the previous case.

This completes the case distinction. Let ω be the inference directly above ρ, then $\mathrm{RR}(\pi_m, \rho, \sigma) = \mathrm{RR}(\pi_m, \omega, \sigma)$. Permute ρ down over σ in the same way as above and apply Lemma 9 to the resulting proof. This yields a proof π'_m such that $\mathrm{RR}(\pi'_m, \omega, \sigma) = 0$ and, because ρ is now below σ, $\mathbf{BLOCK}_{\pi'_m}(\mathbf{S}_k) < \mathbf{BLOCK}_{\pi}(\mathbf{S}_k)$. □

Theorem 3 (Soundness). *Let π be a $\mathbf{LK}_{\mathrm{sk}}$-proof of S. Then there exists a cut-free $\mathbf{LK}$-proof of S.*

Proof. We apply Lemma 9 and Lemma 13 to obtain a sequentially pruned $\mathbf{LK}_{\mathrm{sk}}$-proof π' of S where all inferences are correctly placed.

For the rest of this proof, we allow $\forall\colon r$ and $\exists\colon l$ inferences in $\mathbf{LK}_{\mathrm{sk}}$-proofs (with the usual eigenvariable condition). By induction on the number of strong labelled quantifier inferences in π', we construct sequentially pruned $\mathbf{LK}_{\mathrm{sk}}$-proofs π'' where all inferences are correctly placed, containing strictly less strong labelled quantifier inferences inferences than π'.

Let ρ

$$\dfrac{\overset{(\psi)}{\Gamma \vdash \Delta, \left\langle \overline{\mathbf{FS}} \right\rangle^{\ell}}}{\Gamma \vdash \Delta, \left\langle \forall \mathbf{F} \right\rangle^{\ell}}\,\forall^{sk}\colon r$$

be a $\forall^{sk}\colon r$ inference in π' such that $\mathbf{S}$ is a $>$-maximal Skolem term in π' (the case for ρ being an $\exists^{sk}\colon l$ inference is analogous).

Assume that $\mathbf{S}$ occurs in $\Gamma \cup \Delta \cup \ell$. As π' is an $\mathbf{LK}_{\mathrm{sk}}$-proof, S does not contain Skolem symbols and so a descendent of $\mathbf{S}$ must be eliminated by a labelled quantifier inference σ below ρ. Distinguish:

1. σ is a strong labelled quantifier inference. As π' is sequentially pruned and weakly regular, the Skolem term $\mathbf{T}$ of σ fulfills $\mathbf{S} \neq \mathbf{T}$. Therefore $\mathbf{S} < \mathbf{T}$, which contradicts the assumption of $>$-maximality of $\mathbf{S}$!
2. σ is a weak labelled quantifier inference. Then ρ is not correctly placed!

Hence $\mathbf{S}$ does not occur in $\Gamma \cup \Delta \cup \ell$. Applying Lemma 5, we obtain $\psi\,[\mathbf{S} \leftarrow \mathbf{Y}]$. We replace ρ in π' by

$$\dfrac{\begin{array}{c}(\psi\,[\mathbf{S} \leftarrow \mathbf{Y}])\\ \Gamma \vdash \Delta, \left\langle \overline{\mathbf{FY}} \right\rangle^{\ell}\end{array}}{\Gamma \vdash \Delta, \langle \forall \mathbf{F} \rangle^{\ell}}\ \forall\colon r$$

We perform this procedure on all source inferences of $\mathbf{S}$ at once. As π' is sequentially pruned, all such inferences are parallel and the substitutions do not interfere with each other. As $\mathbf{Y}$ is new, it does not cause eigenvariable violations in $\psi\,[\mathbf{S} \leftarrow \mathbf{Y}]$. As we apply the same replacement on the homomorphic paths, weak regularity is preserved.

Finally, we obtain a tree consisting of $\mathbf{LK}_{\mathrm{sk}}$ inferences which does not contain $\forall^{sk}\colon r$ and $\exists^{sk}\colon l$ inferences, but contains $\forall\colon r$ and $\exists\colon l$ inferences obeying the eigenvariable condition. We replace the $\mathbf{LK}_{\mathrm{sk}}$ inferences by the respective $\mathbf{LK}$ inferences to obtain the desired $\mathbf{LK}$-proof. □

We can now extend the main theorem on CERES$^{\omega}$:

Theorem 4. *Let π be a regular, proper $\mathbf{LK}_{\mathrm{skc}}$-proof of S such that there exists an $\mathcal{R}_{al}$-refutation of $\mathrm{CS}(\pi)$. Then there a cut-free $\mathbf{LK}$-proof of S.*

Proof. By Theorem 2, there exists an $\mathbf{LK}_{\mathrm{sk}}$-proof of S. By Theorem 3, there exists a cut-free $\mathbf{LK}$-proof of S. □

Completeness of $\mathcal{R}_{al}$ implies completeness of the cut-elimination method:

Theorem 5. *Assume completeness of $\mathcal{R}_{al}$. Let π be an $\mathbf{LK}$-proof of a proper sequent S. Then there exists a cut-free $\mathbf{LK}$-proof of S.*

Proof. π can be transformed into a regular $\mathbf{LK}$-proof of S. By Lemma 1, there exists a regular $\mathbf{LK}_{\mathrm{skc}}$-proof of S. Let $\mathrm{CS}_R(\pi)$ be the reduct of $\mathrm{CS}(\pi)$. By Proposition 10, Proposition 1, and Theorem 1, there exists an $\mathcal{R}$-refutation γ of $F(\mathrm{CS}_R(\pi))$. By deleting some $\rightarrow^T$, $\vee^T$ and $\wedge^F$ inferences from γ, we obtain an $\mathcal{R}$-refutation of $\mathrm{CS}_R(\pi)$. By completeness of $\mathcal{R}_{al}$, we may apply Theorem 4. □

Of course, cut-elimination implies consistency. Hence by Gödel's second incompleteness theorem, at some point in the proof of the theorem above we must use assumptions which can not be proven in type theory. This strength is to be found in the proof of Theorem 1.

The following subsection will be devoted to investigating the relative completeness of $\mathcal{R}_{al}$.

8. Relative completeness of $\mathcal{R}_{al}$

So far, we have not been able to prove relative completeness of $\mathcal{R}_{al}$. We state the following:

Conjecture. *Relative Completeness of $\mathcal{R}_{al}$ holds.*

This subsection will present results which indicate that the conjecture can indeed be resolved positively by studying whether the $\mathcal{R}$ calculus can be sufficiently restricted.

8.1. Restricting $\mathcal{R}$ (towards $\mathcal{R}_{al}$)

In this section, we will consider the following calculus:

Definition 26 (Resolution calculus $\mathcal{R}_a$). We define the calculus $\mathcal{R}_a$ analogously to the calculus $\mathcal{R}_{al}$; it consists of the propositional rules of $\mathcal{R}_a$ where all labels are empty, together with the following rules:

$$\frac{\Gamma \vdash \Delta, \forall \mathbf{A}}{\Gamma \vdash \Delta, \mathbf{AX}}\ \forall^T \qquad \frac{\forall \mathbf{A}, \Gamma \vdash \Delta}{\mathbf{A}(\mathbf{fX}_1 \dots \mathbf{X}_n), \Gamma \vdash \Delta}\ \forall^F \qquad \frac{S}{S\,[\mathbf{X} \leftarrow \mathbf{T}]}\ \text{Sub}$$

$$\frac{\Gamma \vdash \Delta, \mathbf{A}, \dots, \mathbf{A} \quad \mathbf{A}, \dots, \mathbf{A}, \Pi \vdash \Lambda}{\Gamma, \Pi \vdash \Delta, \Lambda}\ \text{mCut}$$

where in $\forall^F$, $\mathbf{X}_1, \dots, \mathbf{X}_n$ are all the free variables occuring in $\mathbf{A}$, and if $\tau(\mathbf{X}_i) = t_i$ for $1 \leq i \leq n$ and $\tau(\mathbf{A}) = t \to o$, then $\mathbf{f} \in \mathcal{K}_{t_1,\dots,t_n,t}$. In mCut, $\mathbf{A}$ is atomic.

Note that $\mathcal{R}_a$ is "in-between" Andrews' $\mathcal{R}$ from [2] and $\mathcal{R}_{al}$: it does not have the Sim^T, Sim^F rules of $\mathcal{R}$, but the $\forall^F$ and $\forall^T$ rules work as they do in $\mathcal{R}$. In this section, we are interested in the question whether $\mathcal{R}_a$ is still complete (with respect to $\mathcal{R}$). The answer will be positive for a fragment of $\mathcal{R}$:

Definition 27. Let γ be an $\mathcal{R}$-deduction such that all Skolem terms of $\forall^F$ inferences in γ are constants. Then γ is called an *$\mathcal{R}_c$-deduction.*

The aim of this section is to prove the following result:

Theorem 6. *Let γ be an $\mathcal{R}_c$-refutation of $\mathcal{C}$. Then there exists an $\mathcal{R}_a$-refutation of $\mathcal{C}$.*

Let γ be an $\mathcal{R}$-deduction, and ρ_1, ρ_2 inferences in γ. Then we say that ρ_1 is a *direct ancestor* of ρ_2 if the conclusion of ρ_1 is a premise of ρ_2. ρ_2 is a *direct descendent* of ρ_1 if ρ_1 is a direct ancestor of ρ_2. Similarily, if S_1, S_2 are sequent occurrences in γ then S_1 is a direct ancestor of S_2 if there exists an inference with premise S_1 and conclusion S_2 in γ, and then S_2 is a direct descendent of S_1. The *proper ancestor (descendent)* relations are the transitive closures of the direct ancestor (direct descendent) relations. The *ancestor (descendent)* relations are the reflexive closures of the proper ancestor (descendent) relations. If S_1 is a descendent of S_2 then we also say that S_1 *depends on* S_2. Furthermore, we say that an inference ρ operates on a formula occurrence ω if ω is an auxiliary or main formula of ρ (note that the Sub rule does not operate on any formula occurrences).

For notational convenience we will refer to Sim^T and Sim^F inferences simply as Sim inferences.

Definition 28. We say that a Sim inference ρ in an $\mathcal{R}$-deduction γ is *locked* if all the direct descendents of ρ operate on the main formula of ρ. Let ω be a formula occurrence in γ. Then a sequence of sequents $S_1, \ldots, S_n$ is a *path starting at* ω if S_1 contains ω and for all $1 \leq i < n$, S_i is a direct ancestor of S_{i+1}. A path p starting at ω is called *uninterrupted* if no inference on p operates on a descendent of ω.

Proposition 13. *Let ω be the occurrence of $\mathbf{F}$ in the sequent $\Gamma \vdash \Delta, \mathbf{F}$ ($\mathbf{F}, \Gamma \vdash \Delta$) in an $\mathcal{R}$-deduction γ, and let p be an uninterrupted path starting at ω. Then all sequents in p are of the form $\Pi \vdash \Lambda, \mathbf{F}\sigma$ ($\mathbf{F}\sigma, \Pi \vdash \Lambda$) for some Π, Λ and substitution σ.*

PROOF. By induction on the length of p. σ is determined by the Sub inferences on p. □

Proposition 14. *Let γ be an $\mathcal{R}$-refutation of $\mathcal{C}$. Then there exists an $\mathcal{R}$-refutation ψ of $\mathcal{C}$ such that all* Sim *inferences in ψ are locked and such that the Skolem terms occuring in γ are exactly those occuring in ψ.*

PROOF. We may assume that there exists a Sim inference ρ in γ that is not locked. W.l.o.g. assume that ρ is a Sim^T inference. We construct an $\mathcal{R}$-refutation γ' of $\mathcal{C}$ such that γ' contains strictly less non-locked Sim inferences than γ, and conclude by induction.

Let $\gamma = S_1, \ldots, S_k$. As γ is an $\mathcal{R}$-refutation, S_k does not contain formula occurrences and hence (1) every formula occurrence ω has a descendent which is an auxiliary formula. Let ω be the main formula of ρ, let $S_i = \Gamma \vdash \Delta, \mathbf{A}, \mathbf{A}$ be the premise of ρ (where the $\mathbf{A}$'s are the auxiliary formulas of ρ), and let $S_j = \Gamma \vdash \Delta, \mathbf{A}$ be the conclusion of ρ. As ρ is not locked and by (1), there exist non-trivial uninterrupted paths $p_1, \ldots, p_n$ from ω to some auxiliary formulas occurring in sequents T_i ($1 \leq i \leq n$). Define $\psi = \Sigma_1, \ldots, \Sigma_{j-1}, \Sigma_{j+1}, \Sigma_k$ where

(1) if S_l occurs on some p_i then by Proposition 13, S_l is of the form $\Pi \vdash \Lambda, \mathbf{A}\sigma$ and we define $\Sigma_l = \Pi \vdash \Lambda, \mathbf{A}\sigma, \mathbf{A}\sigma$,
(2) if S_l is inferred from some T_j then $\Sigma_l = T_j, S_l$,
(3) otherwise $\Sigma_l = S_l$.

ψ is an $\mathcal{R}$-refutation of $\mathcal{C}$: W.l.o.g. we treat the case of S_l being inferred in ψ by a unary inference. In case (1) if S_l is inferred from S_j in γ then we can infer Σ_l from $\Sigma_i = S_i$ in ψ. Otherwise it is inferred from some S_m for which also case (1) holds, and we can infer Σ_l from Σ_m. In case (2), we can infer T_j from Σ_j by Sim^T and S_l from T_j as in γ. In case (3) if S_l was inferred from S_m in γ then Σ_m ends in S_m and we can infer S_l from Σ_m just as S_l was inferred from S_m in γ.

Note that we have only introduced locked Sim inferences, and have removed one non-locked Sim inference. Hence ψ contains strictly less non-locked Sim inferences than γ, which concludes the proof. □

Example 13. Consider the $\mathcal{R}$-deduction γ:

1	$Px \vee Qx, Px \vee Qx \vdash \forall y Ry$	
2	$Px \vee Qx \vdash \forall y Ry$	$\mathrm{Sim}^F : 1$
3	$Px \vee Qx \vdash Rz$	$\forall^F : 2$
4	$Pz \vee Qz \vdash Rz$	$\mathrm{Sub} : 3$
5	$Pz \vdash Rz$	$\vee_l^F : 4$
6	$Pc \vee Qc \vdash Rc$	$\mathrm{Sub} : 4$
7	$Qc \vdash Rc$	$\vee_r^F : 6$

Applying Proposition 14 to γ yields the $\mathcal{R}$-deduction

1	$Px \vee Qx, Px \vee Qx \vdash \forall y Ry$	
2	$Px \vee Qx, Px \vee Qx \vdash Rz$	$\forall^F : 1$
3	$Pz \vee Qz, Pz \vee Qz \vdash Rz$	$\mathrm{Sub} : 2$
4	$Pz \vee Qz \vdash Rz$	$\mathrm{Sim}^F : 3$
5	$Pz \vdash Rz$	$\vee_l^F : 4$
6	$Pc \vee Qc, Pc \vee Qc \vdash Rc$	$\mathrm{Sub} : 3$
7	$Pc \vee Qc \vdash Rc$	$\mathrm{Sim}^F : 6$
8	$Qc \vdash Rc$	$\vee_r^F : 7$

Hence from now on we will focus on the following set of rules:

Definition 29 (Rules for $\mathcal{R}'_a$).

$$\frac{\Gamma \vdash \Delta, \neg\mathbf{A}, \ldots, \neg\mathbf{A}}{\mathbf{A}, \Gamma \vdash \Delta}\ \neg^T \qquad \frac{\neg\mathbf{A}, \ldots, \neg\mathbf{A}, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \mathbf{A}}\ \neg^F \qquad \frac{\Gamma \vdash \Delta, \mathbf{A} \vee \mathbf{B}, \ldots, \mathbf{A} \vee \mathbf{B}}{\Gamma \vdash \Delta, \mathbf{A}, \mathbf{B}}\ \vee^T$$

$$\frac{\mathbf{A} \vee \mathbf{B}, \ldots, \mathbf{A} \vee \mathbf{B}, \Gamma \vdash \Delta}{\mathbf{A}, \Gamma \vdash \Delta}\ \vee_l^F \qquad \frac{\mathbf{A} \vee \mathbf{B}, \ldots, \mathbf{A} \vee \mathbf{B}, \Gamma \vdash \Delta}{\mathbf{B}, \Gamma \vdash \Delta}\ \vee_r^F$$

$$\frac{\Gamma \vdash \Delta, \forall\mathbf{A}, \ldots, \forall\mathbf{A}}{\Gamma \vdash \Delta, \mathbf{A}\mathbf{X}}\ \forall^T \qquad \frac{\forall\mathbf{A}, \ldots, \forall\mathbf{A}, \Gamma \vdash \Delta}{\mathbf{A}(\mathbf{f}\mathbf{X}_1 \ldots \mathbf{X}_n), \Gamma \vdash \Delta}\ \forall^F \qquad \frac{\Gamma \vdash \Delta}{(\Gamma \vdash \Delta)\,[\mathbf{X} \leftarrow \mathbf{T}]}\ \mathrm{Sub}$$

$$\frac{\Gamma \vdash \Delta, \mathbf{A}, \ldots, \mathbf{A} \quad \mathbf{A}, \ldots, \mathbf{A}, \Pi \vdash \Lambda}{\Gamma, \Pi \vdash \Delta, \Lambda}\ \mathrm{mCut}$$

with conditions on mCut, $\forall^F$ as in Definition 26 (rules of $\mathcal{R}_a$). Rules for the connectives $\rightarrow, \wedge, \exists$ are defined analogously. An inference is called *singular* if it has at most one auxiliary formula.

Hence the following follows immediately from Proposition 14:

Proposition 15. *Let γ be an $\mathcal{R}$-refutation of $\mathcal{C}$. Then there exists an $\mathcal{R}'_a$-refutation ψ of $\mathcal{C}$ such that the Skolem terms occuring in γ are exactly those occuring in ψ.*

Note that an $\mathcal{R}'_a$-deduction γ is an $\mathcal{R}_a$-deduction iff all inferences in γ except mCut are singular. We introduce some notions regarding the status of inferences in $\mathcal{R}'_a$ deductions:

Definition 30. An inference is called *relevant* if it is not an mCut, $\forall^F$, or $\exists^T$ inference. Let ρ be an $\forall^F$ or $\exists^T$ inference. ρ is called *prefinished* if all inferences operating on a proper ancestor of an auxiliary formula of ρ are singular. ρ is called *finished* if it is prefinished and singular.

Example 14. Consider the $\mathcal{R}'_a$-deduction

1	$A \vee \forall x Px, A \vee \forall x Px \vdash$	
2	$\forall x Px \vdash$	$\vee^F_r : 1$
3	$Ps \vdash$	$\forall^F : 2$

Then inference 3 is not prefinished since inference 2 operates on a proper ancestor of the auxiliary formula of 3, and 2 is not singular. Now consider

1	$A \vee \forall x Px, A \vee \forall x Px \vdash$	
2	$A \vee \forall x Px, \forall x Px \vdash$	$\vee^F_r : 1$
3	$\forall x Px, \forall x Px \vdash$	$\vee^F_r : 2$
4	$Ps \vdash$	$\forall^F : 3$

Here, inference 4 is prefinished but not finished since it is not singular.

Definition 31. Let $S = \mathbf{F}_1, \ldots, \mathbf{F}_n \vdash \mathbf{G}_1, \ldots, \mathbf{G}_m$ be a sequent. If there exist $k_1, \ldots, k_n, \ell_1, \ldots, \ell_m \in \mathbb{N}$ such that

$$S' = k_1 \times \mathbf{F}_1, \ldots, k_n \times \mathbf{F}_n \vdash \ell_1 \times \mathbf{G}_1, \ldots, \ell_m \times \mathbf{G}_m,$$

then S' is a *multiple of* S, where the notation $k_i \times \mathbf{F}_i$ means "k_i occurrences of $\mathbf{F}_i$". Abusing notation, we write $\mathbf{F}_1, \ldots, \mathbf{F}_n \vdash_m \mathbf{G}_1, \ldots, \mathbf{G}_m$ for S' if S' is a multiple of S.

If all relevant inferences in an $\mathcal{R}'_a$-deduction γ are singular, then we say that γ is *singular*. We define $\mathrm{NF}(\gamma)$ to be the number of $\forall^F$ and $\exists^T$ inferences in γ which are not finished (i.e. not prefinished or not singular).

Proposition 16. *Let γ be an $\mathcal{R}'_a$-deduction of $\vdash \Gamma$ from $\mathcal{C}$. Then there exists an $\mathcal{R}'_a$-deduction ψ of $\vdash_m \Gamma$ from $\mathcal{C}$ such that ψ is singular.*

Furthermore, the Skolem terms occuring in ψ are the same as those occuring in γ, and $\mathrm{NF}(\gamma) = \mathrm{NF}(\psi)$.

Proof. Assume γ is not singular. Let $\gamma = S_1, \ldots, S_n$, and let i be the least such that S_i is inferred by a relevant inference ρ such that ρ is not singular. We will construct an $\mathcal{R}'_a$-deduction $\psi = S_1, \ldots, S_{i-1}, \Sigma, S'_{i+1}, \ldots, S'_n$ from $\mathcal{C}$ such that (1) if μ is an inference in ψ with conclusion in $S_1, \ldots, S_{i-1}, \Sigma$, then μ is singular and furthermore, (2) a sequent in ψ is inferred by an $\forall^F$ ($\exists^T$) inference μ iff its corresponding sequent in γ is inferred by an $\forall^F$ ($\exists^T$) inference μ', and

μ is not finished iff μ' is. We may then conclude by induction on $n - i$, where i is defined as above.

$S_1, \ldots, S_{i-1}$ are inferred in ψ as they were in γ. By assumption, all these inferences are singular if they are relevant. Σ is defined as follows: We treat the case of ρ being an $\vee^T$ inference. The other cases are analogous. Let $\Gamma \vdash \Delta, \mathbf{A} \vee \mathbf{B}, \ldots, \mathbf{A} \vee \mathbf{B}$ be the premise of ρ, and let $\Gamma \vdash \Delta, \mathbf{A}, \mathbf{B}$ be the conclusion. Then Σ is the sequence of sequents starting with $\Gamma \vdash \Delta, \mathbf{A} \vee \mathbf{B}, \ldots, \mathbf{A} \vee \mathbf{B}, \mathbf{A}, \mathbf{B}$ and ending with $\Gamma \vdash \Delta, \mathbf{A}, \mathbf{B}, \ldots, \mathbf{A}, \mathbf{B}$, such that every sequent in Σ is inferred from the previous one by the singular version of ρ. The first sequent in Σ can be inferred from the same S_j, $j < i$, as it was in γ, using the singular version of ρ. By construction, (1) holds. For (2), note that by assumption ρ cannot be $\forall^F$ or $\exists^T$, as ρ is relevant. All other inferences are as they were in γ, so (2) holds for this part of ψ.

Now, define S'_j for $i < j \leq n$. Let ω be the main formula of ρ, and let $S_j = \Gamma, \Delta$ where Δ are all the descendents of ω in S_j in γ. Define $S'_j = \Gamma, \Delta, \ldots, \Delta$ if there exists an uninterrupted path starting at ω and ending at S_j in γ (for some suitable number of copies of Δ), and $S'_j = S_j$ otherwise. S'_j can be derived in ψ:

1. If S_j was derived in γ from S_k with $k < i$, then Δ is empty and we can derive $S'_j = S_j$ from S_k.
2. If S_j was derived from S_i in γ, we can derive S'_j from the last element of Σ.
3. If S_j was derived from S_k, with $k > i$, in γ then again we can derive S'_j from S'_k in ψ. If the inference with conclusion S_j is the first inference operating on a descendent of ω in γ, we have to increase the number of auxiliary formulas to derive the correct sequent in ψ. For example, if $S_k = \Gamma \vdash \Delta, \mathbf{A} \vee \mathbf{B}$ and $S_j = \Gamma \vdash \Delta, \mathbf{A}, \mathbf{B}$ is derived by $\vee^T$, then $S'_k = \Gamma \vdash \Delta, \mathbf{A} \vee \mathbf{B}, \ldots, \mathbf{A} \vee \mathbf{B}$ and we derive $S'_j = S_j$ from S'_k by $\vee^T$ in ψ.

For (2), it is clear by construction that S'_j is inferred by $\forall^F$ iff S_j is. Note that inferences from γ are changed iff they operate on descendents of ω, in which case they are not prefinished if they are instances of $\forall^F$ in both γ and ψ. □

The second $\mathcal{R}'_a$-deduction in Example 14 is obtained from the first by applying Proposition 16.

Proposition 17. *Let ρ_1, ρ_2 be $\forall^F$ or $\exists^T$ inferences in an $\mathcal{R}'_a$-deduction such that ρ_1 operates on an ancestor of the main formula of ρ_2. Then if ρ_1 is not finished, ρ_2 is not finished.*

Proof. As ρ_1 is not finished, an inference operating on an ancestor of the main formula ω of ρ_1 is not singular. By assumption ω is an ancestor of the main formula of ρ_2, so ρ_2 is not prefinished and hence not finished. □

For the final results, we will allow the rule of weakening in $\mathcal{R}'_a$-deductions to ease the presentation of the proofs:

$$\frac{\Gamma \vdash \Delta}{\Gamma, \Pi \vdash \Delta, \Lambda} \; weak$$

Proposition 18. *Let γ be an $\mathcal{R}'_a$-refutation of $\mathcal{C}$ using weakening. Then there exists an $\mathcal{R}'_a$-refutation ψ of $\mathcal{C}$ without weakening such that* $\mathrm{NF}(\psi) \leq \mathrm{NF}(\gamma)$.

Proof. By deleting formula occurrences, sequents and inferences. □

Proposition 19. *Let γ be an $\mathcal{R}'_a$-refutation of $\mathcal{C}$ such that all Skolem terms of $\forall^F$ and $\exists^T$ inferences in γ are constants. Then there exists an $\mathcal{R}_a$-refutation of $\mathcal{C}$.*

Proof. Note that if γ is singular and $\mathrm{NF}(\gamma) = 0$, γ is the desired $\mathcal{R}_a$-refutation.

By Proposition 16, we may assume that γ is singular. We proceed by induction on $\mathrm{NF}(\gamma)$, showing that if γ is a singular $\mathcal{R}'_a$-deduction of S from $\mathcal{C}$, then there exists a singular $\mathcal{R}'_a$-deduction ψ of S from $\mathcal{C}$ with $\mathrm{NF}(\psi) = 0$.

If $\mathrm{NF}(\gamma) = 0$, we may take $\psi = \gamma$. Hence assume as inductive hypothesis that for all $\mathcal{R}'_a$-deductions λ of S from $\mathcal{C}$ with $\mathrm{NF}(\lambda) < \mathrm{NF}(\gamma)$, there exists an $\mathcal{R}'_a$-deduction λ' of S from $\mathcal{C}$ with $\mathrm{NF}(\lambda') = 0$.

We say that an $\forall^F$ or $\exists^T$ inference ρ is *uppermost* if all $\forall^F$ or $\exists^T$ inferences operating on a proper ancestor of the auxiliary formula of ρ are prefinished. By assumption, there exists an $\forall^F$ or $\exists^T$ inference in γ that is not finished. Then there exists an uppermost such inference ρ in γ that is not finished. Observe that ρ is prefinished and not singular, as it is uppermost and all relevant inferences are singular. W.l.o.g. let ρ be an $\forall^F$ inference.

Let $\gamma = S_1, \ldots, S_n$, and let the premise of ρ be $S_i = \forall\mathbf{A}, \ldots, \forall\mathbf{A}, \Gamma \vdash \Delta$ (containing $k + 1 \geq 2$ auxiliary formulas), the conclusion be $S_j = \mathbf{Ac}, \Gamma \vdash \Delta$, and denote the main formula of ρ by ω. Note that S_n is the empty sequent since γ is an $\mathcal{R}'_a$-refutation. If S_n does not depend on S_j, then clearly we can simply remove S_j and the sequents that depend on it from γ to obtain a singular $\mathcal{R}'_a$-deduction of S_n from $\mathcal{C}$ containing strictly less $\forall^F$ and $\exists^T$ inferences which are not finished, and we may conclude by the inductive hypothesis. Hence assume S_n depends on S_j. Note that $\mathbf{A}$ does not contain free variables since $\mathbf{c}$ is a constant. Let $\mathbf{c}_1, \ldots, \mathbf{c}_k$ be fresh Skolem constants.

For $1 \leq q \leq k$, we will construct singular $\mathcal{R}'_a$-deductions

1. ψ_0 of $(\Gamma \vdash \Delta) \circ (\mathbf{Ac}_1, \ldots, \mathbf{Ac}_k \vdash_m)$ from $\mathcal{C}$, and
2. ψ_q of $(\Gamma \vdash \Delta) \circ (\mathbf{Ac}_{q+1}, \ldots, \mathbf{Ac}_k \vdash_m)$ from $\mathcal{C} \cup \{(\Gamma \vdash \Delta) \circ (\mathbf{Ac}_q, \ldots, \mathbf{Ac}_k \vdash_m)\}$.

such that for $0 \leq p \leq k$, $\mathrm{NF}(\psi_p) < \mathrm{NF}(\gamma)$. We may then apply the inductive hypothesis to ψ_p to obtain singular $\mathcal{R}'_a$-deductions ψ'_p with $\mathrm{NF}(\psi'_p) = 0$. Hence all inferences except mCut are singular in ψ'_p. We may then rename the Skolem symbols of the ψ'_p such that their sets of Skolem symbols are pairwise disjoint. Then clearly $\psi = \psi'_0, \ldots, \psi'_k$ has $\mathrm{NF}(\psi) = 0$ and is therefore the desired $\mathcal{R}'_a$-refutation.

We start by defining ψ_0. For $j + 1 \leq r \leq n$, if S_r does not depend on S_j then $S'_r = S_r$, and otherwise $S'_r = S_r \circ (\mathbf{Ac}_1, \ldots, \mathbf{Ac}_k \vdash_m)$. Note that $S'_n = \mathbf{Ac}_1, \ldots, \mathbf{Ac}_k \vdash_m$. So let

$$\psi_0 = S_1, \ldots, S_{j-1}, \Sigma, S'_{j+1}, \ldots, S'_n, (\Gamma \vdash \Delta) \circ (\mathbf{Ac}_1, \ldots, \mathbf{Ac}_k \vdash_m),$$

where Σ is a sequence of sequents deriving $\mathbf{Ac}, \mathbf{Ac}_1, \ldots, \mathbf{Ac}_k, \Gamma \vdash \Delta$ from S_i using only singular $\forall^F$. Clearly $S_1, \ldots, S_{j-1}$ can be derived from $\mathcal{C}$ as they were in γ. Since ρ is prefinished, all the $\forall^F$ inferences introduced in deriving Σ are finished. Letting S'_j be the last sequent in Σ, we show that S'_r can be derived in ψ_0 for $j < r \leq n$. Distinguish:

1. If S_r does not depend on S_j, then neither do its premise(s) S_p (S_q). Hence $S'_r = S_r$ and $S'_p = S_p$ (and $S'_q = S_q$) and S'_r can be inferred from S'_p (S'_q) just as it was in γ.
2. If S_r depends on S_j and was inferred by a unary inference μ from S_p, then $p \geq j$ and hence we can infer S'_r from S'_p by the same unary inference. If μ is Sub, remember that $\mathbf{A}$ is closed and hence not affected by the substitution.
3. If S_r depends on S_j and was inferred by mCut from S_p and S_t, then at least one of the premises depends on S_j. Hence we may infer S'_r from S'_p and S'_t by mCut. Note that if both premises depend on S_j, the multiplicities of the $\mathbf{Ac}_q$ increase.

Note that $S'_n = (\mathbf{Ac}_1, \ldots, \mathbf{Ac}_k \vdash_m)$, so the last sequent of ψ_0 can be derived from S'_n by weakening. By construction, for every $\forall^F$ ($\exists^T$) inference in ψ_0 that is not finished there exists a unique $\forall^F$ ($\exists^T$) inference in γ that is not finished, hence $\mathrm{NF}(\psi_0) < \mathrm{NF}(\gamma)$ (because ρ induces only finished inferences in ψ_0). Since all relevant inferences in γ are singular, this is also the case for ψ_0. Hence ψ_0 is as desired.

We turn to the construction of ψ_q for $1 \leq q \leq k$. Let

$$\psi'_q = (\Gamma \vdash \Delta) \circ (\mathbf{Ac}_q, \ldots, \mathbf{Ac}_k \vdash_m), S_{1,q}, \ldots, S_{j-1,q}, S_{j+1,q}, \ldots, S_{n,q}$$

where $S_{r,q}$ is defined in the following way:

1. If S_r does not depend on S_j, then $S_{r,q} = S_r\,[\mathbf{c} \leftarrow \mathbf{c}_q]$.
2. If S_r depends on S_j, denote the inference whose conclusion S_r is by ρ. Distinguish:
   - (a) If no inference in γ on the path from ω to S_r operates on a descendent of ω, then S_r is of the form $\mathbf{Ac}, \Pi \vdash \Lambda$. Then let $S_{r,q} = (\Pi \vdash \Lambda) \circ (\mathbf{Ac}_q, \ldots, \mathbf{Ac}_k \vdash_m)$.
   - (b) ρ is the first inference operating on a descendent of ω. We treat the case where ρ is $\vee^T$, the other cases are similar. So if $S_r = \Pi \vdash \Lambda, \mathbf{B}, \mathbf{C}$ is inferred from $S_\ell = \Pi \vdash \Lambda, \mathbf{B} \vee \mathbf{C}$ then $S_{\ell,q} = (\Pi \vdash \Lambda, \mathbf{B} \vee \mathbf{C}, \ldots, \mathbf{B} \vee \mathbf{C}) \circ (\mathbf{Ac}_{q+1}, \ldots, \mathbf{Ac}_k \vdash_m)$ by the previous case (note that by assumption $\mathbf{Ac}_q = \mathbf{B} \vee \mathbf{C}$). Then let $S_{r,q} = (\Pi \vdash \Lambda, \mathbf{B}, \mathbf{C}) \circ (\mathbf{Ac}_{q+1}, \ldots, \mathbf{Ac}_k \vdash_m)$.
   - (c) Otherwise, $S_{r,q} = S_r \circ (\mathbf{Ac}_{q+1}, \ldots, \mathbf{Ac}_k \vdash_m)$.

For $r \in \{1, \ldots, j-1, j+1, \ldots, n\}$, we show that $S_{r,q}$ can be derived in ψ'_q by distinguishing how S_r is derived in γ:

1. $S_r \in \mathcal{C}$. Then S_r does not contain $\mathbf{c}$ and does not depend on S_j, hence $S_{r,q} \in \mathcal{C}$.

2. If S_r is inferred by Sub with $[\mathbf{X} \leftarrow \mathbf{T}]$ from S_p, then we may use Sub with $[\mathbf{X} \leftarrow \mathbf{T}\,[\mathbf{c} \leftarrow \mathbf{c}_q]]$ to derive $S_{r,q}$ from $S_{p,q}$, again noting that $\mathbf{A}$ is closed.
3. S_r is derived from S_p by a CNF inference. We may use the same inference to infer $S_{r,q}$ from $S_{p,q}$ (In case $S_{r,q}$ is constructed in case 2(b) above, the number of auxiliary formulas of the inference increases).
4. S_r is derived from S_p and S_t by an mCut. We may derive $S_{r,q}$ from $S_{p,q}$ and $S_{t,q}$ using mCut. Again if $S_{r,q}$ is constructed in case 2(b) above, the number of auxiliary formulas of the inference increases. Also, note again that if both premises depend on S_j, then the multiplicities of the $\mathbf{Ac}_\ell$ increase.

By construction, for every $\forall^F$ ($\exists^T$) inference in ψ'_q that is not finished there exists a unique $\forall^F$ ($\exists^T$) inference in γ that is not finished, hence $\mathrm{NF}(\psi'_q) < \mathrm{NF}(\gamma)$ (because ρ does not induce an $\forall^F$ inference in ψ'_q). Note that due to 2(b), also the $\forall^F$ ($\exists^T$) inferences operating on descendents of $\mathbf{Ac}_q$ are not finished, but their corresponding inferences in γ operate on descendents of ω and are hence not finished, too.

Set $\psi''_q = \psi'_q, (\Gamma \vdash \Delta) \circ (\mathbf{Ac}_{q+1}, \ldots, \mathbf{Ac}_k \vdash_m)$ Note that the last sequent of ψ'_q is $S_{n,q} = \mathbf{Ac}_{q+1}, \ldots, \mathbf{Ac}_k \vdash_m$, hence the last sequent of ψ''_q can again be derived by weakening. Finally, we may apply Proposition 16 to ψ''_q to obtain a singular ψ_q such that $\mathrm{NF}(\psi_q) = \mathrm{NF}(\psi''_q) = \mathrm{NF}(\psi'_q) < \mathrm{NF}(\gamma)$. Hence ψ_q is as desired. Finally, we apply Proposition 18 to ψ, which completes the proof. $\square$

Example 15. Consider the $\mathcal{R}'_a$-refutation of $\{\forall x(Px \vee \neg Px), \forall x(Px \vee \neg Px) \vdash\}$:

1	$\forall x(Px \vee \neg Px), \forall x(Px \vee \neg Px) \vdash$	
2	$Ps \vee \neg Ps \vdash$	$\forall^F : 1$
3	$Ps \vdash$	$\vee^F_l : 2$
4	$\neg Ps \vdash$	$\vee^F_r : 2$
5	$\vdash Ps$	$\neg^F : 4$
6	$\vdash$	$\mathrm{mCut} : 5, 3$

In the proof of Proposition 19 we obtain ψ_0

1	$\forall x(Px \vee \neg Px), \forall x(Px \vee \neg Px) \vdash$	
2	$\forall x(Px \vee \neg Px), Ps \vee \neg Ps \vdash$	$\forall^F : 1$
3	$Ps_1 \vee \neg Ps_1, Ps \vee \neg Ps \vdash$	$\forall^F : 2$
4	$Ps_1 \vee \neg Ps_1, Ps \vdash$	$\vee^F_l : 3$
5	$Ps_1 \vee \neg Ps_1, \neg Ps \vdash$	$\vee^F_r : 3$
6	$Ps_1 \vee \neg Ps_1, \vdash Ps$	$\neg^F : 5$
7	$Ps_1 \vee \neg Ps_1, Ps_1 \vee \neg Ps_1 \vdash$	$\mathrm{mCut} : 6, 4$

and ψ'_1

8	$Ps_1 \vee \neg Ps_1, Ps_1 \vee \neg Ps_1 \vdash$	
9	$Ps_1 \vdash$	$\vee^F_l : 8$
10	$\neg Ps_1 \vdash$	$\vee^F_r : 8$
11	$\vdash Ps_1$	$\neg^F : 10$
12	$\vdash$	$\mathrm{mCut} : 9, 11$

ψ_1' is not singular, but after application of Proposition 16 we obtain the singular ψ_1

8	$Ps_1 \vee \neg Ps_1, Ps_1 \vee \neg Ps_1 \vdash$	
9	$Ps_1 \vee \neg Ps_1, Ps_1 \vdash$	$\vee_l^F : 8$
10	$Ps_1, Ps_1 \vdash$	$\vee_l^F : 9$
11	$Ps_1 \vee \neg Ps_1, \neg Ps_1 \vdash$	$\vee_r^F : 8$
12	$\neg Ps_1, \neg Ps_1 \vdash$	$\vee_r^F : 11$
13	$\neg Ps_1 \vdash Ps_1$	$\neg^F : 12$
14	$\vdash Ps_1, Ps_1$	$\neg^F : 13$
15	$\vdash$	$\mathrm{mCut} : 10, 14$

Clearly $\psi = \psi_0, \psi_1$ is the desired $\mathcal{R}_a$-refutation of $\{\forall x(Px \vee \neg Px), \forall x(Px \vee \neg Px) \vdash\}$.

Finally, observe that Theorem 6 follows from Propositions 15 and 19.

9. An example application of CERES$^\omega$

In this section, we apply the method introduced in Section 3 to the analysis of a concrete proof π. π is based on a mathematical proof which consists of two parts: in part (1) we prove that the induction principle IND follows from the least number principle LNP. Part (2) uses IND for proving the sentence A that every number greater than one has a prime divisor. Connecting the two proofs by a cut on the sentence IND results in the proof π which shows that A follows from LNP. By applying cut-elimination on π we obtain a direct proof of A via LNP. This way cut-elimination transforms a proof of A from IND into another one using LNP.

The proof uses usual axioms of arithmetic for $0, 1, *, <, >$ and the predecessor function p. We also define $=$ (of type $\iota \to \iota \to o$) via Leibniz equality. Table 1 lists the symbols we use, along with their types, and the definitions used in the proof. $s_0, \ldots, s_3$ are Skolem symbols.
The shape of π is

$$\dfrac{\dfrac{(\pi_1)}{\mathrm{LNP} \vdash \mathrm{IND}} \quad \dfrac{\dfrac{(\pi_2)}{\cdots}}{\mathrm{IND} \vdash \forall y \exists w(y > 1 \to \mathrm{PD}(w, y))}\ \forall^{sk} : l\ \lambda y.\exists w(y > 1 \wedge \mathrm{PD}(w, y))}{\mathrm{LNP} \vdash \forall y \exists w(y > 1 \to \mathrm{PD}(w, y))}\ cut$$

We indicate which Skolem symbols correspond to which quantifier in the end-sequent of π (with expanded definitions):

$$\forall X(\exists y X(y) \to \overset{s_0}{\exists} y(\forall z(z < y \to \neg X(z)) \wedge X(y))) \vdash$$
$$\underset{s_3}{\forall} y \exists w(y > 1 \to (w > 1 \wedge \underset{s_2}{\forall} z(\underset{s_1}{\exists} q\ z * q = w \to (z = 1 \vee z = w))) \wedge \exists q\ w * q = y)$$

As labels of formulas that do not contain free higher-order variables or quantifiers do not play a role in the machinery of Section 3, we do not write down

Table 1: Symbols and definitions

Symbols	Type	Constant
$*$	$\iota \to \iota \to \iota$	✓
$0, 1, s_3$	ι	✓
$<, >, =$	$\iota \to \iota \to o$	✓
s_0	$(\iota \to o) \to \iota$	✓
s_1, s_2, p	$\iota \to \iota$	✓
$w, x, y, z, \ldots$	ι	
$X, \ldots$	$\iota \to o$	

Symbol	Definition
$x = y$	$(\forall X)(X(x) \to X(y))$
LNP	$\forall X(\exists y X(y) \to \exists y(\forall z(z < y \to \neg X(z)) \land X(y)))$
IND	$\forall X(\forall y(\forall z(z < y \to X(z)) \to X(y)) \to \forall y X(y))$
$\mathrm{D}(x, y)$	$\exists z\ x * z = y$
$\mathrm{PRIME}(x)$	$x > 1 \land \forall z(\mathrm{D}(z, x) \to (z = 1 \lor z = x))$
$\mathrm{PD}(x, y)$	$\mathrm{PRIME}(x) \land \mathrm{D}(x, y)$

such labels in the rest of this paper for readability. The characteristic sequent set of π is[3]

$$
\begin{array}{lll}
\mathrm{CS}(\pi) = & \{ & \\
& C_1: & \langle z_0 < s_0(\lambda x.\neg X_0(x))\rangle^{\lambda x.\neg X_0(x), z_0} \vdash \langle X_0(y_0)\rangle^{\lambda x.\neg X_0(x), y_0}, \\
& & \langle X_0(z_0)\rangle^{\lambda x.\neg X_0(x), z_0}; \\
& C_2: & \langle X_0(s_0(\lambda x.\neg X_0(x)))\rangle^{\lambda x.\neg X_0(x)} \vdash \langle X_0(y_0)\rangle^{\lambda x.\neg X_0(x), y_0}; \\
& C_3: & \vdash y_0 * 1 = y_0; \\
& C_4: & z_0 * z_1 = y_0 \vdash z_0 = 1, z_0 = y_0, z_0 < y_0; \\
& C_5: & z_0 * z_1 = y_0, y_0 > 1 \vdash z_0 = 1, z_0 > 1; \\
& C_6: & \vdash w_0 * (z_1 * z_2) = (w_0 * z_1) * z_2; \\
& C_7: & \vdash s_3 > 1; \\
& C_8: & x_0 > 1, x_0 * y_0 = s_3 \vdash s_2(x_0) * s_1(x_0) = x_0; \\
& C_9: & x_0 > 1, s_2(x_0) = 1, x_0 * y_0 = s_3 \vdash; \\
& C_{10}: & x_0 > 1, s_2(x_0) = x_0, x_0 * y_0 = s_3 \vdash \\
& \} &
\end{array}
$$

The refutation γ of $\mathrm{CS}(\pi)$ is based on the idea to prove that, from the number s_3, we can obtain an infinite strictly decreasing chain of divisors of s_3, which is inductively unsound. Indeed this property can be derived using essentially the clauses $C_7, \ldots, C_{10}$ in $\mathrm{CS}(\pi)$. Formally this argument is realized by replacing

[3] π was formalized using `HLK` (`http://www.logic.at/hlk`) and $\mathrm{CS}(\pi)$ was extracted using the `GAPT` framework (`http://code.google.com/p/gapt/`). The source code for π can be found at `http://www.logic.at/ceres/examples/primediv.html`.

the second-order variable X_0 by $\lambda x.F(x)$ for

$$F(x) \equiv \exists z(\mathrm{D}(z, s_3) \wedge z + x < s_3 \wedge z > 1).$$

Indeed, by $\vdash s_3 > 1$ we can derive (using C_8, C_9, C_{10}):

$$\vdash s_2(s_3) * s_1(s_3) = s_3; \qquad \vdash s_2(s_3) < s_3; \qquad \vdash s_2(s_3) > 1$$

and so $\vdash \mathrm{D}(s_2(s_3), s_3) \wedge s_2(s_3) < s_3 \wedge s_2(s_3) > 1$. Assume now we have already derived

$$(*) \qquad \vdash \mathrm{D}(c, s_3) \wedge c + x < s_3 \wedge c > 1.$$

Then using $\vdash c > 1$ instead of $\vdash s_3 > 1$ we derive

$$\vdash s_2(c) * s_1(c) = c; \qquad \vdash s_2(c) < c; \qquad \vdash s_2(c) > 1$$

so replacing c by $s_2(c)$ we get $\vdash \mathrm{D}(s_2(c), s_3) \wedge s_2(c) + (x + 1) < s_3 \wedge s_2(c) > 1$. $(*)$ for all x leads to a contradiction for $x \leftarrow s_3$.

The proof by LNP obtained via γ can be described informally as follows: We show LNP $\vdash \forall y \exists w(y > 1 \rightarrow \mathrm{PD}(w, y))$. Assume $\neg\forall y \exists w(y > 1 \rightarrow \mathrm{PD}(w, y))$, which is equivalent to $\exists y \forall w(y > 1 \wedge \neg\mathrm{PD}(w, y))$, and assume k is the smallest number s.t. $\forall w(k > 1 \wedge \neg\mathrm{PD}(w, k))$. Using the arguments of γ we get $s_2(k) > 1$, $s_2(k) < k$, $\mathrm{D}(s_2(k), k)$. Hence $\exists w\mathrm{PD}(w, s_2(k))$, so let q be a prime divisor of $s_2(k)$. But then also $\mathrm{D}(q, k)$ and so q is a prime divisor of k, contradiction.

We would like to mention a specific proof-theoretic property of this refutation γ: the proof obtained from γ *cannot* be obtained via the reductive Gentzen method. In fact, in Gentzen's method, X_0 would be replaced by the predicate

$$P\colon \lambda y.\exists w(y > 1 \rightarrow \mathrm{PD}(w, y))$$

which corresponds to the "straightforward" argument. Of course, also this kind of cut-elimination can be obtained by refuting $\mathrm{CS}(\pi)$ via the substitution $X_0 \leftarrow P$. This shows that, by its high flexibility, the CERES^ω method can reveal interesting mathematical arguments unattainable by reductive methods.

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