Dialogue Games for Fuzzy Logic

2. Diplomarbeitsvortrag

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Outline

1. Giles Style Dialogue Games
   - Motivation
   - Description
   - Adequateness for Łukasiewicz Logic

2. t-Norm Based Fuzzy Logics

3. Variants of Giles’s Game for Other Logics

4. Implementation
   - Giles Games
   - Hypersequential Proofs
   - Truth Comparison Games
   - Webgame
Giles Style Dialogue Games

Overview of Giles’s Game I

Motivation

- introduced by Robin Giles in the 1970s
- aim: model reasoning in physical theories
- provide a *tangible meaning* to (compound) propositions
- corresponds to Łukasiewicz Logic

Overview

- atomic propositions are identified with binary experiments
- experiments may show *dispersion*
- at any point in the game each player asserts a (multi)set of propositions
- game is divided into two separate parts:
  - deconstruction of complex propositions
  - evaluation of atomic game states
Giles Style Dialogue Games

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Giles Style Dialogue Games
Overview of Giles’s Game II

### Risk Values
- After playing the game both players have to pay a certain amount of money to each other.
- The expected amount a player has to pay is called his *risk value*.
- Both players aim to minimize their risk.

### Game Interpretation
- Primarily an evaluation game.
- Fixed assignment of probability values to experiments.
- Finite two-player zero-sum game with perfect information.
- Truth of a proposition $F$ is identified with the existence of a winning strategy for a player asserting $F$. 

Giles Style Dialogue Games

Overview of Giles’s Game II

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Game Interpretation
- primarily an evaluation game
- fixed assignment of probability values to experiments
- finite two-player zero-sum game with perfect information
- truth of a proposition $F$ is identified with the existence of a winning strategy for a player asserting $F$
Assume that both players assert only atomic propositions.

**Betting for Positive Results**

Let $a$ be an atomic proposition. He who asserts $a$ agrees to pay his opponent €1 if a trial of $E_a$ yields the outcome "no".

- for each assertion of an atomic proposition a trial of the associated experiment is done
- for an atomic proposition $a$ the corresponding experiment is denoted $E_a$
- the risk value for one player is the expected amount of money he has to pay in this game state
In the following let the players be called *you* and *me*.

**Example**

Let $a$ and $b$ be atomic propositions associated with the experiments $E_a$ and $E_b$ and $\pi(E_a) = 0.3$ and $\pi(E_b) = 0.9$. Assume that you assert $a$ and I assert both $a$ and $b$.

When evaluating this final game state, the experiment $E_a$ is conducted twice and $E_b$ once. In the expected case you have to pay me 0.7 € and I have to pay you 0.8 €. Thus, my risk value for this game state is 0.1 €.
Giles Style Dialogue Games
Decomposing Complex Propositions

Assume that both players assert a (multi)set of arbitrary propositions.

**General Game Rule**

One player chooses a compound proposition asserted by the other one. Either

- he attacks it according to the corresponding dialogue rule. Then the other player has to defend his claim as indicated by the rule.
- or he grants the proposition to his opponent.

Afterwards the proposition is deleted from the game.

- The order in which the players attack each others’ assertions is not specified.

**Implication**

He who asserts $A \rightarrow B$ agrees to assert $B$ if his opponent will assert $A$. 
Giles Style Dialogue Games
Decomposing Complex Propositions

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Implication

*He who asserts* \( A \rightarrow B \) *agrees to assert* \( B \) *if his opponent will assert* \( A \)
Giles Style Dialogue Games

Other Rules

**Disjunction**

*He who asserts* \( A \lor B \) *undertakes to assert either* \( A \) *or* \( B \) *at his own choice if challenged*

**Conjunction**

*He who asserts* \( A \land B \) *undertakes to assert either* \( A \) *or* \( B \) *at his opponent’s choice*

- Negation can be expressed using \( \neg A \equiv A \rightarrow \bot \).
- Other rules suitable for conjunction and disjunction as well.
- Dialogue rules refer to Lorenzen (1960s).
Giles Style Dialogue Games

Łukasiewicz Logic Ł

- many-valued, truth functional fuzzy logic
- domain of truth values: unit interval \([0, 1]\)

Connectives of Łukasiewicz Logic

Connectives: →, &, ∨, ¬ with truth functions:

- \(f_\rightarrow(x, y) = \min(1, 1 - x + y)\),
- \(f_\&(x, y) = \max(0, x + y - 1)\),
- \(f_\land(x, y) = \min(x, y)\),
- \(f_\lor(x, y) = \max(x, y)\),
- \(f_\neg(x) = 1 - x\).

A formula is called *true* in Ł under given interpretation iff it evaluates to 1.
Adequateness of Giles’s Game for Ł

For a fixed assignment of probability values to atomic propositions and a corresponding interpretation, I have a strategy to ensure that my risk is 0 when asserting a formula $A$, if and only if $A$ is true in Łukasiewicz Logic.

Correspondence Between Risk Values and Valuations

Let $\nu$ be an interpretation corresponding to the assignment of probability values to atomic propositions, $A$ be an arbitrary formula, and $\langle A \rangle$ be the risk value (for me) for the game starting with me asserting $A$. Then the valuation of $A$ under $\nu$ in Ł and the inverted risk value $1 - \langle A \rangle$ coincide.
Adequateness of Giles’s Game for Ł

For a fixed assignment of probability values to atomic propositions and a corresponding interpretation, I have a strategy to ensure that my risk is 0 when asserting a formula $A$, if and only if $A$ is true in Łukasiewicz Logic.

Correspondence Between Risk Values and Valuations

Let $v$ be an interpretation corresponding to the assignment of probability values to atomic propositions, $A$ be an arbitrary formula, and $\langle A \rangle$ be the risk value (for me) for the game starting with me asserting $A$. Then the valuation of $A$ under $v$ in Ł and the inverted risk value $1 - \langle A \rangle$ coincide.
t-Norm Based Fuzzy Logics

Definition: t-Norm

Continuous t-norm

A continuous t-norm is a continuous, associative, monotonically increasing function $\ast : [0, 1]^2 \rightarrow [0, 1]$ where $1 \ast x = x \quad \forall x \in [0, 1]$.

Residuum of a continuous t-norm $\ast$

The residuum of $\ast$ is a function $\Rightarrow : [0, 1]^2 \rightarrow [0, 1]$ where $x \Rightarrow y := \max\{z \mid x \ast z \leq y\}$.

- $\ast$ is used as truth function for (strong) conjunction.
- $\Rightarrow$ is used for as truth function implication.
**t-Norm Based Fuzzy Logics**

**Definition: t-Norm**

**Continuous t-norm**

A continuous t-norm is a continuous, associative, monotonically increasing function \( * : [0, 1]^2 \to [0, 1] \) where \( 1 * x = x \ \forall x \in [0, 1] \).

**Residuum of a continuous t-norm \( * \)**

The residuum of \( * \) is a function \( \Rightarrow_* : [0, 1]^2 \to [0, 1] \) where \( x \Rightarrow_* y := \max\{z | x * z \leq y\} \).

- \( * \) is used as truth function for (strong) conjunction.
- \( \Rightarrow_* \) is used for as truth function implication.
The three most important t-norms are:

<table>
<thead>
<tr>
<th>t-Norm</th>
<th>t-Norm</th>
<th>Residuum</th>
</tr>
</thead>
<tbody>
<tr>
<td>Łukasiewicz</td>
<td>$x \ast_L y = \max(0, x + y - 1)$</td>
<td>$x \Rightarrow_L y = \min(1, 1 - x + y)$</td>
</tr>
<tr>
<td>Gödel</td>
<td>$x \ast_G y = \min(x, y)$</td>
<td>$x \Rightarrow_G y = \begin{cases} 1 \text{ if } x \leq y \ y \text{ otherwise} \end{cases}$</td>
</tr>
<tr>
<td>Product</td>
<td>$x \ast\Pi y = x \cdot y$</td>
<td>$x \Rightarrow\Pi y = \begin{cases} 1 \text{ if } x \leq y \ y/x \text{ otherwise} \end{cases}$</td>
</tr>
</tbody>
</table>

- Any continuous t-norm can be constructed from these three ones.
Using $\ast$ and its residuum $\Rightarrow_{\ast}$ a logic $L_{\ast}$ can be defined containing of

- the binary connective $\&$ (strong conjunction),
- the binary connective $\rightarrow$,
- the constant $\bot$.

We can, furthermore, define the following derived connectives:

- $\neg A := A \rightarrow \bot$
- $A \land B := A \& (A \rightarrow B)$
- $A \lor B := ((A \rightarrow B) \rightarrow B) \land ((B \rightarrow A) \rightarrow A)$
Variants for Other Logics

Changing Evaluation Strategy

Joint Bets

A player has to pay 1 € to his opponent, unless all experiments associated with his assertions test positively.

→ Product Logic

Selecting Representatives

Each player picks one of the propositions asserted by his opponent; if the associated experiment tests false, he is paid 1 €.

→ Gödel logic
Variants for Other Logics

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→ Gödel logic
Variants for Other Logics

Changing Dialogue Rules

- just changing the evaluation scheme does not suffice
- introduction of the flag \(\perp\) signalizing that in order to win the game, my risk has to be strictly negative
- dialogue rule for implication has to be adjusted
- loss of uniformity of rules for both players

Implication (by you)

If you assert \(A \rightarrow B\) then, whenever I choose to attack this statement by asserting \(A\), you have the following choice: either you assert \(B\) in reply or you challenge my attack on \(A \rightarrow B\) by replacing the current game with a new one in which the flag \(\perp\) is raised and I assert \(A\) while you assert \(B\).

- also other ways to change the implication rule
Adequateness of Giles’s Game for $G$

For a fixed assignment of probability values to atomic propositions and a corresponding interpretation, I have a winning strategy when asserting a formula $A$, if and only if $A$ is true in Gödel Logic.

Adequateness of Giles’s Game for $\Pi$

For a fixed assignment of probability values to atomic propositions and a corresponding interpretation, I have a winning strategy when asserting a formula $A$, if and only if $A$ is true in Product Logic.
Adequateness of Giles’s Game for G

For a fixed assignment of probability values to atomic propositions and a corresponding interpretation, I have a winning strategy when asserting a formula $A$, if and only if $A$ is true in Gödel Logic.

Adequateness of Giles’s Game for Π

For a fixed assignment of probability values to atomic propositions and a corresponding interpretation, I have a winning strategy when asserting a formula $A$, if and only if $A$ is true in Product Logic.
Other topics the thesis deals with:

- Proofs using relational hypersequents
- Truth comparison games
- Giles’s Game for first order logic
- Devising rules for other connectives
- Using games to prove equivalences
- …
Implementation

Giles Games

A small utility to visualize game trees.

Example: $> \text{giles} \ "a/\(b->c)"$ produces:

```
[ || (a\(b->c))]
```

You choose a

```
[ || a]
```

You choose (b->c)

```
[ || (b->c)]
```

You assert not to attack (b->c)

```
[ || ]
```

You attack by asserting b

```
[b || c]
```

I defend by asserting c
Implementation
Hypersequential Proofs

Similarly, a tool to visualize proofs in the r-hypersequential calculus $rH$. Example: $\$>\ hypseq "a/\(b->c)"\ produces:

\[
\begin{array}{l}
(\text{Atomic}) \\
\leq a \\
\leq a \land (b \to c)
\end{array}
\]

\[
\begin{array}{l}
\leq b \leq c | b \leq c \\
\leq b \to c \\
\leq a \land (b \to c)
\end{array}
\]

$(\to, \leq, r)$

$(\land, \leq, r)$

$\$>\ hypseq "a/\(b->c)"\ produces:
A utility to find winning strategies for the proponent for a truth comparison game.

Example: $\texttt{tcgame } \text{" (a \lor b) \rightarrow (b \lor a) \text{" produces:}$

```
P ((a \land b) \rightarrow (b \land a))
```

```
O \{((a \land b) \rightarrow (b \land a)) < \top\}
```

```
P ((a \land b) \rightarrow (b \land a)) < \top
```

```
O \{(b \land a) < (a \land b), (b \land a) < \top\}
```

```
P (b \land a) < (a \land b)
```

```
O \{(b \land a) < \top, b < (a \land b)\}
```

```
O \{(b \land a) < \top, a < (a \land b)\}
```

```
P b < (a \land b)
```

```
O \{(b \land a) < \top, b < (a \land b)\}
```

```
O \{(b \land a) < \top, a < (a \land b)\}
```

```
P a < (a \land b)
```

```
O \{(b \land a) < \top, a < (a \land b)\}
```

```
O \{(b \land a) < \top, a < (a \land b), a < b\}
```

Implementation

Webgame

A web page where you can actually play Giles style dialogue games. Features:

- multiple undo and redo
- includes variants for Product and Gödel Logic
- elimination of connectives
- simulation of dispersive evaluation
- online at http://www.logic.at/staff/roschger/thesis/webgame/
- ...

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Dialogue Games for Fuzzy Logic
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Implementation
Webgame - Screenshots

>> Enter the Formula(s)

Initial state: \([(a \rightarrow b) \land (c \lor \neg a)]\)

Another Formula: [FOR ME]

Eliminate connectives [OK]

Use the following characters:

"&" for (min) conjunction
"&&" for strong conjunction
"|" for (max) disjunction
"||" for strong disjunction
"->" for implication
"-' for negation
[a-z] for atoms
"0" for falsum
"1" for verum
"("","")" to group expressions

Note: implication is associative to the right.

Finished
Implementation
Webgame - Screenshots

Start a Dialogue

<table>
<thead>
<tr>
<th>Your tenet:</th>
<th>My tenet:</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>((a \rightarrow b) \land (c \lor \neg a))</td>
<td></td>
</tr>
</tbody>
</table>

Available moves:

**Your Actions:**

- [ ] Attack \((a \rightarrow b) \land (c \lor \neg a)\):  Choose \(a \rightarrow b\)  Choose \(c \lor \neg a\)

[Undo last move] [Redo move] [Next Step: Evaluation]
## Implementation

### Webgame - Screenshots

**Start a Dialogue**

<table>
<thead>
<tr>
<th>Your tenet:</th>
<th>My tenet:</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a → b) ∧ (c ∨ ¬a)</td>
<td></td>
</tr>
<tr>
<td>c ∨ ¬a</td>
<td></td>
</tr>
<tr>
<td>¬a</td>
<td></td>
</tr>
</tbody>
</table>

| a  | □ |

Available moves:

**No more Actions left; Game is finished.**

[Undo last move] [Redo move] [Next Step: Evaluation]
Implementation

Webgame - Screenshots

>> Evaluate

Evaluation of Risks

- Final state: [a |⊥]
- Calculating your risk: 0.5
- Calculating my risk: 1
- → You succeed (You gain in average)

Dispersive binary experiments:

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>drawl</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>drawl</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>You win 1 Euro</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>drawl</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>drawl</td>
</tr>
<tr>
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<td>0</td>
<td>You win 1 Euro</td>
</tr>
<tr>
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<td>0</td>
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<td>0</td>
<td>drawl</td>
</tr>
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<tr>
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<td>0</td>
<td>drawl</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>You win 1 Euro</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>drawl</td>
</tr>
</tbody>
</table>

Another Evaluation | Another 10 Ones | Another 100 Ones

☐ Do not display evaluations.

35411 Evaluations done; I have lost 0.5018 Euros in average.
That’s it

Thanks for your attention!

Any questions?