Many-valued logics – a short introduction and case study

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Praeludium

(J.S.BACH, Suite I for Violoncello Solo)

The origin of many-valued logics
Case study: Limits and problems of classical logic

Problems a logician has to face

“I will die tomorrow” — Indetermination

“A chair is no a tree” — local definitions,

\[ A \lor \neg A \]

— pile of sand, first man — Modus ponens,

\[ A, A \supset B \vdash B \]

— liar’s paradox

— program verification — temporal processes, modalities
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Extensions of classical logic

classical logics

- syntax
- modal and temporal

- semantic
- many-valued logics
We are not searching for the one, true logic but for the logic which fits best for a given application.
Fugue

(J.S.BACH, Kunst der Fuge, Contrapunctus II)

Many-valued logics
Development of many-valued logics

The most important stops

- **ARISTOTELES** (De Interpretatione IX), OCKHAM: *future possibilities*, problem of determination vs. fatalism.
- ŁUKASIEWICZ 1920: 3-valued logic of *non-determinism*
- POST 1920: Many-valued logic dealing with functional completion
- GÖDEL 1932: Finite valued logics for approximation of intuitionistic logic
- BOČVAR 1938: Logic of *Paradoxa*
- KLEENE 1952: Logic of the *unknown*
- ZADEH 1965: Fuzzy sets and fuzzy logics
How do we continue?

**Arbitrary finite-valued logics**

For all finite-valued logics with truth-value functions there is an automatic algorithm for generating a sequent calculus, proving completeness etc (MultLog, MultSeq: BAAZ, FERMÜLLER, SALZER, ZACH ET AL. 1996ff).

**Infinite valued logics**

Does it make sense to take truth values from arbitrary partial orderings?

Answer

No, because every logics with substitution property would be a many-valued logic!

Take all sentences as truth values, and all sentences of the logic as designated truth values.
How do we continue?

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Design decisions

Basic requirements
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- \([0, 1]\) as super-set of the truth value set
- functional relation between the truth value of a formula and the one of its sub-formulas.
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Additional ‘natural’ properties of the conjunction

▶ associative ($(A \land B) \land C \Leftrightarrow A \land (B \land C)$)
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Additional ‘natural’ properties of the conjunction

▶ associative \(((A \land B) \land C \iff A \land (B \land C))\)
▶ commutative \((A \land B \iff B \land A)\)
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Additional ‘natural’ properties of the conjunction

▶ associative \(((A \land B) \land C \Leftrightarrow A \land (B \land C))\)
▶ commutative \((A \land B \Leftrightarrow B \land A)\)
▶ order preserving
  If \(A\) is less true than \(b\), then \(A \land C\) is less (or equal) true than \(B \land C\).
▶ continuous
Definition of (continuous) $t$-norms

Definition
A $t$-norm is an associative, commutative, and monotone mapping from $[0, 1]^2 \to [0, 1]$ with $1$ as neutral element.

- $(x \star y) \star z = x \star (y \star z)$
- $x \star y = y \star x$
- $x \leq y \supset x \star z \leq y \star z$
- $1 \star x = x$
- $\star$ is continuous
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**Algebraic view**
$\langle [0, 1], \ast, 1, \leq \rangle$ is a commutative and ordered monoid.
Łukasiewicz $t$-norm

\[ x \star_L y = \max\{0, x + y - 1\} \]
Gödel $t$-norm

\[x \ast _G y = \min\{x, y\}\]
Product $t$-norm

\[ x \star_{\prod} y = xy \]
From $t$-norm to the logic

Every $t$-norm has a residuum

$x \ast z \leq y$ if and only if $x \Rightarrow y = \max \{ z : x \ast z \leq y \}$

Truth functions for operators:

- **strong conjunction $\&$**: defined via the $t$-norm
- **implication $\supset$**: defined via the residuum
- **Negation $\neg$**: $\neg A = A \supset \bot$
- **(weak) disjunction $A \lor B$**: $A \lor B = (A \supset B) \supset B$
- **(weak) conjunction $A \land B$**: $A \land B = \neg (\neg A \lor \neg B)$
- **strong disjunction $A \lor B$**: $A \lor B = \neg (A \supset \neg B)$
From $t$-norm to the logic

The residuum of a $t$-norm

Every $t$-norm has a residuum

$$x \star z \leq y \iff z \leq (x \Rightarrow y)$$

$$x \Rightarrow y : = \max\{z : x \star z \leq y\}$$
From $t$-norm to the logic

The residuum of a $t$-norm

Every $t$-norm has a residuum

\[ x \star z \leq y \iff z \leq (x \Rightarrow y) \]
\[ x \Rightarrow y := \max\{z : x \star z \leq y\} \]

Truth functions for operators

- strong conjunction $\&$: defined via the $t$-norm
- implication $\supset$: defined via the residuum
- Negation: $\neg A := A \supset \bot$
- (weak) disjunction: $A \lor B := (A \supset B) \supset B$
- (weak) conjunction: $A \land B := \neg(\neg A \lor \neg B)$
- strong disjunction: $A \lor B := \neg(\neg A \lor \neg B)$
Properties of $t$-norms

Gödel

non-trivial
idempotent elements
Properties of $t$-norms

Gödel

- non-trivial idempotent elements

Łukasiewicz

- no non-trivial idempotent elements, but zero divisors
Properties of $t$-norms

**Gödel**
- non-trivial idempotent elements

**Łukasiewicz**
- no non-trivial idempotent elements, but zero divisors

**Product**
- no non-trivial idempotent elements, no zero divisors
Representation of $t$-norm

Theorem (McNaughton 1951)

*Every $t$-norm is the ordinal sum of Łukasiewicz $t$-norm and Product $t$-norm.*
Questions and results

- Basic logic: the logic of all \( t \)-norms (Hajek 1998)
- Axiomatizability: propositional logic: easy, first-order: only Gödel logics are axiomatizable (Scarpellini 1962, Horn 1969, Takeuti, Titani 1984, Takano 1987)
- calculi for propositional logic: sequent calculus for Gödel logic (Avron 1991, *hyper sequent calculus*), \( \Pi \)- und \( \mathcal{L} \)-Logik (Gabbay, Metcalfe, Olivetti 2003).
- calculi for first order logic: only for Gödel logic (Baaaz, Zach 2000)
- other questions: automatic theorem proving, size of families, …
Fugue II

(D. Taupin, Prélude et fugue)

Gödel logics and Kripke frames
Definition of Gödel logics

Syntax and Semantics

Usual propositional or first order language, \( \neg A \) is defined as \( A \supset \bot \).

Evaluations:

\[
\begin{align*}
\mathcal{I}(A \land B) &= \min\{\mathcal{I}(A), \mathcal{I}(B)\} \\
\mathcal{I}(A \lor B) &= \max\{\mathcal{I}(A), \mathcal{I}(B)\} \\
\mathcal{I}(A \supset B) &= \begin{cases} 
\mathcal{I}(B) & \text{if } \mathcal{I}(A) > \mathcal{I}(B) \\
1 & \text{if } \mathcal{I}(A) \leq \mathcal{I}(B)
\end{cases}
\end{align*}
\]

This yields the following definition of the semantics of \( \neg \):

\[
\mathcal{I}(\neg A) = \begin{cases} 
0 & \text{if } \mathcal{I}(A) > 0 \\
1 & \text{otherwise}
\end{cases}
\]
Definition of Gödel logics (cont.)

In the first order case the quantifiers are defined as follows:

\[ I(\forall x A(x)) = \inf \text{Distr}_I(A(a)) \]
\[ I(\exists x A(x)) = \sup \text{Distr}_I(A(a)) \]
Definition of Gödel logics (cont.)

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Fix a truth value set \( V \)

\[ \{0, 1\} \subseteq V \subseteq [0, 1] \]

which is closed (as a subset of \( \mathbb{R} \)).

The **Gödel logic based on** \( V \) is defined as the set of formulas valid under all interpretations into \( V \):

\[ A \in G_V \iff G_V \models A \iff \forall I_V : I_V(A) = 1 \]
Projectivity of Gödel logics:
Validity of formulas is not dependent on the absolute values of the truth values, but on their relative position. Every two truth value sets which are continuous and order-preserving bi-embeddable define the same logic.
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Example

\[ V_\uparrow = \{1 - \frac{1}{k} : k \geq 1\} \cup \{1\} \]
\[ V = V_\uparrow \setminus \left\{ \frac{1}{2} \right\} \]
\[ G_{V_\uparrow} = G_V \]
Definition of Kripke frames

Syntax and Semantics

(linear) Kripke frame (linear) partial order $K = (W, R)$

Kripke model $(K, U, \mathrm{val}_K)$ with $\mathrm{val}_K$ is a relation on $W \times \mathcal{A}(L^U)$ such that $w_1 \preceq w_2$ and $\mathrm{val}_K(w_1, \varphi)$ implies $\mathrm{val}_K(w_2, \varphi)$, and $\mathrm{val}_K(w, \bot)$ does not hold for all $w$.

Usual extension to all formulas:

$$\mathrm{val}_K(w, \varphi \land \psi) \text{ iff } \mathrm{val}_K(w, \varphi) \text{ and } \mathrm{val}_K(w, \psi)$$

$$\mathrm{val}_K(w, \varphi \lor \psi) \text{ iff } \mathrm{val}_K(w, \varphi) \text{ or } \mathrm{val}_K(w, \psi)$$

$$\mathrm{val}_K(w, \varphi \supset \psi) \text{ iff for all } v \text{ with } w \preceq v, \mathrm{val}_K(v, \varphi) \text{ implies } \mathrm{val}_K(v, \psi)$$

$$\mathrm{val}_K(w, \forall x \varphi(x)) \text{ iff for all } u \in U, \mathrm{val}_K(w, \varphi(u))$$

$$\mathrm{val}_K(w, \exists x \varphi(x)) \text{ iff there exists } u \in U, \mathrm{val}_K(w, \varphi(u))$$
Structure of $\text{Up}(K)$
From Kripke frames to Gödel logics

**Theorem**

*For every countable linear Kripke frame* $K$ *there is a Gödel set $V_K$ such that $L(K) = G_{V_K}$.*

$\text{Lim}(w)$ denotes that $w$ is a limit world, i.e.

$$(\forall w' > w)(\exists w'' > w)(w'' < w')$$

Define $W'$ by $W \cup \{w^*: \text{Lim}(w)\}$

Extend $\preceq$ to $\preceq'$ where the points from $\text{Lim}(W)$ are split into two.
The logic $L(\mathbb{Q})$

Let $K = \mathbb{Q}$. Every rational number is a supremum. Thus, every rational number is torn apart into two points.
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Take an enumeration of $\mathbb{Q} = \{q_1, q_2, \ldots\}$ and consider the following enumeration induced on $\mathbb{Q}' = \{q_1, q_1^*, q_2, q_2^*, \ldots\}$. 

An embedding of $\mathbb{Q}'$ into $[0, 1]$ preserving the order, infima and suprema will generate a set which is isomorph to the border points of the Cantor middle third set. The closure of this set is the Cantor middle third set. Thus, $L(\mathbb{Q}) = G_{[0, 1]}$. 

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Thus, $L(Q) = G_{C_{[0,1]}} = G_{[0,1]}$
Lemma

For every countable Gödel set $V$ there is a countable linear Kripke frame $K_V$ such that $G_V = L(K_V)$.

$$W_V = V \setminus \text{Sups}(V)$$
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Lemma

Let $V$ be a truth value set with non-empty perfect kernel $P$, and let $W = V \cup [\inf P, 1]$, then the logics induced by $V$ and $W$ are the same, i.e. $G_V = G_W$. 
Lemma
Splitting of Kripke frames: It is possible to split Kripke frames into two, construct the respective truth value sets, and stitch these truth value sets together again. The resulting logics stay the same.

Theorem
For every Gödel set $V$ there is a countable linear Kripke frame $K_V$ such that $G_V = L(K_V)$. 
Thanks

(J.S.BACH, Kunst der Fuge, Contrapunctus II)