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On Some Abstract Termination Criteria

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- Termination by weak termination + ...
  (globally / locally)

  $$P?(\to) \implies SN(\to)$$

  ARSs (+TRSs)

- Termination Criteria based on non-overlapping and/or non-erasing properties

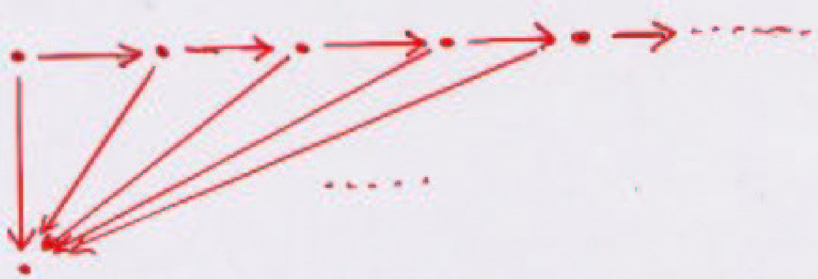
  TRSs

ARS : $(A, \to)$ with $\to \subseteq A \times A$

$WN(\to) \not\Rightarrow SN(\to)$

$$\left.\begin{array}{l} WN(\to) \\ WCR(\to) \end{array}\right\} \not\Rightarrow SN(\to)$$

$$\left.\begin{array}{l} WN(\to) \\ CR(\to) \end{array}\right\} \not\Rightarrow SN(\to)$$



∞

Theorem [Klop'80]

Given: $(A, \to)$, $a \in A$.

If ① $\exists b \in NF(\to) : a \xrightarrow{*} b$ — WN(a)

② $WCR(G(a))$ — loc. confl. below a

③ $\underbrace{\max\{k \mid a \xrightarrow{k} b\}}_{=: n}$ exists — boundedness

then: $SN(a)$

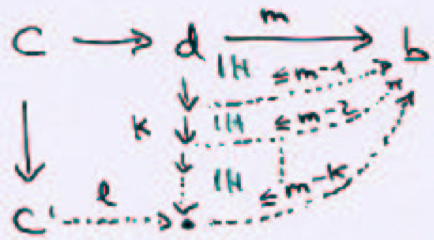
Proof: $B := \{c \mid a \xrightarrow{*} c \xrightarrow{*} b\}$

for $c \in B$: $|c| := \max\{k \mid c \xrightarrow{k} b\}$

(a) $\forall c, c' \in B : c \to c' \Rightarrow |c| > |c'|$

(b) $\left.\begin{array}{l} c \in B \\ c \to c' \end{array}\right\} \Rightarrow c' \in B$ by ind. on $|c|$

$|c| = m+1$:



$1 + m \geq 1 + l + m - k$

$\rightsquigarrow k \geq l$

$\Rightarrow$ all reductions from a have length at most n

$\Rightarrow SN(a) \wedge CR(G(a))$

Theorem [Klop '80]

Given: $(A, \to)$, $a \in A$.

If ① $\exists b \in NF(\to) : a \xrightarrow{*} b$

④ $WCR^1(G(a))$

then: $SN(a)$

WCR^1:

$$\begin{tikzcd}[column sep=small] & \cdot \arrow[dl] \arrow[dr] \arrow[dd, phantom, "+"] & \\ \cdot \arrow[dr, dashed] & & \cdot \arrow[dl, dashed] \\ & \cdot & \end{tikzcd}$$

Proof: $\left.\begin{array}{l} c \in B \\ c \to c' \end{array}\right\} \Rightarrow \begin{array}{l} c' \in B \\ |c'| = |c| - 1 \end{array}$ by ind. on $|c|$

$|c| = m+1$:

$$\begin{tikzcd} c \arrow[r] \arrow[d] \arrow[dr, phantom, "\times"] & d \arrow[r, "m"] \arrow[d, dashed] \arrow[r, phantom, "\mathrm{IH}"'] & b \\ c' \arrow[r, dashed] & \cdot \arrow[ur, dashed, "\leq m-1"'] & \end{tikzcd}$$

Theorem [Toyama'91]

Given: $(A, \to)$, $a \in A$.

If 1. $\exists b \in NF(\to): a \xrightarrow{*} b$

5. $BWCR(G(a))$

then: $SN(a)$

$BWCR(D): \forall a,b,c \in D\ \exists d \in D\ \exists m \in \mathbb{N}:$

$$\begin{tikzcd} a \arrow[r] \arrow[d] & b \arrow[d, dashed, "m"] \\ c \arrow[r, dashed, "m"] & d \end{tikzcd}$$

Proof: ....

Theorem

Given: $(A, \to)$, $a \in A$.

If ① $\exists b \in NF(a) : a \xrightarrow{*} b$

then: $SN(a)$

⑥ BSCR(G(a))

"BSCR"(D): $\forall a,b,c \in D\ \exists d,e \in D\ \exists m,n \in \mathbb{N}$:

$$\begin{tikzcd}
a \arrow[r] \arrow[d] & c \arrow[d, dashed, "n"] \arrow[dd, bend left, dashed, "m"] \\
b \arrow[r, dashed, "m"] \arrow[dr, dashed, "n"'] & d \\
& e
\end{tikzcd}$$

Proof: $\left.\begin{array}{l} c \in B \\ c \to c' \end{array}\right\} \Longrightarrow \begin{array}{l} c' \in B \\ c' \xrightarrow{|c|-1} b \end{array}$

$$\begin{tikzcd}
c \arrow[r] \arrow[d] \arrow[dr, phantom, "\not="] & d \arrow[r, "p"] \arrow[d, dashed, "n"] & b \\
c' \arrow[r, dashed, "m"'] & {} \arrow[ur, dashed, "p-n"'] \arrow[u, phantom, "IH^*", shift left=6]
\end{tikzcd}$$

$1 + m + p - n \le 1 + p$

$m \le n$

Symmetrically:

$1 + n + p - m \le 1 + p$

$n \le m$

hence: $m = n$

$\Longrightarrow c' \xrightarrow{p} b \quad \underbrace{}_{= |c|-1}$

Theorem [Klop'80]

Given: $(A, \to)$, $a \in A$

If ① $\exists b \in NF(\to) : a \xrightarrow{*} b$

② $\underbrace{\max\{k \mid a \xrightarrow{k} b\}}_{=:n}$ exists

⑦ $SCR^{\geq 1}(G(a))$

$SCR^{\geq 1}(D)$:

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\times"] & \cdot \arrow[d, dashed, "\geq 1"] \arrow[dd, bend left, dashed] \\
\cdot \arrow[r, dashed] \arrow[dr, bend right, dashed, "\geq 1"'] & \cdot \\
& \cdot
\end{tikzcd}$$

Proof:

$$\left.\begin{array}{l} c \in B \\ c \to c' \end{array}\right\} \Rightarrow \begin{array}{l} c' \in B \\ c' \xrightarrow{|c|-1} b \end{array}$$

$$\begin{tikzcd}
c \arrow[r] \arrow[d] \arrow[dr, phantom, "\times"] & d \arrow[r, "p"] \arrow[d, dashed, "k \geq 1"] \arrow[r, phantom, "\mathrm{IH}^*"', shift right=3] & b \\
c' \arrow[r, dashed] & \cdot \arrow[ur, bend right, dashed, "p-k"'] &
\end{tikzcd}$$

$p - k + 2 \leq p + 1$

$1 \leq k$

Symmetrically:

$p - 1 + k + 1 \leq p + 1$

$k \leq 1$

$\Rightarrow k = 1$

$c' \xrightarrow{\overbrace{p}^{=|c|-1}} b$

$$WN \Rightarrow [\,WCR^{1} \Leftrightarrow BWCR \Leftrightarrow BSCR \Leftrightarrow SCR^{\geq 1}\,]$$

however:

$$\begin{tikzcd}
WCR^{1} \arrow[rr, Rightarrow] \arrow[dr, Rightarrow] \arrow[dd, Rightarrow] & & WCR^{\leq 1} \arrow[r, Rightarrow] & SCR \\
& SCR^{\geq 1} \arrow[urr, Rightarrow] \arrow[dl, Rightarrow, shift left, "/" marking] & & \\
\text{"BSCR"} \arrow[ur, Rightarrow, shift left, "/" marking] & & &
\end{tikzcd}$$

$\curvearrowright$ maybe useful for refined analysis if $WN(\rightarrow)$ does not hold globally

Non-overlapping (NO) / Non-erasing (NE)

$(*)$ $NO \wedge NE \Rightarrow [WN \Leftrightarrow SN]$ [’92, ’95]

$NO \wedge WIN \Rightarrow SN$ — " —

Theorem $R = R_1 \cup R_2$ [Dershowitz ’95]

$$\left.\begin{array}{ll} ① & R_1, R_2 \ SN \\ ② & R_1 \text{ preserves } R_2\text{-NFs} \\ ③ & R_1 \cup R_2 \ NO \\ ④ & R_1 \cup R_2 \ NE \end{array}\right\} \Rightarrow R \ SN$$

necessary?

Non-overlapping (NO) / Non-erasing

(*) NO $\wedge$ NE $\Rightarrow$ [WN $\Leftrightarrow$ SN] ['92, '95]
NO $\wedge$ WIN $\Rightarrow$ SN

Theorem $R = R_1 \cup R_2$

① R_1, R_2 WIN
② R_1 preserves R_2-NFs
③ $R_1 \cup R_2$ NO

$\Rightarrow$ R SN

Proof: suppose: $\infty_R(s)$

Claim: $\exists s'$: $\overset{\infty_R}{s} \xrightarrow[i(R_2)]{*}_{R_2} \overset{\infty_R}{s'} \in NF(R_2)$

②① $\Rightarrow$ $\exists s''$: $\overset{\infty_R}{s'} \xrightarrow[i(R_1)]{*}_{R_1} s'' \in \underbrace{NF(R_1) \cap NF(R_2)}_{= NF(R)}$

② $\Rightarrow$ ——— $\overset{\infty_R}{s'} \xrightarrow[i(R)]{}_{R_1} s''$ ———

$\Rightarrow$ WIN(s', R)

③(*) $\Rightarrow$ SN(s', R)

$\Rightarrow$ ↯

Non-Erasing Transformations

$$R \left\{ \begin{array}{l} \ldots\ldots \\ p(s(x)) = x \\ fact(0) = s(0) \\ fact(s(x)) = s(x) \cdot fact(\underline{p(s(x))}) \\ \ldots\ldots \end{array} \right.$$

Simplify!

$$SN(R) \Rightarrow SN(R')$$

$$\Leftarrow ?$$

The transformation $R \rightsquigarrow R'$ is sound (wrt. SN) if R is non-overlapping and R' is obtained from R by replacing $l \to r \in R$ with $l \to r' \in R'$

where: $\underbrace{r \xrightarrow[R]{*} r'}_{\text{using only non-erasing rules}}$

this can be generalized (a little bit)

Conclusion

abstract relationships / criteria

+

structural knowledge

may be

very useful

for termination proofs