Relating Innermost, Weak, Uniform and Modular Termination of Term Rewriting Systems

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Abstract. Term rewriting systems play an important role in various areas, e.g. in abstract data type specifications, for automated theorem proving and as a basic computation model for functional programming languages. In theory and practice, one of the most important properties of term rewriting systems is the strong normalization or (finite or uniform) termination property which is undecidable in general. For ensuring this property many sufficient criteria and methods based on well-founded term orderings have been developed. In this paper we follow another line of research and study restricted termination properties of term rewriting systems, in particular weak termination and innermost termination, and their interrelation. New criteria are provided which are sufficient for the equivalence of innermost/weak termination and uniform termination of term rewriting systems. Using these basic results we are also able to prove some new results about modular termination of rewriting. In particular, we show that termination is modular for locally confluent overlay systems. As an easy consequence this result also entails a simplified proof of the fact that completeness is a decomposable property of so-called constructor systems. Interestingly, these modularity results are obtained by means of a proof technique which itself constitutes a *modular* approach.

1 Introduction

Due to the fact that termination is a fundamental property of term rewriting systems (TRSs for short) but undecidable in general, many sufficient criteria, techniques and methods for proving termination have been developed (see [Der87] for a survey). Most practically applicable approaches are based on reduction orderings, i.e. well-founded term orderings which are stable w.r.t. substitutions and monotonic w.r.t. the term structure.

On the other hand, in many rewriting based computation models, e.g. in functional programming languages, the indeterminism of general rewriting is often restricted by imposing some fixed rewriting strategy. For instance, a frequent restriction is innermost reduction, i.e. to require that every reduction step takes place at an innermost position of the term to be reduced. Innermost reduction corresponds closely to the functional evaluation mechanism employed in functional programming

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languages like LISP or ML. Of course, it may be the case that correspondingly restricted computations, i.e. innermost reduction sequences always terminate but arbitrary computations (reduction sequences) do not necessarily terminate. A very simple example illustrating this gap is the following:

Example 1. Let $R = \{f(a) \to f(a), a \to b\}$. Then we have e.g. the infinite reduction sequence $f(a) \to f(a) \to f(a) \to ...$, which uses only non-innermost reduction steps. But of course, every innermost derivation in R (e.g. $f(a) \to f(b)$) is terminating.

Other kinds of restrictions imposed on rewriting steps might also be conceivable according to the intended purpose, e.g. leftmost outermost, top-down, bottom-up or other context-dependent strategies. Unfortunately, very little is known about termination of rewriting under such restrictions and its relation to (uniform) termination. In fact, there is one major exception, namely concerning the important and thoroughly investigated class of so-called *orthogonal* TRSs, i.e. TRSs which are left-linear and non-overlapping (see [Klo90] for a survey of basic ideas, concepts and results about the theory of orthogonal TRSs). It is well-known that any orthogonal TRSs is confluent notwithstanding the fact that it may be non-terminating. For arbitrary TRSs one can conclude in general nothing about confluence[2] or the existence of (unique) normal forms if termination is not guaranteed since these properties are undecidable in the general case.

In the following we shall study in particular under what conditions innermost termination implies (uniform) termination of rewriting. More generally, we shall investigate and develop some extensions and generalizations of results about orthogonal TRSs. This is done by weakening both the no-overlap and the left-linearity requirement. Moreover, in a second part of the paper we shall apply these abstract results in a modular fashion in order to obtain new sufficient criteria for modular termination of TRSs.

The rest of the paper is structured as follows. Firstly, we introduce the basic definitions and notions needed later on. In section 3 we study the innermost, weak and uniform termination properties of TRSs. Then, in section 4, we consider modular properties of TRSs, applying previously developed ideas and results. And finally, related work as well as some open problems are discussed. More details and missing proofs can be found in the extended version of this paper (cf. [Gra92b]).

2 Preliminaries

We briefly recall the basic terminology needed for dealing with TRSs (e.g. [Klo90]). Let $\mathcal{V}$ be a countably infinite set of *variables* and $\mathcal{F}$ be a set of *function symbols* with $\mathcal{V} \cap \mathcal{F} = \emptyset$. Associated to every $f \in \mathcal{F}$ is a natural number denoting its arity. Function symbols of arity 0 are called *constants*. The set $\mathcal{T}(\mathcal{F}, \mathcal{V})$ of *terms* over $\mathcal{F}$ and $\mathcal{V}$ is the smallest set with (1) $\mathcal{V} \subseteq \mathcal{T}(\mathcal{F}, \mathcal{V})$ and (2) if $f \in \mathcal{F}$ has arity n and $t_1, \ldots, t_n \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ then $f(t_1, \ldots, t_n) \in \mathcal{T}(\mathcal{F}, \mathcal{V})$. If some function symbols are allowed to be varyadic then the definition of $\mathcal{T}(\mathcal{F}, \mathcal{V})$ is generalized in an obvious

[2] Of course, this phenomenon is due to the fact that in general confluence and local confluence need not coincide. Hence, the critical pair test for ensuring local confluence is not sufficient for confluence.

way. The set of all *ground terms* (over $\mathcal{F}$), i.e. terms with no variables, is denoted by $\mathcal{T}(\mathcal{F})$. In the following we shall always assume that $\mathcal{T}(\mathcal{F})$ is non-empty, i.e. there is at least one constant in $\mathcal{F}$. Identity of terms is denoted by $\equiv$. The set of variables occurring in a term t is denoted by $V(t)$.

A *context* $C[,\ldots,]$ is a term with 'holes', i.e. a term in $\mathcal{T}(\mathcal{F} \uplus \{\Box\}, \mathcal{V})$ where $\Box$ is a new special constant symbol. If $C[,\ldots,]$ is a context with n occurrences of $\Box$ and $t_1,\ldots,t_n$ are terms then $C[t_1,\ldots,t_n]$ is the term obtained from $C[,\ldots,]$ by replacing from left to right the occurrences of $\Box$ by $t_1,\ldots,t_n$. A context containing precisely one occurrence of $\Box$ is denoted by $C[]$. For the set $\mathcal{T}(\mathcal{F} \uplus \{\Box\}, \mathcal{V})$ we also write $\mathcal{CON}(\mathcal{F}, \mathcal{V})$. A *non-empty* context is a term from $\mathcal{CON}(\mathcal{F}, \mathcal{V}) \setminus \mathcal{T}(\mathcal{F}, \mathcal{V})$ which is different from $\Box$. A term s is a *subterm* of a term t if there exists a context $C[]$ with $t \equiv C[s]$. If in addition $C[] \not\equiv \Box$ then s is a *proper* subterm of t. A substitution σ is a mapping from $\mathcal{V}$ to $\mathcal{T}(\mathcal{F}, \mathcal{V})$ such that its domain $dom(\sigma)$ $\{x \in \mathcal{V} | \sigma x \not\equiv x\}$ is finite. Its homomorphic extension to a mapping from $\mathcal{T}(\mathcal{F}, \mathcal{V})$ to $\mathcal{T}(\mathcal{F}, \mathcal{V})$ is also denoted by σ.

A *term rewriting system (TRS)* is a pair $(\mathcal{R}, \mathcal{F})$ consisting of a signature $\mathcal{F}$ and a set $\mathcal{R} \subseteq \mathcal{T}(\mathcal{F}, \mathcal{V}) \times \mathcal{T}(\mathcal{F}, \mathcal{V})$ of (rewrite) rules (l, r) denoted by $l \to r$ with $l \notin \mathcal{V}$ and $V(r) \subseteq V(l)$. [3] Instead of $(\mathcal{R}, \mathcal{F})$ we also write $\mathcal{R}^{\mathcal{F}}$ or simply $\mathcal{R}$ when $\mathcal{F}$ is clear from the context or irrelevant. Given a TRS $\mathcal{R}^{\mathcal{F}}$ the rewrite relation $\to_{\mathcal{R}^{\mathcal{F}}}$ for terms $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ is defined as follows: $s \to_{\mathcal{R}^{\mathcal{F}}} t$ if there exists a rule $l \to r \in \mathcal{R}$, a substitution σ and a context $C[]$ such that $s \equiv C[\sigma l]$ and $t \equiv C[\sigma r]$. We also write $\to_{\mathcal{R}}$ or simply $\to$ when $\mathcal{F}$ or $\mathcal{R}^{\mathcal{F}}$ is clear from the context, respectively. The symmetric, transitive and transitive-reflexive closures of $\to$ are denoted by $\leftrightarrow$, $\to^+$ and $\to^*$, respectively. By $s \to^m t$ we mean that s is reduced to t in m steps. Two terms s, t are *joinable in* $\mathcal{R}$, denoted by $s \downarrow_{\mathcal{R}} t$, if there exists a term u with $s \to^*_{\mathcal{R}} u \,{}^*_{\mathcal{R}}\!\leftarrow t$. A term s is *irreducible* or a *normal form* if there is no term t with $s \to t$. A TRS $\mathcal{R}$ is *terminating* or *strongly normalizing (SN)* if $\to$ is noetherian, i.e. if there is no infinite reduction sequence $s_1 \to s_2 \to s_3 \to \cdots$. It is said to be *weakly terminating* or *weakly normalizing (WN)* if for every term there exists a normal form. Positions or occurrences of subterms of a term consist of sequences of natural numbers and are compared by the usual lexicographic ordering (which we shall ambiguously denote by $\leq$). Two uncomparable positions p and q are said to be *disjoint* which is denoted by $p|q$. If $p \leq q$ we say that p is above q or q is below p. A reduction step $s \to t$ by applying some rule of $\mathcal{R}$ at position p in s (denoted by $s \to_p t$) is *innermost* if every proper subterm of s/p is irreducible. In that case we also write $s \stackrel{i}{\to} t$. $\mathcal{R}$ is *innermost terminating* or *innermost normalizing (IN)* if every sequence of innermost reduction steps terminates. It is *weakly innermost terminating* or *weakly innermost normalizing (WIN)* if for every term s there exists a terminating sequence of innermost reduction steps starting with s. By $\infty(s)$ we denote the property that there exists an infinite ($\mathcal{R}$-) derivation starting with s. Accordingly, $\neg\infty(s)$ means that every derivation starting with s is finite. By $\infty_i(s)$ we denote the property that there exists an infinite innermost derivation starting with s.

[3] This restriction of excluding variable left-hand sides and right-hand side extra-variables is not a severe one. In particular, concerning termination of rewriting it only excludes trivial cases.

A *partial ordering* $>$ on a set D is a transitive and irreflexive binary relation on D. A partial ordering $>$ on $\mathcal{T}(\mathcal{F},\mathcal{V})$ is said to be *monotonic (w.r.t. the term structure)* if it possesses the *replacement property* $s > t \implies C[s] > C[t]$ for all $s, t, C[]$. It is *stable (w.r.t. substitutions)* if $s > t \implies \sigma s > \sigma t$ for all s, t, σ. A *term ordering on* $\mathcal{T}(\mathcal{F},\mathcal{V})$ is a monotonic and stable partial ordering on $\mathcal{T}(\mathcal{F},\mathcal{V})$. A *reduction ordering* is a well-founded term ordering.

A TRS is *confluent (CONF)* if ${}^* \leftarrow \circ \rightarrow^* \subseteq \rightarrow^* \circ {}^* \leftarrow$ and *locally confluent* if $\leftarrow \circ \rightarrow \subseteq \rightarrow^* \circ {}^* \leftarrow$.[4] A confluent and terminating TRS is said to be *convergent* or *complete (COMP)*. If $l_1 \rightarrow r_1$, $l_2 \rightarrow r_2$ are two rules[5] of $\mathcal{R}$ with p some non-variable position of l_2 such that l_1 and l_2/p are unifiable with most general unifier σ then $(\sigma(l_2[p \leftarrow r_1]), \sigma(r_2))$ is said to be a *critical pair (CP)* of $\mathcal{R}$ (obtained by overlapping $l_1 \rightarrow r_1$ with $l_2 \rightarrow r_2$ at position p). It is well-known that for terminating TRSs local confluence is equivalent to joinability of all critical pairs *(JCP)*. If every critical pair of a TRS $\mathcal{R}$ is obtained by an *overlay*, i.e. by overlapping left hand sides of rules at top position then $\mathcal{R}$ is said to be an *overlay system (OS)*. A TRS $\mathcal{R}$ is said to be *non-overlapping (NO)* if there is no critical pair between rules of $\mathcal{R}$. It is *left-linear (LL)* if every variable occurs at most once in every left hand side of an $\mathcal{R}$-rule. $\mathcal{R}$ is *orthogonal (ORTH)*[6] if it is left-linear and non-overlapping. It is *non-erasing (NE)* if $V(r) = V(l)$ for every rule $l \rightarrow r \in \mathcal{R}$. For properties P and Q of TRSs we write $P + Q$ for denoting the conjunction of P and Q. By $P(\mathcal{R})$ we mean that the TRSs $\mathcal{R}$ has property P. Moreover we also ambiguously use the notation $P(t)$ for terms t provided there is a sensible local interpretation for $P(t)$. For instance, $CONF(t)$ is to denote the property that whenever we have $t \rightarrow^* v$ and $t \rightarrow^* w$ then there exists a term s with $v \rightarrow^* s$ and $w \rightarrow^* s$.

3 Restricted Termination and Confluence Properties of Term Rewriting Systems

In the following we shall study under which conditions various restricted kinds of termination imply (uniform) termination (and possibly also confluence under some additional assumptions). Firstly we consider the no-overlap case, i.e. where no critical pair exists. Then we generalize the analysis by admitting restricted kinds of overlaps. Moreover, where possible, we give examples showing that no precondition of results obtained can be dropped.

It is well-known that under the assumption of termination the confluence property of (finite) TRSs can be easily tested by simply investigating joinability of all critical pairs (cf. [Hue80]). If termination is not guaranteed it is in general much more difficult to establish confluence. The problem is that for non-terminating systems joinability of critical pairs is only equivalent to local confluence but the latter property is not equivalent to confluence any more. There are some sufficient criteria for confluence which do not depend on the termination property but instead on rather strong conditions concerning the syntactical form of the rules and on how

[4] Here, '$\circ$' denotes relation composition.

[5] W.l.o.g. we assume that they are variable disjoint.

[6] In the literature this orthogonality property is sometimes called 'regularity'.

joinability of critical pairs has to be possible (see e.g. [Hue80], [Klo90]). Due to the strong preconditions the practical applicability of these criteria is rather limited.

Perhaps the most important and fundamental result concerning confluence of (possibly) non-terminating TRSs is the following: Any orthogonal, i.e. left-linear and non-overlapping TRS, is confluent (cf. e.g. [Ros73]). In particular, for orthogonal TRSs one also knows some sufficient criteria for termination which are formulated in terms of restricted termination properties (cf. e.g. [Klo90]). But as soon as the orthogonality requirement is weakened, either by allowing critical overlaps or by admitting non-left-linear rules the main results (at least concerning confluence) do not hold any more, in particular the so-called 'parallel moves lemma' which is the technical key lemma for inferring confluence for orthogonal TRSs.

Our overall goal in the following will be to establish (uniform) termination[7] by finding sufficient conditions involving restricted termination properties.

Let us start with the following result about properties of innermost reductions in non-overlapping TRSs.

Lemma 1. *Let R be a TRS with $NO(\mathcal{R})$. Then the following properties hold:*

(a) If $s_0, \ldots, s_k, t_0$ are terms with $s_0 \xrightarrow[i]{} s_1 \xrightarrow[i]{} s_2 \xrightarrow[i]{} \ldots \xrightarrow[i]{} s_{k-1} \xrightarrow[i]{} s_k$ and $s_0 \xrightarrow[i]{} t_0$, then there exist terms $t_1, t_2, \ldots, t_k$ and $m_1, m_2, \ldots, m_k, n_1, n_2, \ldots, n_k \in \mathbf{N}$ such that[8]

$$\begin{array}{ccccccccc} s_0 & \xrightarrow[i]{} & s_1 & \xrightarrow[i]{} & s_2 & \xrightarrow[i]{} & \ldots \xrightarrow[i]{} & s_{k-1} \xrightarrow[i]{} & s_k \\ \downarrow_i & & {\scriptstyle m_1}\downarrow_i & & {\scriptstyle m_2}\downarrow_i & & \cdots & {\scriptstyle m_{k-1}}\downarrow_i & {\scriptstyle m_k}\downarrow_i \\ t_0 & \xrightarrow[i]{}{}^{n_1} & t_1 & \xrightarrow[i]{}{}^{n_2} & t_2 & \xrightarrow[i]{}{}^{n_3} & \ldots \xrightarrow[i]{}{}^{n_{k-1}} & t_{k-1} \xrightarrow[i]{}{}^{n_k} & t_k \end{array}$$

with: $\forall j, 1 \leq j \leq k : n_j = m_j = 0 \ \vee \ n_j = m_j = 1$.

(b) If s, t and u are terms with $s \xrightarrow[i]{}{}^{m} t$ and $s \xrightarrow[i]{}{}^{n} u$ then there exist a term v and $m', n' \in \mathbf{N}$ with $m' \leq m$ and $n' \leq n$ such that

$$\begin{array}{ccc} s & \xrightarrow[i]{}{}^{m} & t \\ {\scriptstyle n}\downarrow_i & & {\scriptstyle n'}\downarrow_i \\ u & \xrightarrow[i]{}{}^{m'} & v \end{array}$$

(c) R is innermost confluent, i.e. $\xrightarrow[i]{}$ is confluent.

Proof. (a) By an easy case analysis of possible innermost reduction steps using the assumption $NO(\mathcal{R})$. (b) and (c) are straightforward consequences of (a). ■

Our next result shows that for non-overlapping systems the existence of an innermost normal form for some term t implies that any derivation initiated by t is finite.

Lemma 2. *Let $\mathcal{R}$ be a TRS with $NO(\mathcal{R})$. Then the following properties hold:*

(a) $WIN(\mathcal{R}) \iff IN(\mathcal{R})$.
(b) $\forall t : [\, WIN(t) \iff IN(t) \,]$.

[7] and hence also enable a (finite) test for the confluence property of (finite) TRSs
[8] $\mathbf{N}$ stands for the set of natural numbers.

Proof. Property (b) implies (a). Hence it suffices to prove the stronger local variant (b). For a proof by contradiction let t be a term with $\infty_i(t)$. By the assumption $WIN(t)$ we know that there is some innermost derivation

$$D : t = t_0 \xrightarrow[i]{} t_1 \xrightarrow[i]{} t_2 \xrightarrow[i]{} \cdots \xrightarrow[i]{} t_{n-1} \xrightarrow[i]{} t_n$$

with t_n irreducible. Obviously we have $\neg\infty_i(t_n)$. Let k, $1 \leq k < n$ be the minimal index with $\infty_i(t_k)$ and $\neg\infty_i(t_{k+1})$. Moreover let $s_{k+1}, s_{k+2}, \ldots$ be given such that

$$D' : t = t_0 \xrightarrow[i]{} t_1 \xrightarrow[i]{} t_2 \xrightarrow[i]{} \cdots \xrightarrow[i]{} t_{k-1} \xrightarrow[i]{} t_k \xrightarrow[i]{} s_{k+1} \xrightarrow[i]{} s_{k+2} \xrightarrow[i]{} \cdots$$

is an infinite innermost derivation. Then, using Lemma 1(a) and the above minimality assumption it is easy to construct an infinite innermost derivation

$$D'' : t_{k+1} \xrightarrow[i]{} s'_{k+1} \xrightarrow[i]{} s'_{k+2} \xrightarrow[i]{} \cdots$$

initiated by t_{k+1}. But this contradicts the assumption $\neg\infty_i(t_{k+1})$. ■

The next result states that in a non-overlapping and innermost terminating TRS $\mathcal{R}$ every reduction step can be expressed in terms of (parallel) innermost reduction steps

Lemma 3. *Let R be a TRS with $NO(\mathcal{R})$ and $IN(\mathcal{R})$ and s, t be terms with $s \to t$. Then there exist terms s', t' with*

$$\begin{array}{ccc} s & \to & t \\ * \downarrow & & * \downarrow \\ s' & \xrightarrow[i]{} & t' \end{array}$$

Proof. If the reduction step $s \to t$ by some rule $l \to r$ is innermost we are done. Otherwise, using the assumptions $NO(\mathcal{R})$ and $IN(\mathcal{R})$, s' and t' are obtained by (parallel) innermost normalizing[9] those subterms of s and t, respectively, which correspond to variables in the rule $l \to r$. ■

Lemma 4. *For any TRS R we have: $NO(\mathcal{R}) \wedge IN(\mathcal{R}) \Longrightarrow CONF(\mathcal{R})$.*

Proof. From Lemma 1(c) we know that $\xrightarrow[i]{}$ is confluent. Applying Lemma 3 we get $\to \subseteq \overset{*}{\underset{i}{\leftrightarrow}}$. From $\xrightarrow[i]{} \subseteq \to$ it finally follows that $\to$ is confluent, too. ■

If we omit the condition $IN(\mathcal{R})$ in the above lemma then $\mathcal{R}$ might not be confluent any more. Consider for instance

Example 2. Let $\mathcal{R}$ be the TRS given by the rules $f(x,x) \to a$, $f(x, g(x)) \to b$, $c \to g(c)$. Clearly, $\mathcal{R}$ is non-overlapping but not confluent. We have e.g. the derivations $f(c,c) \to a$ and $f(c,c) \to f(c, g(c)) \to b$. Here a and b are in normal from. Note that $\mathcal{R}$ is obviously non-terminating and even not innermost terminating.

Concerning termination properties we have the following first main result.

Theorem 5. *For any TRS R we have:*
(a) $NO(\mathcal{R}) \wedge IN(\mathcal{R}) \Longrightarrow SN(\mathcal{R})$.

[9] Note that the assumption $IN(\mathcal{R})$ of the lemma is crucial for ensuring the existence of an innermost normal form!

(b) $NO(\mathcal{R}) \implies [\forall s : IN(s) \implies SN(s)]$.

Proof. We won't give a proof here but simply refer to the proof of the more general Theorem 9. ■

As an easy consequence of this result we obtain

Corollary 6. *For any TRS* $\mathcal{R}$ *we have:* $NO(\mathcal{R}) \wedge IN(\mathcal{R}) \implies COMP(\mathcal{R})$.

Let us now consider some known results concerning termination properties of orthogonal TRSs.

Theorem 7. *(cf. e.g. [Klo90]) Let* R *be a TRS with* $ORTH(\mathcal{R})$. *Then the following properties hold:*
(i) $CONF(\mathcal{R})$,
(ii) $NE(\mathcal{R}) \implies [WN(\mathcal{R}) \iff SN(\mathcal{R})]$,
(iii) $(W)IN(\mathcal{R}) \iff SN(\mathcal{R})$, *and*
(iv) $\forall t : [(W)IN(t) \iff SN(t)]$.

Obviously, Theorem 5(a) and (b) correspond to Theorem 7 (iii)and (iv), respectively. Indeed, it is also possible to prove the following generalization of Theorem 7(ii).

Theorem 8. *For any TRS* $\mathcal{R}$ *the following holds:*
$NO(\mathcal{R}) \wedge WN(\mathcal{R}) \wedge NE(\mathcal{R}) \implies WIN(\mathcal{R}) \implies IN(\mathcal{R}) \implies SN(\mathcal{R})$.

Proof. This is proved again by contradiction and using the idea of minimal counterexamples. ■

Let us reconsider now Theorem 5. Essentially, what we have done there is to drop the precondition $LL(\mathcal{R})$ in Theorem 7(iii), i.e. to require only $NO + IN$ instead of $NO + LL + IN$ for deriving $SN(+CONF)$. Even stronger we shall prove now that the rather restrictive property NO may be further weakened by restricting possible overlaps and guaranteeing local confluence.

Theorem 9. *For any TRS* R *we have:*
(a) $OS(\mathcal{R}) \wedge JCP(\mathcal{R}) \wedge IN(\mathcal{R}) \implies SN(\mathcal{R}) \wedge CONF(\mathcal{R})$,
(b) $OS(\mathcal{R}) \wedge JCP(\mathcal{R}) \implies [\forall s : [IN(s) \implies SN(s) \wedge CONF(s)]]$,
i.e. any locally confluent and innermost terminating overlay system is terminating and complete (part (a)) which also holds in the localized version (b).

Proof. It suffices to prove (b) since (a) follows from it. Due to lack of space we only give a rough sketch of the proof idea and structure. We perform a proof by contradiction and assume that $\mathcal{R}$ is a TRS with $OS(\mathcal{R})$, $JCP(\mathcal{R})$ and $IN(s_0)$ but not $SN(s_0)$ for some term s_0. Then we consider a minimal counterexample for s_0, i.e. an infinite $\mathcal{R}$-derivation $D: \quad s_0 \to s_1 \to s_2 \to \cdots$ which is minimal in the sense that it satisfies
(1) $\forall j \geq 0 : \infty(s_j)$, and
(2) $\forall j \geq 0 \, \forall s'_{j+1} : [s_j \to_p s_{j+1} \wedge s_j \to_q s'_{j+1}$ with $q > p \,] \implies \neg\infty(s'_{j+1})$,

i.e. reduction steps in D are performed at deepest positions. This means in particular that innermost reduction steps are preferred as long as possible in the infinite derivation D. Now let n be the minimal index in D with $\infty(s_n)$ and $s_n \to s_{n+1}$, let's say at position p using the rule $l \to r$, but not $s_n \to_i s_{n+1}$ (i.e. $s_n \to s_{n+1}$ is the first non-innermost reduction step in D). Due to the minimality property (2) above we know that reducing s_n at some position q with $p < q$ yields some term s'_n with $\neg\infty(s'_n)$, hence any derivation starting from (or passing by) s'_n must be finite. This finiteness property also holds for every proper subterm s_n/p. Hence, using the local version of the equivalence of local confluence and confluence under the termination assumption[10] we know that every proper subterm of s_n/p has a unique normal form which may be computed by innermost reduction. The global assumption $OS(\mathcal{R})$ now implies that every reduction of s_n properly below p, let's say with $l' \to r'$, must correspond to a variable overlap between $l' \to r'$ and $l \to r$. Thus we can choose the subterms of s_n at all positions $p_1, \ldots, p_m \geq p$ which correspond to variable positions of the applied rule $l \to r$ and normalize them using innermost reduction. Let s'_n be the resulting term. Then we can apply the rule $l \to r$ in s'_n, too, but this time as an innermost reduction step, i.e. we have $s'_n \to_i s'_{n+1}$ for some s'_{n+1}. On the other hand we can obtain s'_{n+1} from s_{n+1} by innermost normalizing the subterms at those positions of s_{n+1} which correspond to variable positions of r (when considering the step $s_n \to s_{n+1}$ using $l \to r$). Moreover, $s_{n+1} \equiv s'_{n+1}$ is impossible due to the assumptions $\infty(s_{n+1})$ and $\neg\infty(s'_{n+1})$. Summarizing we have found s'_n and s'_{n+1} with

$$s_n \to s_{n+1} \to_i^+ s'_{n+1} \text{ and } s_n \to_i^+ s'_n \to_i s'_{n+1}$$

such that the reduction sequences $s_{n+1} \to_i^+ s'_{n+1}$ and $s_n \to_i^+ s'_n$ are performed by (parallel) innermost normalizing some proper subterms. By a similar kind of reasoning we can now find a next index $n_1 > n_0 := n$ and terms s'_{n_1}, s'_{n_1+1} such that $s_{n_0} \to_i^+ s'_{n_0} \to_i s'_{n_0+1} \to_i^* s'_{n_1} \to_i s'_{n_1+1}$ and $s_{n_0} \to s_{n_0+1} \to^* s_{n_1} \to s_{n_1+1}$ with $s_{n_1+1} \to_i^+ s'_{n_1+1}$ again by steps of (parallel) innermost normalizing some some subterms of s_{n_1+1}. By induction we finally conclude that there exists an infinite innermost derivation starting with s_0 of the form

$$D: \quad s_0 \to_i^* s_{n_0} \to_i^+ s'_{n_0} \to_i^+ s'_{n_0+1} \to_i^* s'_{n_1} \to_i^+ s'_{n_1+1} \to_i^* s'_{n_2} \to_i^+ s'_{n_2+1} \cdots.$$

But this is a contradiction to the assumption $IN(s_0)$ Hence we are done. ■

Note that we cannot weaken the precondition of Theorem 9 by omitting the requirement $JCP(\mathcal{R})$. To wit, consider

Example 3. [11] Let $\mathcal{R}$ be the TRS consisting of the rules $f(a,b,x) \to f(x,x,x)$, $G(x,y) \to x$ and $G(x,y) \to y$. Obviously, $\mathcal{R}$ is an innermost terminating overlay system but it is not strongly normalizing as can be seen from the infinite (cyclic) derivation $f(a,b,G(a,b)) \to f(G(a,b),G(a,b),G(a,b)) \to f(a,G(a,b),G(a,b)) \to f(a,b,G(a,b)) \to \cdots$.

[10] This means the following: For every locally confluent TRS and for every term t the implication $SN(t) \Longrightarrow CONF(t)$ holds.

[11] Note that by partitioning these 3 rules into $\mathcal{R}_1 := \{f(a,b,x) \to f(x,x,x)\}$ and $\mathcal{R}_2 := \{G(x,y) \to x, G(x,y) \to y\}$ we obtain Toyama's counterexample to modularity of termination (cf. [Toy87a]).

According to Theorem 7(iii) $(NO+LL+IN)$ implies SN. Theorem 5 says that even $(NO+IN)$ implies SN. Moreover, from Theorem 9 we know that the weaker property $(OS+JCP+IN)$ implies SN. Hence, another interesting conjecture derived from known results about orthogonal systems would be the following: $(CONF+LL+IN) \Longrightarrow SN$. But again this is not true in general.

Example 4. Let $\mathcal{R}$ be given by the rules $f(a) \to f(a)$ and $a \to b$. Obviously, this system is confluent, left-linear and innermost normalizing but is clearly non-terminating.

4 Modularity Results for Disjoint Unions of TRSs and Combinations of Constructor Systems

We shall now consider modular properties of TRSs and apply our previous results in order to derive some interesting new results and provide simplified proofs of known ones. Let us first give a brief (and incomplete) overview of known results in this field. Modular properties of term rewriting systems have attracted an increasing attention within the last few years. A simple and natural way of such 'modular' constructions is given by the concept of 'direct sum' ([Toy87b]) or 'disjoint union'.[12] Two TRSs $\mathcal{R}_1$ and $\mathcal{R}_2$ over signatures $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively, are said to be *disjoint* if $\mathcal{F}_1$ and $\mathcal{F}_2$ are disjoint, i.e. $\mathcal{F}_1 \cap \mathcal{F}_2 = \emptyset$ (in that case the rule sets of $\mathcal{R}_1$ and $\mathcal{R}_2$ are necessarily disjoint, too). The (disjoint) union of two disjoint TRSs $\mathcal{R}_1$, $\mathcal{R}_2$ is denoted by $R_1 \oplus R_2$. A property P of TRSs is said to be *modular* if the following holds for all disjoint TRSs R_1, R_2: $R_1 \oplus R_2$ has property P iff both R_1 and R_2 have property P. Toyama [Toy87b] has shown that confluence is modular. The termination property, however, is in general not modular as witnessed by Example 3 above. Other, more complicated examples by Klop & Barendregt as well as by Toyama gathered in [Toy87a] show that $\mathcal{R}_1 \oplus \mathcal{R}_2$ may be non-terminating even if R_1 and R_2 are both terminating, confluent and interreduced. As proved in [Rus87] termination is modular for the class of collapse-free TRSs as well as for the class of non-duplicating systems. In [Mid89] it is shown that the disjoint union $\mathcal{R}_1 \oplus \mathcal{R}_2$ of two terminating TRSs $\mathcal{R}_1$, $\mathcal{R}_2$ is again terminating whenever one of the systems is non-duplicating and collapse-free. Moreover, as shown in [KO90a], simple termination[13] is a modular property of (finite) TRSs. A unified approach to modular termination of rewriting is provided in [Gra91] where still more general sufficient conditions for modularity of termination are presented. Some extensions and generalizations of these known results on modular termination to the case of conditional TRSs as well as to some restricted classes of non-disjoint unions of TRSs can be found in [Mid90], [KO90b], [Gra91] and [Gra92a].

Another interesting result is due to [TKB89] where it is shown that termination (and hence completeness) is modular for left-linear and confluent TRSs.

The main link between our abstract results proved in the previous section and the problem of modularity of termination is provided by the following easy but fundamental result.

[12] Roughly spoken, the concept of 'direct sum' as defined in [Toy87b] is slightly more general than that of 'disjoint union' because it allows for renaming function symbols.

[13] A TRS is *simply terminating* if its termination can be proved by a simplification ordering.

Lemma 10. *(cf. e.g. [TKB89]) Innermost termination (IN) is a modular property of TRSs.*

Combining this result with Theorem 5 we obtain

Theorem 11. *Termination (and hence completeness) is modular for the class of non-overlapping TRSs.*

Proof. Let $\mathcal{R}_1$, $\mathcal{R}_2$ be two non-overlapping and terminating TRSs. Hence we have $(NO+SN)(\mathcal{R}_1)$ and $(NO+SN)(\mathcal{R}_2)$ implying $(NO+IN)(\mathcal{R}_1)$ and $(NO+IN)(\mathcal{R}_2)$. Since NO is obviously modular and IN, too, due to Lemma 10, we conclude $(NO+IN)(\mathcal{R}_1 \oplus \mathcal{R}_2)$. By applying Theorem 5 we finally obtain $(NO+SN)(\mathcal{R}_1 \oplus \mathcal{R}_2)$ and hence $(SN+CONF)(\mathcal{R}_1 \oplus \mathcal{R}_2)$ as desired. ■

Similarly, Theorem 9(a) yields the following generalized modularity result.

Theorem 12. *Termination (and hence completeness) is modular for the class of locally confluent overlay systems.*

Proof. Analagous to the proof of Theorem 11 using Lemma 10 and Theorem 9(a). ■

Note that this result is in a sense a variation of the main result of [TKB89] which says that $(LL+CONF+SN)$ is modular. Theorem 12 states that the left-linearity restriction can be dropped if a stronger kind of confluence property is satisfied, namely, every critical pair must constitute a joinable overlay. In other words, $(OS+JCP+SN)$ is modular. Moreover, example 3 shows that dropping JCP is not possible because in general $(OS+SN)$ is not modular. Finally let us mention that Theorem 9(a) can be used to give a substantially simplified proof of the main result from [MT91]. For a precise formulation we need some terminology from [MT91].

Definition 13. ([MT91]) A *constructor system (CS)* is a TRS $\mathcal{R}^{\mathcal{F}}$ with the property that $\mathcal{F}$ can be partitioned into $\mathcal{F} = \mathcal{C} \uplus \mathcal{D}$ such that every left hand side $f(s_1, \ldots, s_n)$ of a rewrite rule from $\mathcal{R}^{\mathcal{F}}$ satisfies $f \in \mathcal{D}$ and $s_1, \ldots, s_n \in \mathcal{T}(\mathcal{C}, \mathcal{V})$.[14] Function symbols in $\mathcal{D}$ are called *defined symbols* and those in $\mathcal{C}$ *constructors*.

Middeldorp and Toyama have shown in [MT91] that completeness is preserved under the union of constructor systems with disjoint sets of defined symbols (and common set of constructor symbols). In fact, a slightly more general result is proved in [MT91]. Since any constructor system is by definition a special overlay system we easily obtain from Theorem 9 the following

Corollary 14. *For any TRS R we have $CS(\mathcal{R}) \wedge IN(\mathcal{R}) \wedge JCP(\mathcal{R}) \Longrightarrow SN(\mathcal{R}) \wedge CONF(\mathcal{R})$, i.e. any innermost terminating and locally confluent constructor system is terminating and confluent, hence complete.*

Moreover we need the following result (which has a simple proof).

[14] This definition of constructor system corresponds to what is usually meant when one speaks of a constructor discipline (for specifying functions).

Lemma 15. *(see [MT91]) The union of innermost terminating constructor systems which do not share defined symbols is again an innermost terminating constructor system.*

Proof. The proof is essentially the same as in [MT91] where it is shown (by innermost reductions) that the union of weakly normalizing constructor systems which do not share defined symbols is again weakly normalizing. ■

Now it is easy to give a substantially simplified and modular proof of

Theorem 16. *([MT91]) The union of complete constructor systems with disjoint sets of defined function symbols is again complete.*[15]

Proof. Straightforward by combining Corollary 14 and Lemma 15. ■

5 Discussion and Open Problems

Before concluding let us finally discuss the general idea underlying the approach presented and some open problems. In fact, what has been done in section 3, is on the one hand side an abstract analysis of the interrelation between restricted and uniform termination under some additional assumptions. On the other hand the goal of the analysis and the results obtained may be considered to be a kind of 'modular' approach to modular properties of TRSs in the following sense. We wanted to find properties P such that e.g. $IN + P \implies SN\ (+P)$ holds. The additional knowledge available was that IN is modular. Generalizing this situation we get the following abstract 'modular' approach for obtaining sufficient conditions for the modularity of some property Q.

Assume that Q is not modular. Then look for other modular properties $P_1, \ldots, P_n$ with[16] $(P_1 + \cdots + P_n) \implies Q$. Then the property $(P_1 + \cdots + P_n + Q)$ is obviously modular, hence Q is modular for the class of TRSs satisfying $(P_1 + \cdots + P_n)$.

Unfortunately, this kind of modular approach is not easily applicable to the above mentioned result, namely

Theorem 17. *[TKB89] $(LL + CONF + SN)$ is a modular property of TRS.*

For obtaining a modular proof here we would have to find some property P (probably a kind of restricted termination property) satisfying

(1) $(LL + CONF + P)$ is modular , and
(2) $(LL + CONF + P) \implies SN$.

Since P cannot be IN, this is an interesting open problem. But note that even for the case that such a (hopefully easy to understand) property P exists it is not clear a priori that the proofs of (1) and (2) would be simple.

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[15] One may allow here common defined function symbols under the additional assumption that for such symbols the corresponding definition rules have to coincide in both systems. See [MT91] for more details and a precise definition of 'composable' constructor systems and 'decomposable' properties of constructor systems.

[16] Slightly weaker, it even suffices that only the conjunction $(P_1 + \cdots + P_n)$ is modular.

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