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Faculty of Informatics

# Cut-elimination by resolution in intuitionistic logic

DISSERTATION

zur Erlangung des akademischen Grades

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eingereicht von

**Giselle Reis**

Matrikelnummer 1029572

an der
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Betreuung: Univ.Prof.Dr.phil. Alexander Leitsch

Diese Dissertation haben begutachtet:

(Univ.Prof.Dr.phil. Alexander Leitsch) (Univ.Prof.Dr.phil. Rosalie Iemhoff)

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(Giselle Reis)

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by

**Giselle Reis**
Registration Number 1029572

to the Faculty of Informatics
at the Vienna University of Technology

Advisor: Univ.Prof.Dr.phil. Alexander Leitsch

The dissertation has been reviewed by:

(Univ.Prof.Dr.phil. Alexander Leitsch) (Univ.Prof.Dr.phil. Rosalie Iemhoff)

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(Giselle Reis)

# Erklärung zur Verfassung der Arbeit

Giselle Reis
Favoritenstraße 9-11, 1040 Wien

Hiermit erkläre ich, dass ich diese Arbeit selbständig verfasst habe, dass ich die verwendeten Quellen und Hilfsmittel vollständig angegeben habe und dass ich die Stellen der Arbeit - einschließlich Tabellen, Karten und Abbildungen -, die anderen Werken oder dem Internet im Wortlaut oder dem Sinn nach entnommen sind, auf jeden Fall unter Angabe der Quelle als Entlehnung kenntlich gemacht habe.

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# Acknowledgements

Upon finishing this PhD thesis I realize that a thesis does not come even close to represent all the things I learned during these 3 years. I would need maybe infinitely many pages to be able to express this in words. In three years I learned not only about cut-elimination and intuitionistic logic. There was also cut introduction, resolution, higher-order logic and how to implement $\lambda$-calculus and generate *eigenvariables*. On top of that, there was learning (very very basic) German, time management to get supermarkets open, not falling on a snowboard, dressing three layers of clothes (and undressing them each time you enter a room), yoga and basic botanics.

It's hard to list all the people that influenced me and kept me going through this experience, but I will try to name a few.

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# Abstract

This work is about eliminating lemmas from constructive proofs. Proofs are represented using intuitionistic sequent calculus and lemmas are cut inferences. So in the end we are actually eliminating cuts from intuitionistic sequent calculus proofs. The way we do this is via a method based on resolution. Why do we want to eliminate lemmas? Because it is important for proof analysis. Different proofs of the same theorem might give us different insights on the theorem itself. We use intuitionistic sequent calculus because we are interested in constructive proofs, which, in principle, give us more information than a non-constructive proof. A constructive proof can actually provide a concrete algorithm for solving a problem. We use a method based on resolution inspired by CERES, which is cut-elimination by resolution for classical logic. CERES can perform better than reductive cut-elimination, complexity-wise speaking, on some types of proofs. Also, the resulting proof after eliminating the cuts via CERES can be different than the ones reductive cut-elimination will yield, which means more information for mathematical proof analysis. Our main goal is to develop a CERES-like method for intuitionistic logic such that it has the same advantages as CERES for classical logic.

The CERES method, as is, for classical logic cannot be straightforwardly applied to intuitionistic calculi. The main difficulty comes from the fact that intuitionistic calculi are often defined by some structural restriction on the sequent. These restrictions are bound to be violated when performing some of CERES' operations and because of this the resulting proof tends to be classical. We need to modify the method such that it accommodates these restrictions and yields intuitionistic cut-free proofs. This change can be done in many ways along the steps of CERES, and several methods were developed.

Our first solution is based on the application of extra inferences to avoid the violations. This turned out to work only for a subclass of intuitionistic logic. It involves using a new resolution calculus and redefining the elements of CERES. The second solution works for the same subclass, but with the original elements of CERES and a simple post-processing of the final proof.

The third approach required an extensive study of rule permutations in the sequent calculus for intuitionistic logic **LJ**. It is based on the observation that if the input proof is of a specific shape, the final proof can be transformed into an intuitionistic proof by an operation we define. The rule permutations are necessary to transform the input proof into this shape, and interestingly enough, some unsoundness is allowed. This unsoundness comes from the permutation of strong quantifier inferences. Usually, these are not allowed for they might generate eigenvariable violations, but in our case, we can live with the violations and guarantee that they will not occur on the final proof.

The fourth and fifth methods are not methods *per se*. The former defines a new way to assemble the final proof in CERES and the latter is a resolution refinement that approximates CERES to reductive cut-elimination. We conjecture that, by combining these operations we obtain a cut-elimination method by resolution for intuitionistic logic (without strong quantifiers in the theorem to be proven).

This thesis summarizes the main results in developing a method of cut-elimination by resolution for intuitionistic logic. Besides the methods already mentioned for sub-classes of this logic, we also discuss the relation between CERES' operations and intuitionistic logic, and pinpoint exactly where the problem lies. We finish the thesis with a conjecture that such a method is possible. The proof of which we leave as future work.

# Kurzfassung

Diese Arbeit behandelt die Elimination von Lemmata in konstruktiven Beweisen. Beweise werden im ituitionistischen Sequentialkalkül dargestellt, Lemmata durch Gebrauch der Schnittregel. Somit handelt diese Arbeit über Schnittelimination im intuitionistischen Sequentialkalkül. Die dafür verwendete Methode beruht auf dem Resolutionsprinzip. Warum sollten wir Lemmata überhaupt eliminieren? Der Grund dafür liegt in der Bedeutung für die Beweisanalyse. Verschiedene Beweise des selben Satzes können uns verschiedene Einsichten in den Satz selbst vermitteln. Wir verwenden den intuitionistischen Sequentialkalkül weil wir an konstruktiven Beweisen interessiert sind, welche uns im Prinzip mehr Information liefern können als nicht-konstruktive. Aus einem konstruktiver Beweis kann auch einen konkreter Algorithmus zur Lösung eines Problems gewonnen werden. Wir verwenden eine resolutions-basierte Methode inspiriert durch CERES, welches eine Schnitteliminationsmethode für die klassische Logik auf Basis des Resolutionskalküls ist. Auf bestimmten Beweistypen ist CERES der reduktiven Schnitteliminationsmethode in komplexitätstheoretischer Hinsicht überlegen. Auch die durch CERES gewonnenen Beweise können von denen durch reduktive Methoden erhaltenen verschieden sein, und daher mehr Information in der mathematischen Beweisanalyse liefern. Die Hauptaufgabe dieser Arbeit ist es eine CERES-Typ Methode für die intuitionistische Logik zu entwickeln welche die selben Vorteile bietet wie CERES in der klassischen Logik.

Die CERES-Methode, so wie sie für die klassische Logik definiert ist, kann nicht einfach auf intuitionistische Kalküle übertragen werden. Das Hauptproblem kommt daher dass intuitionistische Kalküle vielfach über strukturelle Einschränkungen der Sequente definiert werden. Diese Einschränkungen werden typischerweise durch die CERES-Operationen verletzt wodurch ein nur klassischer Beweis entsteht. Aus diesem Grund muss die Methode modifiziert werden sodass sie diese Einschränkungen beachtet und ein intuitionistischer Beweis resultiert. Diese Modifikation (an den Schritten von CERES) kann auf verschiedene Weise geschehen und führt zur Entwicklung verschiedener neuer Methoden.

Die erste Lösung besteht in der Anwendung zusätzlicher Inferenzen um die Regelverletzungen zu vermeiden. Dieser Ansatz stellte sich für eine Unterklasse intuitionitischer Beweise als sinnvoll heraus. Hier wird ein neuer Resolutionskalül entwickelt und die einzelnen Phasen von CERES neu definiert. Die zweite Lösung, welche für die selbe Subklasse gilt, verwendet die ursprüngliche CERES-Methode und wendet ein Postprocessing auf den resultierenden Beweis an.

Der dritte Ansatz erforderte eine ausführliche Untersuchung der Regelpermutationen im Sequentialkalkül **LJ** für die intuitionistiche Logik. Der Ansatz beruht auf der Beobachtung, dass für gewisse Beweisformen der CERES-Beweis durch Postprocessing in einen intuitionistischen

Beweis transformiert werden kann (mit Transformationen die in dieser Arbeit definiert werden). Die Regelpermutationen sind dabei nötig um die Beweise in diese Formen überzuführen, wobei interessanterweise auch inkorrekte Beweistransformationen erlaubt sind. Die Inkorrektheit entsteht dabei durch Permutation starker Quantorenregeln. Diese Inkorrektheit ist natürlich im Allgemeinen inakzeptabel, in diesem Falle aber harmlos, da gezeigt werden kann dass diese im Endbeweis wieder verschwinden.

Die vierte und fünfte Methode sind nicht Methoden *per se*. Die vierte liefert eine neue Methode um den Endbeweis in CERES zusammen zu setzen, die fünfte ist eine Resolutionsverfeinerung welche mittels CERES die reduktive Schnitteliminationsmethode approximiert. Wir vermuten dass, durch Kombination von vier und fünf, eine vollständige Schnitteliminationsmethode für intuitionistischen Beweise von skolemisierten Sätzen entsteht.

Diese Arbeit fasst die Hauptresultate in der Entwicklung einer resolutions-basierten Methode für die intuitionistische Logik zusammen. Außer den beschriebenen Methoden für Subklassen der intuitionistischen Logik beschreiben wir auch das Verhalten zwischen CERES-Operationen und der intuitionistischen Logik, und zeigen auf, wo genau die Probleme liegen. Am Ende der Arbeit wird die Vermutung definiert, dass (wie oben beschrieben) eine vollständige Methode für die intuitionistische Logik gewonnen werden kann; der Beweis davon ist Aufgabe zukünftiger Forschung.

# Contents

# List of Theorems

# CHAPTER 1

# The Big Picture

"What is your PhD?" This is probably the first thing we get asked when people find out we are doing a PhD. Unfortunately, this is also the most dreaded question of almost every PhD student I know. I think it is because we get so involved in the work, and we worry so much about the details to make something work, that we end up forgetting what that something is. And when we get asked this question, we can immediately think of a million tiny things that need to be solved, and that is effectively what we are doing in a daily basis, but that is not the answer the other person is expecting. They want the big picture.

It is important not to loose track of what is the big picture. We must be able to understand the purpose of what we are doing and to explain this to other people. Otherwise, how can we convince funding agencies, employers, relatives and even ourselves of the importance of our work? This chapter is dedicated to the big picture of my PhD.

## 1.1 Cut-elimination

Mathematicians are exact and precise people. Maybe because mathematics is an exact and precise science. Problems need to be well defined, as well as solutions. Theorems need to be proved. A correct proof is the utmost guarantee that a statement is true, while just one counter-example is enough to state that a statement is false. We can say that mathematics revolves around theorems and its proofs. This work is about proofs.

Proofs can be seen as mathematical objects themselves. Like any decent mathematical object, there are some operations that can be performed with them. The problem with proofs is that they are generally presented to people not as mathematical objects, but as a text in natural language proving stuff about other mathematical objects. So it can be a bit hard to convince them that we can work with proofs as we work with natural numbers, real numbers, graphs, trees, etc. But take my word for it, they are also mathematical objects.

Have you ever thought how hard it would be to multiply one hundred and twenty three by forty four? This looks much easier:

$$\begin{array}{r} 123 \\ \times 44 \\ \hline \end{array}$$

Defining operations to a text might be a bit hard, so we have a formal language to represent these proofs. This language is logic. We present it in Chapter 2.

So what can we do with proofs? Well, lots of things! We can for example develop computer programs that will try to prove statements (semi-)automatically or that will check if a proof is correct [1, 2]. We can compare proofs and identify common structures that might become new mathematical concepts [29]. We can transform them in a systematic way and have new insights from a different proof of a theorem [7]. We can develop ways of finding counter-examples of a theorem from a failed proof [26]. We can extract algorithms from it [32]. Just to cite a few. The research field that deals with formal proofs is called *proof theory*.

This work concerns proof transformation. More specifically, we try to develop an operation that will eliminate the use of lemmas in a proof. Lemmas are *helper statements* that are used to prove something. Suppose for example that you want to prove $(x + y)^2 = x^2 + 2xy + y^2$. You can use the definition of exponentiation and do the following:

$$(x + y)(x + y) = xx + xy + yx + yy$$

Again using the definition of exponentiation, we obtain $x^2$ and $y^2$. Now we can use the fact that multiplication is commutative as a lemma, and get $2xy$. Formally, we would have to prove that multiplication is commutative, and then carry on the proof of our main theorem.

The thing about lemmas is that they can be anything. They don't even need to be within the same theory of the theorem you are proving. We can, for example, prove something about number theory using lemmas of topology [3]. Math can be tricky. By removing the lemmas of a proof, we guarantee that this proof will be "pure", meaning that it will use statements only of the theory we are working with. Although the "lemmaless" proofs can be much longer, they can also be conceptually simpler.

In the formal language we are using, lemmas corresponds to structures called *cuts*. Therefore, *cut-elimination* will stand for *lemma elimination*.

## 1.2 Intuitionistic logic

We said before that, in order to define operations on proofs, we must represent them using a formal language. In our case, this language is logic. But this is less exact than it sounds. There are in fact hundreds of logics scattered around, so we have to pick one. We choose intuitionistic logic.

"Why?", you ask. Because intuitionistic proofs are special. Well, actually, *constructive* proofs are special, and intuitionistic logic has a nice constructive flavor. One can argue that there are better logics to deal with constructivism, but intuitionistic logic is one that has a solid mathematical justification and it makes sense to choose a more established and well-founded logic. Although not the best constructive logic, the motivation for intuitionistic logic was Brouwer's philosophy of *intuitionism*, in which a proposition is valid only if it can be constructed from a set of axioms. This is the definition of constructivism itself.

But why is constructiveness so important? In general, constructive proofs are more informative than non-constructive proofs. Take for example the triangle inequality theorem:

**Theorem 1.** *Let $a$, $b$ and $c$ be the lenghts of the sides of an arbitrary triangle such that $c$ is the length of the side opposite to the biggest angle. Then $c < a + b$.*

There is a non-constructive and a constructive proof for that.

*Non-constructive Proof.* Assume that $c \geq a + b$. We know that $a, b, c > 0$, since these are lengths of a triangle's sides. Then we can square both sides:

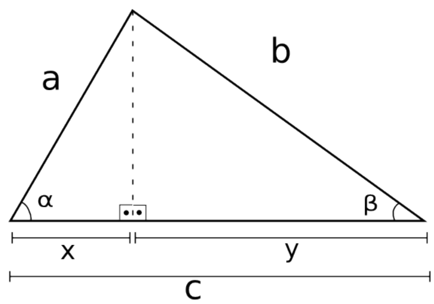
$$c^2 \geq a^2 + 2ab + b^2 > a^2 + b^2$$

From the transitivity of the operators, we conclude that $c^2 > a^2 + b^2$.

But note that the Pythagorean theorem states that, for right triangles, $c^2 = a^2 + b^2$, where $c$ is the hypotenuse, i.e., the side opposite to the biggest angle.

By assuming the contrary of the theorem, we reached a contradiction, therefore the theorem holds. $\square$

*Constructive Proof.* Suppose we have the following triangle:




Then we know that $c = x + y$. Using the trigonometric functions we can compute:

$$x = a \cos \alpha$$
$$y = b \cos \beta$$

We also know that $0^o < \alpha, \beta < 90^o$, therefore $0 < \cos \alpha, \cos \beta < 1$ and $a \cos \alpha < a$ and $b \cos \beta < b$. Using the equation we obtain:

$$c = x + y = a \cos \alpha + b \cos \beta < a + b$$

We can conclude then that the inequality holds. $\square$

Note how different these proofs are. Although the constructive one seems a bit more complicated, it proves the inequality by actually showing how to compute the value for $c$, and then concluding that it holds. That's the kind of information we like about constructive proofs. From the non-constructive one, we simply know that the theorem holds, and nothing more.

Intuitionistic logic is an attempt to formalize this constructivism in a deductive system. Another reason why we choose this logic is the *Curry-Howard isomorphism*, which relates mathematical proofs to computer programs. In principle, we could extract algorithms from proofs of statements. In our example, we can use the constructive proof to obtain an algorithm that computes $c$ in a triangle.

So there. We want to eliminate lemmas from constructive proofs. From now on we start going into the details. We will show the formal language with which we represent proofs, then we present a well-known method for eliminating these lemmas in this formalism (Chapter 2). Then we go on to present another method to eliminate lemmas, on which our solution is based (Chapter 3). The following chapters are dedicated to studying proof transformations inspired by that of Chapter 3 which can work for a fragment of intuitionistic logic.

You might notice that nothing more will look constructive or like a proof at all, but let's try to keep the big picture in mind.

CHAPTER 2

# Logic

In order to facilitate, and possibly automate, the analysis of proofs, it is necessary to represent them using a formal language. We are aware that using this language does not necessarily make proofs more readable for humans, but it is essential for automation. In our case, we will use the language of first-order logic.

## 2.1 Basic concepts

We start introducing basic concepts of logic. These are the building blocks of formal proofs. Propositions (statements or theories) are represented using formulas, steps of reasoning are represented with inference rules and proofs are a set of structured inferences. We will explain each of these elements now.

### Formulas

The definition of formulas in first-order logic depends on the *signature* of the theory being considered. This signature is simply a set of symbols and their roles. First of all, there is a set of *constant symbols*, usually represented by lowercase letters from the beginning of the alphabet, such as $a$, $b$ and $c$. There is also a set of *function symbols*, which need to be specified with their arities. These are represented by lowercase letters such as $f$, $g$ and $h$. *Variable symbols* use letters from the end of the alphabet, like $x$, $y$ and $z$. Finally, there is a set of *relation symbols* and their respective arities, represented by uppercase letters.

The most basic elements of a formula are terms.

**Definition 1** (Term). *A term is defined as follows:*

- *Any variable is a term.*
- *Any constant is a term.*

- *If $f$ is an n-ary function symbol, and $t_1, ..., t_n$ are terms, then $f(t_1, ..., t_n)$ is a term.*

Since we are working on *first-order* logic, all terms have type $\iota$, meaning that they are of one simple elementary type, and not composed type such as $\iota \to \iota$. That said, $\iota$ could be instantiated as any type, e.g., natural number, real number, fruits, animals, etc.

Using the terms, we can construct formulas.

**Definition 2** (Formula)**.** *A formula is defined as follows:*

- *If $R$ is an n-ary relation symbol and $t_1, ..., t_n$ are terms, then $R(t_1, ..., t_n)$ is a formula (also called* atomic *formula).*
- *If $F$ is a formula and $x$ is a variable symbol, then $\forall x.Fx$ and $\exists x.Fx$ are formulas.*
- *If $F$ is a formula, then $\neg F$ is a formula.*
- *If $F_1$ and $F_2$ are formulas, then $F_1 \wedge F_2$, $F_1 \vee F_2$ and $F_1 \to F_2$ are formulas.*

Each connective $\forall, \exists, \neg, \wedge, \vee, \to$ takes its usual meaning, i.e., universal quantification, existential quantification, negation, conjunction, disjunction and implication, respectively.

Note that the only difference between function and relation symbols is that function symbols are applied to a tuple of terms of type $\iota$ forming an expression of type $\iota$, while relation symbols are applied to tuples of terms of type $\iota$ but form an expression of type $o$ (i.e. *true* or *false*).

In general, we will construct formulas using uppercase letters which are simply placeholders for other formulas (possibly more complex).

Given a formula, it will be useful in the future to classify its sub-formulas as positive or negative occurrences. For that we define the *polarity* of sub-formulas.

**Definition 3** (polarity in a formula)**.** *Let $F$ be a formula and $F'$ a sub-formula of $F$. Then we can define the* polarity *of $F'$ in $F$, i.e., $F'$ can be* positive *or* negative *in $F$, according to the following criteria:*

- *If $F \equiv F'$, then $F'$ is* positive *in $F$.*
- *If $F \equiv A \wedge B$ or $F \equiv A \vee B$ or $F \equiv \forall x.A$ or $F \equiv \exists x.A$ and $F'$ is* positive *(*negative*) in $A$ or $B$, then $F'$ is* positive *(*negative*) in $F$.*
- *If $F \equiv A \to B$ and $F'$ is* positive *(*negative*) in $B$, then $F'$ is* positive *(*negative*) in $F$.*
- *If $F \equiv A \to B$ and $F'$ is* positive *(*negative*) in $A$, then $F'$ is* negative *(*positive*) in $F$.*
- *If $F \equiv \neg A$ and $F'$ is* positive *(*negative*) in $A$, then $F'$ is* negative *(*positive*) in $F$.*

Another important concept that will be mentioned frequently is that of *strong* and *week* quantifiers.

**Definition 4** (Strong and weak quantifiers)**.** *Let $F$ be a formula. If $\forall x$ occurs positively (negatively) in $F$, then $\forall x$ is called a* strong (weak) quantifier. *If $\exists x$ occurs positively (negatively) in $F$, then $\exists x$ is called a* weak (strong) quantifier. *Let $A_1, ..., A_n \vdash B_1, ..., B_m$ be a sequent. A quantifier is called strong (weak) on this sequent if it is strong (weak) on the corresponding formula $A_1 \wedge ... \wedge A_n \to B_1 \vee ... \vee B_m$.*

Using formulas we can represent propositions such as "the sum of even numbers is an even number":

$$\forall x.\forall y.(even(x) \wedge even(y) \to even(x + y))$$

The signature of the theory in which this formula is being considered would need to have $x$ and $y$ as variable symbols, $+$ as a binary function symbol and $even$ as a unary relational symbol. Moreover, we could infer that $\iota$, the type of terms, is the type of integers.

## Sequents

Now that we can represent propositions using formulas, we need something to represent that from a set of propositions other propositions can be derived (or proven). This is the role of *sequents*.

**Definition 5.** *A* sequent *is of the form $\Gamma \vdash \Delta$, where $\Gamma$ and $\Delta$ are multi-sets of formulas. We say that $\Gamma$ is the antecedent while $\Delta$ is the succedent.*

Intuitively, it denotes that the conjunction of the formulas in $\Gamma$ entails (or implies, represented by the symbol $\vdash$) the disjunction of the formulas in $\Delta$. This means that a sequent:

$$A_1, ..., A_n \vdash B_1, ..., B_m$$

where $A_i$ and $B_i$ are formulas, can actually be represented by the formula:

$$A_1 \wedge ... \wedge A_n \to B_1 \vee ... \vee B_m$$

Why use sequents then? Because sequents (and not formulas) are used to specify deduction steps that compose a proof in sequent calculus, which is presented next.

Analogous to the formula case, we can also define polarity of formulas within a sequent.

**Definition 6** (polarity in a sequent)**.** *Let $S = A_1, ..., A_n \vdash B_1, ..., B_m$ be a sequent. A formula is said be* positive *(*negative*) in $S$ if it is* positive *(*negative*) in the corresponding formula: $A_1 \wedge ... \wedge A_n \to B_1 \vee ... \vee B_m$.*

## Sequent Calculus

Sequent calculus is a deduction system in which proofs can be represented. The system is a set of *inference rules* that operate on sequents. An example of such a rule is:

$$\frac{\Gamma \vdash \Delta, A \quad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B} \wedge_r$$

Bottom-up this rule can be read as: in order to prove $A \wedge B$ we need to prove $A$ and we need to prove $B$. Alternatively, we can read this rule top-down as: if we could prove $A$ and we could prove $B$, then we can certainly assume $A \wedge B$.

$\Gamma$ and $\Delta$ are called the left and right contexts respectively. The sequent below the horizontal line is called the conclusion and the sequents above the horizontal line are the premises. In principle a sequent calculus rule can have any number of premises, but the systems used in this work have only 0-, 1- or 2-premise rules. Moreover, $A \to B$ is referred to as the *main formula*, while $A$ and $B$ in the premises are *auxiliary formulas*.

In these rules, it is always the case that a proof of the premises implies in a proof of the conclusion. Additionally, for some rules, the converse also holds, meaning that a proof of the conclusion implies in a proof of the premises. We call these bidirectional rules *invertible*, while the others are *non-invertible*.

Most of the rules in a sequent calculus system are what we call *logical* rules, meaning that they operate on a formula and its sub-formulas, like the $\wedge_r$ rule presented previously. But there are also *structural* rules, like contraction (copies a formula - bottom-up), weakening (erases a formula - bottom-up) and the famous cut (which will be explained soon with more details). Here are left contraction, left weakening and the cut:

$$\frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta}\ c_l \quad \frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}\ w_l \quad \frac{\Gamma \vdash \Delta, A \quad A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta}\ cut$$

The other type of rule in sequent calculus systems is the *axiom*, which is a 0-premise rule:

$$\frac{}{A \vdash A}$$

Observe that the 2-premise, or binary, rules presented ($\wedge_r$ and $cut$) have the same left and right contexts in the premises and conclusion. This is because they are *additive* rules. We can also specify them as *multiplicative* rules:

$$\frac{\Gamma \vdash \Delta, A \quad \Gamma' \vdash \Delta', B}{\Gamma, \Gamma' \vdash \Delta, \Delta', A \wedge B}\ \wedge_r \quad \frac{\Gamma \vdash \Delta, A \quad A, \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\ cut$$

Note that in multiplicative rules the contexts from the conclusion are split among the premises. When the structural rules of weakening and contraction are available in the calculus, it does not matter if we use additive or multiplicative rules. In this work, we use mostly multiplicative calculi.

Depending on which rules are available in a sequent calculus systems, some formulas are derivable (meaning that they can be proved) while others are not. So naturally, different logics have different sequent calculus systems. Of course the relation is not one to one. There are sets of rules that do not represent any logic, while there are different sets of rules that represent the same logic.

### Proofs

So we have formulas that represent propositions, they can be organized in sequents to which rules of a sequent calculus are applied. How does this make a proof?

A proof of a sequent $\Gamma \vdash \Delta$ in a sequent calculus $S$ is a tree composed by rules of $S$ having $\Gamma \vdash \Delta$ as the root, or *end-sequent*, and axioms as the leaves. In general, this "tree" is upside-down of the usual way trees are represented in computer science. Here's a proof of a simple statement:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{even(2) \vdash even(2)} \quad \dfrac{}{even(4) \vdash even(4)}}{even(2), even(4) \vdash even(2) \wedge even(4)} \wedge_r \quad \dfrac{}{even(2+4) \vdash even(2+4)}}{even(2), even(4), even(2) \wedge even(4) \rightarrow even(2+4) \vdash even(2+4)} \rightarrow_l}{even(2), even(4), \forall y.(even(2) \wedge even(y) \rightarrow even(2+y)) \vdash even(2+4)} \forall_l}{even(2), even(4), \forall x.\forall y.(even(x) \wedge even(y) \rightarrow even(x+y)) \vdash even(2+4)} \forall_l
$$

Sometimes we will use a double line to indicate that more than one inference rule was applied.

One thing I learned in the past years is that people read proofs differently. Some read them bottom-up, others read them top-down, and it is often unclear to refer to an inference application "before" or "after" another. I also learned that both directions have their merits, and in this thesis there are some situations in which one or the other makes sense. I try to make it always clear in which direction I am working.

## 2.2 Logics

Earlier I said that different sequent calculus systems correspond to different logics. Yes, logic**s**, in plural. You'd think that a logic is what is logical, it is correct reasoning, so how can there be more than one? Well, first of all, you might have found out by now that there's no such thing as *the correct* reasoning. Secondly, the logic as we learn in school (ands and ors operations, negations, etc.) is *not* the logic our brains reasons with. There is research in this direction though [31], they try to find what is the brain's logic basically.

A logic can be defined in many ways. And "what is a logic" can be a very philosophical question. Since I did not study philosophy, I apologize to the philosophers in advance, and define a logic as I think suits best for this work.

Since a logic can be represented by different sequent calculi, this is obviously not a good way to define it. At the same time, it is not all that bad. But in this work there are different calculi for the same logic, so it is not a reasonable definition. First of all, a logic has a set of operators. All the logics we are dealing with in this work use the operators: $\wedge$, $\vee$, $\neg$, $\rightarrow$, $\forall$ and $\exists$. When we work with formulas that do not contain the quantifiers $\forall$ and $\exists$, we say that we are in *propositional logic*. If the quantifiers are present, we call it *predicate logic*. Then there is a meaning for each of those operators, what we call *semantics*. The semantics might seem clear to you now, but there are subtle differences which result in different logics. Apart from the operators, a logic has also a set of truth values, which can be two, three, finitely or even infinitely many. In our case, we have two truth values: *true* and *false*.

Given a signature and these operators, we can construct well-defined formulas in a logic. Depending on the semantics of the operators, the *interpretation* given to each symbol and the *domain* considered, these formulas might be *satisfiable*, *valid* or neither. We say that the interpretation of the symbols and the domain compose a *model*. A formula is satisfiable if it is evaluated to true in *at least one* model, and it is valid if it is evaluated to true in *every* model. Take for example the formula:

$$\forall x.(A(x) \to B(x))$$

If we interpret $A(x)$ as $x$ *is natural* and $B(x)$ as $x$ *is an integer* and take $\mathbb{Z}$ as the domain, then the formula is of course true (considering the most natural semantics of $\forall$ and $\to$). Therefore, the formula is satisfiable. If, on the other hand, we interpret $A(x)$ as $x$ *is an integer* and $B(x)$ as $x$ *is natural*, then the formula will be false if we consider the domain $\mathbb{Z}$. We can conclude then that it is not valid.

In contrast to that, the formula $\forall x.(A(x) \to A(x))$ is valid. Note that this will be true independently of how we interpret $A(x)$ and which domain we choose. We will define a logic as the set of valid formulas. Moreover, this set should be closed under consequence, i.e., if $A_1, ..., A_n$ are valid formulas and they entail another formula $B$, then $B$ must also be valid.

In this work we will use two logics: *classical* and *intuitionistic*. We can say that the set of valid first-order formulas in intuitionistic logic is a subset of valid first-order formulas in classical logic. The same holds for proofs. In the next sessions we define each logic in detail.

## 2.3 Classical Logic

Classical logic is probably the logic that non-logicians have in mind when I say that I work with logic. Its sequent calculus, called **LK**, is depicted in Figure 2.1 and every valid formula has a proof using these rules. Classical logic can be seen as the logic of *stable truth*, in the sense that every sentence is always either true or false. One evidence of this is the fact that the *principle of excluded middle* holds in this logic. This is a proof of this principle in the sequent calculus **LK**:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{A \vdash A}}{\vdash A, \neg A}\,\neg_r}{\vdash A, A \vee \neg A}\,\vee_{r2}}{\vdash A \vee \neg A, A \vee \neg A}\,\vee_{r1}}{\vdash A \vee \neg A}\,c_r$$

The operators of classical logic have a most natural semantics, which use exactly their natural language counterparts. In order to describe the semantics of logics, we will use the *satisfiability* relation $\models$. The expression $\mathcal{M} \models F$ denotes that $F$ is evaluated to true in model $\mathcal{M}$ (remember that a model is simply an interpretation $\mathcal{I}$ of the symbols and a domain $\mathcal{D}$). Then the semantics of classical logic can be stated as:

$$
\frac{}{A \vdash A}\,[\mathrm{I}] \qquad \frac{\Gamma_1 \vdash \Delta_1, P \quad \Gamma_2, P \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\,[\mathrm{Cut}]
$$

$$
\frac{\Gamma \vdash \Delta, P}{\Gamma, \neg P \vdash \Delta}\,[\neg_l] \qquad \frac{\Gamma, P \vdash \Delta}{\Gamma \vdash \Delta, \neg P}\,[\neg_r]
$$

$$
\frac{P_i, \Gamma \vdash \Delta}{P_1 \wedge P_2, \Gamma \vdash \Delta}\,[\wedge_{li}] \qquad \frac{\Gamma_1 \vdash \Delta_1, P \quad \Gamma_2 \vdash \Delta_2, Q}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, P \wedge Q}\,[\wedge_r]
$$

$$
\frac{P, \Gamma_1 \vdash \Delta_1 \quad Q, \Gamma_2 \vdash \Delta_2}{P \vee Q, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\,[\vee_l] \qquad \frac{\Gamma \vdash \Delta, P_i}{\Gamma \vdash \Delta, P_1 \vee P_2}\,[\vee_{ri}]
$$

$$
\frac{\Gamma_1 \vdash \Delta_1, P \quad Q, \Gamma_2 \vdash \Delta_2}{P \to Q, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\,[\to_l] \qquad \frac{\Gamma, P \vdash \Delta, Q}{\Gamma \vdash \Delta, P \to Q}\,[\to_r]
$$

$$
\frac{P\{x \leftarrow \alpha\}, \Gamma \vdash \Delta}{\exists x.P, \Gamma \vdash \Delta}\,[\exists_l] \qquad \frac{\Gamma \vdash \Delta, P\{x \leftarrow t\}}{\Gamma \vdash \Delta, \exists x.P}\,[\exists_r]
$$

$$
\frac{P\{x \leftarrow t\}, \Gamma \vdash \Delta}{\forall x.P, \Gamma \vdash \Delta}\,[\forall_l] \qquad \frac{\Gamma \vdash \Delta, P\{x \leftarrow \alpha\}}{\Gamma \vdash \Delta, \forall x.P}\,[\forall_r]
$$

$$
\frac{P, P, \Gamma \vdash \Delta}{P, \Gamma \vdash \Delta}\,[C_l] \qquad \frac{\Gamma \vdash \Delta, P, P}{\Gamma \vdash \Delta, P}\,[C_r]
$$

$$
\frac{\Gamma \vdash \Delta}{P, \Gamma \vdash \Delta}\,[W_l] \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, P}\,[W_r]
$$

**Figure 2.1: LK**: Sequent calculus for classical logic. It is assumed that $\alpha$ is a variable not contained in $P$, $\Gamma$ or $\Delta$, $t$ does not contain variables bound in $P$ and $i \in \{1, 2\}$.

$$
\begin{aligned}
\mathcal{M} \vDash \neg F &\text{ iff it is not the case that } \mathcal{M} \vDash F \\
\mathcal{M} \vDash F_1 \wedge F_2 &\text{ iff } \mathcal{M} \vDash F_1 \text{ and } \mathcal{M} \vDash F_2 \\
\mathcal{M} \vDash F_1 \vee F_2 &\text{ iff } \mathcal{M} \vDash F_1 \text{ or } \mathcal{M} \vDash F_2 \\
\mathcal{M} \vDash F_1 \to F_2 &\text{ iff if } \mathcal{M} \vDash F_1 \text{ then } \mathcal{M} \vDash F_2 \\
\mathcal{M} \vDash \forall x.F(x) &\text{ iff for every } d \in \mathcal{D} \text{ then } \mathcal{M} \vDash F[x \leftarrow d] \\
\mathcal{M} \vDash \exists x.F(x) &\text{ iff there exists a } d \in \mathcal{D} \text{ such that } \mathcal{M} \vDash F[x \leftarrow d]
\end{aligned}
$$

A formula $F$ is valid in classical logic if it is satisfied by every model, i.e., for all $\mathcal{M}$, $\mathcal{M} \vDash F$. Observe that $F[x \leftarrow d]$ stands for interpreting $x$ with $d$ in $F$.

For untrained eyes, this might seem a lot of formalism for something very simple (almost stupid). But I hope you will be convinced otherwise when the semantics of intuitionistic logic is presented. A very small change in the interpretation of these symbols turn out to make a huge difference.

Note that the semantic description of the operators of a logic might not be unique.

## 2.4 Intuitionistic Logic

If classical logic can be seen as the logic of stable truth, intuitionistic logic can be seen as the logic of *stable proof*. In this logic, a sentence is only considered to be valid if there is a proof of it, and we cannot emphasize enough that a lack of proof of the negation of a sentence is *not* a proof of the sentence. As you may suspect, the principle of excluded middle does not hold in intuitionistic logic, and you will not be able to prove it with either calculi presented here (Figures 2.2 and 2.3). For this reason, intuitionistic logic has some interesting properties.

If one wants an intuitionistic proof of a disjunction of facts $A_1 \vee ... \vee A_n$, then this proof will indicate exactly which disjunct(s) is(are) true. This is called the *disjunctive property*. Also, from an intuitionistic proof (without $\vee_l$) of an existential statement, such as $\exists x.P(x)$, one can obtain a term $a$ for which $P(a)$ holds. This is called the *witness property*. Although every intuitionistic proof is also a classical proof, these properties of course do not always hold in classical logic. For this reason, intuitionistic logic is often referred to as a *constructive* logic[1]. This constructive flavor is useful in a number of situations, for example, when we want not only to find whether something *has* a solution but also *what* is the solution and *how* to obtain it. In the book “A Course in Constructive Algebra”, by Mines, Richman and Ruitenburg, there is a nice comparison of the two worlds:

> The classical view of mathematics is essentially descriptive: we try to describe the facts about a static mathematical universe. Thus, for example, we report that every polynomial of odd degree has a root, and that there is a digit that occurs infinitely often in the decimal expansion of $\pi$. In opposition to this is the constructive view of mathematics, which focuses the attention to the dynamic interaction of the individual with the mathematical universe; in the words of Hao Wang, it is a mathematics of doing, rather then a mathematics of being. The constructive mathematician must show how to construct a root of a polynomial of odd degree, and how to find a digit that occurs infinitely many often in the decimal expansion of $\pi$.

The change from classical to intuitionistic truth can be observed on the *Brouwer-Heyting-Kolmogorov intepretation*, which states what is intended to be a proof of a formula:

$$
\begin{aligned}
F_1 \wedge F_2 &: \text{is a pair } \langle p_1, p_2 \rangle\text{, where } p_1 \text{ is a proof of } F_1 \text{ and } p_2 \text{ is a proof of } F_2\\
F_1 \vee F_2 &: \text{is a proof } p \text{ of either } F_1 \text{ or } F_2 \text{ (with an indication as to which conjunct } p \text{ proves)}\\
F_1 \rightarrow F_2 &: \text{is a function that transforms a proof of } F_1 \text{ into a proof of } F_2\\
\neg F &: \text{interpreted as } F \rightarrow \bot\\
\forall x.F &: \text{is a function that converts any element } a \text{ of the domain into a proof of } F(a)\\
\exists x.F &: \text{is an element } a \text{ of the domain and a proof } p \text{ of } F(a)\\
\text{atomic } F &: \text{the interpretation of primitive propositions is assumed to be known from the context}\\
\bot &: \text{there is no proof of absurdity}
\end{aligned}
$$

[1]But this is disputable

We can observe, using this interpretation, how the logical connectives are independent in intuitionistic logic. In classical logic, $A \vee B$ is equivalent to $\neg(\neg A \wedge \neg B)$, hence the validity of DeMorgan's rules. Using the BHK-interpretation, $A \vee B$ indicates either a proof of $A$ or a proof of $B$, while $\neg(\neg A \wedge \neg B)$ indicates a function capable of transforming a pair of proofs of $\neg A$ and $\neg B$ into a proof of a contradiction. Since these formulas have different interpretations, we can expect that the DeMorgan rules are no longer valid in intuitionistic logic.

Note an interesting thing about this interpretation: $\forall$ and $\rightarrow$ (and consequently $\neg$) are interpreted as *functions*. This already hints to a connection between intuitionistic logic and computation which is formalized by the *Curry-Howard isomorphism*. Also known as the *proofs-as-programs* interpretation, it describes the relationship between computer programs and mathematical proofs. This relationship relies heavily on *type theory*, developed by Russel on the beginning of the $20^{th}$ century.

In the Curry-Howard isomorphism, proofs are seen as programs and formulas are interpreted as types, while the logical connectives are operations between types (e.g., $\rightarrow$ is the functional type, as expected, $\wedge$ is the product type, such as that of pairs, tuples or lists, and $\vee$ is the sum type, in which the conjuncts are eliminated via pattern matching). Using this interpretation, interesting connections were found. Relevant to this work is the correspondence between intuitionistic natural deduction and simply typed $\lambda$-calculus, which is a computation model. This is what makes intuitionistic logic so interesting for computer science.

As we mentioned before, the semantics of intuitionistic logic is slightly different from the classical one. We present here the *Kripke semantics*, also called *possible world semantics*, which is based on the existence of *worlds* containing models that can satisfy formulas. These worlds are connected via an *accessibility relation* denoted by $\preceq$. Let's say that $w$ and $v$ are worlds, then $w \preceq v$ means that $v$ can be reached from $w$. We can see these worlds and relations as a directed graph. This graph will be a *Kripke structure* if it satisfies some properties. First of all, the set of worlds plus the accessibility relation must be a partially ordered set, i.e., the relation $\preceq$ has the following properties:

$$
\begin{aligned}
w &\preceq w && \text{(reflexivity)} \\
w \preceq v \text{ and } v \preceq w &\text{ then } w = v && \text{(antisymmetry)} \\
w \preceq v \text{ and } v \preceq u &\text{ then } w \preceq u && \text{(transitivity)}
\end{aligned}
$$

Moreover, the structure should be monotone on the domain available and on the truth value assignment. Let $\mathcal{D}_w$ denote the domain of the model in world $w$. Then:

$$
\begin{aligned}
&w \preceq v \Rightarrow \mathcal{D}_w \subseteq \mathcal{D}_v \\
&w \preceq v \wedge w \vDash F \Rightarrow v \vDash F
\end{aligned}
$$

This characteristic can be interpreted as an ever growing knowledge. Every step further in the graph, the information you have is at least what you had in the previous state, possibly more.

Now, given the Kripke structure, we define the Kripke semantics for intuitionistic logic:

| | |
|---|---|
| $w \vDash F_1 \wedge F_2$ | iff $w \vDash F_1$ and $w \vDash F_2$ |
| $w \vDash F_1 \vee F_2$ | iff $w \vDash F_1$ or $w \vDash F_2$ |
| $w \vDash F_1 \rightarrow F_2$ | iff for every $v$ such that $w \preceq v$, if $v \vDash F_1$ then $v \vDash F_2$ |
| $w \vDash \neg F$ | iff there is no $v$, $w \preceq v$, such that $v \vDash F$ |
| $w \vDash \forall x.F(x)$ | iff for every $v$, $w \preceq v$, and every $d \in \mathcal{D}_v$ then $v \vDash F(d)$ |
| $w \vDash \exists x.F(x)$ | iff there exists a $d \in \mathcal{D}$ such that $w \vDash F(d)$ |

Observe that the semantics of $\rightarrow$, $\neg$ and $\forall$ depends not only on this world, but on all the followings.

A formula $F$ is valid in intuitionistic logic if and only if it is satisfied in *every* world of *every* possible Kripke structure. On the other hand, to show that a formula is not valid, we need only to show a Kripke structure in which it is not satisfied in every world.

As a deductive system for intuitionistic logic, we will use two different sequent calculi. One of them, probably the most well-known sequent calculus for this logic, is called **LJ**. If you compare this to **LK**, you will see that they are very similar. In fact, **LJ** is basically **LK** with the restriction that the consequent of the sequent can have at most one formula. This calculus is depicted in Figure 2.2.

For some reasons that we will point out later on, we will also use what is called a multi-conclusion calculus for intuitionistic logic. This is depicted in Figure 2.3. You can notice that **LJ'** is somewhere between **LK** and **LJ**, with some rules having multiple formulas on the right while other only allow the main formula, namely $\rightarrow_r$, $\neg_r$ and $\forall_r$.

As mentioned before, the principle of excluded middle, $A \vee \neg A$, is not valid in intuitionistic logic. You can check that it is not derivable in either of the calculi presented.

## 2.5 Cut-elimination

An important proof transformation for proof analysis is the elimination of unnecessary lemmas for obtaining potentially simpler (or at least more direct) proofs. When a mathematical proof is formalized in the sequent calculus **LK**, **LJ** or **LJ'**, the application of lemmas correspond to cuts.

Throughout this document we will use the following configurations of the cut rule:

$$\frac{\Gamma \vdash \Delta, A \quad A, \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\ cut \qquad \text{or} \qquad \frac{\Gamma \vdash A \quad A, \Gamma' \vdash C}{\Gamma, \Gamma' \vdash C}\ cut$$

Note that it has no main formula.

*Cut-elimination* is the procedure of removing the cut rules, and therefore, removing intermediate lemmas. A proof without cuts has the subformula property: all formulas on the proof are (instances of) subformulas of end-sequent formulas. Consequently, cut-free proofs of a theorem will use only the theorem's theory itself.

$$\frac{}{A \vdash A}\,[\mathrm{I}] \qquad \frac{\Gamma_1 \vdash P \quad \Gamma_2, P \vdash C}{\Gamma_1, \Gamma_2 \vdash C}\,[\mathrm{Cut}]$$

$$\frac{\Gamma \vdash P}{\Gamma, \neg P \vdash}\,[\neg_l] \qquad \frac{\Gamma, P \vdash}{\Gamma \vdash \neg P}\,[\neg_r]$$

$$\frac{P_i, \Gamma \vdash C}{P_1 \wedge P_2, \Gamma \vdash C}\,[\wedge_{li}] \qquad \frac{\Gamma \vdash P \quad \Gamma \vdash Q}{\Gamma \vdash P \wedge Q}\,[\wedge_r]$$

$$\frac{P, \Gamma \vdash C \quad Q, \Gamma \vdash C}{P \vee Q, \Gamma \vdash C}\,[\vee_l] \qquad \frac{\Gamma \vdash P_i}{\Gamma \vdash P_1 \vee P_2}\,[\vee_{ri}]$$

$$\frac{\Gamma_1 \vdash P \quad Q, \Gamma_2 \vdash C}{P \rightarrow Q, \Gamma_1, \Gamma_2 \vdash C}\,[\rightarrow_l] \qquad \frac{\Gamma, P \vdash Q}{\Gamma \vdash P \rightarrow Q}\,[\rightarrow_r]$$

$$\frac{P\{x \leftarrow \alpha\}, \Gamma \vdash C}{\exists x.P, \Gamma \vdash C}\,[\exists_l] \qquad \frac{\Gamma \vdash P\{x \leftarrow t\}}{\Gamma \vdash \exists x.P}\,[\exists_r]$$

$$\frac{P\{x \leftarrow t\}, \Gamma \vdash C}{\forall x.P, \Gamma \vdash C}\,[\forall_l] \qquad \frac{\Gamma \vdash P\{x \leftarrow \alpha\}}{\Gamma \vdash \forall x.P}\,[\forall_r]$$

$$\frac{P, P, \Gamma \vdash C}{P, \Gamma \vdash C}\,[C_l] \qquad \frac{\Gamma \vdash C}{P, \Gamma \vdash C}\,[W_l] \qquad \frac{\Gamma \vdash}{\Gamma \vdash P}\,[W_r]$$

**Figure 2.2:** **LJ**: Sequent calculus for intuitionistic logic. It is assumed that $\alpha$ is a variable not contained in $P$, $\Gamma$ or $C$ and $t$ does not contain variables bound in $P$. Also, $C$ stands for one formula or the empty set.

The main result on cut-elimination - the *Hauptsatz* - was proven by Gentzen [15, 16] in 1935. It states that the cut rule is admissible in **LK** and **LJ**. This means that every proof with cuts can be transformed into a cut-free proof. Gentzen's proof of the *Hauptsatz* actually contains an algorithm for removing the cuts from a proof. The method and some of its variants are often referred to as *reductive cut-elimination*, because they are based on local proof-rewriting rules that gradually reduce the complexity of the cut formula until reaching atomic cuts, which can be completely removed. The rewriting rules for the systems **LK** and **LJ** are listed in the Appendix A and we distinguish three types:

**Definition 7** (Elimination rule)**.** *We call* elimination rules *the rewriting operation that removes a cut because one of the auxiliary formulas was weakened or used in an axiom (see example in Figure 2.6).*

**Definition 8** (Rank reduction)**.** *We call* rank reduction *the rewriting operation that permutes the cut rule over other rules in a proof (see example in Figure 2.5).*

**Definition 9** (Grade reduction)**.** *We call* grade reduction *the rewriting operation that transforms a cut on a formula into a cut (or cuts) on simpler formulas, i.e. formulas with less connectives (see example in Figure 2.4).*

$$\dfrac{\Gamma \vdash A, \Delta \quad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \to B \vdash \Delta, \Delta'}\ \to_l \qquad \dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \to B, \Delta}\ \to_r$$

$$\dfrac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ \wedge_l \qquad \dfrac{\Gamma \vdash A, \Delta \quad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \wedge B, \Delta, \Delta'}\ \wedge_r$$

$$\dfrac{\Gamma, A \vdash \Delta \quad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \vee B \vdash \Delta, \Delta'}\ \vee_l \qquad \dfrac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ \vee_r$$

$$\dfrac{\Gamma, A\{t/x\} \vdash \Delta}{\Gamma, \forall x\, A \vdash \Delta}\ \forall_l \qquad \dfrac{\Gamma \vdash A\{c/x\}}{\Gamma \vdash \Delta, \forall x\, A}\ \forall_r$$

$$\dfrac{\Gamma, A\{c/x\} \vdash \Delta}{\Gamma, \exists x\, A \vdash \Delta}\ \exists_l \qquad \dfrac{\Gamma \vdash \Delta, A\{t/x\}}{\Gamma \vdash \Delta, \exists x\, A}\ \exists_r$$

$$\dfrac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\ \neg_l \qquad \dfrac{\Gamma, A \vdash}{\Gamma \vdash \neg A, \Delta}\ \neg_r$$

$$\dfrac{}{A \vdash A}\ \text{init} \qquad \dfrac{\Gamma \vdash B, \Delta \quad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\ \text{cut}$$

$$\dfrac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta}\ c_l \qquad \dfrac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}\ c_r$$

$$\dfrac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}\ w_l \qquad \dfrac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta}\ w_r$$

**Figure 2.3:** The multi-conclusion intuitionistic sequent calculus, **LJ'**, with multiplicative rules.

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma_1 \vdash P} \quad \overset{\varphi_2}{\Gamma_2, Q \vdash A}}{\Gamma, P \to Q \vdash A}\ \to_l \qquad \overset{\varphi_3}{A, \Gamma' \vdash \Delta}}{\Gamma, P \to Q, \Gamma' \vdash \Delta}\ \mathit{cut} \quad \rightsquigarrow \quad \dfrac{\overset{\varphi_1}{\Gamma_1 \vdash P} \qquad \dfrac{\overset{\varphi_2}{\Gamma_2, Q \vdash A} \quad \overset{\varphi_3}{A, \Gamma' \vdash \Delta}}{\Gamma_2, Q, \Gamma' \vdash \Delta}\ \mathit{cut}}{\Gamma, P \to Q, \Gamma' \vdash \Delta, \Delta'}\ \to_l$$

**Figure 2.4:** Rewriting rule that permutes a cut over a $\to_l$ inference.

Using the rewriting rules, reductive cut-elimination methods perform the following operations iteratively:

- Permute the cut rule until it becomes *principal* (the inferences immediately above the cut are those that operate on the cut-formula) using rank reduction rules.
- Reduce the complexity of this cut using grade reduction rules.
- Remove the cut using elimination rules.

This is based on local operations, and since there is no unique normal form for proofs, different cut-free proofs can be obtained if the rewriting is done in a different order.

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash A_1} \quad \overset{\varphi_2}{\Gamma' \vdash A_2}}{\Gamma, \Gamma' \vdash A_1 \wedge A_2}\,\wedge_r \qquad \dfrac{\overset{\varphi_3}{A_i, \Gamma'' \vdash \Delta}}{A_1 \wedge A_2, \Gamma'' \vdash \Delta}\,\wedge_{li}}{\Gamma, \Gamma', \Gamma'' \vdash \Delta}\,\mathit{cut} \quad \rightsquigarrow \quad \dfrac{\overset{\varphi_1}{\Gamma \vdash A_1} \qquad \dfrac{\overset{\varphi_2}{\Gamma' \vdash A_2} \quad \dfrac{\overset{\varphi_3}{A_i, \Gamma'' \vdash \Delta}}{A_1, A_2, \Gamma'' \vdash \Delta}\,w_l}{A_1, \Gamma', \Gamma'' \vdash \Delta}\,\mathit{cut}}{\Gamma, \Gamma', \Gamma'' \vdash \Delta}\,\mathit{cut}$$

**Figure 2.5:** Rewriting rule that reduces the complexity of the cut in a proof.

$$\dfrac{\dfrac{}{A \vdash A} \quad \overset{\varphi}{A, \Gamma \vdash \Delta}}{\Gamma, A \vdash \Delta}\,\mathit{cut} \quad \rightsquigarrow \quad \overset{\varphi}{A, \Gamma \vdash \Delta}$$

**Figure 2.6:** Rewriting rule that eliminates an atomic cut.

If we are interested in eliminating cuts from proofs, we need to make sure that the cut is actually "eliminable", i.e., admissible in the calculi we are working with. Gentzen's Hauptsatz tells us this is the case for **LK** and **LJ**. But we are also working with another calculus: **LJ'**.

**Theorem 2.** *The cut rule is admissible in **LJ'**.*

The proof of cut-elimination for **LJ'** is not exactly like the one for the other calculi. Since the rules vary in single and multiple conclusion, some permutations are not possible, and this can be a bit tricky. In order to prove the theorem, we use some lemmas.

First of all, we need to state when the permutations of the cut work and when they don't. This concerns rank reduction rules.

**Lemma 1.** *The cut rule permutes over all rules on the left branch.*

**Lemma 2.** *The cut rule permute over all rules on the right branch* except $\to_r$*,* $\forall_r$ *and* $\neg_r$*.*

We will also need to deal with permutations of other rules.

**Lemma 3.** *Every rule permutes over invertible rules.*

Using these lemmas we can prove cut-elimination for **LJ'**.

*Proof.* By Lemma 1 we conclude that it is always possible to make the cut principal on the left branch. If the rule operating on it is an axiom or weakening, the cut can be removed and we are done. If, on the other hand, this rule is a logical rule, we need to make the cut principal also on the right to apply one of the grade reduction rules.

From Lemma 2 we know that it is not always possible to permute the cut on the right branch. Assume we've reached a $\to_r$, $\forall_r$ or $\neg_r$ on the right side. Now we have to distinguish on what cut-formula we have.

If this cut formula has $\rightarrow$, $\neg$ or $\forall$ as the topmost connective, then the context of the left branch is empty (given that the cut is principal on the left). In this case, the cut can be permuted over the rule on the right, like it is done in **LJ**.

If the cut formula has $\vee$, $\wedge$ or $\exists$ as its main connective, then the rule operating on the cut formula on the right branch will be $\vee_l$, $\wedge_l$ or $\exists_l$. But these rules are invertible. By Lemma 3, we can guarantee that it is possible to push this rule down the proof until it reaches the cut, making it principal on both sides.

Using these operations, we can always reduce the complexity of the cut formula until eventually eliminating it completely. Thus, the cut rule is admissible in **LJ'**. $\square$

CHAPTER 3

# The CERES method

The method CERES [10] (cut elimination by resolution) is an alternative to reductive cut-elimination, and it is proven to have a non-elementary speed up over the latter. CERES was first developed for first order classical logic, and then extended to second and higher order logic [18, 19]. It has also been adapted to multi-valued logics [11] and Gödel logic [5]. The method has been implemented[1] and applied successfully to proofs of moderate size, such as the tape proof [6] and the lattice proof [20], in fully automatic mode. Also, Fürstenberg's proof of the infinitude of primes was successfully transformed, semi-automatically, into Euclid's argument of prime construction using CERES [7].

In contrast to reductive methods, which remove each cut step by step, the CERES method removes all cuts simultaneously. It performs global operations in the proof, extracting relevant data which are later joined in a new proof containing only atomic cuts[2]. The advantage of this approach is its non-elementary speed-up over reductive methods.

## 3.1 CERES in drawings

Let $\varphi$ be an **LK** proof of a sequent $S$. Then, the method consists of:

1. **Skolemizing $\varphi$.**

   The method requires that the end-sequent contains no occurrence of $\forall$ on the right and $\exists$ on the left (because of the eigenvariable condition associated with the rules for these connectives). In order to achieve this, the proof needs to be skolemized, which means that eigenvariables are replaced by skolem terms. After obtaining the final proof, this operation is reversed, resulting in a proof of the original end-sequent.

[1] http://code.google.com/p/gapt/

[2] The presence of atomic cuts is not a problem, since the complexity of removing those is considerably smaller than that of removing arbitrary cuts.

2. **Constructing the *characteristic clause set* $CL(\varphi)$.**

   Whenever a cut rule is applied, a new formula is introduced in the proof (bottom-up interpretation). This cut formula is then decomposed, and some of the atoms which came from this formula go into leaves of the proof. Figure 3.1 illustrates this situation. Consider that the yellow triangle is $\varphi$. Each red trace represents a cut (on its vertices) and the cut ancestors used trough out the proof. These cut ancestors may end up in axioms[3] $C_i \circ S_i$, where $C_i$ is the part of the sequent composed of cut ancestors and $S_i$ is the part of the sequent composed of end-sequent ancestors. The characteristic clause set contains these[4] $C_i$.

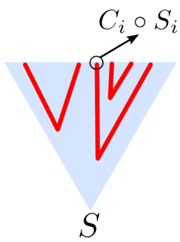



**Figure 3.1:** $\varphi$ with cut and cut-ancestor highlighted.

3. **Constructing a *projection* $\pi_i$ for each $C_i \in CL(\varphi)$.**

   Considering that each $C_i$ is a subsequent of one or more axioms of the proof, it is possible to rebuild the proof from those axioms using only the rules that operate on end-sequent ancestors, thus leaving the atoms that came from cut-formulas unchanged. This will result in a proof of $S \circ C_i$, where $S$ is the end-sequent of $\varphi$ (one may need to apply some weakenings to get $S$). This proof is the projection $\pi_i$ of $C_i$ and it is shown in Figure 3.2.

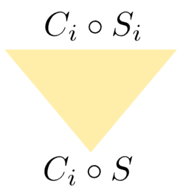



**Figure 3.2:** Projection of clause $C_i$ from the clause set.

4. **Finding a *resolution refutation* $R$ of $CL(\varphi)$.**

   A crucial property of the characteristic clause set is that it is refutable. This means that there is a resolution refutation of this set of clauses, which is just a derivation of the empty sequent ($\vdash$) from the clauses of the characteristic clause set. The resolution refutation is represented in Figure 3.3, and each $C_i$ might be used zero or more times.

---

[3] $(\Gamma \vdash \Delta) \circ (\Gamma' \vdash \Delta') = \Gamma, \Gamma' \vdash \Delta, \Delta'$

[4] Please keep in mind that these are combined in a particular way, and one $C_i$ might come from several leaves.

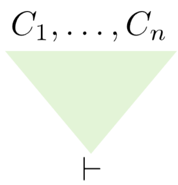



**Figure 3.3:** Resolution refutation of the characteristic clause set.

5. **Merging the projections $\pi_i$ and the resolution refutation $R$.**

   The last step consists of merging the previous elements. The projection for each $C_i$ will be put immediately above the leaf of the resolution refutation that used the same $C_i$, the proper substitutions will be made and the end-sequent is merged down this branch until the root. After plugging in all the projections, a proof of $S$ with only atomic cuts is obtained. Note that each resolution step is translated into an atomic cut, and the other rules of the resolution calculus are also rules in **LK**. Figure 3.4 represents the final proof with only atomic cuts, which in fact only occur in the lower triangle.

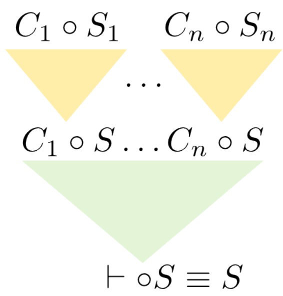



**Figure 3.4:** Final proof with atomic cuts obtained after merging the projections with the resolution refutation.

The CERES method, as it is, can in fact be applied to **LJ** proofs. The problem is that the final proof obtained, with only atomic cuts, is often an **LK** proof, which is an undesirable effect. This happens mainly because of the transformations of removing some rules to build the projections and the merging of elements (projections and resolution refutation) to obtain the final proof, which results on some sequents having more than one formula on the right side.

## 3.2 CERES in details

Now that you have a general idea of what the CERES method is, let's explain it in detail.

For proof skolemization we refer the reader to [9] (Proposition 4.2), but the procedure consists basically on skolemizing the end-sequent and transforming the proof by removing the strong-quantifier inferences. The reason why skolemization is needed will be clear in the example we use throughout this section.

Let $\varphi$ be the following proof, in which cut-formulas and its ancestors are colored in green:

\begin{prooftree}
\AxiomC{$P\alpha \vdash P\alpha$}
\RightLabel{$\neg_l$}
\UnaryInfC{$\neg P\alpha, P\alpha \vdash$}
\RightLabel{$\neg_r$}
\UnaryInfC{$\neg P\alpha \vdash \neg P\alpha$}
\AxiomC{$Q\alpha \vdash Q\alpha$}
\RightLabel{$\vee_l$}
\BinaryInfC{$\neg P\alpha \vee Q\alpha \vdash \neg P\alpha, Q\alpha$}
\doubleLine
\RightLabel{$\vee_r \times 2$}
\UnaryInfC{$\neg P\alpha \vee Q\alpha \vdash \neg P\alpha \vee Q\alpha, \neg P\alpha \vee Q\alpha$}
\RightLabel{$c_r$}
\UnaryInfC{$\neg P\alpha \vee Q\alpha \vdash \neg P\alpha \vee Q\alpha$}
\RightLabel{$\forall_l$}
\UnaryInfC{$\forall x.(\neg Px \vee Qx) \vdash \neg P\alpha \vee Q\alpha$}
\RightLabel{$\exists_r$}
\UnaryInfC{$\forall x.(\neg Px \vee Qx) \vdash \exists y.(\neg P\alpha \vee Qy)$}
\RightLabel{$\forall_r$}
\UnaryInfC{$\forall x.(\neg Px \vee Qx) \vdash \forall x.\exists y.(\neg Px \vee Qy)$}
\AxiomC{$Pa \vdash Pa$}
\RightLabel{$\neg_l$}
\UnaryInfC{$\neg Pa, Pa \vdash$}
\RightLabel{$\neg_r$}
\UnaryInfC{$\neg Pa \vdash \neg Pa$}
\AxiomC{$Q\beta \vdash Q\beta$}
\RightLabel{$\vee_l$}
\BinaryInfC{$\neg Pa \vee Q\beta \vdash \neg Pa, Q\beta$}
\doubleLine
\RightLabel{$\vee_r \times 2$}
\UnaryInfC{$\neg Pa \vee Q\beta \vdash \neg Pa \vee Q\beta, \neg Pa \vee Q\beta$}
\RightLabel{$c_r$}
\UnaryInfC{$\neg Pa \vee Q\beta \vdash \neg Pa \vee Q\beta$}
\doubleLine
\RightLabel{$\exists_r \times 2$}
\UnaryInfC{$\neg Pa \vee Q\beta \vdash \exists x.\exists y.(\neg Px \vee Qy)$}
\RightLabel{$\exists_l$}
\UnaryInfC{$\exists y.(\neg Pa \vee Qy) \vdash \exists x.\exists y.(\neg Px \vee Qy)$}
\RightLabel{$\forall_l$}
\UnaryInfC{$\forall x.\exists y.(\neg Px \vee Qy) \vdash \exists x.\exists y.(\neg Px \vee Qy)$}
\RightLabel{$cut$}
\BinaryInfC{$\forall x.(\neg Px \vee Qx) \vdash \exists x.\exists y.(\neg Px \vee Qy)$}
\end{prooftree}

We start thus with the extraction of the clause set.

**Definition 10** (Clause-set)**.** *The clause set is built recursively from the leaves of the proof until the end sequent. Let $\nu$ be a sequent in this proof. Then:*

- *If $\nu$ is an axiom, then $CL(\nu)$ contains the sub-sequent of $\nu$ composed only of cut ancestors.*
- *If $\nu$ is the result of the application of a unary rule on a sequent $\mu$, then $CL(\nu) = CL(\mu)$*
- *If $\nu$ is the result of the application of a binary rule on sequents $\mu_1$ and $\mu_2$, then we distinguish two cases:*
  - *If the rule is applied to ancestors of the cut formula, then $CL(\nu) = CL(\mu_1) \cup CL(\mu_2)$*
  - *If the rule is applied to ancestors of the end sequent, then $CL(\nu) = CL(\mu_1) \times CL(\mu_2)$*

*Where*[5]*:*

$$CL(\mu_1) \times CL(\mu_2) = \{C \circ D | C \in CL(\mu_1), D \in CL(\mu_2)$$

The clause set of $\varphi$ is:

$$(\{P\alpha \vdash\} \times \{\vdash Q\alpha\}) \cup \{\vdash Pa\} \cup \{Q\beta \vdash\}$$

Which reduces to:

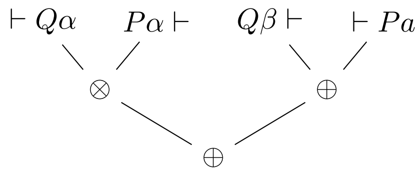
$$CL(\varphi) = \{P\alpha \vdash Q\alpha \ ; \ \vdash Pa \ ; \ Q\beta \vdash\}$$

Alternatively, we can work with a clause-set struct.

[5] The operation $\circ$ represents the merging of sequents, i.e., $(\Gamma \vdash \Delta) \circ (\Gamma' \vdash \Delta') = \Gamma, \Gamma' \vdash \Delta, \Delta'$.

**Definition 11** (Clause-set struct)**.** *The clause set struct is built analogously to a clause set, except that instead of applying the operators $\cup$ and $\times$, we use $\oplus$ and $\otimes$ respectively.*

The clause-set struct of $\varphi$, in a tree representation, is:




Projections are derivations built for each of the clauses in the clause set. Intuitively, projections are derivations from the axioms from where the clause was collected until the end sequent considering only the rules applied to end sequent ancestors.

**Definition 12** (Projection)**.** *Let $C$ be a clause which contains atoms from the axiom set $\mathcal{A}$. Let $\xi$ be an inference on the path between the end sequent and axioms $\mathcal{A}$. We define the projection $\pi(C)$ as the derivation built recursively from the axioms $\mathcal{A}$:*

- *If $\xi$ is an axiom, the derivation is $\xi$ itself.*
- *If $\xi$ is a unary rule:*
  - *If $\xi$ operates on a cut ancestor, the derivation does not change.*
  - *If $\xi$ operates on an end sequent ancestor, the derivation will be increased by the application of this rule.*
- *If $\xi$ is a binary rule:*
  - *If $\xi$ operates on a cut ancestor, then each branch generated different clauses (observe how clauses are built). In this case, two derivations will be generated, one for each branch, without applying $\xi$. These derivations will have the same rules applied from this sequent down, but the sequents will have different cut ancestor formulas.*
  - *If $\xi$ operates on an end sequent ancestor, the derivation will be increased by the application of this rule.*

*In each step, it might be necessary to weaken the auxiliary formulas of an inference. Moreover, if all formulas of the end-sequent are not present after constructing the projection, they are weakened as well.*

*Note that cut-ancestors (from the clause) are just dragged down in the sequent, without changes.*

The clause set of $\varphi$ has three clauses. Consequently, there are three projections:

$$\pi(\vdash Pa)$$

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$Pa \vdash Pa$}
\RightLabel{$\neg_r$}
\UnaryInfC{$\vdash Pa, \neg Pa$}
\RightLabel{$\vee_r$}
\UnaryInfC{$\vdash Pa, \neg Pa \vee Q\beta$}
\doubleLine
\RightLabel{$\exists_r \times 2$}
\UnaryInfC{$\vdash Pa, \exists x.\exists y.(\neg Px \vee Qy)$}
\RightLabel{$w_l$}
\UnaryInfC{$\forall x.(\neg Px \vee Qx) \vdash Pa, \exists x.\exists y.(\neg Px \vee Qy)$}
\end{prooftree}
$$

$$\pi(Q\beta \vdash)$$

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$Q\beta \vdash Q\beta$}
\RightLabel{$\vee_r$}
\UnaryInfC{$Q\beta \vdash \neg Pa \vee Q\beta$}
\doubleLine
\RightLabel{$\exists_r \times 2$}
\UnaryInfC{$Q\beta \vdash \exists x.\exists y.(\neg Px \vee Qy)$}
\RightLabel{$w_l$}
\UnaryInfC{$\forall x.(\neg Px \vee Qx), Q\beta \vdash \exists x.\exists y.(\neg Px \vee Qy)$}
\end{prooftree}
$$

$$\pi(P\alpha \vdash Q\alpha)$$

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$P\alpha \vdash P\alpha$}
\RightLabel{$\neg_l$}
\UnaryInfC{$\neg P\alpha, P\alpha \vdash$}
\AxiomC{}
\UnaryInfC{$Q\alpha \vdash Q\alpha$}
\RightLabel{$\vee_l$}
\BinaryInfC{$\neg P\alpha \vee Q\alpha, P\alpha \vdash Q\alpha$}
\RightLabel{$\forall_l$}
\UnaryInfC{$\forall x.(\neg Px \vee Qx), P\alpha \vdash Q\alpha$}
\RightLabel{$w_r$}
\UnaryInfC{$\forall x.(\neg Px \vee Qx), P\alpha \vdash Q\alpha, \exists x.\exists y.(\neg Px \vee Qy)$}
\end{prooftree}
$$

On the derivation $\pi(P\alpha \vdash Q\alpha)$ we can illustrate why proofs need to be skolemized. Imagine that, instead of $\forall_l$, we needed to apply a $\forall_r$ inference at this point. Now, $\forall_r$ has the side condition that the variable used to instantiate the formula need to be an *eigenvariable*, i.e., a fresh variable different from the ones already occurring in the sequent. Although it would be ok in the input proof, this might cause problems in the projections (because no inferences are applied to cut-ancestors). So the variable $\alpha$, which in the input proof was introduced after (above) $\forall_r$, is now present in the sequent. That would require the we instantiate the formula with another variable, but in this case, the sequent would not be provable. That's why we require that the input proof has no strong-quantifier inferences.

O-projections avoid the application of some weakenings and allow the end-sequent to contain only a subset of formulas of the input proof's end-sequent.

**Definition 13** (O-projections)**.** *O-projections are constructed like regular projections (Definition 12), except that no weakenings are added in the end to obtain all the formulas of the input proof's end-sequent.*

The o-projections of $\varphi$ are the following:

$$\pi(\vdash Pa)$$

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$Pa \vdash Pa$}
\RightLabel{$\neg_r$}
\UnaryInfC{$\vdash Pa, \neg Pa$}
\RightLabel{$\vee_r$}
\UnaryInfC{$\vdash Pa, \neg Pa \vee Q\beta$}
\doubleLine
\RightLabel{$\exists_r \times 2$}
\UnaryInfC{$\vdash Pa, \exists x.\exists y.(\neg Px \vee Qy)$}
\end{prooftree}
$$

$$\pi(Q\beta \vdash)$$

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$Q\beta \vdash Q\beta$}
\RightLabel{$\vee_r$}
\UnaryInfC{$Q\beta \vdash \neg Pa \vee Q\beta$}
\doubleLine
\RightLabel{$\exists_r \times 2$}
\UnaryInfC{$Q\beta \vdash \exists x.\exists y.(\neg Px \vee Qy)$}
\end{prooftree}
$$

$$\frac{\Delta \vdash \Gamma, A \quad \Delta', A' \vdash \Gamma'}{\Delta\sigma, \Delta'\sigma \vdash \Gamma, \sigma\Gamma'\sigma}\, R \qquad \frac{\Delta, A, A' \vdash \Gamma}{\Delta\sigma, A\sigma \vdash \Gamma\sigma}\, C_l \qquad \frac{\Delta \vdash \Gamma, A, A'}{\Delta\sigma \vdash \Gamma\sigma, A\sigma}\, C_r$$

**Figure 3.5:** Resolution calculus

$$\pi(P\alpha \vdash Q\alpha)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{P\alpha \vdash P\alpha}}{\neg P\alpha, P\alpha \vdash}\,\neg_l \qquad \dfrac{}{Q\alpha \vdash Q\alpha}}{\neg P\alpha \lor Q\alpha, P\alpha \vdash Q\alpha}\,\lor_l}{\forall x.(\neg Px \lor Qx), P\alpha \vdash Q\alpha}\,\forall_l$$

For the next step in the CERES method, we need to compute a resolution refutation of the clause set. The resolution calculus is depicted in Figure 3.5 and a resolution refutation is simply the derivation of the empty clause from the clauses in the set.

In the case of $\varphi$, we can obtain the following refutation:

$$\dfrac{\dfrac{\vdash Pa \quad P\alpha \vdash Q\alpha}{\vdash Qa}\, R\{\alpha \leftarrow a\} \qquad Q\beta \vdash}{\vdash}\, R\{\beta \leftarrow a\}$$

In order to use this refutation for a sequent calculus proof, we need to *ground* it. This means we get a resolution refutation where the resolution inferences involve no unification.

$$\dfrac{\dfrac{\vdash Pa \quad Pa \vdash Qa}{\vdash Qa}\, R \qquad Qa \vdash}{\vdash}\, R$$

We need still to remember the substitutions used, so that we can use them to instantiate the variables in the projections. To get a final proof, we need to plug the projections on top of their respective clauses in the refutations. When a unification was involved, as in the case of the clauses $P\alpha \vdash Q\alpha$ and $Q\beta \vdash$, we use an instantiation of the projections, i.e., in this case we use $\pi(P\alpha \vdash Q\alpha)\{\alpha \leftarrow a\}$ and $\pi(Q\beta)\{\beta \leftarrow a\}$.

The final proof is referred to as a *ACNF*, for *a*tomic *c*ut *n*ormal *f*orm. Note that, however, there is not a unique normal form, since it will depend on the resolution refutation used.

The ACNF for $\varphi$, considering the resolution refutation shown before and using o-projections, is:

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$Pa \vdash Pa$}
\RightLabel{$\neg_r$}
\UnaryInfC{$\vdash Pa, \neg Pa$}
\RightLabel{$\vee_r$}
\UnaryInfC{$\vdash Pa, \neg Pa \vee Q\beta$}
\doubleLine
\RightLabel{$\exists_r \times 2$}
\UnaryInfC{$\vdash Pa, \exists x.\exists y.(\neg Px \vee Qy)$}
\AxiomC{}
\UnaryInfC{$Pa \vdash Pa$}
\RightLabel{$\neg_l$}
\UnaryInfC{$\neg Pa, Pa \vdash$}
\AxiomC{}
\UnaryInfC{$Q\alpha \vdash Q\alpha$}
\RightLabel{$\vee_l$}
\BinaryInfC{$\neg Pa \vee Qa, Pa \vdash Qa$}
\RightLabel{$\forall_l$}
\UnaryInfC{$\forall x.(\neg Px \vee Qx), Pa \vdash Qa$}
\RightLabel{$cut$}
\BinaryInfC{$\forall x.(\neg Px \vee Qx) \vdash Qa, \exists x.\exists y.(\neg Px \vee Qy)$}
\AxiomC{}
\UnaryInfC{$Qa \vdash Qa$}
\RightLabel{$\vee_r$}
\UnaryInfC{$Qa \vdash \neg Pa \vee Qa$}
\doubleLine
\RightLabel{$\exists_r \times 2$}
\UnaryInfC{$Qa \vdash \exists x.\exists y.(\neg Px \vee Qy)$}
\RightLabel{$cut$}
\BinaryInfC{$\forall x.(\neg Px \vee Qx) \vdash \exists x.\exists y.(\neg Px \vee Qy), \exists x.\exists y.(\neg Px \vee Qy)$}
\RightLabel{$c_r$}
\UnaryInfC{$\forall x.(\neg Px \vee Qx) \vdash \exists x.\exists y.(\neg Px \vee Qy)$}
\end{prooftree}
$$

Note that there are now two atomic cuts, one on $Pa$ and another on $Qa$. Eliminating these is fairly easy and can be done in linear time.

The main goal of this work was to develop a CERES-like method for intuitionistic logic, in the sense that all cuts are eliminated at once and a resolution refutation is used as a helper to find the necessary atomic cuts.

CHAPTER 4

# The problem

In this Chapter we describe the problems in applying CERES to intuitionistic proofs and some possible solutions. Unfortunately, we could not yet define a method such that it encompasses full intuitionistic logic, but we hope that the analysis presented here will convince the reader that this is not a trivial problem. We also hope that the insights obtained by the partial solutions contribute to the development of a CERES variant that will work for full intuitionistic logic (see Chapter 10).

Let's start by defining properly the system with which we will be working. In Chapter 2 we presented two calculi for intuitionistic logic, namely **LJ** and **LJ'**. We also noted that all rules of **LJ** are particular cases of **LJ'** rules. Although we are aware that **LJ** is the calculus that comes to mind when intuitionistic logic is mentioned, we will allow resulting proofs to be in **LJ'**. There are three reasons for that: (1) every **LJ**-proof is an **LJ'**-proof, trivially, and every **LJ'**-proof can be translated into an **LJ** proof using $\vee_l$ rules [33]; (2) the CERES method relies on a sequent merging operation $\circ$, which would constantly violate the structural restriction of **LJ** (the sequents may have at most one formula on the right side) but would pose less problems for **LJ'**; and (3) CERES uses resolution on atoms, which can be easily translated to an **LJ'**-proof, analogously to the classical case. The input proof is still in **LJ** though. The only restriction we make to these proofs is that they are *skolemized*, i.e., there are no strong-quantifier inferences ($\forall_r$ and $\exists_l$) on end-sequent ancestors. The reason for working with skolemized proofs in the intuitionistic case is the same as for the classical case: eigenvariable violations. Now, we know of course that skolemization for intuitionistic logic is not as simple as for classical logic, but there are some methods developed [8], [17], [25]. Moreover, developing a CERES method for non-skolemized proofs would be a major problem on its own, so we decided to live with this small restriction. Therefore our aim is to analyse the CERES method and possible variations applied to a (skolemized) **LJ**-proof, the result of this operation being an **LJ** or **LJ'**-proof.

## 4.1 What is the problem?

What happens when CERES is applied to an intuitionistic proof? First of all, let's remember the steps of the CERES method:

1. Computation of the clause set.
2. Resolution refutation of the clause set.
3. Construction of the projections.
4. Merging of projections and resolution refutation (final proof with only atomic cuts).

Now we'll analyse, for each step, what is the result when an **LJ**-proof is given as input.

**Clause set** The clause set of an intuitionistic proof will be the same as in the classical case, i.e., composed by atomic sequents possibly having more than one atom on the right. Note that this would be a structural violation for **LJ**, but in **LJ'** the sequents are ok.

**Resolution** The rules of the resolution calculus can be applied unchanged to the clauses in the clause set. Resolution and factoring rules will later be translated, respectively, into cuts and contractions for the final proof. Considering that cuts and contractions impose no structural restrictions in **LJ'**, resolution can be used with no concerns. Observe again that using **LJ** and its single conclusion sequents here would be quite a problem.

**Projections** Projections seem to be the biggest of our problems. During their construction, cut ancestors are just "dragged" down without changes. This is no problem in the classical calculus **LK**, but remember that, as much multi-conclusion as **LJ'** can be, it still has a few restrictive rules that require the right formula to be alone. Namely, $\rightarrow_r$, $\neg_r$ and $\forall_r$:

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B} \rightarrow_r \qquad \frac{\Gamma, A \vdash}{\Gamma \vdash \neg A} \neg_r \qquad \frac{\Gamma \vdash A\{c/x\}}{\Gamma \vdash \forall x\, A} \forall_r$$

When a cut ancestor on the right side has to be merged with a sequent of such a rule, we have a problem. Of course that, when working with skolemized proofs with cuts only on atoms, inferences $\forall_r$ do not occur, but still the other two rules are problematic.

Projections can obviously be built for intuitionistic proofs, the problem is that they might not be intuitionistic themselves. Since these are part of the final proof, classical projections mean classical proof, which is an undesirable effect of the method.

If you are not yet convinced that allowing the final proof to be in a multi-conclusion calculus is better for our problem, take a look at Table 4.1.

The attentive reader might have observed that structural restrictions on sequents is the major issue here. Why not use a calculus for intuitionistic logic that imposes no such restriction then? Are there such calculi? Well, of course there are.

| | **LJ** | **LJ'** |
|---|---|---|
| Clause set | ✘ | ✔ |
| Projections | ✘ | ✘ |
| Resolution | ✘ | ✔ |

**Table 4.1:** Elements of CERES for different intuitionistic calculi.

## 4.2 Trying other calculi

### The labelled calculus attempt

Labelled deductive systems were formalized by Dov Gabbay [14]. Following his ideas, Sara Negri and Roy Dyckhoff developed a labelled calculus for intuitionistic logic [13], which happens to be multi-conclusion (in all rules). In this calculus each formula is given a label, which can be interpreted as a world in the Kripke semantics. There is also a predicate $\leq$ that operate on these labels, and this can be interpreted as the accessibility relation among worlds. Take for example the rule for $\rightarrow_r$:

$$\frac{x \leq y, y : A, \Gamma \vdash \Delta, y : B}{\Gamma \vdash \Delta, x : A \rightarrow B} \rightarrow_r$$

Now remember the Kripke semantics for $A \rightarrow B$: *$A \rightarrow B$ is true in the world $x$ if for every world $y$ accessible from $x$, $A$ true in $y$ implies that $B$ is true in $y$.* The universal quantification over $y$ is captured in the inference rule by that fact that $y$ is a *fresh* label. The problem in applying CERES to such a calculus is exactly the "freshness" requirement of some labels. These rules then have the same effect as strong quantifier rules, and "eigenvariable" violations occur exactly on those labels. Now, we have agreed already that the problem with strong quantifiers would not be a topic of this thesis, so we decided not to pursue this direction further. It is worth noting though, that if a method is ever developed in which strong quantifiers do not cause problems, such method can be used also in the labelled calculus for intuitionistic logic.

### The decorated sequent calculus attempt

Another less popular, but also interesting, approach is the *decorated* sequent calculus for intuitionistic logic [12]. A decorated sequent has the form:

$$A_1(n_1), ..., A_k(n_k) \vdash B_1/S_1, ..., B_m/S_m$$

Where $A_i$ and $B_j$ are formulas, $n_i$ is a natural number and $S_j$ is a set of natural numbers. Each $n_i$ is the *index* of formula $A_i$, and $n_i \neq n_j$ for $i \neq j$. Each $S_j$ is the *dependency set* of formula $B_j$. Intuitively, the set $S_j$ contains the indices of the antecedent formulas that $B_j$ depends on. Naturally there are restrictions on the indices for some rules. Naturally these restrictions were violated when the CERES method was applied to decorated proofs. It was an interesting idea, but unfortunately we got no new insights from it. Being such an unconventional calculus, with harder manipulation because of the indices, we decided to abandon the idea.

It is curious to observe that the rule for $\rightarrow_r$ invariably causes problems in the calculi considered. This seems, on the one hand, a bit odd since the properties of IL are due to disjunctions and existential quantifiers. But it is according to the Kripke semantics and BHK interpretation, where implication (and universal quantifier) is indeed a particular case.

In both these approaches, we considered that the output proof would be in the same calculus as the input proof. Had we wanted to have an **LJ**-proof as output, another non-trivial problem would have to be solved, namely, the translation of labelled and decorated sequent calculus proofs into **LJ**. As of today, there are no methods developed for such translation.

## 4.3 Back to LJ

So let's get back to the original idea with the multi-conclusion calculus **LJ'**. We've seen that the main problem is the violation of $\rightarrow_r$ and $\neg_r$ on the projections[1]. These can occur in two situations. The first one is when cuts occur above these inferences in the input proof[2]:

$$\dfrac{\dfrac{\dfrac{\vdots}{\Gamma, F \vdash} \quad \dfrac{\vdots}{\Gamma', A \vdash B, F}}{\Gamma, \Gamma', A \vdash B}\,cut}{\Gamma, \Gamma' \vdash A \rightarrow B}\rightarrow_r \quad \xrightarrow{projection} \quad \dfrac{\dfrac{\vdots}{\Gamma', A \vdash B, F'}}{\Gamma' \vdash A \rightarrow B, F'}\rightarrow_r$$

In this example, $F'$ is an atomic ancestor of $F$, which does not cause problems in the input proof. But when the projections are built, inferences operating on cut ancestors, and thus the cut itself, are not applied, leaving possibly some of its atomic ancestors on the right side at the application point of $\rightarrow_r$.

The second situation in which violations might occur is when cut ancestors shift sides during a proof above $\rightarrow_r$ and $\neg_r$ inferences:

$$\dfrac{\dfrac{\dfrac{\vdots}{\Gamma, A \vdash P} \quad \dfrac{\vdots}{\Gamma', Q \vdash B}}{\Gamma, \Gamma', P \rightarrow Q, A \vdash B}\rightarrow_l}{\Gamma, \Gamma', P \rightarrow Q \vdash A \rightarrow B}\rightarrow_r \quad \xrightarrow{projection} \quad \dfrac{\dfrac{\dfrac{\vdots}{\Gamma, A \vdash P'}}{\Gamma, A \vdash P', B}\,w_r}{\Gamma \vdash A \rightarrow B, P'}\rightarrow_r$$

In this example, $P \rightarrow Q$ is a cut ancestor which occurs on the left side when the inference $\rightarrow_r$ is applied. But when the $\rightarrow_l$ inference is applied to this formula (bottom-up), the ancestor $P$ is shifted to the right side. When the projection for the left branch is built, considering no more shiftings happened, there might be cut ancestors $P'$ on the right side, thus violating the application of $\rightarrow_r$.

You can observe how this violation occurs in a real proof on Example 1.

[1]Remember that, being the input proof skolemized, there are no $\forall_r$ inferences in the projections.

[2]We will use green for cut ancestors in an intuitionistic proof and red for cut ancestors which make the proof classical.

**Example 1.** *Let $\varphi$ be the proof:*

$$
\dfrac{\dfrac{\dfrac{}{A \vdash A}}{A, \neg A \vdash}\neg_l}{\neg A \vdash \neg A}\neg_r
\quad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}}{B, A \vdash A} w_l}{\neg A, B, A \vdash}\neg_l}{\neg A, B \vdash \neg A}\neg_r}{\neg A \vdash B \to \neg A}\to_r
\;\;
\dfrac{}{\neg A \vdash B \to \neg A}\, cut
$$

*We will compute its projections and see why they are no longer intuitionistic proofs. The clause set of $\varphi$ is very simple: $CL(\varphi) = \{A \vdash;\ \vdash A\}$. The projections for each of these clauses are:*

$$
\dfrac{\dfrac{\dfrac{}{A \vdash A}}{A, \neg A \vdash}\neg_l}{A, \neg A \vdash B \to \neg A} w_r
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}}{A, B \vdash A} w_l}{B \vdash A, \neg A}\neg_r}{\vdash A, B \to \neg A}\to_r}{\neg A \vdash A, B \to \neg A} w_l
$$

*The left projection is ok, but the right projection contains two inferences, namely $\to_r$ and $\neg_r$, which are no longer intuitionistic. In this particular case, the violation happened because the cut ancestor $A$ changed sides on the original proof by a negation rule.*

You might have observed that not every application of $\to_r$ and $\neg_r$ will cause problems. Some of them might be actually fine. Now we define precisely which are the inferences that will be problematic.

**Definition 14** (Critical inferences)**.** *Let $\xi$ be a $\to_r$, $\neg_r$ or $\forall_r$ inference on an end-sequent ancestor in an **LJ**-proof. We say that $\xi$ is a* critical inference *if the clause set computed in the derivation of the conclusion of $\xi$ from the axioms has one or more positive atoms.*

Intuitively, a critical inference is one that will be classical in at least one projection. Note that this does not mean that it will be classical in every projection. Observe also that the positive atoms will be exactly the "extra" formulas when the inference is performed in a projection.

We say that a critical rule occur in its *intuitionistic* form if the context on the right side of the sequent is empty[3]:

$$
\dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\to_r
\quad
\dfrac{\Gamma, A \vdash}{\Gamma \vdash \neg A}\neg_r
\quad
\dfrac{\Gamma \vdash A[x/\alpha]}{\Gamma \vdash \forall x. Ax}\forall_r
$$

Otherwise, we say that it occurs on its *classical* form:

[3]What is this $\forall_r$ rule doing there? I know we don't use it on the projections, but I am listing it for completeness purposes.

$$\frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta} \to_r^K \quad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta} \neg_r^K \quad \frac{\Gamma \vdash A[x/\alpha], \Delta}{\Gamma \vdash \forall x.Ax, \Delta} \forall_r^K$$

Although we call this the classical form, these rules are not the same as the ones for **LK**. They have the particularity that $\Delta$ contains only atomic cut ancestors.

**Lemma 4.** *If a critical inference occurs in its classical form in a projection, then the formulas in $\Delta$ are all atomic cut ancestors.*

*Proof.* We can safely assume that this inference occurred in its intuitionistic form in the input proof, since this is an **LJ** proof. When the projections are being built, the inferences applied to end-sequent ancestors remain the same, thus these formulas are exactly as they occur on the input proof. The only different formulas in the sequent are the atomic cut ancestors, which can potentially occur on the right side when, in the original proof, they occurred on the left, or not occurred at all. □

**Definition 15** (†-rules)**.** *We define as †-rules the following*

$$\frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta} \to_r^\dagger \quad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta} \neg_r^\dagger \quad \frac{\Gamma \vdash A[x/\alpha], \Delta}{\Gamma \vdash \forall x.Ax, \Delta} \forall_r^\dagger$$

*Where $\Delta$ consists of atomic cut ancestors.*

Therefore, the result of applying CERES to an **LJ**-proof might be a classical proof only because of †-inferences. Now all we have to do is find a way to transform †-inferences into intuitionistic inferences.

## 4.4 "Trivial" classes

Before thinking about this transformation, though, we ask ourselves whether there are classes of **LJ**-proofs such that the application of CERES yields **LJ'**-proofs. The answer is of course positive.

For one thing, we can consider the class of skolemized proofs in which there is no occurrence of the inferences $\to_r$ and $\neg_r$ on *end-sequent ancestors*. We can still relax this restriction, but there are some interesting observations about this class of proofs. If the rules $\to_r$, $\neg_r$ and $\forall_r$ are not used on end-sequent ancestors, cut-free **LK**- and **LJ'**-proofs are undistinguishable. In fact, even when atomic cuts are used, the proofs will be the same in both calculi. In this fragment, classical and intuitionistic logics coincide after cut-elimination, and any cut-elimination method will yield an **LJ'**-proof, regardless whether the inferences operating on the cut formulas were classical or not. We can formally characterize this class of proofs by a syntactical restriction on the end sequent.

**Definition 16** (KJ-sequent)**.** *A sequent $S$ is called a* KJ-sequent *if every formula whose main connective is $\to$, $\forall$ and $\neg$ is negative in $S$ (Definition 6).*

**Definition 17 (LJK).** *We define **LJK** as the class of **LK**- or **LJ'**-proofs such that the end sequent is a KJ-sequent.*

**Theorem 3.** *The CERES method applied to an **LJK**-proof yields an **LJ'**-proof.*

But as mentioned before, we can loosen this restriction. On Section 4.1 we observed that not every $\rightarrow_r$ and $\neg_r$ causes problems, only the *critical* ones (Definition 14). Thus we can allow these inferences to occur, as long as they are not critical. Unfortunately we could not yet find a syntactic restriction on the end-sequent that would characterize this class of proofs, since it depends also on the cut formulas and inferences applied to them.

**Theorem 4.** *If a proof contains no critical inferences, then the CERES method yields an **LJ'**-proof.*

## 4.5 Decidable atoms

You might be wondering by now why don't we just apply some negation inferences to change the side of the bothering atomic cut ancestors. Well, it is not that simple, and this is in fact a big part of iCERES, the first method we present. In Chapter 6 we discuss what can be solved using resolution on negated atoms. For now, let's assume we apply a negation inference only to the atom that is violating a critical rule. If this is done in Example 1, its clause set would then be $CL(\varphi) = \{A \vdash; \;\; \neg A \vdash\}$ and the projections become:

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$A \vdash A$}
\RightLabel{$\neg_l$}
\UnaryInfC{$A, \neg A \vdash$}
\RightLabel{$w_r$}
\UnaryInfC{$A, \neg A \vdash B \rightarrow \neg A$}
\end{prooftree}
\qquad
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$A \vdash A$}
\RightLabel{$\neg_l$}
\UnaryInfC{$A, \neg A \vdash$}
\RightLabel{$w_l$}
\UnaryInfC{$A, B \neg A \vdash$}
\RightLabel{$\neg_r$}
\UnaryInfC{$B, \neg A \vdash \neg A$}
\RightLabel{$\rightarrow_r$}
\UnaryInfC{$\neg A \vdash B \rightarrow \neg A$}
\RightLabel{$w_l$}
\UnaryInfC{$\neg A, \neg A \vdash B \rightarrow \neg A$}
\end{prooftree}
$$

Which is cool, because now they are intuitionistic. Note though that resolution does not really work for the clause set. Unless we can resolve $A \vdash$ with $\neg A \vdash$. When can we do this? We can do this if $A \vee \neg A$ holds. Now, we know very well this is not the case in general for intuitionistic logic, but in this setting we need the principle of excluded middle to hold only for atomic $A$. So when we deal with theories in which atomic formulas and grounded predicates are decidable (such as intuitionistic arithmetic for example, where we have to decide on equalities), we can use this "degenerated" resolution rule:

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma', \neg A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}$$

Which is replaced in the final proof by:

$$\cfrac{\cfrac{\vdash A, \neg A \quad \Gamma', \neg A \vdash \Delta'}{\Gamma' \vdash \Delta', A}\ cut \qquad \Gamma, A \vdash \Delta}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\ cut$$

Note that the leftmost premise of this derivation can only be proved if $A \vee \neg A$ is decidable, for atomic $A$.

One important observation is that, when a proof is skolemized, atoms which were previoulsy decidable might no longer have this property. Take for example $(2+2 = 4) \vee (2+2 \neq 4)$. Easy to decide which one is true, right? But with skolemization we might have $(2+c = 4) \vee (2+c \neq 4)$. Not so easy now. Nevertheless, this is likely to cause no problem for the CERES method, since at no point we need to actually make this decision. We already know which are the axioms from the input proof.

Being able to run the CERES method in arithmetic gives us a huge range of possible proofs to work with.

# CHAPTER 5

# The limits

Before going into the possible solutions on having a CERES-like method for intuitionistic logic, we will make clear what are our limits. You should remember now that the CERES method relies heavily on a resolution refutation. Not only that, it can be *any* resolution refutation. In this chapter we show that we can't have such liberty in intuitionistic logic.

We present here two proofs, one in propositional and another in predicate intuitionistic logic. We try to apply the CERES method to both of them and show that, by using the clause set and refutations as defined before, it is impossible to obtain intuitionistic proofs.

## 5.1 Intuitionistic propositional logic

So, let's try to apply CERES to the following **LJ** proof, in which the cut formulas and cut ancestors are marked in green:

$$
\dfrac{
\dfrac{
\dfrac{\dfrac{}{P \vdash P}}{P \vdash P \vee \neg P}\,\vee_r
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{}{P \vdash P}}{P, \neg P \vdash}\,\neg_l}{\neg P \vdash \neg P}\,\neg_r}{\neg P \vdash P \vee \neg P}\,\vee_r
}{P \vee \neg P \vdash P \vee \neg P}\,\vee_l
\qquad
\dfrac{
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P \vdash P}}{\neg P, P \vdash}\,\neg_l}{\neg P \vdash \neg P}\,\neg_r}{\neg P, \neg\neg P \vdash}\,\neg_l}{\neg P, \neg\neg P \vdash P}\,w_r}{\neg P \vdash \neg\neg P \to P}\,\to_r
\qquad
\dfrac{\dfrac{\dfrac{}{P \vdash P}}{P, \neg\neg P \vdash P}\,w_l}{P \vdash \neg\neg P \to P}\,\to_r
}{P \vee \neg P \vdash \neg\neg P \to P}\,\vee_l
}{P \vee \neg P \vdash \neg\neg P \to P}\,cut
$$

First of all, we need to extract a clause set from this proof. This is the following:

$$CL = \{\vdash P \;\; ; \;\; P \vdash\}$$

In fact, if you observe closely, you will see that the cut ancestors from the left branch of the cut were not considered. This is because they would result in the following tautological clause: $P \vdash P$, and in resolution, tautological clauses can be ignored. Remember this. It will be important.

Now, there is only one possible resolution refutation of such a simple clause set. It is this one:

$$\dfrac{\vdash P \quad P \vdash}{\vdash}\ R$$

The next step is to construct the projections of the clauses. We will use o-projections to facilitate the analysis. The only difference to the regular projections is the lack of a weakening of the right side formula $P \vee \neg P$ in the end.

$\pi(\vdash P)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{P \vdash \color{red}{P}}}{\vdash \color{red}{P}, \neg P}\ \neg_r^\dagger}{\neg\neg P \vdash \color{red}{P}}\ \neg_l}{\neg\neg P \vdash \color{red}{P}, P}\ w_r}{\vdash \color{red}{P}, \neg\neg P \to P}\ \to_r^\dagger$$

$\pi(P \vdash)$

$$\dfrac{\dfrac{\overline{\color{green}{P} \vdash P}}{\color{green}{P}, \neg\neg P \vdash P}\ w_l}{\color{green}{P} \vdash \neg\neg P \to P}\ \to_r$$

Note the presence of †-inferences on one of the projections. This makes it a classical derivation.

We can put the projections and resolution together into this final (classical) proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{P \vdash \color{red}{P}}}{\vdash \color{red}{P}, \neg P}\ \neg_r^\dagger}{\neg\neg P \vdash \color{red}{P}}\ \neg_l}{\neg\neg P \vdash \color{red}{P}, P}\ w_r}{\vdash \color{red}{P}, \neg\neg P \to P}\ \to_r^\dagger \quad \dfrac{\dfrac{\overline{\color{green}{P} \vdash P}}{\color{green}{P}, \neg\neg P \vdash P}\ w_l}{\color{green}{P} \vdash \neg\neg P \to P}\ \to_r}{\vdash \neg\neg P \to P, \neg\neg P \to P}\ cut}{\vdash \neg\neg P \to P}\ c_r}{P \vee \neg P \vdash \neg\neg P \to P}\ w_l$$

The proof is classical, but we were aiming for an intuitionistic one. Can we transform this proof into an intuitionistic proof (in **LJ** or **LJ'**)? Well, the end-sequent is certainly intuitionistic, since our input proof is in **LJ**. But the very next sequent, after weakening the left side formula, is not! You can try as much as you want, but you won't find an intuitionistic proof of $\vdash \neg\neg P \to P$. If the full projections are used, the effect is not so obvious, but it is the same. The left formula $P \vee \neg P$ would be weakened in the projections and not used in the proof of $\neg\neg P \vdash P$ at all.

So you see, after applying CERES we got a proof with atomic cuts, but of a classical formula. This is, and always will be, a classical proof.

## 5.2 Intuitionistic predicate logic

Let's show now that the same happens in predicate intuitionistic logic. For that we use the following proof:

$$
\dfrac{
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{Pf\alpha \vdash Pf\alpha} \quad \dfrac{\overline{P\alpha \vdash P\alpha}}{\neg P\alpha, P\alpha \vdash}\neg_l}{Pf\alpha \to P\alpha, \neg P\alpha, Pf\alpha \vdash}\to_l}{\forall y.(Pfy \to Py), \neg P\alpha, Pf\alpha \vdash}\forall_l}{\forall y.(Pfy \to Py), \neg P\alpha \vdash \neg Pf\alpha}\neg_r}{\forall y.(Pfy \to Py) \vdash \neg P\alpha \to \neg Pf\alpha}\to_r}{\forall y.(Pfy \to Py) \vdash \forall y.(\neg Py \to \neg Pfy)}\forall_r
\quad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{Pt \vdash Pt}}{\neg Pt, Pt \vdash}\neg_l}{\neg Pt \vdash \neg Pt}\neg_r \quad \dfrac{\dfrac{\overline{Pft \vdash Pft}}{\neg Pft, Pft \vdash}\neg_l}{\neg Pft \vdash \neg Pft}\neg_r}{\neg Pt \to \neg Pft, \neg Pt \vdash \neg Pft}\to_l}{\neg Pt \to \neg Pft \vdash \neg Pt \to \neg Pft}\to_r}{\neg Pt \to \neg Pft \vdash \exists y.(\neg Py \to \neg Pfy)}\exists_r}{\forall y.(\neg Py \to \neg Pfy) \vdash \exists y.(\neg Py \to \neg Pfy)}\forall_l
}{\forall y.(Pfy \to Py) \vdash \exists y.(\neg Py \to \neg Pfy)}\mathit{cut}
$$

Observe this proof makes no use of the $\vee_l$ rule. The **LJ** fragment without $\vee_l$ is considered a bit special because Herbrand's theorem (or midsequent theorem) holds in it [33]. This theorem also holds for full **LK**. Because of this, there was always an intuition that CERES would work, without changing much, in **LJ** without $\vee_l$. Unfortunately, this example shows us the contrary.

The characteristic clause set of this proof is the following:

$$CL = \{Pf\alpha \vdash P\alpha \;\; ; \;\; Pt \vdash \;\; ; \;\; \vdash Pft\}$$

Which can be refuted by using only two out of the three clauses:

$$
\dfrac{\vdash Pft \quad \dfrac{Pt \vdash}{Pft \vdash}\, t \leftarrow ft}{\vdash}\, R
$$

Now we construct the o-projections of the clauses involved in the resolution, already instantiating $t$ to $ft$ in one of them.

$$\pi(\vdash Pft)$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\overline{Pft \vdash Pft}}{\vdash Pft, \neg Pft}\neg_r^\dagger}{\neg Pt \vdash Pft, \neg Pft}w_l}{\vdash Pft, \neg Pt \to \neg Pft}\to_r^\dagger}{\vdash Pft, \exists y.(\neg Py \to \neg Pfy)}\exists_r
$$

$$\pi(Pt \vdash)[t \leftarrow ft]$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\overline{Pft, \vdash Pft}}{Pft, \neg Pft \vdash}\neg_l}{Pft, \neg Pft \vdash \neg Pfft}w_r}{Pft \vdash \neg Pft \to \neg Pfft}\to_r}{Pft \vdash \exists y.(\neg Py \to \neg Pfy)}\exists_r
$$

Putting the pieces together into an ACNF:

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$Pft \vdash Pft$}
\RightLabel{$\neg_r^\dagger$}
\UnaryInfC{$\vdash Pft, \neg Pft$}
\RightLabel{$w_l$}
\UnaryInfC{$\neg Pt \vdash Pft, \neg Pft$}
\RightLabel{$\rightarrow_r^\dagger$}
\UnaryInfC{$\vdash Pft, \neg Pt \rightarrow \neg Pft$}
\RightLabel{$\exists_r$}
\UnaryInfC{$\vdash Pft, \exists y.(\neg Py \rightarrow \neg Pfy)$}
\AxiomC{}
\UnaryInfC{$Pft, \vdash Pft$}
\RightLabel{$\neg_l$}
\UnaryInfC{$Pft, \neg Pft \vdash$}
\RightLabel{$w_r$}
\UnaryInfC{$Pft, \neg Pft \vdash \neg Pfft$}
\RightLabel{$\rightarrow_r$}
\UnaryInfC{$Pft \vdash \neg Pft \rightarrow \neg Pfft$}
\RightLabel{$\exists_r$}
\UnaryInfC{$Pft \vdash \exists y.(\neg Py \rightarrow \neg Pfy)$}
\RightLabel{$cut$}
\BinaryInfC{$\vdash \exists y.(\neg Py \rightarrow \neg Pfy), \exists y.(\neg Py \rightarrow \neg Pfy)$}
\RightLabel{$c_r$}
\UnaryInfC{$\vdash \exists y.(\neg Py \rightarrow \neg Pfy)$}
\RightLabel{$w_l$}
\UnaryInfC{$\forall y.(Pfy \rightarrow Py) \vdash \exists y.(\neg Py \rightarrow \neg Pfy)$}
\end{prooftree}
$$

And voilà! We have the same problem as in the propositional case. The sequent $\vdash \exists y.(\neg Py \rightarrow \neg Pfy)$ is not provable in intuitionistic logic.

## 5.3 Conclusion

Now what? Should we loose our hopes? Not just yet. During the next chapters of this thesis we will try to work around this limitation. In Chapter 6 we use a different resolution calculus that involves resolution on negated atoms. This method will work for a subclass of intuitionistic logic. Chapter 7 presents an alternative method, covering the same subclass, but using the negative resolution refinement. Chapters 8 and 9 show operations that can be used in the final classical ACNF to transform this into an intuitionistic proof when possible. Finally, Chapter 10 presents a resolution refinement which can possibly be used to obtain ACNF's that can always be transformed into intuitionistic proofs by some of the previous operations.

Now starts the hard work!

CHAPTER 6

# iCERES

We have already seen what the problem is in applying the CERES method to **LJ'**- or **LJ**-proofs: atomic cut ancestors on the right side of the sequent violate the structural restriction of some inference rules. So why not move this atom from the right to the left? How can this be done? Well, negating it!

In this Chapter we describe the very first approach for developing a CERES-like method to intuitionistic logic [24]. It is based on shifting all atomic cut ancestors that appear on the right side of the sequent to the left. We call this method iCERES. For this we must redefine the concepts of clause set and projections, as well as extend the resolution calculus for resolving negated atoms. Please keep in mind that this method is developed to have as input and output **LJ**-proofs, i.e., single-conclusion sequents. In the end we discuss how it can be adapted to a multi-conclusion setting, and the problems that arise there.

We will also describe another method that works for the same class of proofs and involves less changes in the original CERES. It works with the regular clause set and projections, and the same resolution calculus, although a refinement is used.

## 6.1 iCERES

We'll start by redefining clauses, the clause set and projections. The clauses now can contain literals, i.e., atoms or negated atoms, all with negative polarity (they only occur on the left side of the sequent). The clause set is now composed of these new clauses, and we redefine how to extract it from an **LJ**-proof. The projections are basically the same, with the only difference that they will sometimes contain an extra $\neg_l$ rule applied to a cut ancestor.

**Definition 18** (Intuitionistic Clause)**.** *An* intuitionistic clause *is a sequent composed only of atoms or negated atoms on the left side and at most one atom or negated atom on the right.*

**Definition 19** (Intuitionistic Clause Set with Negations)**.** *The* intuitionistic characteristic clause set *is built analogously to the usual characteristic clause set, except that all the formulas on the right hand side are negated and added to the left hand side:*

- *If $\nu$ is an axiom, then $CL^I(\nu)$ is the set containing the sub-sequent composed only of the formulas that are cut-ancestors, such that all the formulas that would appear on the right-hand side are negated and added to the left-hand side.*
- *If $\nu$ is the result of the application of a unary rule on $\mu$, then $CL^I(\nu) = CL^I(\mu)$*
- *If $\nu$ is the result of the application of a binary rule on $\mu_1$ and $\mu_2$, we have to distinguish two cases:*
  - *If the rule is applied to ancestors of a cut formula, then $CL^I(\nu) = CL^I(\mu_1) \cup CL^I(\mu_2)$.*
  - *If the rule is applied to ancestors of the end-sequent, then $CL^I(\nu) = CL^I(\mu_1) \times CL^I(\mu_2)$.*

Although we defined the intuitionistic clauses having at most one literal on the right side, the intuitionistic clause set, as defined, will contain only clauses with no literals on the right.

Now, it is very very important for the method that this different clause set is still unsatisfiable. Let's make sure of that.

**Theorem 5** (Refutability of the Intuitionistic Clause Set)**.** *The intuitionistic clause set is **LJ**-refutable.*

*Proof.* Let $\varphi$ be an **LJ** proof and $CL^I(\varphi)$ be its intuitionistic clause set built according to Definition 19 and $CL(\varphi)$ be its classical clause set obtained by the classical version of CERES. For each clause $C_i = A^i_1, ..., A^i_{n_i} \vdash B^i_1, ..., B^i_{m_i}$ of the classical clause set, we build the closed formula $F_i = \forall \overline{x}. \neg(A^i_1 \wedge ... \wedge A^i_{n_i} \wedge \neg B^i_1 \wedge ... \wedge \neg B^i_{m_i})$.

We know that there is an **LK**-refutation $\psi$ of $CL(\varphi)$:

$$\dfrac{\dfrac{}{C_1} \quad \vdots \quad ... \quad \vdots \quad \dfrac{}{C_k}}{\vdash}$$

By merging each formula $F_i$ to its corresponding clause $C_i$ and propagating it down the refutation, we obtain an **LK** proof $\psi_1$ from the formulas $F_i, A^i_1, ..., A^i_{n_i} \vdash B^i_1, ..., B^i_{m_i}$ of the end-sequent $F_1, ..., F_k \vdash$:

$$\dfrac{\dfrac{\varphi_1}{F_1 \circ C_1} \quad \vdots \quad ... \quad \vdots \quad \dfrac{\varphi_k}{F_k \circ C_k}}{F_1, ..., F_k \vdash}$$

Where each $\varphi_i$ is a derivation of $F_i \circ C_i$ from tautological axioms. We can transform the proof $\psi_1$ into a proof $\psi_2$ of $\vdash \neg(F_1 \wedge ... \wedge F_k)$:

$$
\dfrac{\dfrac{\dfrac{\psi_1}{F_1, ..., F_k \vdash}}{F_1 \wedge ... \wedge F_k \vdash} \wedge_l \times (k-1)}{\vdash \neg(F_1 \wedge ... \wedge F_k)} \neg_r
$$

Since the axioms of this proof are tautological, we can transform this into an **LJ** proof $\psi_3$ via the following negative translation [22]:

$$
\begin{array}{rl}
A & \rightarrow \neg\neg A^* \\
A^* & \rightarrow A \text{ (if A is an atom)} \\
(\neg A)^* & \rightarrow \neg A^* \text{ (if A is an atom)} \\
(A \Box B)^* & \rightarrow (A^* \Box B^*), \Box \in \{\wedge, \vee, \Rightarrow\} \\
(\exists x.A)^* & \rightarrow \exists x.A^* \\
(\forall x.A)^* & \rightarrow \forall x.\neg\neg A^*
\end{array}
$$

The end-sequent of $\psi_3$ is $\vdash \neg(\tilde{F}_1 \wedge ... \wedge \tilde{F}_k)$, where each $\tilde{F}_i$ is the negative translation of $F_i$. Note that $\vdash \neg\neg\neg A$ is **LJ**-equivalent to $\vdash \neg A$, so there is still only one negation on this end-sequent.

From the proof $\psi_3$, we can construct the proof $\psi_4$:

$$
\dfrac{\dfrac{\psi_3}{\vdash \neg(\tilde{F}_1 \wedge ... \wedge \tilde{F}_k)} \qquad \dfrac{\dfrac{\begin{array}{ccc} \Xi_1 & & \Xi_n \\ \vdash \tilde{F}_1 & ... & \vdash \tilde{F}_k \end{array}}{\vdash \tilde{F}_1 \wedge ... \wedge \tilde{F}_k} \wedge_r \times k}{\neg(\tilde{F}_1 \wedge ... \wedge \tilde{F}_k) \vdash} \neg_l}{\vdash} \mathit{cut}
$$

Note that the end-sequent of each derivation $\Xi_i$ is of the form:

$$
\vdash \neg\neg\forall x_1 .... \neg\neg\forall x_r . \neg\neg\neg(A_1^i \wedge ... \wedge A_{n_i}^i \wedge \neg B_1^i \wedge ... \wedge \neg B_{m_i}^i)
$$

And each $\Xi_i$ is:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A_1^i, ..., A_{n_i}^i, \neg B_1^i, ..., \neg B_{m_i}^i \vdash}{A_1^i \wedge ... \wedge A_{n_i}^i \wedge \neg B_1^i \wedge ... \wedge \neg B_{m_i}^i \vdash} \wedge_l \times (n_i + m_i - 1)}{\vdash \neg\neg\neg(A_1^i \wedge ... \wedge A_{n_i}^i \wedge \neg B_1^i \wedge ... \wedge \neg B_{m_i}^i)} \neg_r, \neg_l, \neg_r}{\vdots}}{\vdash \neg\neg\forall x_2 .... \neg\neg\forall x_r . \neg\neg\neg(A_1^i \wedge ... \wedge A_{n_i}^i \wedge \neg B_1^i \wedge ... \wedge \neg B_{m_i}^i)} \forall_r, \neg_l, \neg_r}{\vdash \neg\neg\forall x_1 .... \neg\neg\forall x_r . \neg\neg\neg(A_1^i \wedge ... \wedge A_{n_i}^i \wedge \neg B_1^i \wedge ... \wedge \neg B_{m_i}^i)} \forall_r, \neg_l, \neg_r
$$

So, we obtain an **LJ**-refutation of the clauses $A_1^i, ..., A_{n_i}^i, \neg B_1^i, ..., \neg B_{m_i}^i \vdash$ for every $i$, which are exactly the elements of $CL^I(\varphi)$. $\Box$

Ok, the intuitionistic clause set is refutable, but we don't really want to go through all this trouble for constructing a refutation every time, right? So we'll define our own new resolution calculus for that, one able to deal with negated atoms, containing negation rules and with sequents with at most one formula on the right.

**Definition 20 ($\mathbf{R}^{\neg}$).** *The $\mathbf{R}^{\neg}$ calculus is a resolution calculus with the following rules:*

$$\frac{\Gamma \vdash A \quad \Gamma', A' \vdash \Delta}{\Gamma\sigma, \Gamma'\sigma \vdash \Delta\sigma}\ R \qquad \frac{\Gamma \vdash \neg A \quad \Gamma', \neg A' \vdash \Delta}{\Gamma\sigma, \Gamma'\sigma \vdash \Delta\sigma}\ R\neg$$

$$\frac{A, A' \vdash \Delta}{A\sigma \vdash \Delta\sigma}\ C \qquad \frac{\neg A, \neg A' \vdash \Delta}{\neg A\sigma \vdash \Delta\sigma}\ C\neg$$

$$\frac{\Gamma, A \vdash}{\Gamma \vdash \neg A}\ \neg_r \qquad \frac{\Gamma \vdash A}{\Gamma, \neg A \vdash}\ \neg_l$$

*Where $\Delta$ is a multi-set with at most one formula and $\sigma$ is the most general unifier of $A$ and $A'$.*

Given this resolution calculus, is it the case that we can actually find a resolution refutation of a set of intuitionistic clauses with it? Yes, it is. To prove this we need to show that every **LJ**-refutation (i.e., **LJ**-proofs having $\vdash$ as the end-sequent) of intuitionistic clauses $\mathcal{S}$ can be transformed into a refutation of the same (or instances of) $\mathcal{S}$ with rules corresponding to the rules of $\mathbf{R}^{\neg}$. Let's do that step by step.

First step is to show that, in a refutation, we can restrict the cuts to cuts on atoms and negated atoms.

**Lemma 5.** *If $\varphi$ is an **LJ**-refutation of a set of intuitionistic clauses (Definition 18) $\mathcal{S}$ and $\varphi'$ is a normal form of $\varphi$ with respect to reductive cut-elimination, then any cut-formula in $\varphi'$ is either an atom or a negated atom.*

*Proof.* Assume, for the sake of contradiction, that $\varphi'$ contains a cut whose cut-formula $F$ is neither an atom nor a negated atom. Since the axioms of $\varphi'$ contain only atoms or negated atoms, it must be the case that the left and right occurrences of $F$ in this cut are introduced, respectively, by inferences $\rho_l$ and $\rho_r$ occurring somewhere in $\varphi'$. Two cases can be distinguished:

1. Both $\rho_l$ and $\rho_r$ occur immediately above the cut: in this case, either a grade reduction rule (see Definition 9) can be applied, if both $\rho_l$ and $\rho_r$ are logical inferences, or an elimination rule over weakening (see Definition 7), if at least one of them is a weakening.

2. At least one of $\rho_l$ and $\rho_r$ does not occur immediately above the cut: in this case, a rank reduction rule (see Definition 8) can be applied.

In both cases, the assumption contradicts the fact that $\varphi'$ is in normal form. Therefore, it must be the case that all cut-formulas in $\varphi'$ are either atoms or negated atoms. □

Now that the cut formulas are restricted, we show that the actual inference rule we need are only $\neg_l$, $\neg_r$, $cut$ and $c_l$.

**Lemma 6.** *If $\varphi$ is an **LJ**-refutation of a set of intuitionistic clauses $\mathcal{S}$ and $\varphi'$ is a normal form of $\varphi$ with respect to reductive cut-elimination, then the only inference rules used in $\varphi'$ are $\neg_l$, $\neg_r$, cut and left contraction.*

*Proof.* Assume, for the sake of contradiction, that there is an inference $\rho$ in $\varphi'$ that is neither a $\neg_l$, nor a $\neg_r$, nor a $cut$ inference, nor a left contraction, and let $F$ be its main formula. Since $\varphi'$ is an **LJ**-refutation, its end-sequent is empty. Hence, $F$ must be the ancestor of a cut-formula, and since $F$ is neither an atom nor a negated atom, its descendant cut-formula is also neither an atom nor a negated atom. However, this contradicts Lemma 5, according to which any cut-formula in $\varphi'$ must be either an atom or a negated atom. Therefore, inferences that are neither $\neg_l$, nor $\neg_r$, nor $cut$, nor left contraction cannot occur in $\varphi'$. $\square$

All logical inferences that are neither $\neg_l$ nor $\neg_r$ disappear when $\varphi$ is rewritten to $\varphi'$ due to grade reduction rules. This is exemplified below for the conjunction case:

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma_1 \vdash A} \quad \overset{\varphi_2}{\Gamma_2 \vdash B}}{\Gamma_1, \Gamma_2 \vdash A \wedge B}\,\wedge_r \quad \dfrac{\overset{\varphi_3}{\Gamma_3, A, B \vdash \Delta}}{\Gamma_3, A \wedge B \vdash \Delta}\,\wedge_l}{\Gamma_1, \Gamma_2, \Gamma_3 \vdash \Delta}\,cut \quad \Rightarrow \quad \dfrac{\overset{\varphi_1}{\Gamma_1 \vdash A} \quad \dfrac{\overset{\varphi_2}{\Gamma_2 \vdash B} \quad \overset{\varphi_3}{\Gamma_3, A, B \vdash \Delta}}{\Gamma_2, \Gamma_3, A \vdash \Delta}\,cut}{\Gamma_1, \Gamma_2, \Gamma_3 \vdash \Delta}\,cut$$

The same cannot be done with negation inferences. Observe that, as usual, the grade reduction for negation requires the cut-formulas to be introduced by $\neg_l$ and $\neg_r$:

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma_1, A \vdash}}{\Gamma_1 \vdash \neg A}\,\neg_r \quad \dfrac{\overset{\varphi_2}{\Gamma_2 \vdash A}}{\Gamma_2, \neg A \vdash}\,\neg_l}{\Gamma_1, \Gamma_2 \vdash}\,cut \quad \Rightarrow \quad \dfrac{\overset{\varphi_1}{\Gamma_1, A \vdash} \quad \overset{\varphi_2}{\Gamma_2 \vdash A}}{\Gamma_1, \Gamma_2 \vdash}\,cut$$

However, since the intuitionistic clause can have negated atoms, it may be the case that, (at least) one of the cut-formulas was directly introduced by an axiom, as shown in the example proof below:

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma_1, A \vdash}}{\Gamma_1 \vdash \neg A}\,\neg_r \quad \dfrac{}{\Gamma_2, \neg A \vdash}}{\Gamma_1, \Gamma_2 \vdash}\,cut$$

In such cases, the grade reduction rule for negation cannot be applied, and hence the negation inference and the cut with a negated atomic formula remain.

So far we have shown that an **LJ**-refutation can be reduced to one with cuts only on literals and inferences $\neg_l$, $\neg_r$, $cut$ and $c_l$. But do the axioms remain the same? The next lemma proves that these transformations to normal form alter the axioms only by substituting some variables.

**Lemma 7.** *If $\varphi$ is an **LJ**-refutation of an unsatisfiable set of intuitionistic clauses $\mathcal{S}$ and $\varphi'$ is a normal form of $\varphi$ with respect to reductive cut-elimination, then the axioms of $\varphi'$ are instances of the clauses of $\mathcal{S}$.*

*Proof.* On applying the rewriting rules for cut-elimination, the initial sequents are not altered, except for the quantifier grade reduction rules, shown below:

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma_1 \vdash F(\alpha)}}{\Gamma_1 \vdash \forall x.F(x)}\,\forall_r \qquad \dfrac{\overset{\varphi_2}{\Gamma_2, F(t) \vdash \Delta}}{\Gamma_2, \forall x.F(x) \vdash \Delta}\,\forall_l}{\Gamma_1, \Gamma_2 \vdash \Delta}\,cut \quad \Rightarrow \quad \dfrac{\overset{\varphi_1\{\alpha \leftarrow t\}}{\Gamma_1 \vdash F(t)} \qquad \overset{\varphi_2}{\Gamma_2, F(t) \vdash \Delta}}{\Gamma_1, \Gamma_2 \vdash \Delta}\,cut$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma_1 \vdash F(t)}}{\Gamma_1 \vdash \exists x.F(x)}\,\exists_r \qquad \dfrac{\overset{\varphi_2}{\Gamma_2, F(\alpha) \vdash \Delta}}{\Gamma_2, \exists x.F(x) \vdash \Delta}\,\exists_l}{\Gamma_1, \Gamma_2 \vdash \Delta}\,cut \quad \Rightarrow \quad \dfrac{\overset{\varphi_1}{\Gamma_1 \vdash F(t)} \qquad \overset{\varphi_2\{\alpha \leftarrow t\}}{\Gamma_2, F(t) \vdash \Delta}}{\Gamma_1, \Gamma_2 \vdash \Delta}\,cut$$

In order to eliminate the quantifier of the cut formula, the instantiated version of the formulas must be used. But this imposes no problem, since we can apply the substitution $\sigma = \{\alpha \leftarrow t\}$ on the proof.

If $X$ is an axiom clause in $\varphi_2$, $X\{\alpha \leftarrow t\}$ will be an axiom clause in $\varphi_2\{\alpha \leftarrow t\}$. Finally, $\varphi'$ will have axioms that are, in fact, instances of the clauses in $\mathcal{S}$.

□

Now what? After all these transformations, the axioms did not remain the same. They are actually instances of the original axioms. Can we get them back? Yes, the "lifting" lemma shows that. Intuitively, this lemma guarantees that if there is a resolution of instantiated terms, it is possible to transform ("lift") this into a resolution of the same terms with variables and a substitution.

**Definition 21.** *Let $X$ and $Y$ be clauses, then $X \leq_s Y$ if and only if there exists a substitution $\Theta$ with $X\Theta = Y$.*

**Lemma 8** (Lifting)**.** *Let $C$ and $D$ be clauses with $C \leq_s C'$ and $D \leq_s D'$. Assume that $C'$ and $D'$ have a $R^\neg$-resolvent $E'$. Then, there exists a $R^\neg$-resolvent $E$ of $C$ and $D$ such that $E \leq_s E'$.*

The proof of Lemma 8 is analogous to the one for the ordinary resolution calculus and will not be described here [23].

There, now we are ready to prove the completeness of our resolution calculus.

**Theorem 6** (Completeness of $\mathbf{R}^\neg$)**.** *Let $\mathcal{S}$ be an **LJ**-refutable set of intuitionistic clauses. Then $\mathcal{S}$ is $\mathbf{R}^\neg$-refutable.*

*Proof.* Let $\varphi$ be an **LJ**-refutation of $\mathcal{S}$. By applying Gentzen's proof-rewriting rules for cut-elimination exhaustively, $\varphi$ is rewritten to a normal form $\varphi'$, whose existence is guaranteed by the fact that Gentzen's proof-rewriting system is terminating (see Gentzen's Hauptsatz [15,16]). By Lemmas 5 and 6, $\varphi'$ has only $\neg_l$, $\neg_r$, $cut$ and left contraction inferences. As these inference rules correspond, respectively, to the rules $\neg_l$, $\neg_r$, $\{R, R\neg\}$ and $\{C, C\neg\}$ (without unification) of the $\mathbf{R}^\neg$ calculus, $\varphi'$ can be immediately converted to a ground $\mathbf{R}^\neg$-refutation $\delta$. By Lemma 7, the axioms of $\varphi'$ and hence also of $\delta$ are instances of the clauses in $\mathcal{S}$. Therefore, by the lifting lemma (Lemma 8), $\delta$ can be lifted into an $\mathbf{R}^\neg$-refutation $\delta^*$ of $\mathcal{S}$. □

In fact, due to the way the intuitionistic clause set is constructed, all the clauses have no formula on the right hand side. This means that the rule $\neg_l$ can be dropped from $\mathbf{R}^{\neg}$ and the clause sets used in our scenario will still be refutable. Also, the resolution rule on non-negated atoms could also be eliminated in our case, since we could always replace any (non-negated) resolution by negation inferences and negated resolution. But we will keep it to make the resolutions shorter.

That was a lot of work! But now we have a clause set and a way to refute it. We only need the projections and we are (more or less) done.

**Definition 22** (Intuitionistic Projection)**.** *An* intuitionistic projection *is built analogously to a usual projection, except that all the cut ancestors on the right side are negated and added to the left side.*

*Let $\varphi$ be a proof in **LJ** and $C \in CL^I(\varphi)$. Then the **LJ**-proof $\varphi(C)$ is called an* intuitionistic projection *and it is build inductively on the number of inferences of $\varphi$. Let $\nu$ be a node in $\varphi$ and $\varphi_\nu(C)$ the projection for clause $C$ until node $\nu$:*

1. *$\nu$ is a leaf: then $\varphi_\nu(C)$ is the derivation consisting of applying a negation rule ($\neg_l$) to the atoms which are cut-ancestors in order to shift them from the right to the left side (if there is a cut-ancestor on the right).*

2. *$\nu$ is the result of a unary rule $\xi$ applied to $\mu$:*

   2.a. *$\xi$ operates on a cut ancestor: $\varphi_\nu(C) = \varphi_\mu(C)$*

   2.b. *$\xi$ operates on an end sequent ancestor: $\varphi_\nu(C)$ is $\varphi_\mu(C)$ plus the application of $\xi$ to its end-sequent*

3. *$\nu$ is the result of a binary rule $\xi$ applied to $\mu_1$ and $\mu_2$:*

   3.a. *$\xi$ operates on a cut ancestor: $\varphi_\nu(C)$ is $\varphi_{\mu_i}(C)$ ($i$ depends on which branch $C$ is coming from) plus some weakenings to obtain formulas that were used in the other branch.*

   3.b. *$\xi$ operates on an end sequent ancestor: $\varphi_\nu(C)$ is the result of applying $\xi$ to the end-sequents of $\varphi_{\mu_1}(C)$ and $\varphi_{\mu_2}(C)$.*

I am sure you must be wondering now (at least I am!) why is it not a problem to merge branches of the proof in step 3.b. Because, you see, if these branches have a formula on the right, and the rule merging them is applied to formulas on the left, the conclusion sequent will have two formulas on the right. Since we are working still in **LJ**, this is not nice. Fortunately, we can guarantee that this never happens, exactly because the input is an **LJ** proof. You see, **LJ** does not have contraction of the right formula, right? More or less. The only case the right formula is duplicated is when a $\vee_l$ inference is applied (remember this is purely multiplicative **LJ**). Anyway, when the projections are being built and a binary inference needs to be applied, it is applied to premises that contain, at most, all the formulas in the premises of the original inference in the input proof. I mean, suppose $\xi$ is a binary inference, and the its occurrence in the input proof is on the left and its occurrence in a projection is on the right:

$$\frac{\Gamma_2 \vdash \Delta_2 \quad \Gamma_3 \vdash \Delta_3}{\Gamma_1 \vdash \Delta_1}\ \xi \qquad \frac{\Gamma_2', \Gamma_{c1} \vdash \Delta_2' \quad \Gamma_3', \Gamma_{c2} \vdash \Delta_3'}{\Gamma_1', \Gamma_{c1}, \Gamma_{c2} \vdash \Delta_1'}\ \xi$$

Then we know (by the definition of projections) that $\Gamma_i' \subseteq \Gamma_i$ and $\Delta_i' \subseteq \Delta_i$ for $i \in \{1, 2, 3\}$. $\Gamma_{c1}$ and $\Gamma_{c2}$ contains the atomic cut ancestors for the clause set. Since this rule application was valid on the left, it is certainly valid on the right.

So we have an **LJ**-refutation and **LJ**-projections. Sweet! All that is left is to put these things together. Now comes the problem. You didn't think it would be that easy, right? The problem arises because of the right-side formula of the input proof's end-sequent. Let's call it $R$. So, $R$ will be in the end-sequent of each of the projections, because that's how projections are (built). Imagine now that each **LJ**-projection corresponding to a clause set will be put on top of the **LJ**-resolution refutation. More specifically, it will replace the axiom of the refutation that corresponds to its clause set. Then we will have not only the literals there, but a lot of other formulas as well, belonging to the end-sequent. We've seen how this construction goes (Chapter 3), these formulas are merely copied from premise to conclusion in each rule of the resolution refutation. See where I'm getting? If the formula $R$ occurred in more than one projection, at one point there will be a sequent with more than one copy of $R$! These extra copies of formulas are usually contracted in the end, so that the original end-sequent is obtained. Indeed it also occurs with formulas on the left, but this is not a problem in **LJ**. The copying of $R$, which is on the right, is a problem.

What about the multi-conclusion calculus **LJ'**? You ask yourself. We will get there, let's try to define this for **LJ** first.

We cannot guarantee that $R$ will occur in only one projection. One the one hand, it might have suffered an implicit contraction via the $\vee_l$ rule. On the other hand, binary inferences operating on end-sequent ancestors generate redundant clause sets, since the operation $\otimes$ is the Cartesian product of clauses. So, the branch that used $R$, even if only one, might be used in several projections. It's a shame, but we need to restrict the class of **LJ**-proofs we can deal with this method.

**Definition 23** (**LJ**$^-$). ***LJ**$^-$ is the set of **LJ** proofs of end-sequents with no formula on the right side and no strong quantifiers.*

No formula on the right side in the end-sequent means no problem.

Note that this condition can actually be satisfied simply by applying a $\neg_l$ inference rule at the bottom of any proof containing a formula on the right side. Then this formula will occur negated on the left side and the method can thus be applied. This transformation actually represents proofs by contradiction: suppose we want to prove $\Gamma \vdash F$. A proof by contradiction will assume that $F$ is false and show that this does not hold, i.e., show $\Gamma, \neg F \vdash$.

Now, one might wonder whether this is a "trivial" class of proofs, i.e., a class in which intuitionistic and classical provability coincide. For non-skolemized proofs, the class is not trivial. Take for example the sequent $\neg \exists x. \forall y. (Px \to Py) \vdash$, which is classicaly provable but not intuitionistically. For skolemized proofs, though, the class is trivial, i.e., every skolemized sequent $\Gamma \vdash$ which is classicaly provable is also intuitionistically provable. This would mean that we could apply classical CERES to remove the cuts of this proof, but notice that it could result on a classical proof, which is not the aim here. Therefore, this remains an interesting class. Other interesting remarks on **LJ**$^-$ can be found in Chapter 11, Section 11.2.

With the restriction on the input proofs, we can define a method that results in an intuitionistic proof.

**Definition 24** (NACNF). *A proof is said to be in negated atomic cut normal form (NACNF) when all the cuts are on atoms or negated atoms.*

**Definition 25** (iCERES). *Let $\varphi$ be a proof in **LJ** of a sequent $S$, $CL^I(\varphi)$ its intuitionistic clause set (Definition 19) and $\pi_1, ..., \pi_n$ the intuitionistic projections (Definition 22) of the clauses of $CL^I(\varphi)$. By Theorems 5 and 6, there exists a grounded refutation $\varphi^*$ of $CL^I(\varphi)$. We define iCERES as the procedure of computing the elements $CL^I(\varphi)$, $\pi_1, ..., \pi_n$, and $\varphi^*$ from $\varphi$ and then merging (instances of) $\pi_1, ..., \pi_n$ with $\varphi^*$ in the following way:*

*Let $C_i$ be the clause of a leaf in $\varphi^*$. Then, $C_i$ is replaced by the projection $\pi_i$ (with the proper substitution of variables), which is actually a derivation of $C_i \circ S$. Moreover, the formulas of $S$ are propagated down the refutation.*

**Theorem 7.** *Let $\varphi$ be a proof in **LJ**$^-$ (Definition 23). Then iCERES, applied to $\varphi$, produces an intuitionistic negated atomic cut normal form.*

*Proof.* From Definition 25, we can observe that the result of applying iCERES to an **LJ**-proof consists of the resolution refutation in $R^{\neg}$ merged with the projections. These last elements have no cuts and are derivations in **LJ** by definition. The refutation has resolution rules on atoms and negated atoms, which will be the cuts on the final proof. Since the projections have no formula on the right side of their end sequents, and the resolution sequents have no more than one formula on the right side of each sequent, the final proof is an **LJ**-proof of an end-sequent equal to the one of $\varphi$ up to some contractions on the left. $\square$

It is true that we can transform every **LJ**-proof into an **LJ**$^-$-proof, so why can't we define the method for full **LJ**? The problem is that the end-sequent of the final proof is $\Gamma, \neg F \vdash$, and not the original $\Gamma \vdash F$. So the proof we obtain in the end is a proof of something else (very similar indeed, but not the same) and it is not trivial to transform that into a proof of $\Gamma \vdash F$.

## Example

Let's see this method working! We will apply it to the following **LJ**$^-$ proof[1]:

[1] Which is also an **LJK** proof, but we will use to illustrate the method.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{$I$}
\UnaryInfC{$P\alpha^{\star} \vdash P\alpha$}
\AxiomC{}
\RightLabel{$I$}
\UnaryInfC{$Pf\alpha \vdash Pf\alpha$}
\AxiomC{}
\RightLabel{$I$}
\UnaryInfC{$Pf^2\alpha \vdash Pf^2\alpha^{\star}$}
\RightLabel{$\to_l$}
\BinaryInfC{$Pf\alpha, Pf\alpha \to Pf^2\alpha \vdash Pf^2\alpha^{\star}$}
\RightLabel{$\to_l$}
\BinaryInfC{$P\alpha^{\star}, P\alpha \to Pf\alpha, Pf\alpha \to Pf^2\alpha \vdash Pf^2\alpha^{\star}$}
\doubleLine
\RightLabel{$\forall_l$}
\UnaryInfC{$P\alpha^{\star}, \forall x.(Px \to Pfx), \forall x.(Px \to Pfx) \vdash Pf^2\alpha^{\star}$}
\RightLabel{$\to_r$}
\UnaryInfC{$\forall x.(Px \to Pfx), \forall x.(Px \to Pfx) \vdash (P\alpha \to Pf^2\alpha)^{\star}$}
\RightLabel{$c_l$}
\UnaryInfC{$\forall x.(Px \to Pfx) \vdash (P\alpha \to Pf^2\alpha)^{\star}$}
\RightLabel{$\forall_r$}
\UnaryInfC{$\forall x.(Px \to Pfx) \vdash \forall x.(Px \to Pf^2x)^{\star}$}
\AxiomC{}
\RightLabel{$I$}
\UnaryInfC{$Pa \vdash Pa^{\star}$}
\AxiomC{}
\RightLabel{$I$}
\UnaryInfC{$Pf^2a^{\star} \vdash Pf^2a^{\star}$}
\AxiomC{}
\RightLabel{$I$}
\UnaryInfC{$Pf^4a^{\star} \vdash Pf^4a$}
\RightLabel{$\to_l$}
\BinaryInfC{$Pf^2a^{\star}, (Pf^2a \to Pf^4a)^{\star} \vdash Pf^4a$}
\RightLabel{$\to_l$}
\BinaryInfC{$Pa, (Pa \to Pf^2a)^{\star}, (Pf^2a \to Pf^4a)^{\star} \vdash Pf^4a$}
\doubleLine
\RightLabel{$\forall_l$}
\UnaryInfC{$Pa, \forall x.(Px \to Pf^2x)^{\star}, \forall x.(Px \to Pf^2x)^{\star} \vdash Pf^4a$}
\RightLabel{$c_l$}
\UnaryInfC{$Pa, \forall x.(Px \to Pf^2x)^{\star} \vdash Pf^4a$}
\RightLabel{$\exists_r$}
\UnaryInfC{$Pa, \forall x.(Px \to Pf^2x)^{\star} \vdash \exists z.Pf^4z$}
\RightLabel{$cut$}
\BinaryInfC{$Pa, \forall x.(Px \to Pfx) \vdash \exists z.Pf^4z$}
\RightLabel{$\neg_l$}
\UnaryInfC{$Pa, \forall x.(Px \to Pfx), \neg\exists z.Pf^4z \vdash$}
\end{prooftree}
$$

Note that the cut formulas and cut ancestors are superscribed with $\star$. By removing the inferences applied to end-sequent ancestors and merging the branches as was described on Definition 19, the intuitionistic clause set obtained is:

$$CL^I(\varphi) = \{P\alpha, \neg Pf^2\alpha \vdash \ ; \ \neg Pa \vdash \ ; \ Pf^4a \vdash\}$$

As was proved previously, there is a resolution refutation of this clause set on $R^{\neg}$:

$$
\begin{prooftree}
\AxiomC{$P\alpha, \neg Pf^2\alpha \vdash$}
\RightLabel{$\neg_r$}
\UnaryInfC{$\neg Pf^2\alpha \vdash \neg P\alpha$}
\AxiomC{$\neg Pa \vdash$}
\RightLabel{$R^{\neg}\{\alpha \leftarrow a\}$}
\BinaryInfC{$\neg Pf^2a \vdash$}
\AxiomC{$Pf^4a \vdash$}
\RightLabel{$\neg_r$}
\UnaryInfC{$\vdash \neg Pf^4a$}
\AxiomC{$P\alpha, \neg Pf^2\alpha \vdash$}
\RightLabel{$R^{\neg}\{\alpha \leftarrow f^2a\}$}
\BinaryInfC{$Pf^2a \vdash$}
\RightLabel{$\neg_r$}
\UnaryInfC{$\vdash \neg Pf^2a$}
\RightLabel{$R^{\neg}$}
\BinaryInfC{$\vdash$}
\end{prooftree}
$$

The projections of the three clauses of $CL^I$ are:

$\pi_1[\alpha]$ :

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{$I$}
\UnaryInfC{$P\alpha \vdash P\alpha$}
\AxiomC{}
\RightLabel{$I$}
\UnaryInfC{$Pf\alpha \vdash Pf\alpha$}
\AxiomC{}
\RightLabel{$I$}
\UnaryInfC{$Pf^2\alpha \vdash Pf^2\alpha$}
\RightLabel{$\neg_l$}
\UnaryInfC{$\neg Pf^2\alpha, Pf^2\alpha \vdash$}
\RightLabel{$\to_l$}
\BinaryInfC{$Pf\alpha, \neg Pf^2\alpha, Pf\alpha \to Pf^2\alpha \vdash$}
\RightLabel{$\to_l$}
\BinaryInfC{$P\alpha, \neg Pf^2\alpha, P\alpha \to Pf\alpha, Pf\alpha \to Pf^2\alpha \vdash$}
\RightLabel{$\forall_l \times 2$}
\UnaryInfC{$P\alpha, \neg Pf^2\alpha, \forall x.(Px \to Pfx), \forall x.(Px \to Pfx) \vdash$}
\RightLabel{$c_l$}
\UnaryInfC{$P\alpha, \neg Pf^2\alpha, \forall x.(Px \to Pfx) \vdash$}
\RightLabel{$w \times 2$}
\UnaryInfC{$P\alpha, \neg Pf^2\alpha, Pa, \forall x.(Px \to Pfx) \vdash \exists z.Pf^4z$}
\RightLabel{$\neg_l$}
\UnaryInfC{$P\alpha, \neg Pf^2\alpha, Pa, \forall x.(Px \to Pfx), \neg\exists z.Pf^4z \vdash$}
\end{prooftree}
$$

$\pi_2$ :

$$\dfrac{\dfrac{\dfrac{\overline{Pa \vdash Pa}\ I}{\neg Pa, Pa \vdash}\ \neg_l}{\neg Pa, Pa, \forall x.(Px \to Pfx) \vdash \exists z.Pf^4z}\ w \times 2}{\neg Pa, Pa, \forall x.(Px \to Pfx), \neg \exists z.Pf^4z \vdash}\ \neg_l$$

$\pi_3$ :

$$\dfrac{\dfrac{\dfrac{\overline{Pf^4a \vdash Pf^4a}\ I}{Pf^4a \vdash \exists z.Pf^4z}\ \exists_r}{Pf^4a, Pa, \forall x.(Px \to Pfx) \vdash \exists z.Pf^4z}\ w \times 2}{Pf^4a, Pa, \forall x.(Px \to Pfx), \neg \exists z.Pf^4z \vdash}\ \neg_l$$

By merging the appropriate instances of the projections on the resolution refutation, we obtain the final proof, depicted in Figure 6.1. The projections are colored accordingly. The projection $\pi_1$ used on the left side had $\alpha$ replaced with $a$ and the one used on the right side had $\alpha$ replaced with $f^2a$. Note that this proof is in NACNF, containing only cuts on atoms or negated atoms, and it is still a proof in **LJ**.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{$I$}
\UnaryInfC{$Pa \vdash Pa$}
\AxiomC{}
\RightLabel{$I$}
\UnaryInfC{$Pfa \vdash Pfa$}
\AxiomC{}
\RightLabel{$I$}
\UnaryInfC{$Pf^2a \vdash Pf^2a$}
\RightLabel{$\neg_l$}
\UnaryInfC{$\neg Pf^2a, Pf^2a \vdash$}
\RightLabel{$\to_l$}
\BinaryInfC{$Pfa, \neg Pf^2a, Pfa \to Pf^2a \vdash$}
\RightLabel{$\to_l$}
\BinaryInfC{$Pa, \neg Pf^2a, Pa \to Pfa, Pfa \to Pf^2a \vdash$}
\RightLabel{$\forall_l \times 2$}
\UnaryInfC{$Pa, \neg Pf^2a, \forall x.(Px \to Pfx), \forall x.(Px \to Pfx) \vdash$}
\RightLabel{$c_l$}
\UnaryInfC{$Pa, \neg Pf^2a, \forall x.(Px \to Pfx) \vdash$}
\RightLabel{$w \times 2$}
\UnaryInfC{$Pa, \neg Pf^2a, Pa, \forall x.(Px \to Pfx) \vdash \exists z.Pf^4z$}
\RightLabel{$\neg_l$}
\UnaryInfC{$Pa, \neg Pf^2a, Pa, \forall x.(Px \to Pfx), \neg\exists z.Pf^4z \vdash$}
\RightLabel{$\neg_r$}
\UnaryInfC{$\neg Pf^2a, Pa, \forall x.(Px \to Pfx), \neg\exists z.Pf^4z \vdash \neg Pa$}
\AxiomC{}
\RightLabel{$I$}
\UnaryInfC{$Pa \vdash Pa$}
\RightLabel{$\neg_l$}
\UnaryInfC{$\neg Pa, Pa \vdash$}
\RightLabel{$w \times 2$}
\UnaryInfC{$\neg Pa, Pa, \forall x.(Px \to Pfx) \vdash \exists z.Pf^4z$}
\RightLabel{$\neg_l$}
\UnaryInfC{$\neg Pa, Pa, \forall x.(Px \to Pfx), \neg\exists z.Pf^4z \vdash$}
\RightLabel{$cut$}
\BinaryInfC{$\neg Pf^2a, Pa, Pa, \forall x.(Px \to Pfx), \forall x.(Px \to Pfx), \neg\exists z.Pf^4z, \neg\exists z.Pf^4z \vdash$}
\AxiomC{}
\RightLabel{$I$}
\UnaryInfC{$Pf^4a \vdash Pf^4a$}
\RightLabel{$\exists_r$}
\UnaryInfC{$Pf^4a \vdash \exists z.Pf^4z$}
\RightLabel{$w \times 2$}
\UnaryInfC{$Pf^4a, Pa, \forall x.(Px \to Pfx) \vdash \exists z.Pf^4z$}
\RightLabel{$\neg_l$}
\UnaryInfC{$Pf^4a, Pa, \forall x.(Px \to Pfx), \neg\exists z.Pf^4z \vdash$}
\RightLabel{$\neg_r$}
\UnaryInfC{$Pa, \forall x.(Px \to Pfx), \neg\exists z.Pf^4z \vdash \neg Pf^4a$}
\AxiomC{}
\RightLabel{$I$}
\UnaryInfC{$Pf^2a \vdash Pf^2a$}
\AxiomC{}
\RightLabel{$I$}
\UnaryInfC{$Pf^3a \vdash Pf^3a$}
\AxiomC{}
\RightLabel{$I$}
\UnaryInfC{$Pf^4a \vdash Pf^4a$}
\RightLabel{$\neg_l$}
\UnaryInfC{$\neg Pf^4a, Pf^4a \vdash$}
\RightLabel{$\to_l$}
\BinaryInfC{$Pf^3a, \neg Pf^4a, Pf^3a \to Pf^4a \vdash$}
\RightLabel{$\to_l$}
\BinaryInfC{$Pf^2a, \neg Pf^4a, Pf^2a \to Pf^3a, Pf^3a \to Pf^4a \vdash$}
\RightLabel{$\forall_l \times 2$}
\UnaryInfC{$Pf^2a, \neg Pf^4a, \forall x.(Px \to Pfx), \forall x.(Px \to Pfx) \vdash$}
\RightLabel{$c_l$}
\UnaryInfC{$Pf^2a, \neg Pf^4a, \forall x.(Px \to Pfx) \vdash$}
\RightLabel{$w \times 2$}
\UnaryInfC{$Pf^2a, \neg Pf^4a, Pa, \forall x.(Px \to Pfx) \vdash \exists z.Pf^4z$}
\RightLabel{$\neg_l$}
\UnaryInfC{$Pf^2a, \neg Pf^4a, Pa, \forall x.(Px \to Pfx), \neg\exists z.Pf^4z \vdash$}
\RightLabel{$cut$}
\BinaryInfC{$Pf^2a, Pa, Pa, \forall x.(Px \to Pfx), \forall x.(Px \to Pfx), \neg\exists z.Pf^4z, \neg\exists z.Pf^4z \vdash$}
\RightLabel{$\neg_r$}
\UnaryInfC{$Pa, Pa, \forall x.(Px \to Pfx), \forall x.(Px \to Pfx), \neg\exists z.Pf^4z, \neg\exists z.Pf^4z \vdash \neg Pf^2a$}
\RightLabel{$cut$}
\BinaryInfC{$Pa, Pa, Pa, Pa, \forall x.(Px \to Pfx), \forall x.(Px \to Pfx), \forall x.(Px \to Pfx), \forall x.(Px \to Pfx), \neg\exists z.Pf^4z, \neg\exists z.Pf^4z, \neg\exists z.Pf^4z, \neg\exists z.Pf^4z \vdash$}
\RightLabel{$c \times 9$}
\UnaryInfC{$Pa, \forall x.(Px \to Pfx), \neg\exists z.Pf^4z \vdash$}
\end{prooftree}
$$

**Figure 6.1:** Final proof after applying the method to an LJ-proof of an end-sequent with no formula on the right.

## Trying to extend iCERES

Section 4 of [24] mentions a possible way for fixing NACNF's obtained when iCERES is applied to proofs whose end-sequents have a formula on the right. At the time, we believed that reductively eliminating the atomic cuts would yield an **LJ** proof. What happened there? We found a counter example.

Let $\varphi$ be the proof:

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$A \vdash A$}
\RightLabel{$\neg_l$}
\UnaryInfC{$\neg A, A \vdash$}
\RightLabel{$\neg_r$}
\UnaryInfC{$\neg A \vdash \neg A$}
\RightLabel{$\neg_l$}
\UnaryInfC{$\neg A, \neg\neg A \vdash$}
\AxiomC{}
\UnaryInfC{$A \vdash A$}
\RightLabel{$\neg_l$}
\UnaryInfC{$A, \neg A \vdash$}
\RightLabel{$\neg_r$}
\UnaryInfC{$A \vdash B, \neg\neg A$}
\RightLabel{$cut$}
\BinaryInfC{$\neg A, A \vdash B$}
\RightLabel{$\rightarrow_r$}
\UnaryInfC{$\neg A \vdash A \rightarrow B$}
\AxiomC{}
\UnaryInfC{$C \vdash C$}
\AxiomC{}
\UnaryInfC{$C \vdash C$}
\RightLabel{$cut$}
\BinaryInfC{$C \vdash C$}
\RightLabel{$\wedge_r$}
\BinaryInfC{$\neg A, C \vdash (A \rightarrow B) \wedge C$}
\end{prooftree}
$$

Then the intuitionistic characteristic clause set of $\varphi$ is:

$$CL^I(\varphi) = \{A, \neg C \vdash;\quad A, C \vdash;\quad \neg A, \neg C \vdash;\quad C, \neg A \vdash\}$$

And a possible $\mathbf{R}^{\neg}$ refutation is the following:

$$
\begin{prooftree}
\AxiomC{$A, C \vdash$}
\RightLabel{$\neg_r$}
\UnaryInfC{$A \vdash \neg C$}
\AxiomC{$A, \neg C \vdash$}
\RightLabel{$R$}
\BinaryInfC{$A, A \vdash$}
\RightLabel{$c_l$}
\UnaryInfC{$A \vdash$}
\RightLabel{$\neg_r$}
\UnaryInfC{$\vdash \neg A$}
\AxiomC{$C, \neg A \vdash$}
\RightLabel{$\neg_r$}
\UnaryInfC{$\neg A \vdash \neg C$}
\AxiomC{$\neg A, \neg C \vdash$}
\RightLabel{$R$}
\BinaryInfC{$\neg A, \neg A \vdash$}
\RightLabel{$c_r$}
\UnaryInfC{$\neg A \vdash$}
\RightLabel{$R$}
\BinaryInfC{$\vdash$}
\end{prooftree}
$$

The intuitionistic projections for each of these clauses are:

$\pi(A, \neg C \vdash)$ :

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$A \vdash A$}
\RightLabel{$\neg_l$}
\UnaryInfC{$\neg A, A \vdash$}
\doubleLine
\RightLabel{$w*$}
\UnaryInfC{$\neg A, A, A \vdash B$}
\RightLabel{$\rightarrow_r$}
\UnaryInfC{$\neg A, A \vdash A \rightarrow B$}
\AxiomC{}
\UnaryInfC{$C \vdash C$}
\RightLabel{$\neg_l$}
\UnaryInfC{$C, \neg C \vdash$}
\RightLabel{$w_r$}
\UnaryInfC{$C, \neg C \vdash C$}
\RightLabel{$\wedge_r$}
\BinaryInfC{$\neg A, C, A, \neg C \vdash (A \rightarrow B) \wedge C$}
\end{prooftree}
$$

$\pi(A, C \vdash)$ :

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$A \vdash A$}
\RightLabel{$\neg_l$}
\UnaryInfC{$\neg A, A \vdash$}
\doubleLine
\RightLabel{$w*$}
\UnaryInfC{$\neg A, A, A \vdash B$}
\RightLabel{$\rightarrow_r$}
\UnaryInfC{$\neg A, A \vdash A \rightarrow B$}
\AxiomC{}
\UnaryInfC{$C \vdash C$}
\RightLabel{$\wedge_r$}
\BinaryInfC{$\neg A, A, C \vdash (A \rightarrow B) \wedge C$}
\RightLabel{$w_l$}
\UnaryInfC{$\neg A, C, A, C \vdash (A \rightarrow B) \wedge C$}
\end{prooftree}
$$

$$\pi(\neg A, \neg C \vdash):$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}}{\neg A, A \vdash}\,\neg_l}{\neg A, A \vdash B}\,w_r}{\neg A \vdash A \to B}\,\to_r \quad \dfrac{\dfrac{\dfrac{}{C \vdash C}}{C, \neg C \vdash}\,\neg_l}{C, \neg C \vdash C}\,w_r}{\dfrac{C, \neg A, \neg C \vdash (A \to B) \wedge C}{\neg A, C, \neg A, \neg C \vdash (A \to B) \wedge C}\,w_l}\,\wedge_r$$

$$\pi(C, \neg A \vdash):$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}}{\neg A, A \vdash}\,\neg_l}{\neg A, A \vdash B}\,w_r}{\neg A \vdash A \to B}\,\to_r \quad \dfrac{}{C \vdash C}}{\dfrac{C, \neg A \vdash (A \to B) \wedge C}{\neg A, C, C, \neg A \vdash (A \to B) \wedge C}\,w_l, w_l}\,\wedge_r$$

The ACNF is the following proof:

$$\dfrac{\dfrac{\varphi_l \quad \varphi_r}{\neg A, \neg A, \neg A, \neg A, C, C, C, C \vdash (A \to B) \wedge C, (A \to B) \wedge C, (A \to B) \wedge C, (A \to B) \wedge C}\,cut}{\neg A, C \vdash (A \to B) \wedge C}\,c^*$$

Where $\varphi_l$ is:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}}{\neg A, A \vdash}\,\neg_l}{\neg A, A, A \vdash B}\,w*}{\neg A, A \vdash A \to B}\,\to_r \quad \dfrac{}{C \vdash C}}{\neg A, A, C \vdash (A \to B) \wedge C}\,\wedge_r}{\neg A, C, A, C \vdash (A \to B) \wedge C}\,w_l}{\neg A, C, A \vdash (A \to B) \wedge C, \neg C}\,\neg_r^\dagger \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}}{\neg A, A \vdash}\,\neg_l}{\neg A, A, A \vdash B}\,w*}{\neg A, A \vdash A \to B}\,\to_r \quad \dfrac{\dfrac{\dfrac{}{C \vdash C}}{C, \neg C \vdash}\,\neg_l}{C, \neg C \vdash C}\,w_r}{\neg A, C, A, \neg C \vdash (A \to B) \wedge C}\,\wedge_r}{\dfrac{\dfrac{\neg A, \neg A, C, C, A, A \vdash (A \to B) \wedge C, (A \to B) \wedge C}{\neg A, \neg A, C, C, A \vdash (A \to B) \wedge C, (A \to B) \wedge C}\,c_l}{\neg A, \neg A, C, C \vdash (A \to B) \wedge C, (A \to B) \wedge C, \neg A}\,\neg_r^\dagger}\,cut$$

and $\varphi_r$ is:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}}{\neg A, A \vdash}\,\neg_l}{\neg A, A \vdash B}\,w_r}{\neg A \vdash A \to B}\,\to_r \quad \dfrac{}{C \vdash C}}{C, \neg A \vdash (A \to B) \wedge C}\,\wedge_r}{\neg A, C, C, \neg A \vdash (A \to B) \wedge C}\,w_l, w_l}{\neg A, C, \neg A \vdash (A \to B) \wedge C, \neg C}\,\neg_r^\dagger \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}}{\neg A, A \vdash}\,\neg_l}{\neg A, A \vdash B}\,w_r}{\neg A \vdash A \to B}\,\to_r \quad \dfrac{\dfrac{\dfrac{}{C \vdash C}}{C, \neg C \vdash}\,\neg_l}{C, \neg C \vdash C}\,w_r}{C, \neg A, \neg C \vdash (A \to B) \wedge C}\,\wedge_r}{\neg A, C, \neg A, \neg C \vdash (A \to B) \wedge C}\,w_l}{\dfrac{\neg A, \neg A, C, C, \neg A, \neg A \vdash (A \to B) \wedge C, (A \to B) \wedge C}{\neg A, \neg A, C, C, \neg A \vdash (A \to B) \wedge C, (A \to B) \wedge C}\,c_r}\,cut$$

If you try reductively eliminating the cuts in this proof (belive me, I tried), you will see that the cuts on $C$ will go away with no problems, but eliminating the cut on $A$ will result on the following situation:

$$\dfrac{\dfrac{\vdots}{\neg A, \neg A, C \vdash (A \to B) \wedge C, (A \to B) \wedge C, \neg A}\ \neg_r^{\dagger} \qquad \dfrac{\dfrac{\vdots}{\neg A, A \vdash B}}{\neg A \vdash A \to B}\ \to_r}{\neg A, \neg A, C \vdash (A \to B) \wedge C, (A \to B) \wedge C, A \to B}\ cut$$

And we already know, by Lemma 2, that this permutation is not possible. The resulting proof without cuts will have violations in the $\to_r$ inferences and, therefore, will be classical.

### Summary

iCERES:

- Input: **LJ** proof with cuts (end-sequent has no formula on the right)
- Output: **LJ** proof with literal cuts
- Resolution: $\mathbf{R}^{\neg}$

### In LJ'

I am sure you are wondering now what happens if we use **LJ'**, since there multiple formulas on the right side are allowed. Well, there are two ways of "using" **LJ'**: (1) the input and output are **LJ'** proofs or (2) the input is an **LJ**-proof and the output is an **LJ'**-proof. But both cases present the same problem.

When the end-sequent of the input proof contains one (or more) formula(s) on the right, it is unavoidable that multiple-succedent sequents appear on the construction of the NACNF. In principle this would not be an issue if we allow the output to be an **LJ'**-proof, but if you think a little about it, remember that the resolution proof contains negation inferences. It can contain, in particular $\neg_r$ inferences. And remember also that this is one of the single-conclusion inferences in **LJ'**. So there, we cannot unrestrictedly allow multiple conclusion sequents in the NACNF. Unless no $\neg_r$ inferences occur, i.e., unless there are no negated atoms in the clause set, i.e., unless there were no structural violations of the atoms in the projections, but then, well, then we can just apply CERES.

CHAPTER 7

# Negative resolution CERES

From Chapter 6 we saw that negating atoms turned out not to be the greatest of ideas. In either case, **LJ** or **LJ'**, having negation rules, specially $\neg_r$, in the resolution part of the (N)ACNF causes problems. In Chapter 4 we showed that the resolution part, with only cuts and contractions, is fine if we consider a multi-conclusion calculus. The problem with not negating the atoms is on the projections.

The method presented now is based on a simple intuition: if the formulas violating the **LJ'** inferences are atomic cut ancestors, what happens when we actually get rid of these atomic cuts? Of course the proof cannot be fixed that easily, but it will work for the same class of proofs as iCERES with the advantage that the regular resolution calculus can be used.

## 7.1 Counter-example

Since I already gave up the ending, let's go ahead and show an example that reductive cut-elimination does not work to transform †-inferences into intuitionistic inferences. Conveniently, this is the same proof as used before to show how reductive cut-elimination fails in the attempt to extend iCERES.

Let $\varphi$ be the proof:

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$A \vdash A$}
\RightLabel{$\neg_l$}
\UnaryInfC{$\neg A, A \vdash$}
\RightLabel{$\neg_r$}
\UnaryInfC{$\neg A \vdash \neg A$}
\RightLabel{$\neg_l$}
\UnaryInfC{$\neg A, \neg\neg A \vdash$}
\AxiomC{}
\UnaryInfC{$A \vdash A$}
\RightLabel{$\neg_l$}
\UnaryInfC{$A, \neg A \vdash$}
\RightLabel{$\neg_r$}
\UnaryInfC{$A \vdash B, \neg\neg A$}
\RightLabel{$cut$}
\BinaryInfC{$\neg A, A \vdash B$}
\RightLabel{$\rightarrow_r$}
\UnaryInfC{$\neg A \vdash A \rightarrow B$}
\AxiomC{}
\UnaryInfC{$C \vdash C$}
\AxiomC{}
\UnaryInfC{$C \vdash C$}
\RightLabel{$cut$}
\BinaryInfC{$C \vdash C$}
\RightLabel{$\wedge_r$}
\BinaryInfC{$\neg A, C \vdash (A \rightarrow B) \wedge C$}
\end{prooftree}
$$

Then the characteristic clause set of $\varphi$ is:

$$CL(\varphi) = \{A \vdash C;\quad A, C \vdash;\quad \vdash A, C;\quad C \vdash A\}$$

And (one of) its refutation:

$$\dfrac{\dfrac{\dfrac{\vdash A, C \quad C \vdash A}{\vdash A, A}\,R}{\vdash A}\,c_r \qquad \dfrac{\dfrac{A, C \vdash \quad A \vdash C}{A, A \vdash}\,R}{A \vdash}\,c_l}{\vdash}\,R$$

The projections for each of these clauses are:

$\pi(A \vdash C)$ :

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}}{\neg A, A \vdash}\,\neg_l}{\neg A, A, A \vdash B}\,w*}{\neg A, A \vdash A \to B}\,\to_r \qquad \dfrac{\dfrac{}{C \vdash C}}{C \vdash C, C}\,w_r}{\neg A, C, A \vdash (A \to B) \wedge C, C}\,\wedge_r$$

$\pi(A, C \vdash)$ :

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}}{\neg A, A \vdash}\,\neg_l}{\neg A, A, A \vdash B}\,w*}{\neg A, A \vdash A \to B}\,\to_r \qquad \dfrac{}{C \vdash C}}{\neg A, A, C \vdash (A \to B) \wedge C}\,\wedge_r}{\neg A, C, A, C \vdash (A \to B) \wedge C}\,w_l$$

$\pi(\vdash A, C)$ :

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}}{A \vdash B, A}\,w_r}{\vdash A \to B, A}\,\to_r^\dagger \qquad \dfrac{\dfrac{}{C \vdash C}}{C \vdash C, C}\,w_r}{C \vdash A, C, (A \to B) \wedge C}\,\wedge_r}{\neg A, C \vdash A, C, (A \to B) \wedge C}\,w_l$$

$\pi(C \vdash A)$ :

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}}{A \vdash A, B}\,w_r}{\vdash A, A \to B}\,\to_r^\dagger \qquad \dfrac{}{C \vdash C}}{C \vdash A, (A \to B) \wedge C}\,\wedge_r}{\neg A, C, C \vdash A, (A \to B) \wedge C}\,w_l, w_l$$

Note that there are $\to_r^\dagger$ inferences in the final proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}}{A \vdash B, A}\,w_r}{\vdash A \to B, A}\,\to_r^\dagger \quad \dfrac{\dfrac{}{C \vdash C}}{C \vdash C, C}\,w_r}{C \vdash A, C, (A \to B) \wedge C}\,\wedge_r}{\neg A, C \vdash A, C, (A \to B) \wedge C}\,w_l \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}}{A \vdash A, B}\,w_r}{\vdash A, A \to B}\,\to_r^\dagger \quad \dfrac{}{C \vdash C}}{C \vdash A, (A \to B) \wedge C}\,\wedge_r}{\neg A, C, C \vdash A, (A \to B) \wedge C}\,w_l, w_l}{\neg A, \neg A, C, C \vdash A, A, (A \to B) \wedge C, (A \to B) \wedge C}\,cut}{\neg A, \neg A, C, C \vdash A, (A \to B) \wedge C, (A \to B) \wedge C}\,c_r \qquad \varphi_i}{\neg A, \neg A, \neg A, \neg A, C, C, C, C \vdash (A \to B) \wedge C, (A \to B) \wedge C, (A \to B) \wedge C, (A \to B) \wedge C}\,cut}{\neg A, C \vdash (A \to B) \wedge C}\,c*$$

Where $\varphi_i$ is:

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$A \vdash A$}
\RightLabel{$\neg_l$}
\UnaryInfC{$\neg A, A \vdash$}
\doubleLine
\RightLabel{$w*$}
\UnaryInfC{$\neg A, A, A \vdash B$}
\RightLabel{$\to_r$}
\UnaryInfC{$\neg A, A \vdash A \to B$}
\AxiomC{}
\UnaryInfC{$C \vdash C$}
\RightLabel{$\wedge_r$}
\BinaryInfC{$\neg A, A, C \vdash (A \to B) \wedge C$}
\RightLabel{$w_l$}
\UnaryInfC{$\neg A, C, A, C \vdash (A \to B) \wedge C$}
\AxiomC{}
\UnaryInfC{$A \vdash A$}
\RightLabel{$\neg_l$}
\UnaryInfC{$\neg A, A \vdash$}
\doubleLine
\RightLabel{$w*$}
\UnaryInfC{$\neg A, A, A \vdash B$}
\RightLabel{$\to_r$}
\UnaryInfC{$\neg A, A \vdash A \to B$}
\AxiomC{}
\UnaryInfC{$C \vdash C$}
\RightLabel{$w_r$}
\UnaryInfC{$C \vdash C, C$}
\RightLabel{$\wedge_r$}
\BinaryInfC{$\neg A, C, A \vdash (A \to B) \wedge C, C$}
\RightLabel{$cut$}
\BinaryInfC{$\neg A, \neg A, C, C, A, A \vdash (A \to B) \wedge C, (A \to B) \wedge C$}
\RightLabel{$c_l$}
\UnaryInfC{$\neg A, \neg A, C, C, A \vdash (A \to B) \wedge C, (A \to B) \wedge C$}
\end{prooftree}
$$

Note that the problem is only with the cut on $A$, which is the lower most. So what happens when we try to eliminate this atomic cut using reductive cut-elimination? Well, long story short, after applying some rank reduction rules, we get to the following point (I am just showing the interesting part of the proof):

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$A \vdash A$}
\RightLabel{$w_r$}
\UnaryInfC{$A \vdash B, A$}
\RightLabel{$\to_r^{\dagger}$}
\UnaryInfC{$\vdash A \to B, A$}
\AxiomC{$\varphi_i$}
\RightLabel{$cut$}
\BinaryInfC{$\neg A, \neg A, C \vdash (A \to B) \wedge C, (A \to B) \wedge C, A \to B$}
\noLine
\UnaryInfC{$\vdots$}
\end{prooftree}
$$

And it is finally time to permute the cut over the $\to_r^{\dagger}$ inference. "Will it fix the proof? What happens then?", you ask eagerly. This happens:

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$A \vdash A$}
\RightLabel{$w_r$}
\UnaryInfC{$A \vdash B, A$}
\AxiomC{$\varphi_i$}
\RightLabel{$cut$}
\BinaryInfC{$\neg A, \neg A, C, A \vdash (A \to B) \wedge C, (A \to B) \wedge C, B$}
\RightLabel{$\to_r^{K}$}
\UnaryInfC{$\neg A, \neg A, C \vdash (A \to B) \wedge C, (A \to B) \wedge C, A \to B$}
\noLine
\UnaryInfC{$\vdots$}
\end{prooftree}
$$

Ah now that's a shame. Not only did it not fix the proof, but it made it worse! The $\to_r^K$ rule now is actually classical, since the extra formulas on the right are not cut ancestors anymore, but end-sequent ancestors. We certainly didn't want that.

## 7.2 Analysis

When *does* it work then? Let's analyse abstractly what happened in this example. After performing the rank reduction rules with the cut containing one of the violating atoms, we get to the following point:

$$\dfrac{\dfrac{\Gamma, P \vdash \Delta, Q, A}{\Gamma \vdash \Delta, P \to Q, A}\to_r^\dagger \quad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta', P \to Q}\,cut$$

Remember that, this being a †-inference, $\Delta$ contains other atomic cut ancestors, but $\Delta'$ contains arbitrary formulas. When we try to apply $\to_r^\dagger$ before cutting, these formulas in $\Delta'$ will make this inference no longer a † one a classical one, thus resulting in an **LK** proof.

But if $\Delta'$ is empty, we can safely permute the cut over $\to_r^\dagger$ and reduce the number of extra atomic cut-ancestors on the right side by one. Look:

$$\dfrac{\dfrac{\Gamma, P \vdash \Delta, Q, A}{\Gamma \vdash \Delta, P \to Q, A}\to_r^\dagger \quad \Gamma', A \vdash}{\Gamma, \Gamma' \vdash \Delta, P \to Q}\,cut \quad \rightsquigarrow \quad \dfrac{\dfrac{\Gamma, P \vdash \Delta, Q, A \quad \Gamma', A \vdash}{\Gamma, \Gamma', P \vdash \Delta, Q}\,cut}{\Gamma, \Gamma' \vdash \Delta, P \to Q}\to_r^\dagger$$

If we can permute the other cuts on atoms in $\Delta$ over $\to_r^\dagger$ in this way, we will eventually get to a point where there will be no extra formula on the right, and the proof will be again in **LJ**. But can we always permute the cut? No.

**Theorem 8.** *A cut rule in **LJ'** permutes over every rule on its left and right branches, except †-rules (with or without a right context).*

*Proof.* Since all rules which are not †-rules are the same as in **LK**, the permutation holds. The problem with †-rules is that they require the right context to contain only atomic cut ancestors. When such permutation is done:

$$\dfrac{\Gamma \vdash \Delta, A \quad \dfrac{A, \Gamma', P \vdash \Delta', Q}{A, \Gamma' \vdash \Delta', P \to Q}\to_r^\dagger}{\Gamma, \Gamma' \vdash \Delta, \Delta', P \to Q}\,cut \quad \rightsquigarrow \quad \dfrac{\dfrac{\Gamma \vdash \Delta, A \quad A, \Gamma', P \vdash \Delta', Q}{\Gamma, \Gamma', P \vdash \Delta, \Delta', Q}\,cut}{\Gamma, \Gamma' \vdash \Delta, \Delta', P \to Q}\to_r^?$$

$$\dfrac{\dfrac{\Gamma, P \vdash \Delta, Q, A}{\Gamma \vdash \Delta, P \to Q, A}\to_r^\dagger \quad A, \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta', P \to Q}\,cut \quad \rightsquigarrow \quad \dfrac{\dfrac{\Gamma, P \vdash \Delta, Q, A \quad A, \Gamma' \vdash \Delta'}{\Gamma, \Gamma', P \vdash \Delta, \Delta', Q}\,cut}{\Gamma, \Gamma' \vdash \Delta, \Delta', P \to Q}\to_r^?$$

It might be the case that the right context $\Delta, \Delta'$ contains other formulas. Note that this happens independently of whether the †-inference is on the left or right branch. □

The non-permutation of cut over †-inferences on the left branch is not really a problem, since we only need to permute it up until this point. And then, being $\Delta'$ empty, we can permute the cut over the †-inference, reducing the number of atomic cut-ancestors on its right context by one.

The non-permutation over these inferences on the right, though, is a problem. In order to fix the proof using the operation shown before, the cut must have as a premise a sequent with an empty right side. Now, how do you get to that? Since we cannot freely permute the cut until reaching such a sequent (if it exists), we will require that the end-sequent of a projection contains no end-sequent ancestors (of the input proof) on the right side.

Given the projections as in Definition 12, the only case in which its end-sequent contains only atomic cut ancestors on the right is if the end-sequent of the input proof contains no formula on the right. This is exactly the same class for which iCERES (Definition 25) works. Using $\mathbf{LJ}^-$ proofs gives us a chance of not having end-sequent ancestors on the right side of the sequent. What about cut ancestors? We can guarantee that they will not occur on the right side if we use a refinement of resolution called *negative resolution*. For this we define the following:

**Definition 26** (Negative clause)**.** *A clause is* negative *if it contains atoms only on the left side of its sequent representation.*

**Definition 27** (Negative projection)**.** *A projection of a negative clause is called a* negative projection.

One important thing to note about negative projections is that they don't have †-inferences with formulas in the right context. Therefore these projections are intuitionistic by definition.

**Theorem 9.** *All †-inferences occurring in negative projections have an empty right context.*

*Proof.* The proof is obvious from Lemma 4 and the definition of negative projections. □

Now we can explain what negative resolution is all about.

**Definition 28** (Negative resolution)**.** *A resolution refutation is called* negative *if every resolution rule is applied to at least one negative clause.*

It is known that negative resolution is complete [23]. Therefore:

**Theorem 10.** *If a clause set is refutable, then it contains at least one negative clause.*

*Proof.* Assume that there is no negative clause. Then there exists an obvious model satisfying the clause set, in which every atom is interpreted as true (remember that ? $\vdash T$ is valid). Therefore, the clause set is not refutable, which is a contradiction. □

**Lemma 9.** *If CERES is applied to an* $\boldsymbol{LJ}^-$ *proof using negative resolution, then †-inferences can be transformed into intuitionistic inferences by permuting the atomic cuts as in the rank reduction rules (Definition 8).*

*Proof.* The proof is an induction on the number of atomic cut ancestors in the right context of each †-inference.

Let $\rho$ be a †-inference occurring in the proof with right context $\Delta^\dagger = \Delta, A$, $|\Delta^\dagger| = n$. Assume without loss of generality that there are no other †-inferences between $\rho$ and the cut on $A$. Then by Theorem 8 the cut on $A$ can be permuted until $\rho$.

If $n = 0$, then this †-inference is already on its intuitionistic form and nothing needs to be done[1].

Induction hypothesis: if $n = k$, then the proof can be fixed.

[1] Also note that in this case there are no other †-inferences with formulas on the right context somewhere above.

Let $n = k+1$. Given that negative resolution was used, the other premise of the cut contains no formulas on the right side. Therefore, the following operation can be performed:

$$\dfrac{\dfrac{\Gamma, P \vdash Q, \Delta, \color{red}{A}}{\Gamma \vdash P \to Q, \Delta, \color{red}{A}} \to_r^{\dagger} \quad \Gamma', A \vdash}{\Gamma, \Gamma' \vdash \Delta, P \to Q} \; cut \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\Gamma, P \vdash \Delta, Q, \color{red}{A} \quad \Gamma', A \vdash}{\Gamma, \Gamma', P \vdash \Delta, Q} \; cut}{\Gamma, \Gamma' \vdash \Delta, P \to Q} \to_r^{\dagger}$$

The resulting proof has $n = k$ and thus we can apply the induction hypothesis. □

**Theorem 11** (Rank reducing)**.** *Let $\varphi$ be the ACNF resulting from applying CERES with negative resolution to an **LJ**$^-$ proof. Then eliminating the atomic cuts from $\varphi$ using rank-reduction rules yields an **LJ'** proof.*

*Proof.* The result follows from Lemma 9 and the fact that, if classical inferences occur in the ACNF, then they are †-inferences. □

Why can't we just require that negative projections have no formulas on the right side? This seems to make the method more general, right? Well, we thought so for a very long (too long!) time. But there is a simple counter argument for that. Suppose this is your negative resolution:

$$\dfrac{\vdash A \quad \dfrac{A \vdash B \quad B \vdash}{A \vdash} \; R}{\vdash} \; R$$

This contains only one negative clause, namely $B \vdash$, and consequently one negative projection. So with this relaxed requirement, the projection (or o-projection, Definition 13) of $B \vdash$ should have no formulas on the right, but we cannot say anything about the others. Now, suppose there is a critical rule in the projection of $\vdash A$. Of course that this is resolved with a negative clause, but this then can have end-sequent ancestors on the right coming from the projection of $A \vdash B$. In this case, we cannot guarantee the emptiness of the right hand side of the sequent $A \vdash$ in the ACNF. See the problem?

Another worthwhile observation is that it seems the same result could be obtained by using regular resolution, instead of a refinement, but the proof of Theorem 11 would become considerably more complicated.

## Summary

Rank reducing CERES:

- Input: **LJ** proof with cuts (end-sequent has no formulas on the right)
- Output: **LJ'** proof with atomic cuts
- Resolution: negative refinement

# CHAPTER 8

# Merging CERES

In the last chapter we saw that it is not possible to permute the cut over †-inferences. This was a problem in particular for the right branch of the cut. Remember that the last method presented relied on the fact that one of premises of the cut inference, the right premise, had the format $A, \Gamma \vdash$, i.e., it contained no formulas on the right side. In order to achieve that, we had to require that the end-sequent of the input proof had no formulas on the right side. If the sequent $A, \Gamma \vdash$ was not the end-sequent, but some other sequent in the middle of the projection, we cannot guarantee that the cut could be permuted until this sequent, exactly because there could be †-inferences on the way (even if these inferences have an empty right context). While investigating this situation, we came across some interesting idea which resulted in the method we present now.

## 8.1 Merging rules

Suppose we want to eliminate violations in †-inferences on the left branch of the cut. We saw in last chapter that it is possible to permute the cut over rules in the left branch until a †-inference. On the right branch, though, we needed to find a sequent such as $A, \Gamma \vdash$. If this was not the end-sequent and we needed to permute the cut over some rules to get to it, we might encounter a †-inference on the way that would not allow this permutation anymore. Then we would have something like this:

$$\dfrac{\dfrac{\Gamma, P \vdash \Delta, Q, A}{\Gamma \vdash \Delta, P \to Q, A} \to_r^\dagger \quad \dfrac{A, \Gamma', R \vdash \Delta', S}{A, \Gamma' \vdash \Delta', R \to S} \to_r^\dagger}{\Gamma, \Gamma' \vdash \Delta, P \to Q, \Delta', R \to S} \; cut$$

Of course, if we use negative resolution, $\Delta'$ would be empty, but that's not really helpful. We still cannot permute the cut over the inference on the right branch. So what *can* we do?

Now, remember the way projections are built. They are composed of inferences on end-sequent ancestors that come from the input proof. This means that many projections have a common set of such inferences. Copies of the same formula can be derived by many projections, and they are contracted afterwards in the ACNF. What if these two †-inferences are instances of the same inference of the input proof? Then the derivation is actually[1]:

$$\dfrac{\dfrac{\Gamma, P \vdash \Delta, Q, A}{\Gamma \vdash \Delta, P \to Q, A}\,{\to_r^\dagger} \qquad \dfrac{A, \Gamma', P \vdash \Delta', Q}{A, \Gamma' \vdash \Delta', P \to Q}\,{\to_r^\dagger}}{\dfrac{\Gamma, \Gamma' \vdash \Delta, P \to Q, \Delta', P \to Q}{\Gamma, \Gamma' \vdash \Delta, \Delta', P \to Q}\,c_r}\,\mathit{cut}$$

Now we can do something! This derivation can be rewritten to:

$$\dfrac{\dfrac{\dfrac{\Gamma, P \vdash \Delta, Q, A \qquad A, \Gamma', P \vdash \Delta', Q}{\Gamma, \Gamma', P, P \vdash \Delta, \Delta', Q, Q}\,\mathit{cut}}{\Gamma, \Gamma', P \vdash \Delta, \Delta', Q}\,c*}{\Gamma, \Gamma' \vdash \Delta, \Delta', P \to Q}\,{\to_r^\dagger}$$

What happened there? We transformed the application of two †-inferences with right contexts $\Delta$, $\Delta'$ and $A$ into an application of one †-inference with context $\Delta$ and $\Delta'$. Looks good. We call this transformation *merging*.

**Definition 29** (Merging)**.**

$$\dfrac{\dfrac{\Gamma, A \vdash C, \Delta}{\Gamma \vdash \neg A, C, \Delta}\,\neg_r \qquad \dfrac{C, \Gamma', A \vdash \Delta'}{C, \Gamma' \vdash \neg A, \Delta'}\,\neg_r}{\Gamma, \Gamma' \vdash \neg A, \neg A, \Delta, \Delta'}\,\mathit{cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\dfrac{\Gamma, A \vdash C, \Delta \qquad C, \Gamma', A \vdash \Delta'}{\Gamma, \Gamma', A, A \vdash \Delta, \Delta'}\,\mathit{cut}}{\Gamma, \Gamma', A \vdash \Delta, \Delta'}\,c_l}{\Gamma, \Gamma' \vdash \neg A, \Delta, \Delta'}\,\neg_r}{\Gamma, \Gamma' \vdash \neg A, \neg A, \Delta, \Delta'}\,w_r$$

$$\dfrac{\dfrac{\Gamma, A \vdash B, C, \Delta}{\Gamma \vdash A \to B, C, \Delta}\,{\to_r} \qquad \dfrac{C, \Gamma', A \vdash B, \Delta'}{C, \Gamma' \vdash A \to B, \Delta'}\,{\to_r}}{\Gamma, \Gamma' \vdash A \to B, A \to B, \Delta, \Delta'}\,\mathit{cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\dfrac{\Gamma, A \vdash B, C, \Delta \qquad C, \Gamma', A \vdash B, \Delta'}{\Gamma, \Gamma', A, A \vdash B, B, \Delta, \Delta'}\,\mathit{cut}}{\Gamma, \Gamma', A \vdash B, \Delta, \Delta'}\,c^*}{\Gamma, \Gamma' \vdash A \to B, \Delta, \Delta'}\,{\to_r}}{\Gamma, \Gamma' \vdash A \to B, A \to B, \Delta, \Delta'}\,w_r$$

*Note that $\Delta$ and $\Delta'$ are sets of atomic cut ancestors.*

Now let's see how we get there. First of all, there should be applications of this †-inference in both projections. How can an inference occur on two or more projections? I'll show you. Let $\rho$ be a binary rule operating on a cut ancestor in the input proof ($\rho$ can actually be the *cut* itself):

$$\dfrac{\overset{\varphi_1}{\Gamma' \vdash \Delta'} \qquad \overset{\varphi_2}{\Gamma'' \vdash \Delta''}}{\Gamma \vdash \Delta}\,\rho$$

[1]Keep in mind that structural rules can be permuted over any rule, and therefore we can reach the situation where we contract $P \to Q$ right below the cut.

The clause sets $CL(\varphi_1)$ and $CL(\varphi_2)$ are combined using $\oplus$, i.e., the union of the clauses. Each of these clauses will have a corresponding projection. Now, everything below $\rho$ will be a common part of every such projection. This means, in fact, that:

**Theorem 12.** *Let $\rho$ be a rule operating on an end-sequent ancestor in the input proof. If $\rho$ occurs below the lower-most cut, then there will be an instance of it in every projection.*

In particular, if the critical inferences occur below the lower most cut, they will occur in every projection. Note that, for this to hold, we *need* to use the regular projections, as in Definition 12, and cannot use o-projections (Definition 13). Interesting, right? If the critical inferences occur in every projection, we can always perform the merging, and we can eventually fix the proof. Here's how:

**Theorem 13.** *If the critical inferences occur in all projections, then the final ACNF can be transformed from classical to intuitionistic by permutation of cuts and the merging operation.*

*Proof.* Lets consider the following induction measures:

$k$ : number of critical inferences in the input proof.

$m_i$ : number of *occurrences* of the †-inference $i$ in the ACNF.

$n^i_j$ : number of atoms violating occurrence $j$ of †-inference $i$.

We know the following:

- If $k = 0$ then $m_i = 0$ and $n^i_j = 0$, and the ACNF is intuitionistic.
- If $n^i_j$ is 0, then $m_i$ is decreased by 1.
- If $m_i$ is 0, then $k$ is decreased by 1.

The cut rule can be permuted over every rule in **LJ**, therefore, we can always permute the cut up until two applications of the same critical rule are reached. At this point, the merging operation is performed and we have two possible situations:

- If $\Delta'$ is empty, then the operation reduces $n^i_j$ by one.
- If $\Delta'$ is not empty, then the operation reduces $m_i$ by one.

□

Now let's worry about how we make these critical inferences occur in every projection. From Theorem 12 we conclude that if we want an inference to occur in every projection, then this inference must occur below the cut in the input proof. If this is not already the case, can we transform the input proof so it is? Maybe. In order to do that we need to permute the critical inferences (and all subsequent rules operating on successors of its main formula) down the (input) proof until they are all below the cut. We analysed the permutations of all rules in

**LJ**, and the results are summarized in Table 8.2. The actual permutations are listed in Appendix B.

Table 8.2 can be interpreted as follows: the status of position $(i, j)$ corresponds to the permutation of rule $i$ down rule $j$. Possible statuses are shown in Table 8.1.

| | |
|---|---|
| ✔ | permutation is ok |
| ✔* | permutation is ok as long as $i$ is applied to both premises of $j$. |
| ✘ | permutation is not possible |
| $\circ$ | the case does not occur in **LJ** |
| ? | the permutation makes no sense. |
| ev | a violation of eigenvariable conditions might occur. |

**Table 8.1:** Status of permutations

Lines that are grey indicate cases that will not occur in our specific input proofs, but they were also analysed for the sake of completeness. Note that all permutations identified in this table are in accordance with Kleene's analysis [21]. The difference in results is due to the fact that we consider structural rules as any regular rule.

| $\cdot$ | $\to_l$ | $\to_r$ | $\wedge_l$ | $\wedge_r$ | $\vee_l$ | $\vee_r$ | $\forall_l$ | $\forall_r$ | $\exists_l$ | $\exists_r$ | $\neg_l$ | $\neg_r$ | $w_l$ | $w_r$ | $c_l$ | $cut$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\to_l$ | ✔ | ✘ | ✔ | ✔ | ✘ | ✔ | ✔ | ✔ | ✔ | ✔ | ✔ | ✘ | ✔ | ✔ | ? | ✔ |
| $\to_r$ | ✔ | $\circ$ | ✔ | $\circ$ | ✔ | $\circ$ | ✔ | $\circ$ | ✔ | $\circ$ | $\circ$ | $\circ$ | ✔ | $\circ$ | ✔ | ✔ |
| $\wedge_l$ | ✔ | ✔ | ✔ | ✔ | ✔ | ✔ | ✔ | ✔ | ✔ | ✔ | ✔ | ✔ | ✔ | ✔ | ✔ | ✔ |
| $\wedge_r$ | ✔ | $\circ$ | ✔ | $\circ$ | ✔* | $\circ$ | ✔ | $\circ$ | ✔ | $\circ$ | $\circ$ | $\circ$ | ✔ | $\circ$ | ? | ✔ |
| $\vee_l$ | ✔ | ✔ | ✔ | ✔ | ✔ | ✔ | ✔ | ✔ | ✔ | ✔ | ✔ | ✔ | ✔ | ✔ | ? | ✔ |
| $\vee_r$ | ✔ | $\circ$ | ✔ | $\circ$ | ✘ | $\circ$ | ✔ | $\circ$ | ✔ | $\circ$ | $\circ$ | $\circ$ | ✔ | $\circ$ | ✔ | ✔ |
| $\forall_l$ | ✔ | ✔ | ✔ | ✔ | ✔ | ✔ | ✔ | ev | ev | ✔ | ✔ | ✔ | ✔ | ✔ | ✔ | ✔ |
| $\forall_r$ | ✔ | $\circ$ | ✔ | $\circ$ | ✔* | $\circ$ | ✔ | $\circ$ | ✔ | $\circ$ | $\circ$ | $\circ$ | ✔ | $\circ$ | ✔ | ✔ |
| $\exists_l$ | ✔ | ✔ | ✔ | ✔ | ✔ | ✔ | ✔ | ✔ | ✔ | ✔ | ✔ | ✔ | ✔ | ✔ | ✔ | ✔ |
| $\exists_r$ | ✔ | $\circ$ | ✔ | $\circ$ | ✘ | $\circ$ | ✔ | $\circ$ | ev | $\circ$ | $\circ$ | $\circ$ | ✔ | $\circ$ | ✔ | ✔ |
| $\neg_l$ | ✔ | $\circ$ | ✔ | $\circ$ | ✔ | $\circ$ | ✔ | $\circ$ | ✔ | $\circ$ | $\circ$ | ✘ | ✔ | ✘ | ✔ | ✔ |
| $\neg_r$ | ✔ | $\circ$ | ✔ | $\circ$ | ✔ | $\circ$ | ✔ | $\circ$ | ✔ | $\circ$ | $\circ$ | $\circ$ | ✔ | $\circ$ | ✔ | ✔ |
| $w_l$ | ✔ | ✔ | ✔ | ✔ | ✔ | ✔ | ✔ | ✔ | ✔ | ✔ | ✔ | ✔ | ✔ | ✔ | ✔ | ✔ |
| $w_r$ | ✔ | $\circ$ | ✔ | $\circ$ | ✔* | $\circ$ | ✔ | $\circ$ | ✔ | $\circ$ | $\circ$ | $\circ$ | ✔ | $\circ$ | ✔ | ✔ |
| $c_l$ | ✔ | ✔ | ✔ | ✔ | ✔ | ✔ | ✔ | ✔ | ✔ | ✔ | ✔ | ✔ | ✔ | ✔ | ✔ | ✔ |
| $cut$ | ✔ | ✘ | ✔ | ✔ | ✘ | ✔ | ✔ | ✔ | ✔ | ✔ | ✔ | ✘ | ✔ | ✔ | ✘ | ✔ |

**Table 8.2:** Permutation of rules in **LJ**

As you can see from the table, most of the times the rules can be permuted. But most of the times is not good enough. Let's discuss some of the cases which are not ok.

We will explain first the ? case. It occurs only when permuting inferences down a contraction. In particular, binary inferences. The problem occurs when the contracted formulas go each to one branch of the binary rule. Take $\wedge_r$ as an example:

$$\dfrac{\dfrac{\begin{matrix}\varphi_1 \\ \Gamma, P \vdash A\end{matrix} \quad \begin{matrix}\varphi_2 \\ \Gamma', P \vdash B\end{matrix}}{\Gamma, \Gamma', P, P \vdash A \wedge B}\ \wedge_r}{\Gamma, \Gamma', P \vdash A \wedge B}\ c_l$$

If we try to apply $\wedge_r$ before $c_l$ (bottom-up), we would have to contract $P$, which is exactly what is done right now. So the permutation really makes no sense.

If a critical inference operates on an ancestor of $A \wedge B$, then the contraction in $P$ will have to be permuted down as well. This is not a problem in principle, since $c_l$ permutes down every rule (see Table 8.2), but issues might occur if $P$ is an ancestor of a more complex formula or if $P$ is a cut ancestor.

The case ✔* indicates that a permutation is possible only if the upper inference is applied to *both* premises of the lower inference. This was presented already in Lemma 8 of [21]. Note that this case only occurs when permuting *right* inferences down $\vee_l$ and it is related to the fact that the right formula suffers an auto-contraction when $\vee_l$ is applied. When the upper inference is not applied to both premises, it is reasonable to assume that this configuration can be reached by permuting the rules. Note that this upper inference might be $\wedge_r$, $\forall_r$ (which actually will not occur in skolemized proofs) or $w_r$, and all these rules permute down the others, so in principle they can be brought to the premise of $\vee_l$.

There is not a lot to say about cases ✘ and $\circ$. The first ones are the cases that break our solution, unfortunately, and the second are cases that are not possible in **LJ** because they would require more than one formula on the right side of the sequent.

Case ev is interesting. In general, these permutations are not allowed because of eigenvariable violations. Look for example the proof of $\forall x.Px \vdash \forall x.Px$:

$$\dfrac{\dfrac{\overline{P\alpha \vdash P\alpha}}{\forall x.Px \vdash P\alpha}\ \forall_l}{\forall x.Px \vdash \forall x.Px}\ \forall_r \quad \rightsquigarrow \quad \dfrac{\dfrac{\overline{P\alpha \vdash P\alpha}}{P\alpha \vdash \forall x.Px}\ \forall_r \ \text{wrong!}}{\forall x.Px \vdash \forall x.Px}\ \forall_l$$

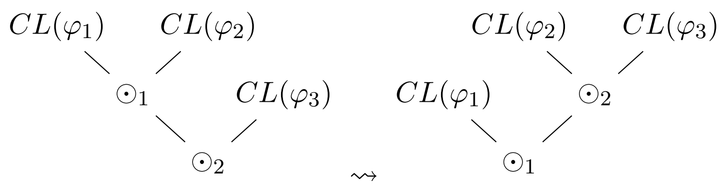
If we apply $\forall_l$ first (bottom-up), and instantiate $x$ with $\alpha$, then the application of $\forall_r$ becomes unsound, since $\alpha$ is no longer a fresh variable. Interestingly enough, we can allow that in our case. Note that eigenvariable violations only occur in strong quantifier rules, i.e., $\forall_r$ and $\exists_l$. Since we work with skolemized proofs, these rules, if they occur, are restricted to cut ancestors. Therefore, when building the projections, they will not be applied and no violations occur. For this reason we will allow this local and temporary unsoundness, if this means we can transform the proof into one in which all critical inferences occur below the cuts.

So far we know that, if all critical inferences in the input proof can be permuted below all cuts, then we can fix the ACNF using the merging operation. We have also analysed the permutation of rules to understand when this is possible and when it isn't. We concluded that some permutations, although unsound, may be allowed. But how does this method work exactly? Obviously the projections have to be computed from the transformed (possibly unsound) proof. You might wonder if we can use the clause set from the original proof, and the answer is no. In what follows we will describe the clause set before and after permuting the critical inferences and show that while in some cases the original resolution refutation can be used, in others it cannot. From this we will conclude that permuting the rules is unavoidable if we want to fix the ACNF using merging.

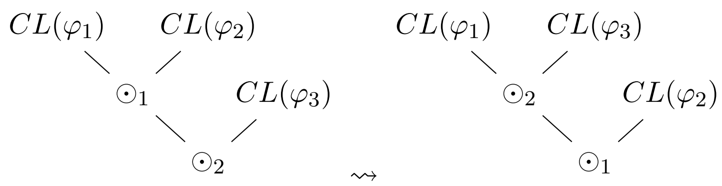
Remember the definition of clause set (Definition 10). The structure of the clause set is determined solely by binary rule applications. This means that permuting unary rules will change absolutely nothing in the clause set, which considerably reduces the number of cases we need to analyse. We could identify 9 schemas of structural change of the clause set, according to the permutation performed, which we define now.

**Definition 30** ($\tau_i(\odot_1, \odot_2)$)**.** *We define $\tau_1(\odot_1, \odot_2), ..., \tau_9(\odot_1, \odot_2)$ as the transformations suffered by the clause set struct (Definition 11) of a proof upon permuting binary inferences. The parameters $\odot_1$ and $\odot_2$ are either $\otimes$ or $\oplus$, depending on whether the binary inference operates in an end-sequent or cut ancestor, respectively. We will fix $\odot_1$ to correspond to the application of the upper rule, which will be permuted down. The transformations are:*

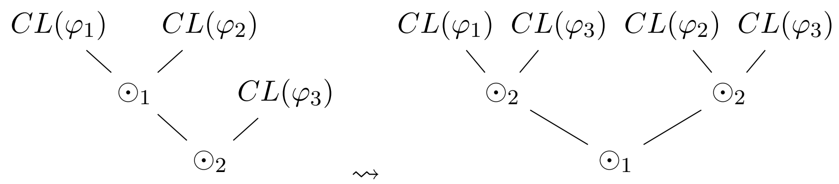
$\tau_1(\odot_1, \odot_2)$ :



$\tau_2(\odot_1, \odot_2)$ :



$\tau_3(\odot_1, \odot_2)$ :



$\tau_4(\odot_1, \odot_2)$ :

$$
\begin{tikzcd}
& CL(\varphi_2) \arrow[dr, dash] & & CL(\varphi_3) \arrow[dl, dash] \\
CL(\varphi_1) \arrow[dr, dash] & & \odot_1 \arrow[dl, dash] & \\
& \odot_2 & &
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
CL(\varphi_1) \arrow[dr, dash] & & CL(\varphi_2) \arrow[dl, dash] & \\
& \odot_2 \arrow[dr, dash] & & CL(\varphi_3) \arrow[dl, dash] \\
& & \odot_1 &
\end{tikzcd}
$$

$\tau_5(\odot_1, \odot_2)$ :

$$
\begin{tikzcd}
& CL(\varphi_2) \arrow[dr, dash] & & CL(\varphi_3) \arrow[dl, dash] \\
CL(\varphi_1) \arrow[dr, dash] & & \odot_1 \arrow[dl, dash] & \\
& \odot_2 & &
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
& CL(\varphi_1) \arrow[dr, dash] & & CL(\varphi_3) \arrow[dl, dash] \\
CL(\varphi_2) \arrow[dr, dash] & & \odot_2 \arrow[dl, dash] & \\
& \odot_1 & &
\end{tikzcd}
$$

$\tau_6(\odot_1, \odot_2)$ :

$$
\begin{tikzcd}
& CL(\varphi_2) \arrow[dr, dash] & & CL(\varphi_3) \arrow[dl, dash] \\
CL(\varphi_1) \arrow[dr, dash] & & \odot_1 \arrow[dl, dash] & \\
& \odot_2 & &
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
CL(\varphi_1) \arrow[dr, dash] & & CL(\varphi_2) \arrow[dl, dash] & & CL(\varphi_1) \arrow[dr, dash] & & CL(\varphi_3) \arrow[dl, dash] \\
& \odot_2 \arrow[drr, dash] & & & & \odot_2 \arrow[dll, dash] & \\
& & & \odot_1 & & &
\end{tikzcd}
$$

$\tau_7(\odot_1, \odot_2)$ :

$$
\begin{tikzcd}
CL(\varphi_1) \arrow[dr, dash] & & CL(\varphi_2) \arrow[dl, dash] & & CL(\varphi_3) \arrow[dr, dash] & & CL(\varphi_4) \arrow[dl, dash] \\
& \odot_1 \arrow[drr, dash] & & & & \odot_1 \arrow[dll, dash] & \\
& & & \odot_2 & & &
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
CL(\varphi_1) \arrow[dr, dash] & & CL(\varphi_3) \arrow[dl, dash] & \\
& \odot_2 \arrow[dr, dash] & & CL(\varphi_4) \arrow[dl, dash] \\
& & \odot_1 &
\end{tikzcd}
$$

$\tau_8(\odot_1, \odot_2)$ :

$$
\begin{tikzcd}
CL(\varphi_1) \arrow[dr, dash] & & CL(\varphi_2) \arrow[dl, dash] & & CL(\varphi_3) \arrow[dr, dash] & & CL(\varphi_4) \arrow[dl, dash] \\
& \odot_1 \arrow[drr, dash] & & & & \odot_1 \arrow[dll, dash] & \\
& & & \odot_2 & & &
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
& CL(\varphi_2) \arrow[dr, dash] & & CL(\varphi_4) \arrow[dl, dash] \\
CL(\varphi_1) \arrow[dr, dash] & & \odot_2 \arrow[dl, dash] & \\
& \odot_1 & &
\end{tikzcd}
$$

$\tau_9(\odot_1, \odot_2)$ :

$$
\begin{tikzcd}
CL(\varphi_1) \arrow[dr, dash] & & CL(\varphi_2) \arrow[dl, dash] & & CL(\varphi_3) \arrow[dr, dash] & & CL(\varphi_4) \arrow[dl, dash] \\
& \odot_1 \arrow[drr, dash] & & & & \odot_1 \arrow[dll, dash] & \\
& & & \odot_2 & & &
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
CL(\varphi_1) \arrow[dr, dash] & & CL(\varphi_3) \arrow[dl, dash] & & CL(\varphi_2) \arrow[dr, dash] & & CL(\varphi_4) \arrow[dl, dash] \\
& \odot_2 \arrow[drr, dash] & & & & \odot_2 \arrow[dll, dash] & \\
& & & \odot_1 & & &
\end{tikzcd}
$$

| | $\mathcal{C}_b$ | $\mathcal{C}_a$ |
|---|---|---|
| $\tau_1(\otimes, \otimes)$ | $(\mathcal{C}_1 \otimes \mathcal{C}_2) \otimes \mathcal{C}_3$ | $\mathcal{C}_1 \otimes (\mathcal{C}_2 \otimes \mathcal{C}_3)$ |
| $\tau_2(\otimes, \otimes)$ | $(\mathcal{C}_1 \otimes \mathcal{C}_2) \otimes \mathcal{C}_3$ | $(\mathcal{C}_1 \otimes \mathcal{C}_3) \otimes \mathcal{C}_2$ |
| $\tau_3(\otimes, \otimes)$ | $(\mathcal{C}_1 \otimes \mathcal{C}_2) \otimes \mathcal{C}_3$ | $(\mathcal{C}_1 \otimes \mathcal{C}_3) \otimes (\mathcal{C}_2 \otimes \mathcal{C}_3)$ |
| $\tau_4(\otimes, \otimes)$ | $\mathcal{C}_1 \otimes (\mathcal{C}_2 \otimes \mathcal{C}_3)$ | $(\mathcal{C}_1 \otimes \mathcal{C}_2) \otimes \mathcal{C}_3$ |
| $\tau_5(\otimes, \otimes)$ | $\mathcal{C}_1 \otimes (\mathcal{C}_2 \otimes \mathcal{C}_3)$ | $\mathcal{C}_2 \otimes (\mathcal{C}_1 \otimes \mathcal{C}_3)$ |
| $\tau_6(\otimes, \otimes)$ | $\mathcal{C}_1 \otimes (\mathcal{C}_2 \otimes \mathcal{C}_3)$ | $(\mathcal{C}_1 \otimes \mathcal{C}_2) \otimes (\mathcal{C}_1 \otimes \mathcal{C}_3)$ |
| $\tau_7(\otimes, \otimes)$ | $(\mathcal{C}_1 \otimes \mathcal{C}_2) \otimes (\mathcal{C}_3 \otimes \mathcal{C}_4)$ | $(\mathcal{C}_1 \otimes \mathcal{C}_3) \otimes \mathcal{C}_4$ |
| $\tau_8(\otimes, \otimes)$ | $(\mathcal{C}_1 \otimes \mathcal{C}_2) \otimes (\mathcal{C}_3 \otimes \mathcal{C}_4)$ | $\mathcal{C}_1 \otimes (\mathcal{C}_2 \otimes \mathcal{C}_4)$ |
| $\tau_9(\otimes, \otimes)$ | $(\mathcal{C}_1 \otimes \mathcal{C}_2) \otimes (\mathcal{C}_3 \otimes \mathcal{C}_4)$ | $(\mathcal{C}_1 \otimes \mathcal{C}_3) \otimes (\mathcal{C}_2 \otimes \mathcal{C}_4)$ |

**Table 8.3:** Transformations $\tau_i(\otimes, \otimes)$

Since in our particular case the inference we wish to permute down operates on an end-sequent ancestor, we can safely consider $\odot_1 = \otimes$. Therefore we need only to analyse $\odot_2 = \otimes$ and $\odot_2 = \oplus$. Tables 8.3 and 8.4 summarizes the operations for each possible $\odot_2$, respectively. We use a linear representation of the clause set struct and already normalize it (i.e., distribute $\otimes$ over $\oplus$). For readability purposes, we denote $CL(\varphi_i)$ as $\mathcal{C}_i$. Also, we use $\mathcal{C}_b$ to denote the clause set before the permutation and $\mathcal{C}_a$ to denote it afterwards.

Let's look carefully what happens to the clause set under each of these transformations. We'll start with $\tau_i(\otimes, \otimes)$ (Table 8.3). Given that $\otimes$ is associative and symmetric, transformations $\tau_1(\otimes, \otimes), \tau_2(\otimes, \otimes), \tau_4(\otimes, \otimes), \tau_5(\otimes, \otimes)$ and $\tau_9(\otimes, \otimes)$ will not alter the clause set. Therefore, in this case, the refutation of $\mathcal{C}_b$ could be used to construct the ACNF, while using the projections computed from the transformed proof.

Transformations $\tau_3(\otimes, \otimes)$ and $\tau_6(\otimes, \otimes)$ are analogous: they will require a branch of the proof to be copied and used twice, thus generating a redundancy on the clause set. Nevertheless, the same clauses will still be available, modulo the contraction of a few atoms. This is easier to see if we use the properties of $\otimes$ and rewrite $\mathcal{C}_a$ for both cases, respectively:

| | | |
|---|---|---|
| $\tau_3(\otimes, \otimes)$ | $(\mathcal{C}_1 \otimes \mathcal{C}_2) \otimes \mathcal{C}_3$ | $(\mathcal{C}_1 \otimes \mathcal{C}_2) \otimes (\mathcal{C}_3 \otimes \mathcal{C}_3)$ |
| $\tau_6(\otimes, \otimes)$ | $\mathcal{C}_1 \otimes (\mathcal{C}_2 \otimes \mathcal{C}_3)$ | $(\mathcal{C}_1 \otimes \mathcal{C}_1) \otimes (\mathcal{C}_2 \otimes \mathcal{C}_3)$ |

Although the clause set of the transformed proof might be exponentially larger, we can use the resolution refutation of $\mathcal{C}_b$ and use a subset of the clauses in $\mathcal{C}_a$ to compute the necessary projections, possibly with contractions of atomic cut ancestors.

The transformations $\tau_7(\otimes, \otimes)$ and $\tau_8(\otimes, \otimes)$ are more interesting. They actually get rid of one branch of the proof, reducing the redundancy of the clause set. The clauses of $\mathcal{C}_a$ subsume those of $\mathcal{C}_b$ (i.e., the former clauses are subsets of literals of the latter clauses), and because of that a resolution refutation using clauses from $\mathcal{C}_b$ can be transformed into another one using clause from $\mathcal{C}_a$ [23].

| | $\mathcal{C}_b$ | $\mathcal{C}_a$ |
|---|---|---|
| $\tau_1(\otimes, \oplus)$ | $(\mathcal{C}_1 \otimes \mathcal{C}_2) \oplus \mathcal{C}_3$ | $(\mathcal{C}_1 \otimes \mathcal{C}_2) \oplus (\mathcal{C}_1 \otimes \mathcal{C}_3)$ |
| $\tau_2(\otimes, \oplus)$ | $(\mathcal{C}_1 \otimes \mathcal{C}_2) \oplus \mathcal{C}_3$ | $(\mathcal{C}_1 \otimes \mathcal{C}_2) \oplus (\mathcal{C}_2 \otimes \mathcal{C}_3)$ |
| $\tau_3(\otimes, \oplus)$ | $(\mathcal{C}_1 \otimes \mathcal{C}_2) \oplus \mathcal{C}_3$ | $(\mathcal{C}_1 \otimes \mathcal{C}_2) \oplus (\mathcal{C}_1 \otimes \mathcal{C}_3) \oplus (\mathcal{C}_2 \otimes \mathcal{C}_3) \oplus (\mathcal{C}_3 \otimes \mathcal{C}_3)$ |
| $\tau_4(\otimes, \oplus)$ | $\mathcal{C}_1 \oplus (\mathcal{C}_2 \otimes \mathcal{C}_3)$ | $(\mathcal{C}_1 \otimes \mathcal{C}_3) \oplus (\mathcal{C}_2 \otimes \mathcal{C}_3)$ |
| $\tau_5(\otimes, \oplus)$ | $\mathcal{C}_1 \oplus (\mathcal{C}_2 \otimes \mathcal{C}_3)$ | $(\mathcal{C}_1 \otimes \mathcal{C}_2) \oplus (\mathcal{C}_2 \otimes \mathcal{C}_3)$ |
| $\tau_6(\otimes, \oplus)$ | $\mathcal{C}_1 \oplus (\mathcal{C}_2 \otimes \mathcal{C}_3)$ | $(\mathcal{C}_1 \otimes \mathcal{C}_1) \oplus (\mathcal{C}_1 \otimes \mathcal{C}_3) \oplus (\mathcal{C}_1 \otimes \mathcal{C}_2) \oplus (\mathcal{C}_2 \otimes \mathcal{C}_3)$ |
| $\tau_7(\otimes, \oplus)$ | $(\mathcal{C}_1 \otimes \mathcal{C}_2) \oplus (\mathcal{C}_3 \otimes \mathcal{C}_4)$ | $(\mathcal{C}_1 \otimes \mathcal{C}_4) \oplus (\mathcal{C}_3 \otimes \mathcal{C}_4)$ |
| $\tau_8(\otimes, \oplus)$ | $(\mathcal{C}_1 \otimes \mathcal{C}_2) \oplus (\mathcal{C}_3 \otimes \mathcal{C}_4)$ | $(\mathcal{C}_1 \otimes \mathcal{C}_2) \oplus (\mathcal{C}_1 \otimes \mathcal{C}_4)$ |
| $\tau_9(\otimes, \oplus)$ | $(\mathcal{C}_1 \otimes \mathcal{C}_2) \oplus (\mathcal{C}_3 \otimes \mathcal{C}_4)$ | $(\mathcal{C}_1 \otimes \mathcal{C}_2) \oplus (\mathcal{C}_2 \otimes \mathcal{C}_3) \oplus (\mathcal{C}_1 \otimes \mathcal{C}_4) \oplus (\mathcal{C}_3 \otimes \mathcal{C}_4)$ |

**Table 8.4:** Transformations $\tau_i(\otimes, \oplus)$

So we conclude that, if $\odot_2$ is $\otimes$, then we can use the resolution refutation of the input proof, and the projections of the (possibly unsound) intermediary proof, in which the critical rules were permuted down. Unfortunately this does not hold if $\odot_2$ is $\oplus$.

Interestingly enough, in cases $\tau_1(\otimes, \oplus)$ to $\tau_6(\otimes, \oplus)$ and $\tau_9(\otimes, \oplus)$ the clauses in the *original* clause set, $\mathcal{C}_b$, subsume those in $\mathcal{C}_a$. This is the exact contrary of cases $\tau_7(\otimes, \otimes)$ and $\tau_8(\otimes, \otimes)$, and it means we could use the refutation of $\mathcal{C}_a$ with the clauses of $\mathcal{C}_b$, but that's not really what we are after, right? Cases $\tau_7(\otimes, \oplus)$ and $\tau_8(\otimes, \oplus)$ are even more strange, since the clause sets before and after the permutation are not comparable.

Because of these cases, we cannot say that the original clause set (and refutation) can be used. We can conclude that, if we want to fix the proof by merging, then all the rules need to be permuted below all the cuts, and the CERES method will be applied to this new transformed proof.

Complexitywise, this is not the best transformation, as you might have noticed. Although there are a some cases in which the clause set is actually simplified (e.g., $\tau_7(\otimes, \otimes)$ and $\tau_8(\otimes, \otimes)$), in others it is made bigger and more redundant (e.g., $\tau_3(\otimes, \oplus)$ and $\tau_6(\otimes, \oplus)$). Because of these "bad" cases, the transformed proof can be, on the worst case, exponentially larger than the original one.

## Example

We give now a (simple) example on how this method would work. Let $\varphi$ be the proof:

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$A \vdash A$}
\RightLabel{$\neg_l$}
\UnaryInfC{$A, \neg A \vdash$}
\RightLabel{$\neg_r$}
\UnaryInfC{$\neg A \vdash \neg A$}
\AxiomC{}
\UnaryInfC{$A \vdash A$}
\RightLabel{$w_l$}
\UnaryInfC{$B, A \vdash A$}
\RightLabel{$\neg_l$}
\UnaryInfC{$\neg A, B, A \vdash$}
\RightLabel{$\neg_r$}
\UnaryInfC{$\neg A, B \vdash \neg A$}
\RightLabel{$\rightarrow_r$}
\UnaryInfC{$\neg A \vdash B \rightarrow \neg A$}
\RightLabel{$cut$}
\BinaryInfC{$\neg A \vdash B \rightarrow \neg A$}
\end{prooftree}
$$

The original clause set and projections are:

$$\{A \vdash \ ; \ \vdash A\}$$

$$\dfrac{\dfrac{\dfrac{}{A \vdash A}}{A, \neg A \vdash} \neg_l}{A, \neg A \vdash B \to \neg A} w_r \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}}{A, B \vdash A} w_l}{B \vdash A, \neg A} \neg_r^\dagger}{\vdash A, B \to \neg A} \to_r^\dagger}{\neg A \vdash A, B \to \neg A} w_l$$

Note that the †-inferences occur in only one of the projections, meaning that we cannot fix the ACNF by merging. So we permute the critical inferences below the cut and we have a new proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}}{A, \neg A \vdash} \neg_l}{\neg A \vdash \neg A} \neg_r \quad \dfrac{\dfrac{\dfrac{}{A \vdash A}}{B, A \vdash A} w_l}{\neg A, B, A \vdash} \neg_l}{\neg A, B, A \vdash} cut}{\neg A, B \vdash \neg A} \neg_r}{\neg A \vdash B \to \neg A} \to_r$$

Since this involved only permutations of unary rules, the clause set remains unchanged. But the projections now are:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}}{A, \neg A \vdash} \neg_l}{A, \neg A, B, A \vdash} w^*}{A, \neg A, B \vdash \neg A} \neg_r}{A, \neg A \vdash B \to \neg A} \to_r \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}}{A, B \vdash A} w_l}{B \vdash A, \neg A} \neg_r^\dagger}{\vdash A, B \to \neg A} \to_r^\dagger}{\neg A \vdash A, B \to \neg A} w_l$$

Using the simple resolution refutation of the clause set, we obtain the (still violated) ACNF:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}}{A, \neg A \vdash} \neg_l}{A, \neg A, B, A \vdash} w^*}{A, \neg A, B \vdash \neg A} \neg_r}{A, \neg A \vdash B \to \neg A} \to_r \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}}{A, B \vdash A} w_l}{B \vdash A, \neg A} \neg_r^\dagger}{\vdash A, B \to \neg A} \to_r^\dagger}{\neg A \vdash A, B \to \neg A} w_l}{\dfrac{\neg A, \neg A \vdash B \to \neg A, B \to \neg A}{\neg A \vdash B \to \neg A} c^*} cut$$

But by permuting the cut up and using the merging operation, we get a nice intuitionistic proof. First we merge $\to_r$:

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$A \vdash A$}
\RightLabel{$\neg_l$}
\UnaryInfC{$A, \neg A \vdash$}
\doubleLine
\RightLabel{$w^*$}
\UnaryInfC{$A, \neg A, B, A \vdash$}
\RightLabel{$\neg_r$}
\UnaryInfC{$A, \neg A, B \vdash \neg A$}
\AxiomC{}
\UnaryInfC{$A \vdash A$}
\RightLabel{$w_l$}
\UnaryInfC{$A, B \vdash A$}
\RightLabel{$\neg_r^\dagger$}
\UnaryInfC{$B \vdash A, \neg A$}
\RightLabel{$cut$}
\BinaryInfC{$\neg A, B, B \vdash \neg A, \neg A$}
\doubleLine
\RightLabel{$c^*$}
\UnaryInfC{$\neg A, B \vdash \neg A$}
\RightLabel{$\to_r$}
\UnaryInfC{$\neg A \vdash B \to \neg A$}
\RightLabel{$w_r$}
\UnaryInfC{$\neg A \vdash B \to \neg A, B \to \neg A$}
\RightLabel{$w_l$}
\UnaryInfC{$\neg A, \neg A \vdash B \to \neg A, B \to \neg A$}
\doubleLine
\RightLabel{$c^*$}
\UnaryInfC{$\neg A \vdash B \to \neg A$}
\end{prooftree}
$$

Then we merge $\neg_r$:

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$A \vdash A$}
\RightLabel{$\neg_l$}
\UnaryInfC{$A, \neg A \vdash$}
\doubleLine
\RightLabel{$w^*$}
\UnaryInfC{$A, \neg A, B, A \vdash$}
\AxiomC{}
\UnaryInfC{$A \vdash A$}
\RightLabel{$w_l$}
\UnaryInfC{$A, B \vdash A$}
\RightLabel{$cut$}
\BinaryInfC{$\neg A, B, B, A, A \vdash$}
\RightLabel{$c_l$}
\UnaryInfC{$\neg A, B, B, A \vdash$}
\RightLabel{$\neg_r$}
\UnaryInfC{$\neg A, B, B \vdash \neg A$}
\RightLabel{$w_r$}
\UnaryInfC{$\neg A, B, B \vdash \neg A, \neg A$}
\doubleLine
\RightLabel{$c^*$}
\UnaryInfC{$\neg A, B \vdash \neg A$}
\RightLabel{$\to_r$}
\UnaryInfC{$\neg A \vdash B \to \neg A$}
\RightLabel{$w_r$}
\UnaryInfC{$\neg A \vdash B \to \neg A, B \to \neg A$}
\RightLabel{$w_l$}
\UnaryInfC{$\neg A, \neg A \vdash B \to \neg A, B \to \neg A$}
\doubleLine
\RightLabel{$c^*$}
\UnaryInfC{$\neg A \vdash B \to \neg A$}
\end{prooftree}
$$

Ok, not so nice, I admit. But it becomes nicer if the redundant structural rules are eliminated (which can be done in linear time, by the way).

## Classes

So far we know that if we can permute all critical rules below the cut, the ACNF can be transformed into an intuitionistic proof by merging. But when exactly can we permute these rules? Table 8.2 gives us an idea.

The rules in the lines of the table are all applied to end-sequent ancestors. If we consider proofs in which there are no $\to_l$, $\neg_l$ and $\wedge_r$ applied to end-sequent ancestors, then we can always fix the ACNF. This gives us a syntactic characterization of (a part of) the proofs which can be dealt with using the merging method.

## Summary

Merging CERES:

- Input: **LJ** proof with cuts (such that critical inferences can be permuted below all cuts)
- Output: **LJ'** proof with atomic cuts
- Resolution: normal on the clause set of the transformed proof

CHAPTER 9

# Joining CERES

In the past chapters we tried to work with Gentzen's rank reduction rules to eliminate the atomic cuts that introduce violating atoms. What if there is another way to eliminate atomic cuts that does not involve rank reduction? In this chapter we explore such possibility.

## 9.1 Joining

Let's explain the idea behind joining. The CERES method collects atomic cut ancestors from the proof, right? Actually, it collects atomic cut ancestors that occur in axioms. Then these are organized in a clause set that happens to be refutable. In the refutation of the clause set, these atoms are resolved. Let's say an atom $A^\star$ was resolved between clauses $C_1$ and $C_2$ from the clause set. This means that the input proof had a cut which had $A^\star$ as an ancestor. Moreover, there were axioms $A \vdash A^\star$ and $A^\star \vdash A$. For the CERES method we also need to build projections for each of the clauses, and in particular, for $C_1$ and $C_2$. These projections will contain the axioms $A \vdash A^\star$ and $A^\star \vdash A$, and the $A^\star$s will be dragged down until the end-sequent of the projections where they are finally cut. Now, instead of cutting these $A^\star$s in the end-sequent, this new method will *join* the branches in which $A^\star$ occurs, from the axiom down, getting to an end-sequent $S$ which contains the formulas from both end-sequents of the projections. After this is done to every atom that is resolved in the resolution refutation, the end-sequent $S$ will be the end-sequent of the input proof, modulo some contractions. How do we join things? We will explain everything formally now.

First of all, let's define which are the paths we need to join.

**Definition 31** (Inference path)**.** *Let $\pi$ be a projection and $A^\star$ an atomic cut ancestor occurring in $\pi$. We define the* inference path $\mathcal{P}(\pi, A^\star)$ *as the sequence of inferences from the axiom in which $A^\star$ occurs until the end-sequent. We define the* length *of the path $|\mathcal{P}(\pi, A^\star)|$ as the number of inferences occuring in it.*

Of course that different inference paths might contain instances of the same input proof inference. When two inferences $\rho_1$ and $\rho_2$ in different projections are the same input proof inference, we say that $\rho_1$ and $\rho_2$ are a match. The matching inferences of two projections must occur in the same relative order to each other.

**Theorem 14.** *Let $A^\star$ be an atomic cut ancestor occurring in projections $\pi_1$ and $\pi_2$, and $P_1 = \mathcal{P}(\pi_1, A^\star)$ and $P_2 = \mathcal{P}(\pi_2, A^\star)$. Then the matching rules in $P_1$ and $P_2$ occur in the same relative order in each path. Moreover, they occur in the same order as in the input proof.*

*Proof.* Trivial from how the projections are constructed. □

What happens during a join is the combination of two inference paths into one, such that the matching inferences are applied only once. When matching critical inferences are merged, as shown in the last chapter, the result can be an inference with less violations, and ultimately an intuitionistic inference.

**Definition 32** (Joining)**.** *Let $A^\star$ be an atomic cut ancestor occurring in projections $\pi_1$ and $\pi_2$ with opposite polarity such that it is resolved in the resolution refutation, and $P_1 = \mathcal{P}(\pi_1, A^\star)$ and $P_2 = \mathcal{P}(\pi_2, A^\star)$. We define the joining $\pi_1 \bowtie_A \pi_2$ inductively on $(|P_1|, |P_2|)$. Whenever $A$ is clear from the context, we omit it in the symbol $\bowtie$.*

*We will not deal with the cases in which $|P_1| = 0$ or $|P_2| = 0$. This means that $A^\star$ does not occur in one or both projections and therefore it makes no sense to join them. Our base case is then $(1, 1)$ (case 0), when the paths contain only the axioms. Their join is trivial:*

$$\frac{}{A \vdash A^\star} \quad \bowtie \quad \frac{}{A^\star \vdash A} \quad = \quad \frac{}{A \vdash A} \tag{0.0}$$

*Now we look at the cases where $|P_1|$ or $|P_2|$ is bigger than 1. In the following derivations, we consider that $\varphi_1 \in P_1$ and $\varphi_2 \in P_2$, while $\varphi_1'$ and $\varphi_2'$ are derivations not contained in $P_1$ and $P_2$, respectively. Moreover $\rho_1$ and $\rho_2$ are the last (lower-most) inferences in $P_1$ and $P_2$, also respectively, and whenever these are matching inferences, they are both denoted by $\rho$.*

*If $|P_1| = 1$ and $|P_2| = k > 1$ (case 1):*

$$\frac{}{A \vdash A^\star} \quad \bowtie \quad \frac{\varphi_2}{S_2}\,\rho_2 \quad = \quad \frac{\left(\overline{A \vdash A^\star} \bowtie \varphi_2\right)}{S}\,\rho_2 \tag{1.1}$$

$$\frac{}{A \vdash A^\star} \quad \bowtie \quad \frac{\varphi_2 \quad \varphi_2'}{S_2}\,\rho_2 \quad = \quad \frac{\left(\overline{A \vdash A^\star} \bowtie \varphi_2\right) \quad \varphi_2'}{S}\,\rho_2 \tag{1.2}$$

$$\frac{}{A \vdash A^\star} \quad \bowtie \quad \frac{\varphi_2' \quad \varphi_2}{S_2}\,\rho_2 \quad = \quad \frac{\varphi_2' \quad \left(\overline{A \vdash A^\star} \bowtie \varphi_2\right)}{S}\,\rho_2 \tag{1.3}$$

*The new join operates on paths of length $(1, k-1)$, therefore we can apply the induction hypothesis.*

*If $|P_1| = k > 1$ and $|P_2| = 1$ (case 2), the cases are analogous:*

$$\frac{\varphi_1}{S_1}\,\rho_1 \quad \bowtie \quad \overline{A^\star \vdash A} \quad = \quad \frac{\left(\overline{A^\star \vdash A} \bowtie \varphi_1\right)}{S}\,\rho_1 \tag{2.1}$$

$$\frac{\varphi_1 \quad \varphi_1'}{S_1}\,\rho_1 \quad \bowtie \quad \overline{A^\star \vdash A} \quad = \quad \frac{\left(\overline{A^\star \vdash A} \bowtie \varphi_1\right) \quad \varphi_1'}{S}\,\rho_1 \tag{2.2}$$

$$\frac{\varphi_1' \quad \varphi_1}{S_1}\,\rho_1 \quad \bowtie \quad \overline{A^\star \vdash A} \quad = \quad \frac{\varphi_1' \quad \left(\overline{A^\star \vdash A} \bowtie \varphi_1\right)}{S}\,\rho_1 \tag{2.3}$$

*This time the induction hypothesis is applied to* $(k-1, 1)$.

*If* $|P_1| = j > 1$ *and* $|P_2| = k > 1$ *(case 3), we need to distinguish a few cases.*
*If* $\rho_1$ *and* $\rho_2$ *are matching inferences (case 3.1), then we can apply only one instance of it:*

$$\frac{\varphi_1}{S_1}\,\rho \quad \bowtie \quad \frac{\varphi_2}{S_2}\,\rho \quad = \quad \frac{\left(\varphi_1 \bowtie \varphi_2\right)}{S}\,\rho \tag{3.1.1}$$

$$\frac{\varphi_1 \quad \varphi_1'}{S_1}\,\rho \quad \bowtie \quad \frac{\varphi_2 \quad \varphi_2'}{S_2}\,\rho \quad = \quad \frac{\left(\varphi_1 \bowtie \varphi_2\right) \quad \varphi_1'}{S}\,\rho \tag{3.1.2}$$

$$\frac{\varphi_1' \quad \varphi_1}{S_1}\,\rho \quad \bowtie \quad \frac{\varphi_2' \quad \varphi_2}{S_2}\,\rho \quad = \quad \frac{\varphi_1' \quad \left(\varphi_1 \bowtie \varphi_2\right)}{S}\,\rho \tag{3.1.3}$$

$$\frac{\varphi_1 \quad \varphi_1'}{S_1}\,\rho \quad \bowtie \quad \frac{\varphi_2' \quad \varphi_2}{S_2}\,\rho \quad = \quad \frac{\varphi_2' \quad \varphi_1'}{S}\,\rho \tag{3.1.4}$$

$$\frac{\varphi_1' \quad \varphi_1}{S_1}\,\rho \quad \bowtie \quad \frac{\varphi_2 \quad \varphi_2'}{S_2}\,\rho \quad = \quad \frac{\varphi_1' \quad \varphi_2'}{S}\,\rho \tag{3.1.5}$$

*The new joins are on paths of length* $(j-1, k-1)$ *(except on the last two cases, in which there will be no more joins defined), therefore the induction hypothesis holds. It is worth noting that the last four cases might require some weakenings to obtain formulas occurring in the branches that were not used. Also, the choice of* $\varphi_1'$ *on the second and third cases is arbitrary,* $\varphi_2'$ *could be used just as well.*

*If* $\rho_1$ *has a match in* $P_2$ *but this is not* $\rho_2$ *(case 3.2), then we can apply only* $\rho_2$ *and keep* $\rho_1$ *until reaching its match. In the following cases,* $\rho_1$ *could be a unary or binary inference. The join is then defined as:*

$$\dfrac{\varphi_1}{S_1}\,\rho_1 \quad\Join\quad \dfrac{\varphi_2}{S_2}\,\rho_2 \quad=\quad \dfrac{\left(\dfrac{\varphi_1}{S_1}\,\rho_1 \Join \varphi_2\right)}{S}\,\rho_2 \tag{3.2.1}$$

$$\dfrac{\varphi_1}{S_1}\,\rho_1 \quad\Join\quad \dfrac{\varphi_2 \quad \varphi_2'}{S_2}\,\rho_2 \quad=\quad \dfrac{\left(\dfrac{\varphi_1}{S_1}\,\rho_1 \Join \varphi_2\right) \quad \varphi_2'}{S}\,\rho_2 \tag{3.2.2}$$

$$\dfrac{\varphi_1}{S_1}\,\rho_1 \quad\Join\quad \dfrac{\varphi_2' \quad \varphi_2}{S_2}\,\rho_2 \quad=\quad \dfrac{\varphi_2' \quad \left(\dfrac{\varphi_1}{S_1}\,\rho_1 \Join \varphi_2\right)}{S}\,\rho_2 \tag{3.2.3}$$

*The new join is on paths of length* $(j, k-1)$.

*If* $\rho_2$ *has a matching inference in* $P_1$ *but this is not* $\rho_2$ *(case 3.3), the cases are analogous as the previous ones:*

$$\dfrac{\varphi_1}{S_1}\,\rho_1 \quad\Join\quad \dfrac{\varphi_2}{S_2}\,\rho_2 \quad=\quad \dfrac{\left(\varphi_1 \Join \dfrac{\varphi_2}{S_2}\,\rho_2\right)}{S}\,\rho_1 \tag{3.3.1}$$

$$\dfrac{\varphi_1 \quad \varphi_1'}{S_1}\,\rho_1 \quad\Join\quad \dfrac{\varphi_2}{S_2}\,\rho_2 \quad=\quad \dfrac{\left(\varphi_1 \Join \dfrac{\varphi_2}{S_2}\,\rho_2\right) \quad \varphi_1'}{S}\,\rho_1 \tag{3.3.2}$$

$$\dfrac{\varphi_1' \quad \varphi_1}{S_1}\,\rho_1 \quad\Join\quad \dfrac{\varphi_2}{S_2}\,\rho_2 \quad=\quad \dfrac{\varphi_1' \quad \left(\varphi_1 \Join \dfrac{\varphi_2}{S_2}\,\rho_2\right)}{S}\,\rho_1 \tag{3.3.3}$$

*This time the new join is on paths of length* $(j-1, k)$.

*If neither* $\rho_1$ *nor* $\rho_2$ *have matching inferences in* $P_2$ *and* $P_1$ *respectively (case 3.4), then both inferences need to be applied in some order. This order is not clear from the beginning, and some backtracking might be necessary to find the correct one.*

$$\dfrac{\varphi_1}{S_1}\,\rho_1 \quad\bowtie\quad \dfrac{\varphi_2}{S_2}\,\rho_2 \quad=\quad \dfrac{\dfrac{\left(\varphi_1 \bowtie \varphi_2\right)}{S}\,\rho_1}{S'}\,\rho_2 \tag{3.4.1}$$

$$\dfrac{\varphi_1 \quad \varphi_1'}{S_1}\,\rho_1 \quad\bowtie\quad \dfrac{\varphi_2}{S_2}\,\rho_2 \quad=\quad \dfrac{\dfrac{\left(\varphi_1 \bowtie \varphi_2\right) \quad \varphi_1'}{S}\,\rho_1}{S'}\,\rho_2 \tag{3.4.2}$$

$$\dfrac{\varphi_1' \quad \varphi_1}{S_1}\,\rho_1 \quad\bowtie\quad \dfrac{\varphi_2}{S_2}\,\rho_2 \quad=\quad \dfrac{\dfrac{\varphi_1' \quad \left(\varphi_1 \bowtie \varphi_2\right)}{S}\,\rho_1}{S'}\,\rho_2 \tag{3.4.3}$$

$$\dfrac{\varphi_1}{S_1}\,\rho_1 \quad\bowtie\quad \dfrac{\varphi_2 \quad \varphi_2'}{S_2}\,\rho_2 \quad=\quad \dfrac{\dfrac{\left(\varphi_1 \bowtie \varphi_2\right)}{S}\,\rho_1 \qquad \varphi_2'}{S'}\,\rho_2 \tag{3.4.4}$$

$$\dfrac{\varphi_1}{S_1}\,\rho_1 \quad\bowtie\quad \dfrac{\varphi_2' \quad \varphi_2}{S_2}\,\rho_2 \quad=\quad \dfrac{\varphi_2' \qquad \dfrac{\left(\varphi_1 \bowtie \varphi_2\right)}{S}\,\rho_1}{S'}\,\rho_2 \tag{3.4.5}$$

$$\dfrac{\varphi_1 \quad \varphi_1'}{S_1}\,\rho_1 \quad\bowtie\quad \dfrac{\varphi_2 \quad \varphi_2'}{S_2}\,\rho_2 \quad=\quad \dfrac{\dfrac{\left(\varphi_1 \bowtie \varphi_2\right) \quad \varphi_1'}{S}\,\rho_1 \qquad \varphi_2'}{S'}\,\rho_2 \tag{3.4.6}$$

$$\dfrac{\varphi_1 \quad \varphi_1'}{S_1}\,\rho_1 \quad\bowtie\quad \dfrac{\varphi_2' \quad \varphi_2}{S_2}\,\rho_2 \quad=\quad \dfrac{\varphi_2' \qquad \dfrac{\left(\varphi_1 \bowtie \varphi_2\right) \quad \varphi_1'}{S}\,\rho_1}{S'}\,\rho_2 \tag{3.4.7}$$

$$\dfrac{\varphi_1' \quad \varphi_1}{S_1}\,\rho_1 \quad\bowtie\quad \dfrac{\varphi_2 \quad \varphi_2'}{S_2}\,\rho_2 \quad=\quad \dfrac{\dfrac{\varphi_1' \quad \left(\varphi_1 \bowtie \varphi_2\right)}{S}\,\rho_1 \qquad \varphi_2'}{S'}\,\rho_2 \tag{3.4.8}$$

$$\dfrac{\varphi_1' \quad \varphi_1}{S_1}\,\rho_1 \quad\bowtie\quad \dfrac{\varphi_2' \quad \varphi_2}{S_2}\,\rho_2 \quad=\quad \dfrac{\varphi_2' \qquad \dfrac{\varphi_1' \quad \left(\varphi_1 \bowtie \varphi_2\right)}{S}\,\rho_1}{S'}\,\rho_2 \tag{3.4.9}$$

*The new join is on paths of length* $(j-1, k-1)$.

*One last (very!) important observation is regarding weakenings, which must be skipped under certain conditions. Whenever* $\rho_1$*,* $\rho_2$ *or* $\rho$ *is a weakening, we must check the end-sequent of the current join. If the weakened formula is already present (the* same *end-sequent ancestor as in the original proof), then the weakening should not be applied. Otherwise it should be applied.*

We realize the method does not make clear what is the proper inference order application. In some cases it does not matter the order, the only difference is that one final proof will be more redundant then the other. In other cases, the algorithm works with one ordering but not with another. There is some search involved to find out the correct order.

As a side remark, we note that although the definition seems to be constructing the join bottom-up, this is just for induction purposes. In our examples we find it easier to construct them top-down, by first joining the corresponding axioms and taking down from there.

For this method it is worth using o-projections (Definition 13), since this avoids unnecessary weakenings.

## Example

Before analyzing what happens to the critical inferences during the join, let's show an example of joining in a proof to make this clearer. The proof, admittedly, has no critical rules, but it is a non-trivial case of joining. Let $\varphi$ be the following proof with 2 cuts. The cuts are colored in green and blue to differentiate the atoms in the clause set.

$$\dfrac{\dfrac{\dfrac{\overline{A \vdash A}}{A \vdash A \vee C}\vee_r \quad \dfrac{\overline{C \vdash C}}{C \vdash A \vee C}\vee_r}{A \vee C \vdash A \vee C}\vee_l \quad \dfrac{\dfrac{\dfrac{\dfrac{\overline{A \vdash A}}{A \vdash A \vee C}\vee_r \quad \dfrac{\overline{C \vdash C}}{C \vdash A \vee C}\vee_r}{A \vee C \vdash A \vee C}\vee_l \quad \dfrac{\dfrac{\overline{B \vdash B} \quad \overline{A \vdash A}}{B, A \vdash B \wedge A}\wedge_r}{B \wedge A \vdash B \wedge A}\wedge_l}{A \vee C, (A \vee C) \rightarrow (B \wedge A) \vdash B \wedge A}\rightarrow_l \quad \dfrac{\dfrac{\overline{B \vdash B} \quad \overline{A \vdash A}}{B, A \vdash B \wedge A}\wedge_r}{B \wedge A \vdash B \wedge A}\wedge_l}{A \vee C, (A \vee C) \rightarrow (B \wedge A) \vdash B \wedge A}cut}{(A \vee C) \rightarrow (B \wedge A), A \vee C \vdash B \wedge A}cut$$

Its clause set is:

$$CL(\varphi) = \{\vdash A, C \ ; \ B, A \vdash \ ; \ A \vdash B \ ; \ C \vdash B \ ; \ C \vdash A\}$$

The o-projections are:

$$\pi(\vdash A, C): \qquad \dfrac{\overline{A \vdash A} \quad \overline{C \vdash C}}{A \vee C \vdash A, C}\vee_l$$

$$\pi(B, A \vdash): \qquad \dfrac{\overline{B \vdash B} \quad \overline{A \vdash A}}{B, A \vdash B \wedge A}\wedge_r$$

$$\pi(A \vdash B):$$

$$\dfrac{\dfrac{\overline{A \vdash A}}{A \vdash A \vee C}\vee_r \quad \dfrac{\dfrac{\overline{B \vdash B}}{B, A \vdash B}w_l}{B \wedge A \vdash B}\wedge_l}{(A \vee C) \rightarrow (B \wedge A), A \vdash B}\rightarrow_l$$

$\pi(C \vdash B)$ :

$$\dfrac{\dfrac{\dfrac{}{C \vdash C}}{C \vdash A \vee C}\vee_r \qquad \dfrac{\dfrac{\dfrac{}{B \vdash B}}{B, A \vdash B}w_l}{B \wedge A \vdash B}\wedge_l}{(A \vee C) \to (B \wedge A), C \vdash B}\to_l$$

$\pi(C \vdash A)$ :

$$\dfrac{\dfrac{\dfrac{}{C \vdash C}}{C \vdash A \vee C}\vee_r \qquad \dfrac{\dfrac{\dfrac{}{A \vdash A}}{B, A \vdash A}w_l}{B \wedge A \vdash A}\wedge_l}{(A \vee C) \to (B \wedge A), C \vdash A}\to_l$$

Suppose we want to use the following resolution refutation to guide the joins[1]:

$$\dfrac{\dfrac{\vdash A, C \qquad \dfrac{\dfrac{C \vdash A \qquad \dfrac{\dfrac{C \vdash A \quad A \vdash B}{C \vdash B}R \qquad B, A \vdash}{C, A \vdash}R}{C, C \vdash}R}{C \vdash}c_l}{\vdash A}R \qquad \dfrac{\dfrac{\dfrac{\dfrac{\vdash A, C \qquad \dfrac{C \vdash A \quad A \vdash B}{C \vdash B}R}{\vdash A, B}R \qquad A \vdash B}{\vdash B, B}R}{\vdash B}c_r \qquad B, A \vdash}{A \vdash}R}{\vdash}R$$

This means that we need to join projections $\pi(C \vdash A)$ with $\pi(A \vdash B)$, via atom $A$. The result is:

$$\dfrac{\dfrac{\dfrac{}{C \vdash C}}{C \vdash A \vee C}\vee_r \qquad \dfrac{\dfrac{\dfrac{}{B \vdash B}}{B, A \vdash B}w_l}{B \wedge A \vdash B}\wedge_l}{(A \vee C) \to (B \wedge A), C \vdash B}\to_l$$

Note that the branches with $A$, which were actually being joined, disappeared. This corresponds to case (3.1.5).

This resulting derivation has to be joined with $\pi(B, A \vdash)$ via atom $B$. There is some liberty for choosing which inference is applied first, we show here two possible results of this join:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{C \vdash C}}{C \vdash A \vee C}\vee_r \qquad \dfrac{\dfrac{\dfrac{}{B \vdash B}}{B, A \vdash B}w_l}{B \wedge A \vdash B}\wedge_l}{(A \vee C) \to (B \wedge A), C \vdash B}\to_l \qquad \dfrac{}{A \vdash A}}{(A \vee C) \to (B \wedge A), C, A \vdash B \wedge A}\wedge_r \qquad \dfrac{\dfrac{\dfrac{}{C \vdash C}}{C \vdash A \vee C}\vee_r \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{B \vdash B} \quad \dfrac{}{A \vdash A}}{B, A \vdash B \wedge A}\wedge_r}{B, A, A \vdash B \wedge A}w_l}{B \wedge A, A \vdash B \wedge A}\wedge_l}{(A \vee C) \to (B \wedge A), C, A \vdash B \wedge A}\to_l$$

Our analysis indicates that using the right derivation is better in our case, since it duplicates less branches in the proof. The difference is mainly on the order of application of $\to_l$ and $\wedge_r$,

[1]This is complicated on purpose.

the two binary rules. Observe also that $w_l$ still needs to be applied in this case, because the $A$ occurring in the sequent is a cut ancestor.

We choose then the right derivation and join it with $\pi(C \vdash A)$ via $A$, resulting in:

$$\dfrac{\dfrac{\dfrac{}{C \vdash C}}{C \vdash A \lor C}\lor_r \qquad \dfrac{\dfrac{\dfrac{}{B \vdash B} \quad \dfrac{}{A \vdash A}}{B, A \vdash B \land A}\land_r}{B \land A \vdash B \land A}\land_l}{(A \lor C) \to (B \land A), C \vdash B \land A}\to_l$$

Interesting things happened in this step. First of all, the $w_l$ was skipped, since $A$ was available as the correct end-sequent ancestor. Second of all, there is only one occurrence of $C$ in the end-sequent of this derivation. If you look carefully at the resolution refutation that is guiding us, you will see that there are two $C$'s, but this is not the result of the join, since one of the branches from which one $C$ came from was not used. In general, we can say that the induced clause resulting from a join subsumes the resolvent of the two original clauses. We will soon explain about this subsumption property. Before that, let's finish the joins for this proof, it's a good learning exercise.

Next step is to join the last derivation with $\pi(\vdash A, C)$. We will refer to the result as $\varphi_{\vdash A}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A} \quad \dfrac{}{C \vdash C}}{A \lor C \vdash A, C}\lor_l}{A \lor C \vdash A, A \lor C}\lor_r \qquad \dfrac{\dfrac{\dfrac{}{B \vdash B} \quad \dfrac{}{A \vdash A}}{B, A \vdash B \land A}\land_r}{B \land A \vdash B \land A}\land_l}{A \lor C, (A \lor C) \to (B \land A) \vdash B \land A, A}\to_l$$

Now let's deal with the other branch of the resolution refutation. First we join $\pi(C \vdash A)$ with $\pi(A \vdash B)$ via $A$:

$$\dfrac{\dfrac{\dfrac{}{C \vdash C}}{C \vdash A \lor C}\lor_r \qquad \dfrac{\dfrac{\dfrac{}{B \vdash B}}{B, A \vdash B}w_l}{B \land A \vdash B}\land_l}{(A \lor C) \to (B \land A), C \vdash B}\to_l$$

This is again case (3.1.5), in which the joining branch itself is discarded. We now join this with $\pi(\vdash A, C)$ via $C$:

$$\dfrac{\dfrac{\dfrac{}{A \vdash A} \quad \dfrac{\dfrac{}{C \vdash C}}{C \vdash A \lor C}\lor_r}{A \lor C \vdash A, A \lor C}\lor_l \qquad \dfrac{\dfrac{\dfrac{}{B \vdash B}}{B, A \vdash B}w_l}{B \land A \vdash B}\land_l}{A \lor C, (A \lor C) \to (B \land A) \vdash A, B}\to_l$$

When joining these branches one curious thing happens. At first it seems arbitrary if we apply $\lor_l$ before $\lor_r$ or the other way around on the left branch, but once we get to the next join we can only continue if we apply $\lor_r$ before (above) $\lor_l$. If the wrong order is used, there will be one extra atom in the derivation. Don't believe me? Ok, here we go. Imagine that, top-down, the rules in the $A$ branch of the derivation above are $\{\vee_l, \vee_r, \to_l\}$. The next step is to join this branch with $\pi(A \vdash B)$. The first rule in the projection branch is $\vee_r$, which has a match in the current derivation. So we need to apply $\vee_l$ first (top-down) to get to the matching inference, which results in the following derivation:

$$\dfrac{\dfrac{\overline{A \vdash A} \quad \overline{C \vdash C}}{A \vee C \vdash A, C}\,\vee_l}{A \vee C \vdash A, A \vee C}\,\vee_r$$

See the extra $A$ there? We need to be careful with that.

So let's say we use the correct order, the join can be performed with no problems, and the same effect of non-duplication occurs as before. Although the resolvent of this resolution has two $B$'s, the joined derivation has only one. As a plus, we don't have to worry about this contraction. We will skip this step and show already the join with $\pi(B, A)$, which we call $\varphi_{A\vdash}$:

$$\dfrac{\dfrac{\dfrac{\overline{A \vdash A}}{A \vdash A \vee C}\,\vee_r \quad \overline{C \vdash C}}{A \vee C \vdash A \vee C}\,\vee_l \qquad \dfrac{\dfrac{\dfrac{\overline{B \vdash B} \quad \overline{A \vdash A}}{B, A \vdash B \wedge A}\,\wedge_r}{B, A, A \vdash B \wedge A}\,w_l}{B \wedge A, A \vdash B \wedge A}\,\wedge_l}{A \vee C, (A \vee C) \to (B \wedge A), A \vdash B \wedge A}\,\to_l$$

Final step! We must join $\varphi_{\vdash A}$ with $\varphi_{A\vdash}$ via $A$, of course. We will start by applying $\wedge_r$:

$$\dfrac{\overline{B \vdash B} \quad \overline{A \vdash A}}{B, A \vdash B \wedge A}\,\wedge_r$$

Now there is a weakening of $A$ on the left. Do we need this? There is an $A$ already there, but we *do* need the weakening. If you observe, the $A$ which is present is an ancestor of $A \vee C$, so *not* the same weakened $A$. But alas! We don't need this branch after all. This is case (3.1.4), in which the joined branch is discarded. Well, it was a good example for the weakening anyway. The final proof is then:

$$\dfrac{\dfrac{\dfrac{\overline{A \vdash A}}{A \vdash A \vee C}\,\vee_r \quad \overline{C \vdash C}}{A \vee C \vdash A \vee C}\,\vee_l \qquad \dfrac{\dfrac{\overline{B \vdash B} \quad \overline{A \vdash A}}{B, A \vdash B \wedge A}\,\wedge_r}{B \wedge A \vdash B \wedge A}\,\wedge_l}{A \vee C, (A \vee C) \to (B \wedge A) \vdash B \wedge A}\,\to_l$$

## Subsumption

Now let's discuss the subsumption property. You might have noticed that, when doing the join operation, some branches of the proof can disappear. These branches might contain some atomic cut ancestor, which are not necessarily the one on which the join is guided. Can we spare these atoms? Turns out we can, because the new clause subsumes the original one.

**Definition 33** (Subsumption)**.** *A clause $C_1$ subsumes a clause $C_2$, written $C_1 \leq C_2$, if there exists a substitution $\sigma$ such that the literals in $C_1\sigma$ are a subset of the literals in $C_2$.*

In the case of joining, there is no need for a substitution $\sigma$.

**Theorem 15.** *Let $C_1$ and $C_2$ be two clauses resolved in the resolution refutation via an atom $A$ and $C_r$ its resolvent. Let $\varphi$ be $\pi(C_1) \bowtie_A \pi(C_2)$ and $C_i$ be the clause composed of the cut ancestors in $\varphi$'s end-sequent. Then $C_i \leq C_r$.*

*Proof.* From the definition of projections, we know that every literal occurring in $C_r$ is in an axiom of $\pi(C_1)$ or $\pi(C_2)$. When these projections are joined, as in Definition 32, some axioms might not be used anymore simply because a branch of the projection was not used. Therefore, the literals appearing in $C_i$ will be only a subset of those appearing in $C_r$. $\square$

Because of the subsumption principle [23], we know that if $C' \leq C$, then $C$ can be replaced by $C'$ in a resolution refutation. Therefore the simpler clauses obtained by joining can still be used in a new (and possibly simpler) resolution refutation that will guide future joins.

### Quantifiers

One (more) thing we need to be careful about when joining derivations is with quantifier rules. Since we are working with skolemized proofs, there will be no strong quantifiers, which means that we don't need to worry about eigenvariable violations. But we do need to worry when matching inferences are quantifier inferences. If they use the same term in the substitution, we can apply one quantifier with no problem. What if they use different terms?

Suppose two weak quantifiers are matching inferences. This means that they are occurrence of the same inference in the input proof. In the original proof, this quantifier might have been instantiated with a term $t$ which has an eigenvariable from a strong quantifier applied to a cut-ancestor or not.

If $t$ does not contain an eigenvariable, then its occurrence in every projections will be the same, and the rules can be merged in one.

If, on the other hand, $t$ does contain an eigenvariable, then this might have been instantiated to different terms, resulting from the unifications during resolution refutation. In this case, the week quantifier inference needs to be applied twice and the main formula contracted. Observe that this does not pose a problem for **LJ'** because week quantifier inferences do not have structural restrictions.

### Critical inferences

But will this join always result in an intuitionistic proof? Suppose $\rho$ is a violated instance of a critical inference in a projection. Eventually the branch to which $\rho$ belongs will be joined (if not, then it's not used in the final proof and we don't care about this inference anyway. This could happen if there was some subsumed clause, for example.). During joining, a few things might occur:

- The branch where $\rho$ is might disappear. In this case we don't have to worry about anything, since the violated inference will vanish completely.

- $\rho$ has a matching inference, say $\rho'$, on the other branch. Suppose $\rho$ is violated by $n$ atoms and $\rho'$ is violated by $n'$ atoms. The join operation is for sure in an atom $A$ that violated both $\rho$ and $\rho'$. The new $\rho^*$ inference, in the result of the join, will be violated by $n + n' - 2$, worst case scenario. Considering that, after all the joins, these violating atoms will no longer occur, our intuition tells us that there will be no more violations.

- $\rho$ has no matching inference. In this case, it might be crucial the order in which $\rho$ is applied. As we've seen on the previous example, the ordering might determine whether some matching inference will occur or not on the next join, and for critical rules it is very important that matching inferences are found. On the other hand, $\rho$ needs to be applied anyway. Of course the violating atom $A$ will no longer occur, but we need (can we?) to make sure that there are no other formulas on the right side (ancestors of end-sequent). Again in this case the order of rule applications plays a major role.

In fact, in the general case we cannot always fix the critical inferences. If we take the propositional proof from Chapter 5, we see that, no matter how the join is performed, the final proof is still classical. On the next chapter we try to work around this.

CHAPTER 10

# Indexing CERES

Remember I said, a long time ago, in Chapter 5, that ignoring the tautologies from the clause set was important? Well, for intuitionistic logic, *not* ignoring them is what is important. It turns out we will need to use all the clauses for a resolution refutation if we want to transform the final ACNF into an intuitionistic proof.

## 10.1 Indexing

In his thesis [28], Bruno Woltzenlogel Paleo discussed resolution refinements that would reduce the distance between the normal forms generated by the CERES method and the normal forms generated by reductive cut-elimination methods. By restricting the way resolutions are constructed, he argued, CERES would involve less proof search and be closer to a "genuine" proof-transformation. His examples show that, already in classical logic, there are normal forms generated by CERES which cannot be generated by reductive cut-elimination. The refinements aim to bring these normal forms together.

In intuitionistic logic the situation is even more serious. Observe that the ACNF generated by CERES has the characteristic that all cuts and possibly some contractions occur below all logical inferences. Now, intuitionistic proofs with atomic cuts cannot always be transformed into such format. Since there is less liberty on the order of inferences, it is not always the case that the cuts can be permuted down all logical inferences. This is a first indication that the ACNF will eventually turn out classical.

Since we know reductive cut-elimination is possible in the intuitionistic calculi **LJ** and **LJ'**, we will use one of the resolution refinements defined in Bruno's[1] thesis [28] in hope that this will generate ACNF's which can be enventually transformed into intuitionistic proofs.

The refinement we will use is called "atomic cut-linkage" ( [28], page 154, definition 6.2.3). Its main idea is to restrict the resolution refutation in a way that the pairs of resolved atoms are exactly those atoms which would be cut had a reductive cut-elimination method been used. To

[1] In Brazil, we usually refer to people on a first-name basis. I and Bruno are both Brazilians.

enforce this, we will use indices for atoms occuring in cut-ancestors and make sure that only atoms with the same index are resolved in the resolution refutation. First of all, we define how to index these atoms in the input proof.

**Definition 34** (Atom indexing)**.** *Let $\varphi$ be a proof with cuts and $F$ the auxiliary formula of a cut inference. For each atomic sub-formula of $F$ we assign an index $i$ different from the indices used in all other cuts of $\varphi$. It is important to note that the atomic sub-formulas of two auxiliary formulas of the same cut, on both premises, will have* the same *index.*

In the proof below, the atom indexing is represented by superscripts on the green atoms, which are the cut ancestors.

$$
\dfrac{
\dfrac{
\dfrac{\overline{P \vdash P^1}}{P \vdash P^1 \lor \neg P}\,\lor_r
\quad
\dfrac{\dfrac{\dfrac{\overline{P^2 \vdash P}}{P^2, \neg P \vdash}\,\neg_l}{\neg P \vdash \neg P^2}\,\neg_r}{\neg P \vdash P^1 \lor \neg P^2}\,\lor_r
}{P \lor \neg P \vdash P^1 \lor \neg P^2}\,\lor_l
\qquad
\dfrac{
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{P \vdash P^2}}{\neg P^2, P \vdash}\,\neg_l}{\neg P^2 \vdash \neg P}\,\neg_r}{\neg P^2, \neg\neg P \vdash}\,\neg_l}{\neg P^2, \neg\neg P \vdash P}\,w_r}{\neg P^2 \vdash \neg\neg P \to P}\,\to_r
\quad
\dfrac{\dfrac{\overline{P^1 \vdash P}}{P^1, \neg\neg P \vdash P}\,w_l}{P^1 \vdash \neg\neg P \to P}\,\to_r
}{P^1 \lor \neg P^2 \vdash \neg\neg P \to P}\,\lor_l
}{P \lor \neg P \vdash \neg\neg P \to P}\,cut
$$

Given the indexing, we define a resolution refinement that will take these indices into account and only allows atoms with the same index to be resolved, which we will call *indexed resolution*. Observe that, if we perform reductive cut-elimination until having only atomic cuts in a proof, these will be cuts exactly on atoms with the same index. The completeness of this resolution refinement then comes for free.

Note, though, that the completeness is only possible because we have clause sets coming from a proof with cuts. It makes no sense to consider the refinement in a general setting of resolution.

If we now consider the indices as part of the language (part of the atomic predicate, for example), we can use the same resolution calculus to find a refutation of the clause set, and the condition that only atoms with the same index are resolved will be fulfilled. The clause set of our example proof, thus, is the following:

$$CL = \{P^2 \vdash P^1 \;\;;\;\; P^1 \vdash \;\;;\;\; \vdash P^2\}$$

Now, if you remember correctly, had we not used indices, the clause $P^2 \vdash P^1$ would not be part of the clause set, since it would be a tautology. But now it needs to be considered. Even more so because, had we only $P^1 \vdash$ and $\vdash P^2$, the clause set would not be refutable. In the new language, $P^1 \neq P^2$. We get then a different resolution refutation:

$$
\dfrac{\vdash P^2 \qquad \dfrac{P^2 \vdash P^1 \quad P^1 \vdash}{P^2 \vdash}\,R}{\vdash}\,R
$$

And we need to compute the o-projections of the three clauses:

$\pi(\vdash P^2)$

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$P \vdash P^2$}
\RightLabel{$\neg_r^\dagger$}
\UnaryInfC{$\vdash P^2, \neg P$}
\RightLabel{$\neg_l$}
\UnaryInfC{$\neg\neg P \vdash P^2$}
\RightLabel{$w_r$}
\UnaryInfC{$\neg\neg P \vdash P^2, P$}
\RightLabel{$\rightarrow_r^\dagger$}
\UnaryInfC{$\vdash P^2, \neg\neg P \rightarrow P$}
\end{prooftree}
$$

$\pi(P^1 \vdash)$

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$P^1 \vdash P$}
\RightLabel{$w_l$}
\UnaryInfC{$P^1, \neg\neg P \vdash P$}
\RightLabel{$\rightarrow_r$}
\UnaryInfC{$P^1 \vdash \neg\neg P \rightarrow P$}
\end{prooftree}
$$

$\pi(P^2 \vdash P^1)$

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$P \vdash P^1$}
\AxiomC{}
\UnaryInfC{$P^2 \vdash P$}
\RightLabel{$\neg_r$}
\UnaryInfC{$\neg P, P^2 \vdash$}
\RightLabel{$w_r$}
\UnaryInfC{$\neg P, P^2 \vdash P^1$}
\RightLabel{$\vee_l$}
\BinaryInfC{$P \vee \neg P, P^2 \vdash P^1$}
\end{prooftree}
$$

The use of this resolution refinement changes nothing on the projections. They still contain some †-inference. What changes then? Well, with a different resolution refutation, the ACNF changes:

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$P \vdash P^2$}
\RightLabel{$\neg_r^\dagger$}
\UnaryInfC{$\vdash P^2, \neg P$}
\RightLabel{$\neg_l$}
\UnaryInfC{$\neg\neg P \vdash P^2$}
\RightLabel{$w_r$}
\UnaryInfC{$\neg\neg P \vdash P^2, P$}
\RightLabel{$\rightarrow_r^\dagger$}
\UnaryInfC{$\vdash P^2, \neg\neg P \rightarrow P$}
\AxiomC{}
\UnaryInfC{$P \vdash P^1$}
\AxiomC{}
\UnaryInfC{$P^2 \vdash P$}
\RightLabel{$\neg_r$}
\UnaryInfC{$\neg P, P^2 \vdash$}
\RightLabel{$w_r$}
\UnaryInfC{$\neg P, P^2 \vdash P^1$}
\RightLabel{$\vee_l$}
\BinaryInfC{$P \vee \neg P, P^2 \vdash P^1$}
\AxiomC{}
\UnaryInfC{$P^1 \vdash P$}
\RightLabel{$w_l$}
\UnaryInfC{$P^1, \neg\neg P \vdash P$}
\RightLabel{$\rightarrow_r$}
\UnaryInfC{$P^1 \vdash \neg\neg P \rightarrow P$}
\RightLabel{$cut$}
\BinaryInfC{$P \vee \neg P, P^2 \vdash \neg\neg P \rightarrow P$}
\RightLabel{$cut$}
\BinaryInfC{$P \vee \neg P \vdash \neg\neg P \rightarrow P, \neg\neg P \rightarrow P$}
\RightLabel{$c_r$}
\UnaryInfC{$P \vee \neg P \vdash \neg\neg P \rightarrow P$}
\end{prooftree}
$$

It is still classical though. But observe that this ACNF does not have the undesirable effect of an intuitionistically invalid sequent before the cut.

In order to get an intuitionistic proof, we will not try to fix this ACNF, but join the projections using the method of Chapter 9. The resolution refutation that will guide the join will be the indexed one, since using the other simpler refutation will not result in an intuitionistic proof.

The result of joining the projections $\pi(P^2 \vdash P^1)$ and $\pi(P^1 \vdash)$ is[2] the following:

[2] This is just one possible result for the joining.

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$P \vdash P$}
\AxiomC{}
\UnaryInfC{$P^2 \vdash P$}
\RightLabel{$\neg_r$}
\UnaryInfC{$\neg P, P^2 \vdash$}
\RightLabel{$w_r$}
\UnaryInfC{$\neg P, P^2 \vdash P$}
\RightLabel{$\vee_l$}
\BinaryInfC{$P \vee \neg P, P^2 \vdash P$}
\RightLabel{$w_l$}
\UnaryInfC{$P \vee \neg P, P^2, \neg\neg P \vdash P$}
\RightLabel{$\rightarrow_r$}
\UnaryInfC{$P \vee \neg P, P^2 \vdash \neg\neg P \rightarrow P$}
\end{prooftree}
$$

And by joining this derivation with the projection $\pi(\vdash P^2)$ we get the following intuitionistic proof:

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$P \vdash P$}
\AxiomC{}
\UnaryInfC{$P \vdash P$}
\RightLabel{$\neg_l$}
\UnaryInfC{$\neg P, P \vdash$}
\RightLabel{$\neg_r$}
\UnaryInfC{$\neg P \vdash \neg P$}
\RightLabel{$\neg_l$}
\UnaryInfC{$\neg P, \neg\neg P \vdash$}
\RightLabel{$w_r$}
\UnaryInfC{$\neg P, \neg\neg P \vdash P$}
\RightLabel{$\vee_l$}
\BinaryInfC{$P \vee \neg P, \neg\neg P \vdash P$}
\RightLabel{$\rightarrow_r$}
\UnaryInfC{$P \vee \neg P \vdash \neg\neg P \rightarrow P$}
\end{prooftree}
$$

## Predicate Logic

Remember the predicate proof in Chapter 5? Using this resolution refinement plus joining also works for it. Let's show that.

The proof with atom indexing is the following:

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$Pf\alpha^2 \vdash Pf\alpha$}
\AxiomC{}
\UnaryInfC{$P\alpha \vdash P\alpha^1$}
\RightLabel{$\neg_l$}
\UnaryInfC{$\neg P\alpha^1, P\alpha \vdash$}
\RightLabel{$\rightarrow_l$}
\BinaryInfC{$Pf\alpha \rightarrow P\alpha, \neg P\alpha^1, Pf\alpha^2 \vdash$}
\RightLabel{$\forall_l$}
\UnaryInfC{$\forall y.(Pfy \rightarrow Py), \neg P\alpha^1, Pf\alpha^2 \vdash$}
\RightLabel{$\neg_r$}
\UnaryInfC{$\forall y.(Pfy \rightarrow Py), \neg P\alpha^1 \vdash \neg Pf\alpha^2$}
\RightLabel{$\rightarrow_r$}
\UnaryInfC{$\forall y.(Pfy \rightarrow Py) \vdash \neg P\alpha^1 \rightarrow \neg Pf\alpha^2$}
\RightLabel{$\forall_r$}
\UnaryInfC{$\forall y.(Pfy \rightarrow Py) \vdash \forall y.(\neg Py^1 \rightarrow \neg Pfy^2)$}
\AxiomC{}
\UnaryInfC{$Pt^1 \vdash Pt$}
\RightLabel{$\neg_l$}
\UnaryInfC{$\neg Pt, Pt^1 \vdash$}
\RightLabel{$\neg_r$}
\UnaryInfC{$\neg Pt \vdash \neg Pt^1$}
\AxiomC{}
\UnaryInfC{$Pft \vdash Pft^2$}
\RightLabel{$\neg_l$}
\UnaryInfC{$\neg Pft^2, Pft \vdash$}
\RightLabel{$\neg_r$}
\UnaryInfC{$\neg Pft^2 \vdash \neg Pft$}
\RightLabel{$\rightarrow_l$}
\BinaryInfC{$\neg Pt^1 \rightarrow \neg Pft^2, \neg Pt \vdash \neg Pft$}
\RightLabel{$\rightarrow_r$}
\UnaryInfC{$\neg Pt^1 \rightarrow \neg Pft^2 \vdash \neg Pt \rightarrow \neg Pft$}
\RightLabel{$\exists_r$}
\UnaryInfC{$\neg Pt^1 \rightarrow \neg Pft^2 \vdash \exists y.(\neg Py \rightarrow \neg Pfy)$}
\RightLabel{$\forall_l$}
\UnaryInfC{$\forall y.(\neg Py^1 \rightarrow \neg Pfy^2) \vdash \exists y.(\neg Py \rightarrow \neg Pfy)$}
\RightLabel{$cut$}
\BinaryInfC{$\forall y.(Pfy \rightarrow Py) \vdash \exists y.(\neg Py \rightarrow \neg Pfy$}
\end{prooftree}
$$

The clause set with the indices indicated is:

$$
CL = \{Pf\alpha^2 \vdash P\alpha^1 \;\; ; \;\; Pt^1 \vdash \;\; ; \;\; \vdash Pft^2\}
$$

Observe that, using these indices, the resolution that we had before, i.e.:

$$
\begin{prooftree}
\AxiomC{$\vdash Pft^2$}
\AxiomC{$Pt^1 \vdash$}
\RightLabel{$t \leftarrow ft$}
\UnaryInfC{$Pft^1 \vdash$}
\RightLabel{$R$}
\BinaryInfC{$\vdash$}
\end{prooftree}
$$

is not an indexed resolution, since there are two atoms, one with index 1 and the other with index 2, being resolved. The indexed resolution refutation for this clause set is:

$$\dfrac{\dfrac{\vdash Pft^2 \qquad \dfrac{Pf\alpha^2 \vdash P\alpha^1}{Pft^2 \vdash Pt^1}\,\alpha \leftarrow t}{\vdash Pt^1}\,R \qquad Pt^1 \vdash}{\vdash}\,R$$

The o-projections are almost the same, differing on the instantiation of $Pt$. In order to use the indexed resolution we need also the projection of the clause $Pf\alpha \vdash P\alpha$ with $\alpha$ instantiated to $t$.

$$\pi(\vdash Pft^2)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{Pft \vdash Pft^2}}{\vdash Pft^2, \neg Pft}\,\neg_r^\dagger}{\neg Pt \vdash Pft^2, \neg Pft}\,w_l}{\vdash Pft^2, \neg Pt \to \neg Pft}\,\to_r^\dagger}{\vdash Pft^2, \exists y.(\neg Py \to \neg Pfy)}\,\exists_r$$

$$\pi(Pt^1 \vdash)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{Pt^1, \vdash Pt}}{Pt^1, \neg Pt \vdash}\,\neg_l}{Pt^1, \neg Pt \vdash \neg Pft}\,w_r}{Pt^1 \vdash \neg Pt \to \neg Pft}\,\to_r}{Pt^1 \vdash \exists y.(\neg Py \to \neg Pfy)}\,\exists_r$$

$$\pi(Pf\alpha^2 \vdash P\alpha^1)[\alpha \leftarrow t]$$

$$\dfrac{\dfrac{\dfrac{}{Pft^2 \vdash Pft} \qquad \dfrac{}{Pt \vdash Pt^1}}{Pft \to Pt, Pft^2 \vdash Pt^1}\,\to_l}{\forall y.(Pfy \to Py), Pft^2 \vdash Pt^1}\,\forall_l$$

Now, instead of constructing an ACNF, we will join these projections using the indexed refutation as a guidance. We first join $\pi(\vdash Pft^2)$ with $\pi(Pf\alpha \vdash P\alpha)[\alpha \leftarrow t]$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{Pft \vdash Pft} \qquad \dfrac{}{Pt \vdash Pt^1}}{Pft \to Pt, Pft \vdash Pt^1}\,\to_l}{Pft \to Pt \vdash Pt^1, \neg Pft}\,\neg_r}{Pft \to Pt, \neg Pt \vdash Pt^1, \neg Pft}\,w_l}{Pft \to Pt \vdash Pt^1, \neg Pt \to \neg Pft}\,\to_r^\dagger}{Pft \to Pt \vdash Pt^1, \exists y.(\neg Py \to \neg Pfy)}\,\exists_r}{\forall y.(Pfy \to Py) \vdash Pt^1, \exists y.(\neg Py \to \neg Pfy)}\,\forall_l$$

And now we join this derivation with $\pi(Pt^1 \vdash)$ to get the following intuitionistic proof:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{Pft \vdash Pft} \quad \dfrac{\dfrac{}{Pt \vdash Pt}}{Pt, \neg Pt \vdash} \neg_l}{Pft \to Pt, \neg Pt, Pft \vdash} \to_l}{Pft \to Pt, \neg Pt \vdash \neg Pft} \neg_r}{Pft \to Pt \vdash \neg Pt \to \neg Pft} \to_r}{Pft \to Pt \vdash \exists y.(\neg Py \to \neg Pfy)} \exists_r}{\forall y.(Pfy \to Py) \vdash \exists y.(\neg Py \to \neg Pfy)} \forall_l
$$

## 10.2 Conjecture

So it seems that this method works. What did we do? We take the proof with cuts, annotate its atomic cut ancestors with the *atom indexing*, compute the clause set, refute the clause set using indexed resolution, compute the projections and join them according to the refutation. Are we absolutely 100% sure that this method works for every intuitionistic proof? No. That's why we have a conjecture, and not a theorem, here.

**Conjecture 1.** *Let $\varphi$ be a skolemized **LJ** proof with cuts. Then, the computation of the following steps:*

- *Compute the atom indexing of $\varphi$.*
- *Extract the characteristic clause set.*
- *Find a resolution refutation $\mathcal{R}$ using indexed resolution.*
- *Compute the projections.*
- *Join the projections (Definition 32) using $\mathcal{R}$ to guide the joining order.*

*yields an intuitionistic proof.*

CHAPTER 11

# Complexity analysis

One advantage of the CERES method over other reductive cut-elimination methods is the speed-up obtained by the former over the later. Speed-up of what? Of the space in memory needed to remove the cuts by both methods. Note that this is different from the size of the final (cut-free or ACNF) proof. In fact, for the example in which we show this speed-up, the final proof using both methods would be the same. The difference is on the steps in between, where reductive cut-elimination will take a non-elementary amount of space while CERES will not.

We were wondering whether the same results would apply for intuitionistic logic and, it turns out, it does! The proof in which the speed-up is shown is actually intuitionistic *and* belongs to one of the "trivial" classes, in which classical and intuitionistic provability coincide. This means that classical CERES can be applied to these proofs, and the speed-up still holds. We show this in Section 11.1. The changes proposed to the method throughout this thesis do not seem to interfere on this speed-up result, but a more careful analysis is left for the future.

Another interesting complexity issue is the comparison of sizes between classical and intuitionistic proofs. This is related with $\mathbf{LJ}^-$, the class of proofs for which the method of Chapter 6 works. We discuss this in detail here, in Section 11.2.

## 11.1 Complexity of CERES

We introduce now the "power tower"[1] of 2's function, which is a non-elementary function:

$$2_0 = 1$$
$$2_{n+1} = 2^{2_n}$$

For those unfamiliar with it, it is very bad. Really. Worse than exponential, *a lot* worse. Unfortunately, cut-elimination's space complexity is a power tower of 2's. And this holds for *any*

[1] Also called tetration.

cut-elimination method. This follows from two independent results [27, 30] and an explanation can be found here: [4], Corollary 4.3.1.

If the worst case complexity scenario is bad for any method, all we can do is compare them in specific cases. Here we show a proof sequence for which CERES is better than Gentzen's reductive cut-elimination. But first we need to make things more precise.

By Gentzen's cut-elimination I mean reductive cut-elimination (using the rewriting rules in Appendix A) following the strategy of removing the upper-most cuts first.

To explain *better/worse* we need to define a complexity measure. For this example we will deal with space complexity, meaning the space in memory needed to solve a problem. We could also have chosen time complexity, since in Gentzen's cut-elimination algorithm is also $O(2_n)$ in the number of steps. But for CERES we would need to analyze time complexity of resolution. To try and keep things simpler, we will reason only on space complexity, which we can measure just by measuring the biggest proof obtained during reductive cut-elimination and CERES's ACNF size.

The results here were proven in [4] in a slightly different way. Nevertheless, we repeat them since it applies for intuitionistic CERES on the class **LJK** (Definition 17).

One thing you should know (that I will not explain here) is the Statman's proof sequence. We will denote a proof in this sequence as $\gamma_n$ and its end-sequent as $\Delta_n \vdash D_n$, where $D_n$ is one formula. You don't really have to know what it is exactly, but an important property is that eliminating the cuts in theses proofs causes them to blow up in size. The resulting cut-free proof (in fact, *any* resulting cut-free proof) is non-elementary in size with respect to the size of the input proof $\gamma_n$. Also, they are intuitionistic. Keep this in mind.

Let $\varphi$ be the following proof with cuts:

$$\dfrac{\dfrac{\dfrac{}{A \vdash A} \quad \begin{matrix}\gamma_n \\ \Delta_n \vdash D_n\end{matrix}}{A, \Delta_n \vdash A \wedge D_n}\,\wedge_r \qquad \dfrac{\dfrac{\dfrac{}{A \vdash A} \quad \dfrac{}{A \vdash A}}{A, A \to A \vdash A}\,\to_l}{A \wedge D_n, A \to A \vdash A}\,\wedge_l}{A, \Delta_n, A \to A \vdash A}\,cut$$

When using Gentzen's strategy for removing cuts, we start by the upper-most cuts, i.e., those in the Statman proof. As was mentioned before, this blows up the size of the proof, until the very end, when the cut on $A \wedge D_n$ is eliminated and then that huge non-elementary chunk of proof disappears. Still, we need the non-elementary amount of space to hold intermediary steps in this reductive cut-elimination.

Now, what happens when we apply CERES? Well, we first compute the clause set, which is:

$$\begin{aligned} CL(\varphi) &= \{\vdash A\} \cup CL(\gamma_n) \cup \{A \vdash\} \\ &= \{\vdash A \ ; \ A \vdash\} \cup CL(\gamma_n) \end{aligned}$$

Oh, look at that! We don't need the clauses coming for the Statman proof to get a resolution refutation. Which means we don't really need to officially eliminate the cuts from $\gamma_n$ in the first place, since this refutation already indicates we won't need this part of the proof.

$$\frac{\vdash A \quad A \vdash}{\vdash} R$$

By computing the projections and putting it together with the resolution refutation we get the following ACNF, which is obviously elementary in size with respect to $\varphi$.

$$\dfrac{\dfrac{\dfrac{}{A \vdash A} \quad \dfrac{\dfrac{}{A \vdash A} \quad \dfrac{}{A \vdash A}}{A, A \to A \vdash A} \to_l}{A, A \to A \vdash A} cut}{A, \Delta_n, A \to A \vdash A} w_l$$

So by skipping intermediary steps, CERES takes much less space than Gentzen's reductive cut-elimination in this case. And note that all the proofs are intuitionistic, and fall in the trivial class defined previously.

Now, this is one example, and you might be wondering whether there is not another proof such that CERES will take a non-elementary amount of space while Gentzen's method will take only elementary space. The answer is *no*. CERES will be, on the worst case, as bad as Gentzen's method, but never worse. This is also proven in [4], Theorem 6.10.3.

## 11.2 Complexity of $\mathbf{LJ}^-$

In Chapter 6 we discussed briefly about classical and intuitionistic provability of the class $\mathbf{LJ}^-$. Now we will elaborate on that.

First of all, we show that classical proofs can be *much* shorter than intuitionistic proofs. In fact, there exists a sequence of sequents $\mathcal{S}_n$ such that classical proofs of $\mathcal{S}_n$ are recursively bounded while intuitionistic proofs of $\mathcal{S}_n$ have no recursive bound. To see this, we first need to remember that the set of intuitionistic proofs is a subset of the set of classical proofs. This fact alone already indicates that intuitionistic proofs cannot be smaller then classical proofs, since they are classical themselves. Secondly, we show that proofs cannot have a recursive bound in general.

**Theorem 16.** *For every recursive function $f$, there exists a sequence of first order sequents $\mathcal{S}_n^f$ with minimal size proofs $\varphi_n$ such that the size of $\varphi_n$ is bigger than $f(n)$ for infinitely many $n$'s.*

*Proof.* We prove this by contradiction. Assume that there exists a recursive function $f$ such that it bounds every proof sequence $\varphi_n$ of every sequence of sequents $\mathcal{S}_n$. With this bound, we have a decision procedure for the logic, which is simply to search for a proof of $\mathcal{S}_n$ whose size is at most $f(n)$. Since we know that first order logic is undecidable, we reach a contradiction and conclude that the theorem is true. □

Observe that this theorem is valid for both classical and intuitionstic first order logic.

Now we show that classical logic can have a speed-up over intuitionistic logic. Take one of such sequences of sequents $\mathcal{S}_n$ such that the smallest $\mathbf{LJ}$ proofs of this sequence are not recursively bounded (Theorem 16 guarantees that they exist). Say that $\mathcal{S}_n = \Gamma_n \vdash F_n$. Now take a sequence $\mathcal{S}'_n$ which is $\Gamma_n \vdash F_n \vee (A \vee \neg A)$, i.e., a modified $\mathcal{S}_n$. Since $A \vee \neg A$ is not provable in intuitionistic logic, an **LJ** proof will have to discard this part of the sequent by applying an $\vee_r$ rule and go on with the non-recursively bounded proofs of $\Gamma_n \vdash F_n$. On the other hand, a classical proof of such sequents can in fact get rid of $\Gamma_n$ and $F_n$ and prove only $A \vee \neg A$, obtaining a proof sequence which is basically linear on $n$.

This is how much more verbose intuitionistic logic can be. So, in principle, intuitionistic proofs can be super-recursively abbreviated using classical proofs, and classical proofs are, in the worst case, as bad as intuitionistic ones, but never worse.

The interesting thing is that this does not hold for the class of skolemized **LJ**$^-$ proofs! In fact, we can show that, by using CERES with negative resolution (Chapter 7), we obtain recursively bounded intuitionistic proofs from recursively bounded classical proofs of $\Gamma \vdash$.

Let's show how this works on an example first. Let $\varphi$ be the following *classical* proof:

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$Pfa \vdash Pfa$}
\doubleLine
\RightLabel{$w$}
\UnaryInfC{$Pa, Pfa \vdash Pfa, Pffa$}
\RightLabel{$\rightarrow_r$}
\UnaryInfC{$Pa \vdash Pfa, Pfa \rightarrow Pffa$}
\RightLabel{$\rightarrow_r$}
\UnaryInfC{$\vdash Pa \rightarrow Pfa, Pfa \rightarrow Pffa$}
\doubleLine
\RightLabel{$\exists_r$}
\UnaryInfC{$\vdash \exists x.(Px \rightarrow Pfx), \exists x.(Px \rightarrow Pfx)$}
\RightLabel{$c_r$}
\UnaryInfC{$\vdash \exists x.(Px \rightarrow Pfx)$}
\RightLabel{$\neg_l$}
\UnaryInfC{$\neg\exists x.(Px \rightarrow Pfx) \vdash$}
\end{prooftree}
$$

Using $\varphi$ we can construct $\Xi$ which proves the same end-sequent but has a full intuitionistic proof on the right branch of the cut:

$$
\begin{prooftree}
\AxiomC{$\varphi$}
\noLine
\UnaryInfC{$\neg\exists x.(Px \rightarrow Pfx) \vdash$}
\RightLabel{$\neg_r$}
\UnaryInfC{$\vdash \neg\neg\exists x.(Px \rightarrow Pfx)$}
\AxiomC{}
\UnaryInfC{$P\alpha \vdash P\alpha$}
\AxiomC{}
\UnaryInfC{$Pf\alpha \vdash Pf\alpha$}
\RightLabel{$\rightarrow_l$}
\BinaryInfC{$P\alpha \rightarrow Pf\alpha, P\alpha \vdash Pf\alpha$}
\RightLabel{$\rightarrow_r$}
\UnaryInfC{$P\alpha \rightarrow Pf\alpha \vdash P\alpha \rightarrow Pf\alpha$}
\RightLabel{$\exists_r$}
\UnaryInfC{$P\alpha \rightarrow Pf\alpha \vdash \exists x.(Px \rightarrow Pfx)$}
\RightLabel{$\exists_l$}
\UnaryInfC{$\exists x.(Px \rightarrow Pfx) \vdash \exists x.(Px \rightarrow Pfx)$}
\RightLabel{$\neg_l$}
\UnaryInfC{$\neg\exists x.(Px \rightarrow Pfx), \exists x.(Px \rightarrow Pfx) \vdash$}
\RightLabel{$\neg_r$}
\UnaryInfC{$\neg\exists x.(Px \rightarrow Pfx) \vdash \neg\exists x.(Px \rightarrow Pfx)$}
\RightLabel{$\neg_l$}
\UnaryInfC{$\neg\exists x.(Px \rightarrow Pfx), \neg\neg\exists x.(Px \rightarrow Pfx) \vdash$}
\RightLabel{$cut$}
\BinaryInfC{$\neg\exists x.(Px \rightarrow Pfx) \vdash$}
\end{prooftree}
$$

This is a proof with cut, so we can apply the CERES method to eliminate it. But let's not apply classical CERES. Although the proof is classical, we choose to apply CERES with negative resolution. The clause set extracted is the following:

$$CL(\Xi) = \{Pfa \vdash Pfa \ ; \ \vdash P\alpha \ ; \ Pf\alpha \vdash\}$$

Of course we can get rid of the tautology $Pfa \vdash Pfa$. The refutation is then very simple:

$$\dfrac{\dfrac{\vdash P\alpha}{\vdash Pfa}\ \alpha \leftarrow fa \qquad \dfrac{Pf\alpha \vdash}{Pfa \vdash}\ \alpha \leftarrow a}{\vdash}\ R$$

And the projections come only from the intuitionistic side of the proof (which is very convenient):

$\pi(\vdash P\alpha)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P\alpha \vdash P\alpha}}{P\alpha \vdash P\alpha, Pf\alpha}\ w_r}{\vdash P\alpha, P\alpha \to Pf\alpha}\ \to_r^{\dagger}}{\vdash P\alpha, \exists x.(Px \to Pfx)}\ \exists_r}{\neg\exists x.(Px \to Pfx) \vdash P\alpha}\ \neg_l$$

$\pi(Pf\alpha \vdash)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{Pf\alpha \vdash Pf\alpha}}{Pf\alpha, P\alpha \vdash Pf\alpha}\ w_l}{Pf\alpha \vdash P\alpha \to Pf\alpha}\ \to_r}{Pf\alpha \vdash \exists x.(Px \to Pfx)}\ \exists_r}{Pf\alpha, \neg\exists x.(Px \to Pfx) \vdash}\ \neg_l$$

Notice that there is a †-inference in one of the projections, but we have already seen how this can be fixed. The ACNF is the following:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{Pfa \vdash Pfa}}{Pfa \vdash Pfa, Pffa}\ w_r}{\vdash Pfa, Pfa \to Pffa}\ \to_r^{\dagger}}{\vdash Pfa, \exists x.(Px \to Pfx)}\ \exists_r}{\neg\exists x.(Px \to Pfx) \vdash Pfa}\ \neg_l \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{Pfa \vdash Pfa}}{Pfa, Pa \vdash Pfa}\ w_l}{Pfa \vdash Pa \to Pfa}\ \to_r}{Pfa \vdash \exists x.(Px \to Pfx)}\ \exists_r}{Pfa, \neg\exists x.(Px \to Pfx) \vdash}\ \neg_l}{\neg\exists x.(Px \to Pfx), \neg\exists x.(Px \to Pfx) \vdash}\ cut}{\neg\exists x.(Px \to Pfx) \vdash}\ c_l$$

And after reductive cut elimination of the atomic cut we have a nice intuitionistic proof!

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{Pfa \vdash Pfa}}{Pfa, Pa \vdash Pfa}\ w_l}{Pfa \vdash Pa \to Pfa}\ \to_r}{Pfa \vdash \exists x.(Px \to Pfx)}\ \exists_r}{\neg\exists x.(Px \to Pfx), Pfa \vdash}\ \neg_l}{\neg\exists x.(Px \to Pfx), Pfa \vdash Pffa}\ w_r}{\neg\exists x.(Px \to Pfx) \vdash Pfa \to Pffa}\ \to_r}{\neg\exists x.(Px \to Pfx) \vdash \exists x.(Px \to Pfx)}\ \exists_r}{\neg\exists x.(Px \to Pfx), \neg\exists x.(Px \to Pfx) \vdash}\ \neg_l}{\neg\exists x.(Px \to Pfx) \vdash}\ c_l$$

From this we can devise a method to obtain intuitionistic proofs from classical proofs of sequents of the type $\Gamma \vdash$. Moreover, we guarantee that these intuitionistic proofs are recursively bounded.

**Theorem 17.** *For every sequence of sequents $\mathcal{S}_n$ of the form $\Gamma \vdash$, if the classical proofs of $\mathcal{S}_n$ are recursively bounded, then so are the intuitionistic proofs.*

*Proof.* In order to prove the theorem we will define a (primitive recursive) transformation of classical proofs of such sequents into intuitionistic ones. Let $\mathcal{S}$ be an arbitrary sequent $A_1, ..., A_k \vdash$ and let $\varphi_c$ be a classical proof of $\mathcal{S}$. Then we can construct $\Xi$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\varphi_c}{A_1, ..., A_k \vdash}}{A_1 \wedge ... \wedge A_k \vdash}\wedge_l}{\vdash \neg(A_1 \wedge ... \wedge A_k)}\neg_r \qquad \dfrac{\varphi_i}{\neg(A_1 \wedge ... \wedge A_k), A_1, ..., A_k \vdash}}{A_1, ..., A_k \vdash}\; cut$$

We can safely assume that $\varphi_i$ is an intuitionistic proof which is linear on the size of the formulas $A_1, ..., A_k$ and, thus, recursively bounded.

The bounded intuitionistic proof of $\mathcal{S}$ will be obtained as the result of applying CERES with negative resolution (Chapter 7) to $\Xi$. The complexity of this transformation is that of reductive cut-elimination, i.e., non-elementary but primitive recursive. Observe that the classical part of this proof contains only cut ancestors. This means that all clauses in the characteristic clause set coming from $\varphi_c$ will be tautologies. These tautologies can be eliminated from the clause set and will not be used for the resolution refutation. As a consequence, the projections that will assemble the ACNF will be all from $\varphi_i$, i.e., intuitionistic with some possible violations. These violations can be eliminated via reductive cut-elimination of atomic cuts in the ACNF, as indicated by the method. This final step is of at most exponential complexity, and therefore primitive recursive.

We can conclude then that there exists a primitive recursive function $f$ such that, for every sequence of proofs $\varphi_n$ of sequents $\Gamma_n \vdash$, we can obtain a sequence of intuitionistic proofs $\varphi'_n$ whose size is bounded by $f$. $\square$

CHAPTER 12

# Conclusion

You have reached the end. Congratulations! I am afraid few people will come this far. Let's review what happened then (and if you are one of those people that just jump to the conclusion, maybe you will become curious and go back to read something).

## 12.1 Wrapping up

The main goal of this work was to develop a method to eliminate lemmas from constructive proofs. We explained in Chapter 1 why we use intuitionistic logic for constructive proofs and how lemmas are cuts in sequent calculi. Chapter 2 was an introduction to the formalism used, and there we also show an existing method for eliminating cuts, namely, Gentzen's reductive cut-elimination. Then we go on explaining CERES, a cut-elimination method based on resolution which is developed for classical logic (among others). Our main aim was to try and develop a CERES-like method for intuitionistic logic.

Although it seems we have not reached the main result, we believe Conjecture 1 can be proved. On the way there, though, there were many interesting insights. One key observation is that we cannot use unrestricted resolution like it is done for classical logic. This is illustrated by the examples in Chapter 5. Therefore, although the practical problem lies mainly on the projections, we cannot expect to transform the classical ACNF into intuitionistic by using *any* resolution refutation.

Given this limitation, Chapters 6, 7 and 8 present some alternatives.

The subclasses of intuitionistic logic covered by the methods presented in Chapters 6 and 7 are the same, but they use different approaches. In iCERES a new resolution calculus is defined which involves resolving negative atoms as well. On top of that, atomic cut ancestors are shifted to the left side of the sequent using negation inferences. This makes projections intuitionistic and with the new resolution calculus a final intuitionistic ACNF can be assembled without problems. Negative resolution CERES, on the other hand, shows that, by using the negative resolution refinement, the final classical ACNF can be made intuitionistic once the atomic cuts are eliminated reductively.

iCERES has the obvious advantage that no post processing is needed, but new elements and calculus have to be defined. Negative resolution CERES needs a post processing, but the elements used are well-known and fewer modifications on the classical CERES are needed.

The method in Chapter 8, merging CERES, is more esoteric. It studies shapes of input proofs on which the CERES method can be applied unchanged, and a post processing can transform the ACNF from classical into intuitionistic. It turns out that the inferences must occur in a particular order, so we studied extensively the permutation of inferences in **LJ** and, when it is possible to permute the rules and reach this order, merging CERES yields an intuitionistic proof. Because the method depends on inference ordering, it is hard to describe the class on which it works syntactically.

Chapter 9, joining CERES, presents a different approach. Instead of constructing the ACNF, the projections are joined via what would be the atomic cut ancestors in a final proof. One can think that they are joined like zippers, with these atoms disappearing in the process. The joining procedure defined in this chapter, is an interesting proof transformation, that can possibly be used for other purposes. There is still, nevertheless, a great deal of non-determinism in joining, and backtracking would be probably necessary if this is to be implemented.

Even using the joining procedure, we cannot obtain a CERES-like method if we don't restrict the resolution refutation. In Chapter 10 we suggest how this can be worked around. The indexing CERES uses a resolution refinement defined previously which approximates CERES to a reductive cut-elimination method. We conjecture that, by using indexed resolution and joining, we obtain a cut-elimination method by resolution for intuitionistic logic.

Chapter 11 is dedicated to a study of the complexity of the CERES method. We show that the speed-up obtained in the classical case also applies to the trivial classes of the intuitionistic case. In the same chapter, we show a result with respect to the complexity of proofs of sequents of the shape $\Gamma \vdash$.

## 12.2 What the future holds

Obviously the main immediate future direction of this work is to try to prove Conjecture 1. There exists already a vague idea on how to do this, but due to time constraints, it will not be possible to include a proof in this thesis.

A middle term goal is the implementation of the indexed resolution refinement and the joining method, with which experiments on constructive proofs could be run automatically.

It would also be interesting to study how the resulting proofs of the algorithm proposed in the conjecture are related to the results of reductive cut-elimination methods.

That's it! Thank you and good night.

# APPENDIX A

# Reductive cut-elimination

In this chapter we show the rewriting rules for performing reductive cut-elimination in **LK** and **LJ**. Each of the following sections have three subsections, namely: cut-elimination, rank reduction and grade reduction.

The *cut-elimination* section contains the rewriting rules that actually eliminate a cut inference from a derivation. The *rank reduction* section presents the rewriting rules that permute the cut over other inferences in the proof. This is necessary to make the cut *principal*, in which case the rewriting rules from section *grade reduction* can be applied and the complexity of the cut-formula is reduced (except on the case of contraction).

In all rewriting rules, the cut formula and its ancestors are marked in green.

## A.1 Rewriting rules for LK

### Cut-elimination

Over axiom inferences:

$$\frac{\overline{A \vdash A} \quad \overset{\varphi}{A, \Gamma \vdash \Delta}}{\Gamma, A \vdash \Delta}\,cut \quad \rightsquigarrow \quad \overset{\varphi}{A, \Gamma \vdash \Delta}$$

$$\frac{\overset{\varphi}{\Gamma \vdash \Delta, A} \quad \overline{A \vdash A}}{\Gamma \vdash \Delta, A}\,cut \quad \rightsquigarrow \quad \overset{\varphi}{\Gamma \vdash \Delta, A}$$

Over weakening:

$$\frac{\dfrac{\overset{\varphi_1}{\Gamma \vdash \Delta}}{\Gamma \vdash \Delta, A}\,w_r \quad \overset{\varphi_2}{A, \Gamma' \vdash \Delta'}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,cut \quad \rightsquigarrow \quad \frac{\overset{\varphi_1}{\Gamma \vdash \Delta}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,w_r^*, w_l^*$$

$$\dfrac{\overset{\varphi_1}{\Gamma \vdash \Delta, A} \qquad \dfrac{\overset{\varphi_2}{\Gamma' \vdash \Delta'}}{A, \Gamma' \vdash \Delta'}\, w_l}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\, \mathit{cut} \quad \rightsquigarrow \quad \dfrac{\overset{\varphi_2}{\Gamma' \vdash \Delta'}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\, w_r^*, w_l^*$$

## Rank reduction

Over unary inferences:

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma^* \vdash \Delta^*, A}}{\Gamma \vdash \Delta, A}\, \rho \qquad \overset{\varphi_2}{A, \Gamma' \vdash \Delta'}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\, \mathit{cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma^* \vdash \Delta^*, A} \qquad \overset{\varphi_2}{A, \Gamma' \vdash \Delta'}}{\Gamma^*, \Gamma' \vdash \Delta^*, \Delta'}\, \mathit{cut}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\, \rho$$

$$\dfrac{\overset{\varphi_1}{\Gamma \vdash \Delta, A} \qquad \dfrac{\overset{\varphi_2}{A, \Gamma'^* \vdash \Delta'^*}}{A, \Gamma' \vdash \Delta'}\, \rho}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\, \mathit{cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash \Delta, A} \qquad \overset{\varphi_2}{A, \Gamma'^* \vdash \Delta'^*}}{\Gamma, \Gamma'^* \vdash \Delta, \Delta'^*}\, \mathit{cut}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\, \rho$$

Over binary inferences:

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma_1 \vdash \Delta_1, A} \qquad \overset{\varphi_2}{\Gamma_2 \vdash \Delta_2}}{\Gamma \vdash \Delta, A}\, \rho \qquad \overset{\varphi_3}{A, \Gamma' \vdash \Delta'}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\, \mathit{cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma_1 \vdash \Delta_1, A} \qquad \overset{\varphi_3}{A, \Gamma' \vdash \Delta'}}{\Gamma_1, \Gamma' \vdash \Delta_1, \Delta'}\, \mathit{cut} \qquad \overset{\varphi_2}{\Gamma_2 \vdash \Delta_2}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\, \rho$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma_1 \vdash \Delta_1} \qquad \overset{\varphi_2}{\Gamma_2 \vdash \Delta_2, A}}{\Gamma \vdash \Delta, A}\, \rho \qquad \overset{\varphi_3}{A, \Gamma' \vdash \Delta'}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\, \mathit{cut} \quad \rightsquigarrow \quad \dfrac{\overset{\varphi_1}{\Gamma_1 \vdash \Delta_1} \qquad \dfrac{\overset{\varphi_2}{\Gamma_2 \vdash \Delta_2, A} \qquad \overset{\varphi_3}{A, \Gamma' \vdash \Delta'}}{\Gamma_2, \Gamma' \vdash \Delta_2, \Delta'}\, \mathit{cut}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\, \rho$$

$$\dfrac{\overset{\varphi_1}{\Gamma \vdash \Delta, A} \qquad \dfrac{\overset{\varphi_2}{A, \Gamma_1' \vdash \Delta_1'} \qquad \overset{\varphi_3}{\Gamma_2' \vdash \Delta_2'}}{A, \Gamma' \vdash \Delta'}\, \rho}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\, \mathit{cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash \Delta, A} \qquad \overset{\varphi_2}{A, \Gamma_1' \vdash \Delta_1'}}{\Gamma, \Gamma_1' \vdash \Delta, \Delta_1'}\, \mathit{cut} \qquad \overset{\varphi_3}{\Gamma_2' \vdash \Delta_2'}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\, \rho$$

$$\dfrac{\overset{\varphi_1}{\Gamma \vdash \Delta, A} \qquad \dfrac{\overset{\varphi_2}{\Gamma_1' \vdash \Delta_1'} \qquad \overset{\varphi_3}{A, \Gamma_2' \vdash \Delta_2'}}{A, \Gamma' \vdash \Delta'}\, \rho}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\, \mathit{cut} \quad \rightsquigarrow \quad \dfrac{\overset{\varphi_2}{\Gamma_1' \vdash \Delta_1'} \qquad \dfrac{\overset{\varphi_1}{\Gamma \vdash \Delta, A} \qquad \overset{\varphi_3}{A, \Gamma_2' \vdash \Delta_2'}}{\Gamma, \Gamma_2' \vdash \Delta, \Delta_2'}\, \mathit{cut}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\, \rho$$

## Grade reduction

Of a cut on an $\wedge$-formula:

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma\vdash\Delta,A_1} \quad \overset{\varphi_2}{\Gamma'\vdash\Delta',A_2}}{\Gamma,\Gamma'\vdash\Delta,\Delta',A_1\wedge A_2}\wedge_r \quad \dfrac{\overset{\varphi_3}{A_i,\Gamma''\vdash\Delta''}}{A_1\wedge A_2,\Gamma''\vdash\Delta''}\wedge_{li}}{\Gamma,\Gamma',\Gamma''\vdash\Delta,\Delta',\Delta''}\mathit{cut} \quad\rightsquigarrow\quad \dfrac{\overset{\varphi_1}{\Gamma\vdash\Delta,A_1}\quad \dfrac{\overset{\varphi_2}{\Gamma'\vdash\Delta',A_2}\quad \dfrac{\overset{\varphi_3}{A_i,\Gamma''\vdash\Delta''}}{A_1,A_2,\Gamma''\vdash\Delta''}w_l}{A_1,\Gamma',\Gamma''\vdash\Delta',\Delta''}\mathit{cut}}{\Gamma,\Gamma',\Gamma''\vdash\Delta,\Delta',\Delta''}\mathit{cut}$$

Of a cut on an $\vee$-formula:

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma\vdash\Delta,A_i}}{\Gamma\vdash\Delta,A_1\vee A_2}\vee_{ri} \quad \dfrac{\overset{\varphi_2}{A_1,\Gamma'\vdash\Delta'}\quad\overset{\varphi_3}{A_2,\Gamma''\vdash\Delta''}}{A_1\vee A_2,\Gamma',\Gamma''\vdash\Delta',\Delta''}\vee_l}{\Gamma,\Gamma',\Gamma''\vdash\Delta,\Delta',\Delta''}\mathit{cut} \quad\rightsquigarrow\quad \dfrac{\dfrac{\dfrac{\overset{\varphi_1}{\Gamma\vdash\Delta,A_i}}{\Gamma\vdash\Delta,A_1,A_2}w_r\quad \overset{\varphi_3}{A_2,\Gamma''\vdash\Delta''}}{\Gamma,\Gamma''\vdash\Delta,\Delta'',A_1}\mathit{cut}\quad \overset{\varphi_2}{A_1,\Gamma'\vdash\Delta'}}{\Gamma,\Gamma',\Gamma''\vdash\Delta,\Delta',\Delta''}\mathit{cut}$$

Of a cut on an $\rightarrow$-formula:

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma,A\vdash\Delta,B}}{\Gamma\vdash\Delta,A\rightarrow B}\rightarrow_r \quad \dfrac{\overset{\varphi_2}{\Gamma'\vdash\Delta',A}\quad\overset{\varphi_3}{B,\Gamma''\vdash\Delta''}}{A\rightarrow B,\Gamma',\Gamma''\vdash\Delta',\Delta''}\rightarrow_l}{\Gamma,\Gamma',\Gamma''\vdash\Delta,\Delta',\Delta''}\mathit{cut} \quad\rightsquigarrow\quad \dfrac{\overset{\varphi_2}{\Gamma'\vdash\Delta',A}\quad \dfrac{\overset{\varphi_1}{A,\Gamma\vdash\Delta,B}\quad\overset{\varphi_3}{B,\Gamma''\vdash\Delta''}}{A,\Gamma,\Gamma''\vdash\Delta,\Delta''}\mathit{cut}}{\Gamma,\Gamma',\Gamma''\vdash\Delta,\Delta',\Delta''}\mathit{cut}$$

Of a cut on an $\neg$-formula:

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma,A\vdash\Delta}}{\Gamma\vdash\Delta,\neg A}\neg_r \quad \dfrac{\overset{\varphi_2}{\Gamma'\vdash\Delta',A}}{\neg A,\Gamma'\vdash\Delta'}\neg_l}{\Gamma,\Gamma'\vdash\Delta,\Delta'}\mathit{cut} \quad\rightsquigarrow\quad \dfrac{\overset{\varphi_2}{\Gamma'\vdash\Delta',A}\quad\overset{\varphi_1}{A,\Gamma\vdash\Delta}}{\Gamma,\Gamma'\vdash\Delta,\Delta'}\mathit{cut}$$

Of a cut on an $\exists$-formula:

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma\vdash\Delta,A[x/t]}}{\Gamma\vdash\Delta,\exists x.Ax}\exists_r \quad \dfrac{\overset{\varphi_2}{A[x/\alpha],\Gamma'\vdash\Delta'}}{\exists x.Ax,\Gamma'\vdash\Delta'}\exists_l}{\Gamma,\Gamma'\vdash\Delta,\Delta'}\mathit{cut} \quad\rightsquigarrow\quad \dfrac{\overset{\varphi_1}{\Gamma\vdash\Delta,A[x/t]}\quad\overset{\varphi_2[\alpha/t]}{A[x/t],\Gamma'\vdash\Delta'}}{\Gamma,\Gamma'\vdash\Delta,\Delta'}\mathit{cut}$$

Of a cut on an $\forall$-formula:

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash \Delta, A[x/\alpha]}}{\Gamma \vdash \Delta, \forall x.Ax}\,\forall_r \qquad \dfrac{\overset{\varphi_2}{A[x/t], \Gamma' \vdash \Delta'}}{\forall x.Ax, \Gamma' \vdash \Delta'}\,\forall_l}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,cut \quad \rightsquigarrow \quad \dfrac{\overset{\varphi_1[\alpha/t]}{\Gamma \vdash \Delta, A[x/t]} \qquad \overset{\varphi_2}{A[x/t], \Gamma' \vdash \Delta'}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,cut$$

Of a cut on a contracted formula:

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, \vdash \Delta, A, A}}{\Gamma \vdash \Delta, A}\,c_r \qquad \overset{\varphi_2}{A, \Gamma' \vdash \Delta'}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,cut \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, \vdash \Delta, A, A} \qquad \overset{\varphi_2}{A, \Gamma' \vdash \Delta'}}{\Gamma, \Gamma' \vdash \Delta, \Delta', A}\,cut \qquad \overset{\varphi_2}{A, \Gamma' \vdash \Delta'}}{\Gamma, \Gamma', \Gamma' \vdash \Delta, \Delta', \Delta'}\,cut}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,c_l^*, c_r^*$$

$$\dfrac{\overset{\varphi_1}{\Gamma, \vdash \Delta, A} \qquad \dfrac{\overset{\varphi_2}{A, A, \Gamma' \vdash \Delta'}}{A, \Gamma' \vdash \Delta'}\,c_l}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,cut \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, \vdash \Delta, A} \qquad \dfrac{\overset{\varphi_1}{\Gamma, \vdash \Delta, A} \qquad \overset{\varphi_2}{A, A, \Gamma' \vdash \Delta'}}{A, \Gamma, \Gamma' \vdash \Delta, \Delta'}\,cut}{\Gamma, \Gamma, \Gamma' \vdash \Delta, \Delta, \Delta'}\,cut}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,c_l^*, c_r^*$$

## A.2 Rewriting rules for LJ

In all rewriting rules for **LJ**, $\Delta$ is at most one formula.

### Cut-elimination

Over axiom inferences:

$$\dfrac{\dfrac{}{A \vdash A} \qquad \overset{\varphi}{A, \Gamma \vdash \Delta}}{\Gamma, A \vdash \Delta}\,cut \quad \rightsquigarrow \quad \overset{\varphi}{A, \Gamma \vdash \Delta}$$

$$\dfrac{\overset{\varphi}{\Gamma \vdash A} \qquad \dfrac{}{A \vdash A}}{\Gamma \vdash A}\,cut \quad \rightsquigarrow \quad \overset{\varphi}{\Gamma \vdash A}$$

Over weakening:

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash}}{\Gamma \vdash A}\,w_r \qquad \overset{\varphi_2}{A, \Gamma' \vdash \Delta}}{\Gamma, \Gamma' \vdash \Delta}\,cut \quad \rightsquigarrow \quad \dfrac{\overset{\varphi_1}{\Gamma \vdash}}{\Gamma, \Gamma' \vdash \Delta}\,w_r^*, w_l^*$$

$$\dfrac{\overset{\varphi_1}{\Gamma \vdash A} \qquad \dfrac{\overset{\varphi_2}{\Gamma' \vdash \Delta}}{A, \Gamma' \vdash \Delta}\,w_l}{\Gamma, \Gamma' \vdash \Delta}\,cut \quad\rightsquigarrow\quad \dfrac{\overset{\varphi_2}{\Gamma' \vdash \Delta}}{\Gamma, \Gamma' \vdash \Delta}\,w_r^*, w_l^*$$

## Rank reduction

Over unary inferences:

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma^* \vdash A}}{\Gamma \vdash A}\,\rho \qquad \overset{\varphi_2}{A, \Gamma' \vdash \Delta}}{\Gamma, \Gamma' \vdash \Delta}\,cut \quad\rightsquigarrow\quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma^* \vdash A} \qquad \overset{\varphi_2}{A, \Gamma' \vdash \Delta}}{\Gamma^*, \Gamma' \vdash \Delta}\,cut}{\Gamma, \Gamma' \vdash \Delta}\,\rho$$

$$\dfrac{\overset{\varphi_1}{\Gamma \vdash A} \qquad \dfrac{\overset{\varphi_2}{A, \Gamma'^* \vdash \Delta^*}}{A, \Gamma' \vdash \Delta}\,\rho}{\Gamma, \Gamma' \vdash \Delta}\,cut \quad\rightsquigarrow\quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash A} \qquad \overset{\varphi_2}{A, \Gamma'^* \vdash \Delta^*}}{\Gamma, \Gamma'^* \vdash \Delta^*}\,cut}{\Gamma, \Gamma' \vdash \Delta}\,\rho$$

Over binary inferences:

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma_1, P \vdash A} \qquad \overset{\varphi_2}{\Gamma_2, Q \vdash A}}{\Gamma, P \vee Q \vdash A}\,\vee_l \qquad \overset{\varphi_3}{A, \Gamma' \vdash \Delta}}{\Gamma, P \vee Q, \Gamma' \vdash \Delta}\,cut \quad\rightsquigarrow\quad \dfrac{\dfrac{\dfrac{\overset{\varphi_1}{\Gamma_1, P \vdash A} \qquad \overset{\varphi_3}{A, \Gamma' \vdash \Delta'}}{\Gamma_1, P, \Gamma' \vdash \Delta}\,cut \qquad \dfrac{\overset{\varphi_2}{\Gamma_2, Q \vdash A} \qquad \overset{\varphi_3}{A, \Gamma' \vdash \Delta'}}{\Gamma_2, Q, \Gamma' \vdash \Delta}\,cut}{\Gamma, P \vee Q, \Gamma', \Gamma' \vdash \Delta}\,\vee_l}{\Gamma, P \vee Q, \Gamma' \vdash \Delta}\,c_l$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma_1 \vdash P} \qquad \overset{\varphi_2}{\Gamma_2, Q \vdash A}}{\Gamma, P \to Q \vdash A}\,\to_l \qquad \overset{\varphi_3}{A, \Gamma' \vdash \Delta}}{\Gamma, P \to Q, \Gamma' \vdash \Delta}\,cut \quad\rightsquigarrow\quad \dfrac{\overset{\varphi_1}{\Gamma_1 \vdash P} \qquad \dfrac{\overset{\varphi_2}{\Gamma_2, Q \vdash A} \qquad \overset{\varphi_3}{A, \Gamma' \vdash \Delta}}{\Gamma_2, Q, \Gamma' \vdash \Delta}\,cut}{\Gamma, P \to Q, \Gamma' \vdash \Delta, \Delta'}\,\to_l$$

$$\dfrac{\overset{\varphi_1}{\Gamma \vdash A} \qquad \dfrac{\overset{\varphi_2}{A, \Gamma_1' \vdash \Delta_1} \qquad \overset{\varphi_3}{\Gamma_2' \vdash \Delta_2}}{A, \Gamma' \vdash \Delta}\,\rho}{\Gamma, \Gamma' \vdash \Delta}\,cut \quad\rightsquigarrow\quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash A} \qquad \overset{\varphi_2}{A, \Gamma_1' \vdash \Delta_1}}{\Gamma, \Gamma_1' \vdash \Delta_1}\,cut \qquad \overset{\varphi_3}{\Gamma_2' \vdash \Delta_2}}{\Gamma, \Gamma' \vdash \Delta}\,\rho$$

$$\dfrac{\overset{\varphi_1}{\Gamma \vdash A} \qquad \dfrac{\overset{\varphi_2}{\Gamma_1' \vdash \Delta_1} \qquad \overset{\varphi_3}{A, \Gamma_2' \vdash \Delta_2}}{A, \Gamma' \vdash \Delta}\,\rho}{\Gamma, \Gamma' \vdash \Delta}\,cut \quad\rightsquigarrow\quad \dfrac{\overset{\varphi_2}{\Gamma_1' \vdash \Delta_1} \qquad \dfrac{\overset{\varphi_1}{\Gamma \vdash A} \qquad \overset{\varphi_3}{A, \Gamma_2' \vdash \Delta_2}}{\Gamma, \Gamma_2' \vdash \Delta_2}\,cut}{\Gamma, \Gamma' \vdash \Delta}\,\rho$$

## Grade reduction

Of a cut on an $\wedge$-formula:

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash A_1} \quad \overset{\varphi_2}{\Gamma' \vdash A_2}}{\Gamma, \Gamma' \vdash A_1 \wedge A_2}\,\wedge_r \quad \dfrac{\overset{\varphi_3}{A_i, \Gamma'' \vdash \Delta}}{A_1 \wedge A_2, \Gamma'' \vdash \Delta}\,\wedge_{li}}{\Gamma, \Gamma', \Gamma'' \vdash \Delta}\,\mathit{cut} \quad \rightsquigarrow \quad \dfrac{\overset{\varphi_1}{\Gamma \vdash A_1} \quad \dfrac{\overset{\varphi_2}{\Gamma' \vdash A_2} \quad \dfrac{\overset{\varphi_3}{A_i, \Gamma'' \vdash \Delta}}{A_1, A_2, \Gamma'' \vdash \Delta}\,w_l}{A_1, \Gamma', \Gamma'' \vdash \Delta}\,\mathit{cut}}{\Gamma, \Gamma', \Gamma'' \vdash \Delta}\,\mathit{cut}$$

Of a cut on an $\vee$-formula:

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash A_1}}{\Gamma \vdash A_1 \vee A_2}\,\vee_{r1} \quad \dfrac{\overset{\varphi_2}{A_1, \Gamma' \vdash \Delta} \quad \overset{\varphi_3}{A_2, \Gamma'' \vdash \Delta}}{A_1 \vee A_2, \Gamma', \Gamma'' \vdash \Delta}\,\vee_l}{\Gamma, \Gamma', \Gamma'' \vdash \Delta}\,\mathit{cut} \quad \rightsquigarrow \quad \dfrac{\overset{\varphi_1}{\Gamma \vdash A_1} \quad \dfrac{\overset{\varphi_2}{A_1, \Gamma' \vdash \Delta}}{A_1, \Gamma', \Gamma'' \vdash \Delta}\,w_r}{\Gamma, \Gamma', \Gamma'' \vdash \Delta}\,\mathit{cut}$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash A_2}}{\Gamma \vdash A_1 \vee A_2}\,\vee_{r2} \quad \dfrac{\overset{\varphi_2}{A_1, \Gamma' \vdash \Delta} \quad \overset{\varphi_3}{A_2, \Gamma'' \vdash \Delta}}{A_1 \vee A_2, \Gamma', \Gamma'' \vdash \Delta}\,\vee_l}{\Gamma, \Gamma', \Gamma'' \vdash \Delta}\,\mathit{cut} \quad \rightsquigarrow \quad \dfrac{\overset{\varphi_1}{\Gamma \vdash A_2} \quad \dfrac{\overset{\varphi_3}{A_2, \Gamma'' \vdash \Delta}}{A_2, \Gamma', \Gamma'' \vdash \Delta}\,w_r}{\Gamma, \Gamma', \Gamma'' \vdash \Delta}\,\mathit{cut}$$

Of a cut on an $\rightarrow$-formula:

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A \vdash B}}{\Gamma \vdash A \rightarrow B}\,\rightarrow_r \quad \dfrac{\overset{\varphi_2}{\Gamma' \vdash A} \quad \overset{\varphi_3}{B, \Gamma'' \vdash \Delta}}{A \rightarrow B, \Gamma', \Gamma'' \vdash \Delta}\,\rightarrow_l}{\Gamma, \Gamma', \Gamma'' \vdash \Delta}\,\mathit{cut} \quad \rightsquigarrow \quad \dfrac{\overset{\varphi_2}{\Gamma' \vdash A} \quad \dfrac{\overset{\varphi_1}{A, \Gamma \vdash B} \quad \overset{\varphi_3}{B, \Gamma'' \vdash \Delta}}{A, \Gamma, \Gamma'' \vdash \Delta}\,\mathit{cut}}{\Gamma, \Gamma', \Gamma'' \vdash \Delta}\,\mathit{cut}$$

Of a cut on an $\neg$-formula:

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A \vdash}}{\Gamma \vdash \neg A}\,\neg_r \quad \dfrac{\overset{\varphi_2}{\Gamma' \vdash A}}{\neg A, \Gamma' \vdash}\,\neg_l}{\Gamma, \Gamma' \vdash}\,\mathit{cut} \quad \rightsquigarrow \quad \dfrac{\overset{\varphi_2}{\Gamma' \vdash A} \quad \overset{\varphi_1}{A, \Gamma \vdash}}{\Gamma, \Gamma' \vdash}\,\mathit{cut}$$

Of a cut on an $\exists$-formula:

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash A[x/t]}}{\Gamma \vdash \exists x.Ax}\,\exists_r \quad \dfrac{\overset{\varphi_2}{A[x/\alpha], \Gamma' \vdash \Delta}}{\exists x.Ax, \Gamma' \vdash \Delta}\,\exists_l}{\Gamma, \Gamma' \vdash \Delta}\,\mathit{cut} \quad \rightsquigarrow \quad \dfrac{\overset{\varphi_1}{\Gamma \vdash A[x/t]} \quad \overset{\varphi_2[\alpha/t]}{A[x/t], \Gamma' \vdash \Delta}}{\Gamma, \Gamma' \vdash \Delta}\,\mathit{cut}$$

Of a cut on an ∀-formula:

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash A[x/\alpha]}}{\Gamma \vdash \forall x.Ax}\ \forall_r \qquad \dfrac{\overset{\varphi_2}{A[x/t], \Gamma' \vdash \Delta}}{\forall x.Ax, \Gamma' \vdash \Delta}\ \forall_l}{\Gamma, \Gamma' \vdash \Delta}\ cut \quad \rightsquigarrow \quad \dfrac{\overset{\varphi_1[\alpha/t]}{\Gamma \vdash A[x/t]} \qquad \overset{\varphi_2}{A[x/t], \Gamma' \vdash \Delta}}{\Gamma, \Gamma' \vdash \Delta}\ cut$$

Of a cut on a contracted formula:

$$\dfrac{\overset{\varphi_1}{\Gamma, \vdash A} \qquad \dfrac{\overset{\varphi_2}{A, A, \Gamma' \vdash \Delta}}{A, \Gamma' \vdash \Delta}\ c_l}{\Gamma, \Gamma' \vdash \Delta}\ cut \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, \vdash A} \qquad \dfrac{\overset{\varphi_1}{\Gamma, \vdash A} \qquad \overset{\varphi_2}{A, A, \Gamma' \vdash \Delta}}{A, \Gamma, \Gamma' \vdash \Delta}\ cut}{\Gamma, \Gamma, \Gamma' \vdash \Delta}\ cut}{\Gamma, \Gamma' \vdash \Delta}\ c_l^*$$

# APPENDIX B

# Permutation of rules in LJ

This Appendix shows the possible permutations of LJ rules down a proof. I used the purely multiplicative version of LJ. Of course, not all cases are possible. The cases in which this fails do not have a derivation on the right hand side, instead, there is a "?". The rule which is supposed to be permuted down is always applied to a formula whose formula variables are $A$ and $B$. It is important to note that this is invariably an end-sequent ancestor. The bottom rule is applied to a formula with formula variables $P$ and $Q$, which could be an end-sequent or cut ancestor. The sequents are represented as multi-sets of formulas (i.e. the order does not matter).

The order in which the rules are presented is: $\to_l$, $\to_r$, $\wedge_l$, $\wedge_r$, $\vee_l$, $\vee_r$, $\forall_l$, $\forall_r$, $\exists_l$, $\exists_r$, $\neg_l$, $\neg_r$, $w_l$, $w_r$, $c_l$, $cut$.

## B.1 Permutation of $\to_l$ down

$\to_l$

$$\cfrac{\cfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \overset{\varphi_2}{\Gamma', B \vdash P}}{\Gamma, \Gamma', A \to B \vdash P}\to_l \quad \overset{\varphi_3}{\Gamma'', Q \vdash F}}{\Gamma, \Gamma', \Gamma'', A \to B, P \to Q \vdash F}\to_l \quad \rightsquigarrow \quad \cfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \cfrac{\overset{\varphi_2}{\Gamma', B \vdash P} \quad \overset{\varphi_3}{\Gamma'', Q \vdash F}}{\Gamma', \Gamma'', P \to Q, B \vdash F}\to_l}{\Gamma, \Gamma', \Gamma'', A \to B, P \to Q \vdash F}\to_l$$

$$\cfrac{\overset{\varphi_1}{\Gamma, \vdash P} \quad \cfrac{\overset{\varphi_2}{\Gamma', Q \vdash A} \quad \overset{\varphi_3}{\Gamma'', B \vdash F}}{\Gamma', \Gamma'', A \to B, Q \vdash F}\to_l}{\Gamma, \Gamma', \Gamma'', A \to B, P \to Q \vdash F}\to_l \quad \rightsquigarrow \quad \cfrac{\cfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \overset{\varphi_2}{\Gamma', Q \vdash A}}{\Gamma, \Gamma', P \to Q, \vdash A}\to_l \quad \overset{\varphi_3}{\Gamma'', B \vdash F}}{\Gamma, \Gamma', \Gamma'', A \to B, P \to Q \vdash F}\to_l$$

$$\cfrac{\overset{\varphi_1}{\Gamma, \vdash P} \quad \cfrac{\overset{\varphi_2}{\Gamma' \vdash A} \quad \overset{\varphi_3}{\Gamma'', Q, B \vdash F}}{\Gamma', \Gamma'', A \to B, Q \vdash F}\to_l}{\Gamma, \Gamma', \Gamma'', A \to B, P \to Q \vdash F}\to_l \quad \rightsquigarrow \quad \cfrac{\overset{\varphi_2}{\Gamma' \vdash A} \quad \cfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \overset{\varphi_3}{\Gamma', Q, B \vdash F}}{\Gamma, \Gamma'', P \to Q, B \vdash F}\to_l}{\Gamma, \Gamma', \Gamma'', A \to B, P \to Q \vdash F}\to_l$$

$\to_r$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P \vdash A} \quad \overset{\varphi_2}{\Gamma', B \vdash Q}}{\Gamma, \Gamma', A \to B, P \vdash Q}\,\to_l}{\Gamma, \Gamma', A \to B \vdash P \to Q}\,\to_r \quad \rightsquigarrow \quad ?$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \overset{\varphi_2}{\Gamma', P, B \vdash Q}}{\Gamma, \Gamma', A \to B, P \vdash Q}\,\to_l}{\Gamma, \Gamma', A \to B \vdash P \to Q}\,\to_r \quad \rightsquigarrow \quad \dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \dfrac{\overset{\varphi_2}{\Gamma', B, P \vdash Q}}{\Gamma', B \vdash P \to Q}\,\to_r}{\Gamma, \Gamma', A \to B \vdash P \to Q}\,\to_l$$

$\wedge_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, Q \vdash A} \quad \overset{\varphi_2}{\Gamma', B \vdash F}}{\Gamma, \Gamma', A \to B, Q \vdash F}\,\to_l}{\Gamma, \Gamma', P \wedge Q, A \to B \vdash F}\,\wedge_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, Q \vdash A}}{\Gamma, P \wedge Q \vdash A}\,\wedge_l \quad \overset{\varphi_2}{\Gamma', B \vdash F}}{\Gamma, \Gamma', P \wedge Q, A \to B \vdash F}\,\to_l$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \overset{\varphi_2}{\Gamma', Q, B \vdash F}}{\Gamma, \Gamma', A \to B, Q \vdash F}\,\to_l}{\Gamma, \Gamma', P \wedge Q, A \to B \vdash F}\,\wedge_l \quad \rightsquigarrow \quad \dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \dfrac{\overset{\varphi_2}{\Gamma', B, Q \vdash F}}{\Gamma', P \wedge Q, B \vdash F}\,\wedge_l}{\Gamma, \Gamma, P \wedge Q, A \to B \vdash F}\,\to_l$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \overset{\varphi_2}{\Gamma', P, B \vdash F}}{\Gamma, \Gamma', A \to B, P \vdash F}\,\to_l}{\Gamma, \Gamma', P \wedge Q, A \to B \vdash F}\,\wedge_l \quad \rightsquigarrow \quad \dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \dfrac{\overset{\varphi_2}{\Gamma', B, P \vdash F}}{\Gamma', P \wedge Q, B \vdash F}\,\wedge_l}{\Gamma, \Gamma, P \wedge Q, A \to B \vdash F}\,\to_l$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P \vdash A} \quad \overset{\varphi_2}{\Gamma', B \vdash F}}{\Gamma, \Gamma', A \to B, P \vdash F}\,\to_l}{\Gamma, \Gamma', P \wedge Q, A \to B \vdash F}\,\wedge_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P \vdash A}}{\Gamma, P \wedge Q \vdash A}\,\wedge_l \quad \overset{\varphi_2}{\Gamma', B \vdash F}}{\Gamma, \Gamma', P \wedge Q, A \to B \vdash F}\,\to_l$$

$\wedge_r$

$$\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \qquad \dfrac{\overset{\varphi_2}{\Gamma' \vdash A} \quad \overset{\varphi_3}{\Gamma'', B \vdash Q}}{\Gamma', \Gamma'', A \to B \vdash Q}\,{\to_l}}{\Gamma, \Gamma', \Gamma'', A \to B \vdash P \wedge Q}\,{\wedge_r} \quad \rightsquigarrow \quad \dfrac{\overset{\varphi_2}{\Gamma' \vdash A} \qquad \dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \overset{\varphi_3}{\Gamma'', B \vdash Q}}{\Gamma, \Gamma'', B \vdash P \wedge Q}\,{\wedge_r}}{\Gamma, \Gamma', \Gamma'', A \to B \vdash P \wedge Q}\,{\to_l}$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \overset{\varphi_2}{\Gamma', B \vdash P}}{\Gamma, \Gamma', A \to B \vdash P}\,{\to_l} \qquad \overset{\varphi_3}{\Gamma'' \vdash Q}}{\Gamma, \Gamma', \Gamma'', A \to B \vdash P \wedge Q}\,{\wedge_r} \quad \rightsquigarrow \quad \dfrac{\overset{\varphi_1}{\Gamma \vdash A} \qquad \dfrac{\overset{\varphi_2}{\Gamma', B \vdash P} \quad \overset{\varphi_3}{\Gamma'' \vdash Q}}{\Gamma', \Gamma'', B \vdash P \wedge Q}\,{\wedge_r}}{\Gamma, \Gamma', \Gamma'', A \to B \vdash P \wedge Q}\,{\to_l}$$

$\vee_l$

$$\dfrac{\overset{\varphi_1}{\Gamma, P \vdash F} \qquad \dfrac{\overset{\varphi_2}{\Gamma' \vdash A} \quad \overset{\varphi_3}{\Gamma'', Q, B \vdash F}}{\Gamma', \Gamma'', A \to B, Q \vdash F}\,{\to_l}}{\Gamma, \Gamma', \Gamma'', A \to B, P \vee Q \vdash F}\,{\vee_l} \quad \rightsquigarrow \quad \dfrac{\overset{\varphi_2}{\Gamma' \vdash A} \qquad \dfrac{\overset{\varphi_1}{\Gamma, P \vdash F} \quad \overset{\varphi_3}{\Gamma'', B, Q \vdash F}}{\Gamma, \Gamma'', P \vee Q, B \vdash F}\,{\vee_l}}{\Gamma, \Gamma', \Gamma'', P \vee Q, A \to B \vdash F}\,{\to_l}$$

$$\dfrac{\overset{\varphi_1}{\Gamma, P \vdash F} \qquad \dfrac{\overset{\varphi_2}{\Gamma', Q \vdash A} \quad \overset{\varphi_3}{\Gamma'', B \vdash F}}{\Gamma', \Gamma'', A \to B, Q \vdash F}\,{\to_l}}{\Gamma, \Gamma', \Gamma'', A \to B, P \vee Q \vdash F}\,{\vee_l} \quad \rightsquigarrow \quad ?$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \overset{\varphi_2}{\Gamma', P, B \vdash F}}{\Gamma, \Gamma', A \to B, P \vdash F}\,{\to_l} \qquad \overset{\varphi_3}{\Gamma'', Q \vdash F}}{\Gamma, \Gamma', \Gamma'', A \to B, P \vee Q \vdash F}\,{\vee_l} \quad \rightsquigarrow \quad \dfrac{\overset{\varphi_1}{\Gamma \vdash A} \qquad \dfrac{\overset{\varphi_2}{\Gamma', B, P \vdash F} \quad \overset{\varphi_3}{\Gamma'', Q \vdash F}}{\Gamma', \Gamma'', P \vee Q, B \vdash F}\,{\vee_l}}{\Gamma, \Gamma', \Gamma'', P \vee Q, A \to B \vdash F}\,{\to_l}$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P \vdash A} \quad \overset{\varphi_2}{\Gamma', B \vdash F}}{\Gamma, \Gamma', A \to B, P \vdash F}\,{\to_l} \qquad \overset{\varphi_3}{\Gamma'', Q \vdash F}}{\Gamma, \Gamma', \Gamma'', A \to B, P \vee Q \vdash F}\,{\vee_l} \quad \rightsquigarrow \quad ?$$

$\vee_r$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \overset{\varphi_2}{\Gamma', B \vdash Q}}{\Gamma, \Gamma', A \to B \vdash Q}\,{\to_l}}{\Gamma, \Gamma', A \to B \vdash P \vee Q}\,{\vee_r} \quad \rightsquigarrow \quad \dfrac{\overset{\varphi_1}{\Gamma \vdash A} \qquad \dfrac{\overset{\varphi_2}{\Gamma', B \vdash Q}}{\Gamma', B \vdash P \vee Q}\,{\vee_r}}{\Gamma, \Gamma', A \to B \vdash P \vee Q}\,{\to_l}$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \overset{\varphi_2}{\Gamma', B \vdash P}}{\Gamma, \Gamma', A \to B \vdash P}\, \to_l}{\Gamma, \Gamma', A \to B \vdash P \vee Q}\, \vee_r \quad \rightsquigarrow \quad \dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \dfrac{\overset{\varphi_2}{\Gamma', B \vdash P}}{\Gamma', B \vdash P \vee Q}\, \vee_r}{\Gamma, \Gamma', A \to B \vdash P \vee Q}\, \to_l$$

$\forall_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \overset{\varphi_2}{\Gamma', P[x/t], B \vdash F}}{\Gamma, \Gamma', A \to B, P[x/t] \vdash F}\, \to_l}{\Gamma, \Gamma', \forall x.Px, A \to B \vdash F}\, \forall_l \quad \rightsquigarrow \quad \dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \dfrac{\overset{\varphi_2}{\Gamma', B, P[x/t] \vdash F}}{\Gamma', \forall x.Px, B \vdash F}\, \forall_l}{\Gamma, \Gamma', \forall x.Px, A \to B \vdash F}\, \to_l$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P[x/t] \vdash A} \quad \overset{\varphi_2}{\Gamma', B \vdash F}}{\Gamma, \Gamma', A \to B, P[x/t] \vdash F}\, \to_l}{\Gamma, \Gamma', \forall x.Px, A \to B \vdash F}\, \forall_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P[x/t] \vdash A}}{\Gamma, \forall x.Px, \vdash A}\, \forall_l \quad \overset{\varphi_2}{\Gamma', B \vdash F}}{\Gamma, \Gamma', \forall x.Px, A \to B \vdash F}\, \to_l$$

$\forall_r$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \overset{\varphi_2}{\Gamma', B \vdash P[x/\alpha]}}{\Gamma, \Gamma, A \to B \vdash P[x/\alpha]}\, \to_l}{\Gamma, \Gamma', A \to B \vdash \forall x.Px}\, \forall_r \quad \rightsquigarrow \quad \dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \dfrac{\overset{\varphi_2}{\Gamma', B \vdash P[x/\alpha]}}{\Gamma', B \vdash \forall x.Px}\, \forall_r}{\Gamma, \Gamma', A \to B \vdash \forall x.Px}\, \to_l$$

$\exists_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \overset{\varphi_2}{\Gamma', P[x/\alpha], B \vdash F}}{\Gamma, \Gamma', A \to B, P[x/\alpha] \vdash F}\, \to_l}{\Gamma, \Gamma', \exists x.Px, A \to B \vdash F}\, \exists_l \quad \rightsquigarrow \quad \dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \dfrac{\overset{\varphi_2}{\Gamma', B, P[x/\alpha] \vdash F}}{\Gamma', \exists x.Px, B \vdash F}\, \exists_l}{\Gamma, \Gamma', \exists x.Px, A \to B \vdash F}\, \to_l$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P[x/\alpha] \vdash A} \quad \overset{\varphi_2}{\Gamma', B \vdash F}}{\Gamma, \Gamma', A \to B, P[x/\alpha] \vdash F}\, \to_l}{\Gamma, \Gamma', \exists x.Px, A \to B \vdash F}\, \exists_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P[x/\alpha] \vdash A}}{\Gamma, \exists x.Px \vdash A}\, \exists_l \quad \overset{\varphi_2}{\Gamma', B \vdash F}}{\Gamma, \Gamma', \exists x.Px, A \to B \vdash F}\, \to_l$$

$\exists_r$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \overset{\varphi_2}{\Gamma', B \vdash P[x/t]}}{\Gamma, \Gamma', A \to B \vdash P[x/t]}\,\to_l}{\Gamma, \Gamma', A \to B \vdash \exists x.Px}\,\exists_r \quad \leadsto \quad \dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \dfrac{\overset{\varphi_2}{\Gamma', B \vdash P[x/t]}}{\Gamma', B \vdash \exists x.Px}\,\exists_r}{\Gamma, \Gamma', A \to B \vdash \exists x.Px}\,\to_l$$

$\neg_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \overset{\varphi_2}{\Gamma', B \vdash P}}{\Gamma, \Gamma', A \to B \vdash P}\,\to_l}{\Gamma, \Gamma', A \to B, \neg P \vdash}\,\neg_l \quad \leadsto \quad \dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \dfrac{\overset{\varphi_2}{\Gamma', B \vdash P}}{\Gamma', \neg P, B \vdash}\,\neg_l}{\Gamma, \Gamma', \neg P, A \to B \vdash}\,\to_l$$

$\neg_r$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \overset{\varphi_2}{\Gamma', P, B \vdash}}{\Gamma, \Gamma', A \to B, P \vdash}\,\to_l}{\Gamma, \Gamma', A \to B \vdash \neg P}\,\neg_r \quad \leadsto \quad \dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \dfrac{\overset{\varphi_2}{\Gamma, B, P \vdash}}{\Gamma', B \vdash \neg P}\,\neg_r}{\Gamma, \Gamma', A \to B \vdash \neg P}\,\to_l$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P \vdash A} \quad \overset{\varphi_2}{\Gamma', B \vdash}}{\Gamma, \Gamma', A \to B, P \vdash}\,\to_l}{\Gamma, \Gamma', A \to B \vdash \neg P}\,\neg_r \quad \leadsto \quad ?$$

$w_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \overset{\varphi_2}{\Gamma', B \vdash F}}{\Gamma, \Gamma', A \to B \vdash F}\,\to_l}{\Gamma, \Gamma', A \to B, P \vdash F}\,w_l \quad \leadsto \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash A}}{\Gamma, P \vdash A}\,w_l \quad \overset{\varphi_2}{\Gamma', B \vdash F}}{\Gamma, \Gamma', A \to B, P \vdash F}\,\to_l$$

$w_r$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \overset{\varphi_2}{\Gamma', B \vdash}}{\Gamma, \Gamma', A \to B \vdash}\,\to_l}{\Gamma, \Gamma', A \to B \vdash P}\,w_r \quad \leadsto \quad \dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \dfrac{\overset{\varphi_2}{\Gamma', B \vdash}}{\Gamma', B \vdash P}\,w_r}{\Gamma, \Gamma', A \to B \vdash P}\,\to_l$$

$c_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P, P \vdash A} \quad \overset{\varphi_2}{\Gamma', B \vdash F}}{\Gamma, \Gamma', A \to B, P, P \vdash F}\,\to_l}{\Gamma, \Gamma', A \to B, P \vdash F}\,c_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P, P \vdash A}}{\Gamma, P \vdash A}\,c_l \quad \overset{\varphi_2}{\Gamma', B \vdash F}}{\Gamma, \Gamma', A \to B, P \vdash F}\,\to_l$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \overset{\varphi_2}{\Gamma', P, P, B \vdash F}}{\Gamma, \Gamma', A \to B, P, P \vdash F}\,\to_l}{\Gamma, \Gamma', A \to B, P \vdash F}\,c_l \quad \rightsquigarrow \quad \dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \dfrac{\overset{\varphi_2}{\Gamma', B, P, P \vdash F}}{\Gamma', P, B \vdash F}\,c_l}{\Gamma, \Gamma', A \to B, P \vdash F}\,\to_l$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P \vdash A} \quad \overset{\varphi_2}{\Gamma', P, B \vdash F}}{\Gamma, \Gamma', A \to B, P, P \vdash F}\,\to_l}{\Gamma, \Gamma', A \to B, P \vdash F}\,c_l \quad \rightsquigarrow \quad \text{adding another contraction makes no sense}$$

$cut$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \overset{\varphi_2}{\Gamma', B \vdash P}}{\Gamma, \Gamma', A \to B \vdash P}\,\to_l \quad \overset{\varphi_3}{\Gamma'', P \vdash F}}{\Gamma, \Gamma', \Gamma'', A \to B \vdash F}\,cut \quad \rightsquigarrow \quad \dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \dfrac{\overset{\varphi_2}{\Gamma', B \vdash P} \quad \overset{\varphi_3}{\Gamma'', P \vdash F}}{\Gamma', \Gamma'', B \vdash F}\,cut}{\Gamma, \Gamma', \Gamma'', A \to B \vdash F}\,\to_l$$

$$\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \dfrac{\overset{\varphi_2}{\Gamma', P \vdash A} \quad \overset{\varphi_3}{\Gamma'', B \vdash F}}{\Gamma', \Gamma'', A \to B, P \vdash F}\,\to_l}{\Gamma, \Gamma', \Gamma'', A \to B \vdash F}\,cut \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \overset{\varphi_2}{\Gamma'', P \vdash A}}{\Gamma, \Gamma' \vdash A}\,cut \quad \overset{\varphi_3}{\Gamma'', B \vdash F}}{\Gamma, \Gamma', \Gamma'', A \to B \vdash F}\,\to_l$$

$$\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \dfrac{\overset{\varphi_2}{\Gamma' \vdash A} \quad \overset{\varphi_3}{\Gamma'', P, B \vdash F}}{\Gamma', \Gamma'', A \to B, P \vdash F}\,\to_l}{\Gamma, \Gamma', \Gamma'', A \to B \vdash F}\,cut \quad \rightsquigarrow \quad \dfrac{\overset{\varphi_2}{\Gamma' \vdash A} \quad \dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \overset{\varphi_3}{\Gamma'', B, P \vdash F}}{\Gamma, \Gamma'', B \vdash F}\,cut}{\Gamma, \Gamma', \Gamma'', A \to B \vdash F}\,\to_l$$

## B.2 Permutation of $\to_r$ down

$\to_l$

$$\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \dfrac{\overset{\varphi_2}{\Gamma', Q, A \vdash B}}{\Gamma', Q \vdash A \to B}\,\to_r}{\Gamma, \Gamma', P \to Q \vdash A \to B}\,\to_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \overset{\varphi_2}{\Gamma', A, Q \vdash B}}{\Gamma, \Gamma', P \to Q, A \vdash B}\,\to_l}{\Gamma, \Gamma', P \to Q \vdash A \to B}\,\to_r$$

$\to_r$

N/A

$\wedge_l$

$$\dfrac{\dfrac{\dfrac{\varphi_1}{\Gamma, Q, A \vdash B}}{\Gamma, Q \vdash A \to B}\ \to_r}{\Gamma, P \wedge Q \vdash A \to B}\ \wedge_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\varphi_1}{\Gamma, A, Q \vdash B}}{\Gamma, P \wedge Q, A \vdash B}\ \wedge_l}{\Gamma, P \wedge Q \vdash A \to B}\ \to_r$$

$$\dfrac{\dfrac{\dfrac{\varphi_1}{\Gamma, P, A \vdash B}}{\Gamma, P \vdash A \to B}\ \to_r}{\Gamma, P \wedge Q \vdash A \to B}\ \wedge_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\varphi_1}{\Gamma, A, P \vdash B}}{\Gamma, P \wedge Q, A \vdash B}\ \wedge_l}{\Gamma, P \wedge Q \vdash A \to B}\ \to_r$$

$\wedge_r$

N/A

$\vee_l$

$$\dfrac{\dfrac{\dfrac{\varphi_1}{\Gamma, P, A \vdash B}}{\Gamma, P \vdash A \to B}\ \to_r \quad \dfrac{\dfrac{\varphi_2}{\Gamma', Q, A \vdash B}}{\Gamma', Q \vdash A \to B}\ \to_r}{\Gamma, \Gamma, P \vee Q \vdash A \to B}\ \vee_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\dfrac{\varphi_1}{\Gamma, A, P \vdash B} \quad \dfrac{\varphi_2}{\Gamma', A, Q \vdash B}}{\Gamma, \Gamma', P \vee Q, A, A \vdash B}\ \vee_l}{\Gamma, \Gamma', P \vee Q, A \vdash B}\ c_l}{\Gamma, \Gamma', P \vee Q \vdash A \to B}\ \to_r$$

$\vee_r$

N/A

$\forall_l$

$$\dfrac{\dfrac{\dfrac{\varphi_1}{\Gamma, P[x/t], A \vdash B}}{\Gamma, P[x/t] \vdash A \to B}\ \to_r}{\Gamma, \forall x.Px \vdash A \to B}\ \forall_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\varphi_1}{\Gamma, A, P[x/t] \vdash B}}{\Gamma, \forall x.Px, A \vdash B}\ \forall_l}{\Gamma, \forall x.Px \vdash A \to B}\ \to_r$$

$\forall_r$

N/A

$\exists_l$

$$\cfrac{\cfrac{\overset{\varphi_1}{\Gamma, P[x/\alpha], A \vdash B}}{\Gamma, P[x/\alpha] \vdash A \to B}\,\to_r}{\Gamma, \exists x.Px \vdash A \to B}\,\exists_l \quad\rightsquigarrow\quad \cfrac{\cfrac{\overset{\varphi_1}{\Gamma, A, P[x/\alpha] \vdash B}}{\Gamma, \exists x.Px, A \vdash B}\,\exists_l}{\Gamma, \exists x.Px \vdash A \to B}\,\to_r$$

$\exists_r$

N/A

$\neg_l$

N/A

$\neg_r$

N/A

$w_l$

$$\cfrac{\cfrac{\overset{\varphi_1}{\Gamma, A \vdash B}}{\Gamma \vdash A \to B}\,\to_r}{\Gamma, P \vdash A \to B}\,w_l \quad\rightsquigarrow\quad \cfrac{\cfrac{\overset{\varphi_1}{\Gamma, A \vdash B}}{\Gamma, P, A \vdash B}\,w_l}{\Gamma, P \vdash A \to B}\,\to_r$$

$w_r$

N/A

$c_l$

$$\cfrac{\cfrac{\overset{\varphi_1}{\Gamma, P, P, A \vdash B}}{\Gamma, P, P \vdash A \to B}\,\to_r}{\Gamma, P \vdash A \to B}\,c_l \quad\rightsquigarrow\quad \cfrac{\cfrac{\overset{\varphi_1}{\Gamma, A, P, P \vdash B}}{\Gamma, P, A \vdash B}\,c_l}{\Gamma, P \vdash A \to B}\,\to_r$$

$cut$

$$\cfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \cfrac{\overset{\varphi_2}{\Gamma', P, A \vdash B}}{\Gamma', P \vdash A \to B}\,\to_r}{\Gamma, \Gamma' \vdash A \to B}\,cut \quad\rightsquigarrow\quad \cfrac{\cfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \overset{\varphi_2}{\Gamma', P, A \vdash B}}{\Gamma, \Gamma', A \vdash B}\,cut}{\Gamma, \Gamma' \vdash A \to B}\,\to_r$$

## B.3 Permutation of $\wedge_l$ down

$\to_l$

$$\dfrac{\begin{array}{c}\varphi_1\\ \Gamma \vdash P\end{array} \quad \dfrac{\begin{array}{c}\varphi_2\\ \Gamma', Q, B \vdash F\end{array}}{\Gamma', A \wedge B, Q \vdash F}\wedge_l}{\Gamma, \Gamma', A \wedge B, P \to Q \vdash F}\to_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{array}{c}\varphi_1\\ \Gamma \vdash P\end{array} \quad \begin{array}{c}\varphi_2\\ \Gamma', B, Q \vdash F\end{array}}{\Gamma, \Gamma', P \to Q, B \vdash F}\to_l}{\Gamma, \Gamma', A \wedge B, P \to Q \vdash F}\wedge_l$$

$$\dfrac{\begin{array}{c}\varphi_1\\ \Gamma \vdash P\end{array} \quad \dfrac{\begin{array}{c}\varphi_2\\ \Gamma', Q, A \vdash F\end{array}}{\Gamma', A \wedge B, Q \vdash F}\wedge_l}{\Gamma, \Gamma', A \wedge B, P \to Q \vdash F}\to_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{array}{c}\varphi_1\\ \Gamma \vdash P\end{array} \quad \begin{array}{c}\varphi_2\\ \Gamma', A, Q \vdash F\end{array}}{\Gamma, \Gamma', P \to Q, A \vdash F}\to_l}{\Gamma, \Gamma', A \wedge B, P \to Q \vdash F}\wedge_l$$

$$\dfrac{\dfrac{\begin{array}{c}\varphi_1\\ \Gamma, B \vdash P\end{array}}{\Gamma, A \wedge B \vdash P}\wedge_l \quad \begin{array}{c}\varphi_2\\ \Gamma', Q \vdash F\end{array}}{\Gamma, \Gamma', A \wedge B, P \to Q \vdash F}\to_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{array}{c}\varphi_1\\ \Gamma, B \vdash P\end{array} \quad \begin{array}{c}\varphi_2\\ \Gamma', Q \vdash F\end{array}}{\Gamma, \Gamma', P \to Q, B \vdash F}\to_l}{\Gamma, \Gamma', A \wedge B, P \to Q \vdash F}\wedge_l$$

$$\dfrac{\dfrac{\begin{array}{c}\varphi_1\\ \Gamma, A \vdash P\end{array}}{\Gamma, A \wedge B \vdash P}\wedge_l \quad \begin{array}{c}\varphi_2\\ \Gamma', Q \vdash F\end{array}}{\Gamma, \Gamma', A \wedge B, P \to Q \vdash F}\to_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{array}{c}\varphi_1\\ \Gamma, A \vdash P\end{array} \quad \begin{array}{c}\varphi_2\\ \Gamma', Q \vdash F\end{array}}{\Gamma, \Gamma', P \to Q, A \vdash F}\to_l}{\Gamma, \Gamma', A \wedge B, P \to Q \vdash F}\wedge_l$$

$\to_r$

$$\dfrac{\dfrac{\begin{array}{c}\varphi_1\\ \Gamma, P, B \vdash Q\end{array}}{\Gamma, A \wedge B, P \vdash Q}\wedge_l}{\Gamma, A \wedge B \vdash P \to Q}\to_r \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{array}{c}\varphi_1\\ \Gamma, B, P \vdash Q\end{array}}{\Gamma, B \vdash P \to Q}\to_r}{\Gamma, A \wedge B \vdash P \to Q}\wedge_l$$

$$\dfrac{\dfrac{\begin{array}{c}\varphi_1\\ \Gamma, P, A \vdash Q\end{array}}{\Gamma, A \wedge B, P \vdash Q}\wedge_l}{\Gamma, A \wedge B \vdash P \to Q}\to_r \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{array}{c}\varphi_1\\ \Gamma, A, P \vdash Q\end{array}}{\Gamma, A \vdash P \to Q}\to_r}{\Gamma, A \wedge B \vdash P \to Q}\wedge_l$$

$\wedge_l$

$$\dfrac{\dfrac{\begin{array}{c}\varphi_1\\ \Gamma, P, A \vdash F\end{array}}{\Gamma, A \wedge B, P \vdash F}\wedge_l}{\Gamma, A \wedge B, P \wedge Q \vdash F}\wedge_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{array}{c}\varphi_1\\ \Gamma, A, P \vdash F\end{array}}{\Gamma, P \wedge Q, A \vdash F}\wedge_l}{\Gamma, A \wedge B, P \wedge Q \vdash F}\wedge_l$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P, B \vdash F}}{\Gamma, A \wedge B, P \vdash F}\,\wedge_l}{\Gamma, A \wedge B, P \wedge Q \vdash F}\,\wedge_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, B, P \vdash F}}{\Gamma, P \wedge Q, B \vdash F}\,\wedge_l}{\Gamma, A \wedge B, P \wedge Q \vdash F}\,\wedge_l$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, Q, A \vdash F}}{\Gamma, A \wedge B, Q \vdash F}\,\wedge_l}{\Gamma, A \wedge B, P \wedge Q \vdash F}\,\wedge_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A, Q \vdash F}}{\Gamma, P \wedge Q, A \vdash F}\,\wedge_l}{\Gamma, A \wedge B, P \wedge Q \vdash F}\,\wedge_l$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, Q, B \vdash F}}{\Gamma, A \wedge B, Q \vdash F}\,\wedge_l}{\Gamma, A \wedge B, P \wedge Q \vdash F}\,\wedge_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, B, Q \vdash F}}{\Gamma, P \wedge Q, B \vdash F}\,\wedge_l}{\Gamma, A \wedge B, P \wedge Q \vdash F}\,\wedge_l$$

$\wedge_r$

$$\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \dfrac{\overset{\varphi_2}{\Gamma', B \vdash Q}}{\Gamma', A \wedge B \vdash Q}\,\wedge_l}{\Gamma, \Gamma', A \wedge B \vdash P \wedge Q}\,\wedge_r \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \overset{\varphi_2}{\Gamma', B \vdash Q}}{\Gamma, \Gamma', B \vdash P \wedge Q}\,\wedge_r}{\Gamma, \Gamma', A \wedge B \vdash P \wedge Q}\,\wedge_l$$

$$\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \dfrac{\overset{\varphi_2}{\Gamma', A \vdash Q}}{\Gamma', A \wedge B \vdash Q}\,\wedge_l}{\Gamma, \Gamma', A \wedge B \vdash P \wedge Q}\,\wedge_r \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \overset{\varphi_2}{\Gamma', A \vdash Q}}{\Gamma, \Gamma', A \vdash P \wedge Q}\,\wedge_r}{\Gamma, \Gamma', A \wedge B \vdash P \wedge Q}\,\wedge_l$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, B \vdash P}}{\Gamma, A \wedge B \vdash P}\,\wedge_l \quad \overset{\varphi_2}{\Gamma' \vdash Q}}{\Gamma, \Gamma', A \wedge B \vdash P \wedge Q}\,\wedge_r \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, B \vdash P} \quad \overset{\varphi_2}{\Gamma' \vdash Q}}{\Gamma, \Gamma', B \vdash P \wedge Q}\,\wedge_r}{\Gamma, \Gamma', A \wedge B \vdash P \wedge Q}\,\wedge_l$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A \vdash P}}{\Gamma, A \wedge B \vdash P}\,\wedge_l \quad \overset{\varphi_2}{\Gamma' \vdash Q}}{\Gamma, \Gamma', A \wedge B \vdash P \wedge Q}\,\wedge_r \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A \vdash P} \quad \overset{\varphi_2}{\Gamma' \vdash Q}}{\Gamma, \Gamma', A \vdash P \wedge Q}\,\wedge_r}{\Gamma, \Gamma', A \wedge B \vdash P \wedge Q}\,\wedge_l$$

$\vee_l$

$$\dfrac{\overset{\varphi_1}{\Gamma, P \vdash F} \qquad \dfrac{\overset{\varphi_2}{\Gamma', Q, B \vdash F}}{\Gamma', A \wedge B, Q \vdash F}\wedge_l}{\Gamma, \Gamma', A \wedge B, P \vee Q \vdash F}\vee_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P \vdash F} \qquad \overset{\varphi_2}{\Gamma', B, Q \vdash F}}{\Gamma, \Gamma', P \vee Q, B \vdash F}\vee_l}{\Gamma, \Gamma', A \wedge B, P \vee Q \vdash F}\wedge_l$$

$$\dfrac{\overset{\varphi_1}{\Gamma, P \vdash F} \qquad \dfrac{\overset{\varphi_2}{\Gamma', Q, A \vdash F}}{\Gamma', A \wedge B, Q \vdash F}\wedge_l}{\Gamma, \Gamma', A \wedge B, P \vee Q \vdash F}\vee_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P \vdash F} \qquad \overset{\varphi_2}{\Gamma', A, Q \vdash F}}{\Gamma, \Gamma', P \vee Q, A \vdash F}\vee_l}{\Gamma, \Gamma', A \wedge B, P \vee Q \vdash F}\wedge_l$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P, B \vdash F}}{\Gamma, A \wedge B, P \vdash F}\wedge_l \qquad \overset{\varphi_2}{\Gamma', Q \vdash F}}{\Gamma, \Gamma', A \wedge B, P \vee Q \vdash F}\vee_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, B, P \vdash F} \qquad \overset{\varphi_2}{\Gamma', Q \vdash F}}{\Gamma, \Gamma', P \vee Q, B \vdash F}\vee_l}{\Gamma, \Gamma', A \wedge B, P \vee Q \vdash F}\wedge_l$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P, A \vdash F}}{\Gamma, A \wedge B, P \vdash F}\wedge_l \qquad \overset{\varphi_2}{\Gamma', Q \vdash F}}{\Gamma, \Gamma', A \wedge B, P \vee Q \vdash F}\vee_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A, P \vdash F} \qquad \overset{\varphi_2}{\Gamma', Q \vdash F}}{\Gamma, \Gamma', P \vee Q, A \vdash F}\vee_l}{\Gamma, \Gamma', A \wedge B, P \vee Q \vdash F}\wedge_l$$

$\vee_r$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, B \vdash Q}}{\Gamma, A \wedge B \vdash Q}\wedge_l}{\Gamma, A \wedge B \vdash P \vee Q}\vee_r \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, B \vdash Q}}{\Gamma, B \vdash P \vee Q}\vee_r}{\Gamma, A \wedge B \vdash P \vee Q}\wedge_l$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, B \vdash P}}{\Gamma, A \wedge B \vdash P}\wedge_l}{\Gamma, A \wedge B \vdash P \vee Q}\vee_r \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, B \vdash P}}{\Gamma, B \vdash P \vee Q}\vee_r}{\Gamma, A \wedge B \vdash P \vee Q}\wedge_l$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A \vdash Q}}{\Gamma, A \wedge B \vdash Q}\wedge_l}{\Gamma, A \wedge B \vdash P \vee Q}\vee_r \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A \vdash Q}}{\Gamma, A \vdash P \vee Q}\vee_r}{\Gamma, A \wedge B \vdash P \vee Q}\wedge_l$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A \vdash P}}{\Gamma, A \wedge B \vdash P}\wedge_l}{\Gamma, A \wedge B \vdash P \vee Q}\vee_r \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A \vdash P}}{\Gamma, A \vdash P \vee Q}\vee_r}{\Gamma, A \wedge B \vdash P \vee Q}\wedge_l$$

$\forall_l$

$$\dfrac{\dfrac{\dfrac{\varphi_1}{\Gamma, P[x/t], B \vdash F}}{\Gamma, A \wedge B, P[x/t] \vdash F}\,\wedge_l}{\Gamma, \forall x.Px, A \wedge B \vdash F}\,\forall_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\varphi_1}{\Gamma, B, P[x/t] \vdash F}}{\Gamma, \forall x.Px, B \vdash F}\,\forall_l}{\Gamma, A \wedge B, \forall x.Px \vdash F}\,\wedge_l$$

$$\dfrac{\dfrac{\dfrac{\varphi_1}{\Gamma, P[x/t], A \vdash F}}{\Gamma, A \wedge B, P[x/t] \vdash F}\,\wedge_l}{\Gamma, \forall x.Px, A \wedge B \vdash F}\,\forall_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\varphi_1}{\Gamma, A, P[x/t] \vdash F}}{\Gamma, \forall x.Px, A \vdash F}\,\forall_l}{\Gamma, A \wedge B, \forall x.Px \vdash F}\,\wedge_l$$

$\forall_r$

$$\dfrac{\dfrac{\dfrac{\varphi_1}{\Gamma, B \vdash P[x/\alpha]}}{\Gamma, A \wedge B \vdash P[x/\alpha]}\,\wedge_l}{\Gamma, A \wedge B \vdash \forall x.Px}\,\forall_r \quad \rightsquigarrow \quad \dfrac{\dfrac{\Gamma, B \vdash P[x/\alpha]}{\Gamma, B \vdash \forall x.Px}\,\forall_r}{\Gamma, A \wedge B \vdash \forall x.Px}\,\wedge_l$$

$$\dfrac{\dfrac{\dfrac{\varphi_1}{\Gamma, A \vdash P[x/\alpha]}}{\Gamma, A \wedge B \vdash P[x/\alpha]}\,\wedge_l}{\Gamma, A \wedge B \vdash \forall x.Px}\,\forall_r \quad \rightsquigarrow \quad \dfrac{\dfrac{\Gamma, A \vdash P[x/\alpha]}{\Gamma, A \vdash \forall x.Px}\,\forall_r}{\Gamma, A \wedge B \vdash \forall x.Px}\,\wedge_l$$

$\exists_l$

$$\dfrac{\dfrac{\dfrac{\varphi_1}{\Gamma, P[x/\alpha], B \vdash F}}{\Gamma, A \wedge B, P[x/\alpha] \vdash F}\,\wedge_l}{\Gamma, \exists x.Px A \wedge B \vdash F}\,\exists_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\varphi_1}{\Gamma, B, P[x/\alpha] \vdash F}}{\Gamma, \exists x.Px, B \vdash F}\,\exists_l}{\Gamma, A \wedge B, \exists x.Px \vdash F}\,\wedge_l$$

$$\dfrac{\dfrac{\dfrac{\varphi_1}{\Gamma, P[x/\alpha], A \vdash F}}{\Gamma, A \wedge B, P[x/\alpha] \vdash F}\,\wedge_l}{\Gamma, \exists x.Px A \wedge B \vdash F}\,\exists_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\varphi_1}{\Gamma, A, P[x/\alpha] \vdash F}}{\Gamma, \exists x.Px, A \vdash F}\,\exists_l}{\Gamma, A \wedge B, \exists x.Px \vdash F}\,\wedge_l$$

$\exists_r$

$$\dfrac{\dfrac{\dfrac{\varphi_1}{\Gamma, B \vdash P[x/t]}}{\Gamma, A \wedge B \vdash P[x/t]}\,\wedge_l}{\Gamma, A \wedge B \vdash \exists x.Px}\,\exists_r \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\varphi_1}{\Gamma, B \vdash P[x/t]}}{\Gamma, B \vdash \exists x.Px}\,\exists_r}{\Gamma, A \wedge B \vdash \exists x.Px}\,\wedge_l$$

$$\dfrac{\dfrac{\dfrac{\varphi_1}{\Gamma, A \vdash P[x/t]}}{\Gamma, A \wedge B \vdash P[x/t]}\,\wedge_l}{\Gamma, A \wedge B \vdash \exists x.Px}\,\exists_r \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\varphi_1}{\Gamma, A \vdash P[x/t]}}{\Gamma, A \vdash \exists x.Px}\,\exists_r}{\Gamma, A \wedge B \vdash \exists x.Px}\,\wedge_l$$

$\neg_l$

$$\dfrac{\dfrac{\dfrac{\varphi_1}{\Gamma, B \vdash P}}{\Gamma, A \wedge B \vdash P}\,\wedge_l}{\Gamma, A \wedge B, \neg P \vdash}\,\neg_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\varphi_1}{\Gamma, B \vdash P}}{\Gamma, \neg P, B \vdash}\,\neg_l}{\Gamma, A \wedge B, \neg P \vdash}\,\wedge_l$$

$$\dfrac{\dfrac{\dfrac{\varphi_1}{\Gamma, A \vdash P}}{\Gamma, A \wedge B \vdash P}\,\wedge_l}{\Gamma, A \wedge B, \neg P \vdash}\,\neg_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\varphi_1}{\Gamma, A \vdash P}}{\Gamma, \neg P, A \vdash}\,\neg_l}{\Gamma, A \wedge B, \neg P \vdash}\,\wedge_l$$

$\neg_r$

$$\dfrac{\dfrac{\dfrac{\varphi_1}{\Gamma, P, B \vdash}}{\Gamma, A \wedge B, P \vdash}\,\wedge_l}{\Gamma, A \wedge B \vdash \neg P}\,\neg_r \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\varphi_1}{\Gamma, B, P \vdash}}{\Gamma, B \vdash \neg P}\,\neg_r}{\Gamma, A \wedge B \vdash \neg P}\,\wedge_l$$

$$\dfrac{\dfrac{\dfrac{\varphi_1}{\Gamma, P, A \vdash}}{\Gamma, A \wedge B, P \vdash}\,\wedge_l}{\Gamma, A \wedge B \vdash \neg P}\,\neg_r \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\varphi_1}{\Gamma, A, P \vdash}}{\Gamma, A \vdash \neg P}\,\neg_r}{\Gamma, A \wedge B \vdash \neg P}\,\wedge_l$$

$w_l$

$$\dfrac{\dfrac{\dfrac{\varphi_1}{\Gamma, A \vdash F}}{\Gamma, A \wedge B \vdash F}\,\wedge_l}{\Gamma, A \wedge B, P \vdash F}\,w_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\varphi_1}{\Gamma, A \vdash F}}{\Gamma, A, P \vdash F}\,w_l}{\Gamma, A \wedge B, P \vdash F}\,\wedge_l$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, B \vdash F}}{\Gamma, A \wedge B \vdash F}\,\wedge_l}{\Gamma, A \wedge B, P \vdash F}\,w_l \quad\rightsquigarrow\quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, B \vdash F}}{\Gamma, B, P \vdash F}\,w_l}{\Gamma, A \wedge B, P \vdash F}\,\wedge_l$$

$w_r$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A \vdash}}{\Gamma, A \wedge B \vdash}\,\wedge_l}{\Gamma, A \wedge B \vdash P}\,w_r \quad\rightsquigarrow\quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A \vdash}}{\Gamma, A \vdash P}\,w_r}{\Gamma, A \wedge B \vdash P}\,\wedge_l$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, B \vdash}}{\Gamma, A \wedge B \vdash}\,\wedge_l}{\Gamma, A \wedge B \vdash P}\,w_r \quad\rightsquigarrow\quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, B \vdash}}{\Gamma, B \vdash P}\,w_r}{\Gamma, A \wedge B \vdash P}\,\wedge_l$$

$c_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P, P, A \vdash F}}{\Gamma, A \wedge B, P, P \vdash F}\,\wedge_l}{\Gamma, A \wedge B, P \vdash F}\,c_l \quad\rightsquigarrow\quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A, P, P \vdash F}}{\Gamma, P, A \vdash F}\,c_l}{\Gamma, A \wedge B, P \vdash F}\,\wedge_l$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P, P, B \vdash F}}{\Gamma, A \wedge B, P, P \vdash F}\,\wedge_l}{\Gamma, A \wedge B, P \vdash F}\,c_l \quad\rightsquigarrow\quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, B, P, P \vdash F}}{\Gamma, P, B \vdash F}\,c_l}{\Gamma, A \wedge B, P \vdash F}\,\wedge_l$$

$cut$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A \vdash P}}{\Gamma, A \wedge B \vdash P}\,\wedge_l \qquad \overset{\varphi_2}{\Gamma', P \vdash F}}{\Gamma, \Gamma', A \wedge B \vdash F}\,cut \quad\rightsquigarrow\quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A \vdash P} \quad \overset{\varphi_2}{\Gamma', P \vdash F}}{\Gamma, \Gamma', A \vdash F}\,cut}{\Gamma, \Gamma', A \wedge B \vdash F}\,\wedge_l$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, B \vdash P}}{\Gamma, A \wedge B \vdash P}\,\wedge_l \qquad \overset{\varphi_2}{\Gamma', P \vdash F}}{\Gamma, \Gamma', A \wedge B \vdash F}\,cut \quad\rightsquigarrow\quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, B \vdash P} \quad \overset{\varphi_2}{\Gamma', P \vdash F}}{\Gamma, \Gamma', B \vdash F}\,cut}{\Gamma, \Gamma', A \wedge B \vdash F}\,\wedge_l$$

$$\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \dfrac{\overset{\varphi_2}{\Gamma', P, A \vdash F}}{\Gamma', A \wedge B, P \vdash F}\,\wedge_l}{\Gamma, \Gamma', A \wedge B \vdash F}\,\mathit{cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \overset{\varphi_2}{\Gamma', A, P \vdash F}}{\Gamma, \Gamma', A \vdash F}\,\mathit{cut}}{\Gamma, \Gamma', A \wedge B \vdash F}\,\wedge_l$$

$$\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \dfrac{\overset{\varphi_2}{\Gamma', P, B \vdash F}}{\Gamma', A \wedge B, P \vdash F}\,\wedge_l}{\Gamma, \Gamma', A \wedge B \vdash F}\,\mathit{cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \overset{\varphi_2}{\Gamma', B, P \vdash F}}{\Gamma, \Gamma', B \vdash F}\,\mathit{cut}}{\Gamma, \Gamma', A \wedge B \vdash F}\,\wedge_l$$

## B.4 Permutation of $\wedge_r$ down

$\to_l$

$$\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \dfrac{\overset{\varphi_2}{\Gamma' \vdash A} \quad \overset{\varphi_3}{\Gamma'', Q \vdash B}}{\Gamma', \Gamma'', Q \vdash A \wedge B}\,\wedge_r}{\Gamma, \Gamma', \Gamma'', P \to Q \vdash A \wedge B}\,\to_l \quad \rightsquigarrow \quad \dfrac{\overset{\varphi_2}{\Gamma' \vdash A} \quad \dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \overset{\varphi_3}{\Gamma'', Q \vdash B}}{\Gamma, \Gamma'', P \to Q \vdash B}\,\to_l}{\Gamma, \Gamma', \Gamma'', P \to Q \vdash A \wedge B}\,\wedge_r$$

$$\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \dfrac{\overset{\varphi_2}{\Gamma', Q \vdash A} \quad \overset{\varphi_3}{\Gamma'' \vdash B}}{\Gamma', \Gamma'', Q \vdash A \wedge B}\,\wedge_r}{\Gamma, \Gamma', \Gamma'', P \to Q \vdash A \wedge B}\,\to_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \overset{\varphi_2}{\Gamma', Q \vdash A}}{\Gamma, \Gamma', P \to Q \vdash A}\,\to_l \quad \overset{\varphi_3}{\Gamma'' \vdash B}}{\Gamma, \Gamma', \Gamma'', P \to Q \vdash A \wedge B}\,\wedge_r$$

$\to_r$

N/A

$\wedge_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \overset{\varphi_2}{\Gamma', Q \vdash B}}{\Gamma, \Gamma', Q \vdash A \wedge B}\,\wedge_r}{\Gamma, \Gamma', P \wedge Q \vdash A \wedge B}\,\wedge_l \quad \rightsquigarrow \quad \dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \dfrac{\overset{\varphi_2}{\Gamma', Q \vdash B}}{\Gamma', P \wedge Q \vdash B}\,\wedge_l}{\Gamma, \Gamma', P \wedge Q \vdash A \wedge B}\,\wedge_r$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, Q \vdash A} \quad \overset{\varphi_2}{\Gamma' \vdash B}}{\Gamma, \Gamma', Q \vdash A \wedge B}\,\wedge_r}{\Gamma, \Gamma', P \wedge Q \vdash A \wedge B}\,\wedge_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, Q \vdash A}}{\Gamma, P \wedge Q \vdash A}\,\wedge_l \quad \overset{\varphi_2}{\Gamma' \vdash B}}{\Gamma, \Gamma', P \wedge Q \vdash A \wedge B}\,\wedge_r$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \overset{\varphi_2}{\Gamma', P \vdash B}}{\Gamma, \Gamma', P \vdash A \wedge B}\wedge_r}{\Gamma, \Gamma', P \wedge Q \vdash A \wedge B}\wedge_l \quad \rightsquigarrow \quad \dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \dfrac{\overset{\varphi_2}{\Gamma', P \vdash B}}{\Gamma', P \wedge Q \vdash B}\wedge_l}{\Gamma, \Gamma', P \wedge Q \vdash A \wedge B}\wedge_r$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P \vdash A} \quad \overset{\varphi_2}{\Gamma' \vdash B}}{\Gamma, \Gamma', P \vdash A \wedge B}\wedge_r}{\Gamma, \Gamma', P \wedge Q \vdash A \wedge B}\wedge_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P \vdash A}}{\Gamma, P \wedge Q \vdash A}\wedge_l \quad \overset{\varphi_2}{\Gamma' \vdash B}}{\Gamma, \Gamma', P \wedge Q \vdash A \wedge B}\wedge_r$$

$\wedge_r$

N/A

$\vee_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \overset{\varphi_2}{\Gamma', P \vdash B}}{\Gamma, \Gamma', P \vdash A \wedge B}\wedge_r \quad \dfrac{\overset{\varphi_3}{\Gamma'' \vdash A} \quad \overset{\varphi_4}{\Gamma''', Q \vdash B}}{\Gamma'', \Gamma''', Q \vdash A \wedge B}\wedge_r}{\Gamma, \Gamma', \Gamma'', \Gamma''', P \vee Q \vdash A \wedge B}\vee_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \dfrac{\overset{\varphi_2}{\Gamma', P \vdash B} \quad \overset{\varphi_4}{\Gamma''', Q \vdash B}}{\Gamma', \Gamma''', P \vee Q \vdash B}\vee_l}{\Gamma, \Gamma', \Gamma''', P \vee Q \vdash A \wedge B}\wedge_r}{\Gamma, \Gamma', \Gamma'', \Gamma''', P \vee Q \vdash A \wedge B}w_l^*$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P \vdash A} \quad \overset{\varphi_2}{\Gamma' \vdash B}}{\Gamma, \Gamma', P \vdash A \wedge B}\wedge_r \quad \dfrac{\overset{\varphi_3}{\Gamma'' \vdash A} \quad \overset{\varphi_4}{\Gamma''', Q \vdash B}}{\Gamma'', \Gamma''', Q \vdash A \wedge B}\wedge_r}{\Gamma, \Gamma', \Gamma'', \Gamma''', P \vee Q \vdash A \wedge B}\vee_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P \vdash A} \quad \dfrac{\overset{\varphi_3}{\Gamma'' \vdash A}}{\Gamma'', Q \vdash A}w_l}{\Gamma, \Gamma'', P \vee Q \vdash A}\vee_l \quad \dfrac{\dfrac{\overset{\varphi_2}{\Gamma' \vdash B}}{\Gamma', P \vdash B}w_l \quad \overset{\varphi_4}{\Gamma''', Q \vdash B}}{\Gamma', \Gamma''', P \vee Q \vdash B}\vee_l}{\Gamma, \Gamma', \Gamma'', \Gamma''', P \vee Q, P \vee Q \vdash A \wedge B}\wedge_r}{\Gamma, \Gamma', \Gamma'', \Gamma''', P \vee Q \vdash A \wedge B}c_l$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P \vdash A} \quad \overset{\varphi_2}{\Gamma' \vdash B}}{\Gamma, \Gamma', P \vdash A \wedge B}\wedge_r \quad \dfrac{\overset{\varphi_3}{\Gamma'', Q \vdash A} \quad \overset{\varphi_4}{\Gamma''' \vdash B}}{\Gamma'', \Gamma''', Q \vdash A \wedge B}\wedge_r}{\Gamma, \Gamma', \Gamma'', \Gamma''', P \vee Q \vdash A \wedge B}\vee_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P \vdash A} \quad \overset{\varphi_3}{\Gamma'', Q \vdash A}}{\Gamma, \Gamma'', P \vee Q \vdash A}\vee_l \quad \overset{\varphi_4}{\Gamma''' \vdash B}}{\Gamma, \Gamma'', \Gamma''', P \vee Q \vdash A \wedge B}\wedge_r}{\Gamma, \Gamma', \Gamma'', \Gamma''', P \vee Q \vdash A \wedge B}w_l^*$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \overset{\varphi_2}{\Gamma', P \vdash B}}{\Gamma, \Gamma', P \vdash A \wedge B}\wedge_r \quad \dfrac{\overset{\varphi_3}{\Gamma'', Q \vdash A} \quad \overset{\varphi_4}{\Gamma''' \vdash B}}{\Gamma'', \Gamma''', Q \vdash A \wedge B}\wedge_r}{\Gamma, \Gamma', \Gamma'', \Gamma''', P \vee Q \vdash A \wedge B}\vee_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash A}}{\Gamma, P \vdash A}w_l \quad \overset{\varphi_3}{\Gamma'', Q \vdash A}}{\Gamma, \Gamma'', P \vee Q \vdash A}\vee_l \quad \dfrac{\overset{\varphi_2}{\Gamma', P \vdash B} \quad \dfrac{\overset{\varphi_4}{\Gamma''' \vdash B}}{\Gamma''', Q \vdash B}w_l}{\Gamma', \Gamma''', P \vee Q \vdash B}\vee_l}{\Gamma, \Gamma', \Gamma'', \Gamma''', P \vee Q, P \vee Q \vdash A \wedge B}\wedge_r}{\Gamma, \Gamma', \Gamma'', \Gamma''', P \vee Q \vdash A \wedge B}c_l$$

$\vee_r$

N/A

$\forall_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \overset{\varphi_2}{\Gamma', P[x/t] \vdash B}}{\Gamma, \Gamma', P[x/t] \vdash A \wedge B}\wedge_r}{\Gamma, \Gamma', \forall x.Px \vdash A \wedge B}\forall_l \quad \rightsquigarrow \quad \dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \dfrac{\overset{\varphi_2}{\Gamma', P[x/t] \vdash B}}{\Gamma', \forall x.Px \vdash B}\forall_l}{\Gamma, \Gamma', \forall x.Px \vdash A \wedge B}\wedge_r$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P[x/t] \vdash A} \quad \overset{\varphi_2}{\Gamma' \vdash B}}{\Gamma, \Gamma', P[x/t] \vdash A \wedge B}\wedge_r}{\Gamma, \Gamma', \forall x.Px \vdash A \wedge B}\forall_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P[x/t] \vdash A}}{\Gamma, \forall x.Px \vdash A}\forall_l \quad \overset{\varphi_2}{\Gamma' \vdash B}}{\Gamma, \Gamma', \forall x.Px \vdash A \wedge B}\wedge_r$$

$\forall_r$

N/A

$\exists_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \overset{\varphi_2}{\Gamma', P[x/\alpha] \vdash B}}{\Gamma, \Gamma', P[x/\alpha] \vdash A \wedge B}\wedge_r}{\Gamma, \Gamma', \exists x.Px \vdash A \wedge B}\exists_l \quad \rightsquigarrow \quad \dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \dfrac{\overset{\varphi_2}{\Gamma', P[x/\alpha] \vdash B}}{\Gamma', \exists x.Px \vdash B}\exists_l}{\Gamma, \Gamma', \exists x.Px \vdash A \wedge B}\wedge_r$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P[x/\alpha] \vdash A} \quad \overset{\varphi_2}{\Gamma' \vdash B}}{\Gamma, \Gamma', P[x/\alpha] \vdash A \wedge B}\wedge_r}{\Gamma, \Gamma', \exists x.Px \vdash A \wedge B}\exists_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P[x/\alpha] \vdash A}}{\Gamma, \exists x.Px \vdash A}\exists_l \quad \overset{\varphi_2}{\Gamma' \vdash B}}{\Gamma, \Gamma', \exists x.Px \vdash A \wedge B}\wedge_r$$

$\exists_r$

N/A

$\neg_l$

N/A

$\neg_r$

N/A

$w_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \overset{\varphi_2}{\Gamma' \vdash B}}{\Gamma, \Gamma' \vdash A \wedge B}\,\wedge_r}{\Gamma, \Gamma', P \vdash A \wedge B}\,w_l \quad\rightsquigarrow\quad \dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \dfrac{\overset{\varphi_2}{\Gamma' \vdash B}}{\Gamma', P \vdash B}\,w_l}{\Gamma, \Gamma', P \vdash A \wedge B}\,\wedge_r$$

$w_r$

N/A

$c_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P, P \vdash A} \quad \overset{\varphi_2}{\Gamma' \vdash B}}{\Gamma, \Gamma', P, P \vdash A \wedge B}\,\wedge_r}{\Gamma, \Gamma', P \vdash A \wedge B}\,c_l \quad\rightsquigarrow\quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P, P \vdash A}}{\Gamma, P \vdash A}\,c_l \quad \overset{\varphi_2}{\Gamma' \vdash B}}{\Gamma, \Gamma', P \vdash A \wedge B}\,\wedge_r$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \overset{\varphi_2}{\Gamma', P, P \vdash B}}{\Gamma, \Gamma', P, P \vdash A \wedge B}\,\wedge_r}{\Gamma, \Gamma', P \vdash A \wedge B}\,c_l \quad\rightsquigarrow\quad \dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \dfrac{\overset{\varphi_2}{\Gamma', P, P \vdash B}}{\Gamma', P \vdash B}\,c_l}{\Gamma, \Gamma', P \vdash A \wedge B}\,\wedge_r$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P \vdash A} \quad \overset{\varphi_2}{\Gamma', P \vdash B}}{\Gamma, \Gamma', P, P \vdash A \wedge B}\,\wedge_r}{\Gamma, \Gamma', P \vdash A \wedge B}\,c_l \quad\rightsquigarrow\quad ?$$

$cut$

$$\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \dfrac{\overset{\varphi_2}{\Gamma', P \vdash A} \quad \overset{\varphi_3}{\Gamma'' \vdash B}}{\Gamma', \Gamma'', P \vdash A \wedge B}\,\wedge_r}{\Gamma, \Gamma', \Gamma'' \vdash A \wedge B}\,cut \quad\rightsquigarrow\quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \overset{\varphi_2}{\Gamma', P \vdash A}}{\Gamma, \Gamma' \vdash A}\,cut \quad \overset{\varphi_3}{\Gamma'' \vdash B}}{\Gamma, \Gamma', \Gamma'' \vdash A \wedge B}\,\wedge_r$$

$$\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \dfrac{\overset{\varphi_2}{\Gamma' \vdash A} \quad \overset{\varphi_3}{\Gamma'', P \vdash B}}{\Gamma', \Gamma'', P \vdash A \wedge B}\,\wedge_r}{\Gamma, \Gamma', \Gamma'' \vdash A \wedge B}\,cut \quad\rightsquigarrow\quad \dfrac{\overset{\varphi_2}{\Gamma' \vdash A} \quad \dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \overset{\varphi_3}{\Gamma'', P \vdash B}}{\Gamma, \Gamma'' \vdash B}\,cut}{\Gamma, \Gamma', \Gamma'' \vdash A \wedge B}\,\wedge_r$$

## B.5 Permutation of $\lor_l$ down

$\to_l$

$$\dfrac{\overset{\varphi_1}{\Gamma\vdash P}\quad \dfrac{\overset{\varphi_2}{\Gamma',Q,A\vdash F}\quad \overset{\varphi_3}{\Gamma'',B\vdash F}}{\Gamma',\Gamma'',A\lor B,Q\vdash F}\lor_l}{\Gamma,\Gamma',\Gamma'',A\lor B,P\to Q\vdash F}\to_l \quad\rightsquigarrow\quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma\vdash P}\quad \overset{\varphi_2}{\Gamma',A,Q\vdash F}}{\Gamma,\Gamma',P\to Q,A\vdash F}\to_l\quad \overset{\varphi_3}{\Gamma'',B\vdash F}}{\Gamma,\Gamma',\Gamma'',P\to Q,A\lor B\vdash F}\lor_l$$

$$\dfrac{\overset{\varphi_1}{\Gamma\vdash P}\quad \dfrac{\overset{\varphi_2}{\Gamma',A\vdash F}\quad \overset{\varphi_3}{\Gamma'',Q,B\vdash F}}{\Gamma',\Gamma'',A\lor B,Q\vdash F}\lor_l}{\Gamma,\Gamma',\Gamma'',A\lor B,P\to Q\vdash F}\to_l \quad\rightsquigarrow\quad \dfrac{\overset{\varphi_2}{\Gamma',A\vdash F}\quad \dfrac{\overset{\varphi_1}{\Gamma\vdash P}\quad \overset{\varphi_3}{\Gamma'',B,Q\vdash F}}{\Gamma,\Gamma'',P\to Q,B\vdash F}\to_l}{\Gamma,\Gamma',\Gamma'',P\to Q,A\lor B\vdash F}\lor_l$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma,A\vdash P}\quad \overset{\varphi_2}{\Gamma',B\vdash P}}{\Gamma,\Gamma',A\lor B\vdash P}\lor_l\quad \overset{\varphi_3}{\Gamma'',Q\vdash F}}{\Gamma,\Gamma',\Gamma'',A\lor B,P\to Q\vdash F}\to_l \quad\rightsquigarrow\quad \dfrac{\dfrac{\dfrac{\overset{\varphi_1}{\Gamma,A\vdash P}\quad \overset{\varphi_3}{\Gamma'',Q\vdash F}}{\Gamma,\Gamma'',P\to Q,A\vdash F}\to_l\quad \dfrac{\overset{\varphi_2}{\Gamma',B\vdash P}\quad \overset{\varphi_3}{\Gamma'',Q\vdash F}}{\Gamma',\Gamma'',P\to Q,B\vdash F}\to_l}{\Gamma,\Gamma',\Gamma'',\Gamma'',P\to Q,P\to Q,A\lor B\vdash F}\lor_l}{\Gamma,\Gamma',\Gamma'',P\to Q,A\lor B\vdash F}c_l^*$$

$\to_r$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma,P,A\vdash Q}\quad \overset{\varphi_2}{\Gamma',B\vdash Q}}{\Gamma,\Gamma',A\lor B,P\vdash Q}\lor_l}{\Gamma,\Gamma',A\lor B\vdash P\to Q}\to_r \quad\rightsquigarrow\quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma,A,P\vdash Q}}{\Gamma,A\vdash P\to Q}\to_r\quad \dfrac{\dfrac{\overset{\varphi_2}{\Gamma',B\vdash Q}}{\Gamma',B,P\vdash Q}w_l}{\Gamma',B\vdash P\to Q}\to_r}{\Gamma,\Gamma',A\lor B\vdash P\to Q}\lor_l$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma,A\vdash Q}\quad \overset{\varphi_2}{\Gamma',P,B\vdash Q}}{\Gamma,\Gamma',A\lor B,P\vdash Q}\lor_l}{\Gamma,\Gamma',A\lor B\vdash P\to Q}\to_r \quad\rightsquigarrow\quad \dfrac{\dfrac{\dfrac{\overset{\varphi_1}{\Gamma,A\vdash Q}}{\Gamma,A,P\vdash Q}w_l}{\Gamma,A\vdash P\to Q}\to_r\quad \dfrac{\overset{\varphi_2}{\Gamma',B,P\vdash Q}}{\Gamma',B\vdash P\to Q}\to_r}{\Gamma,\Gamma',A\lor B\vdash P\to Q}\lor_l$$

$\land_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma,Q,A\vdash F}\quad \overset{\varphi_2}{\Gamma',B\vdash F}}{\Gamma,\Gamma',A\lor B,Q\vdash F}\lor_l}{\Gamma,\Gamma,P\land Q,A\lor B\vdash F}\land_l \quad\rightsquigarrow\quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma,A,Q\vdash F}}{\Gamma,P\land Q,A\vdash F}\land_l\quad \overset{\varphi_2}{\Gamma',B\vdash F}}{\Gamma,\Gamma',P\land Q,A\lor B\vdash F}\lor_l$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A \vdash F} \quad \overset{\varphi_2}{\Gamma', Q, B \vdash F}}{\Gamma, \Gamma', A \lor B, Q \vdash F}\,\lor_l}{\Gamma, \Gamma', P \land Q, A \lor B \vdash F}\,\land_l \quad\rightsquigarrow\quad \dfrac{\overset{\varphi_1}{\Gamma, A \vdash F} \quad \dfrac{\overset{\varphi_2}{\Gamma', B, Q \vdash F}}{\Gamma', P \land Q, B \vdash F}\,\land_l}{\Gamma, \Gamma', P \land Q, A \lor B \vdash F}\,\lor_l$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P, A \vdash F} \quad \overset{\varphi_2}{\Gamma', B \vdash F}}{\Gamma, \Gamma', A \lor B, P \vdash F}\,\lor_l}{\Gamma, \Gamma, P \land Q, A \lor B \vdash F}\,\land_l \quad\rightsquigarrow\quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A, P \vdash F}}{\Gamma, P \land Q, A \vdash F}\,\land_l \quad \overset{\varphi_2}{\Gamma', B \vdash F}}{\Gamma, \Gamma', P \land Q, A \lor B \vdash F}\,\lor_l$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A \vdash F} \quad \overset{\varphi_2}{\Gamma', P, B \vdash F}}{\Gamma, \Gamma', A \lor B, P \vdash F}\,\lor_l}{\Gamma, \Gamma', P \land Q, A \lor B \vdash F}\,\land_l \quad\rightsquigarrow\quad \dfrac{\overset{\varphi_1}{\Gamma, A \vdash F} \quad \dfrac{\overset{\varphi_2}{\Gamma', B, P \vdash F}}{\Gamma', P \land Q, B \vdash F}\,\land_l}{\Gamma, \Gamma', P \land Q, A \lor B \vdash F}\,\lor_l$$

$\land_r$

$$\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \dfrac{\overset{\varphi_2}{\Gamma', A \vdash Q} \quad \overset{\varphi_3}{\Gamma'', B \vdash Q}}{\Gamma', \Gamma'', A \lor B \vdash Q}\,\lor_l}{\Gamma, \Gamma', \Gamma'', A \lor B \vdash P \land Q}\,\land_r \quad\rightsquigarrow\quad \dfrac{\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \overset{\varphi_2}{\Gamma', A \vdash Q}}{\Gamma, \Gamma', A \vdash P \land Q}\,\land_r \quad \dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \overset{\varphi_3}{\Gamma'', B \vdash Q}}{\Gamma, \Gamma'', B \vdash P \land Q}\,\land_r}{\Gamma, \Gamma, \Gamma', \Gamma'', A \lor B \vdash P \land Q}\,\lor_l}{\Gamma, \Gamma', \Gamma'', A \lor B \vdash P \land Q}\,c_l^*$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A \vdash P} \quad \overset{\varphi_2}{\Gamma', B \vdash P}}{\Gamma, \Gamma', A \lor B \vdash P}\,\lor_l \quad \overset{\varphi_3}{\Gamma'' \vdash Q}}{\Gamma, \Gamma', \Gamma'', A \lor B \vdash P \land Q}\,\land_r \quad\rightsquigarrow\quad \dfrac{\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A \vdash P} \quad \overset{\varphi_3}{\Gamma'' \vdash Q}}{\Gamma, \Gamma'', A \vdash P \land Q}\,\land_r \quad \dfrac{\overset{\varphi_2}{\Gamma', B \vdash P} \quad \overset{\varphi_3}{\Gamma'' \vdash Q}}{\Gamma', \Gamma'', B \vdash P \land Q}\,\land_r}{\Gamma, \Gamma', \Gamma'', \Gamma'', A \lor B \vdash P \land Q}\,\lor_l}{\Gamma, \Gamma', \Gamma'', A \lor B \vdash P \land Q}\,c_l^*$$

$\lor_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P, A \vdash F} \quad \overset{\varphi_2}{\Gamma', B \vdash F}}{\Gamma, \Gamma', A \lor B, P \vdash F}\,\lor_l \quad \overset{\varphi_3}{\Gamma'', Q \vdash F}}{\Gamma, \Gamma', \Gamma'', A \lor B, P \lor Q \vdash F}\,\lor_l \quad\rightsquigarrow\quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A, P \vdash F} \quad \overset{\varphi_3}{\Gamma'', Q \vdash F}}{\Gamma, \Gamma'', P \lor Q, A \vdash F}\,\lor_l \quad \overset{\varphi_2}{\Gamma', B \vdash F}}{\Gamma, \Gamma', \Gamma'', A \lor B, P \lor Q \vdash F}\,\lor_l$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A \vdash F} \quad \overset{\varphi_2}{\Gamma', P, B \vdash F}}{\Gamma, \Gamma', A \lor B, P \vdash F}\,\lor_l \quad \overset{\varphi_3}{\Gamma'', Q \vdash F}}{\Gamma, \Gamma', \Gamma'', A \lor B, P \lor Q \vdash F}\,\lor_l \quad\rightsquigarrow\quad \dfrac{\overset{\varphi_1}{\Gamma, A \vdash F} \quad \dfrac{\overset{\varphi_2}{\Gamma', B, P \vdash F} \quad \overset{\varphi_3}{\Gamma'', Q \vdash F}}{\Gamma', \Gamma'', P \lor Q, B \vdash F}\,\lor_l}{\Gamma, \Gamma', \Gamma'', A \lor B, P \lor Q \vdash F}\,\lor_l$$

$$\dfrac{\overset{\varphi_1}{\Gamma, P \vdash F} \qquad \dfrac{\overset{\varphi_2}{\Gamma', Q, A \vdash F} \quad \overset{\varphi_3}{\Gamma'', B \vdash F}}{\Gamma', \Gamma'', A \vee B, Q \vdash F}\,\vee_l}{\Gamma, \Gamma', \Gamma'', A \vee B, P \vee Q \vdash F}\,\vee_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P \vdash F} \quad \overset{\varphi_2}{\Gamma', A, Q \vdash F}}{\Gamma, \Gamma', P \vee Q, A \vdash F}\,\vee_l \qquad \overset{\varphi_3}{\Gamma'', B \vdash F}}{\Gamma, \Gamma', \Gamma'', A \vee B, P \vee Q \vdash F}\,\vee_l$$

$$\dfrac{\overset{\varphi_1}{\Gamma, P \vdash F} \qquad \dfrac{\overset{\varphi_2}{\Gamma', A \vdash F} \quad \overset{\varphi_3}{\Gamma'', Q, B \vdash F}}{\Gamma', \Gamma'', A \vee B, Q \vdash F}\,\vee_l}{\Gamma, \Gamma', \Gamma'', A \vee B, P \vee Q \vdash F}\,\vee_l \quad \rightsquigarrow \quad \dfrac{\overset{\varphi_2}{\Gamma', A \vdash F} \qquad \dfrac{\overset{\varphi_1}{\Gamma, P \vdash F} \quad \overset{\varphi_3}{\Gamma'', B, Q \vdash F}}{\Gamma, \Gamma'', P \vee Q, B \vdash F}\,\vee_l}{\Gamma, \Gamma', \Gamma'', A \vee B, P \vee Q \vdash F}\,\vee_l$$

$\vee_r$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A \vdash Q} \quad \overset{\varphi_2}{\Gamma', B \vdash Q}}{\Gamma, \Gamma', A \vee B \vdash Q}\,\vee_l}{\Gamma, \Gamma', A \vee B \vdash P \vee Q}\,\vee_r \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A \vdash Q}}{\Gamma, A \vdash P \vee Q}\,\vee_l \qquad \dfrac{\overset{\varphi_2}{\Gamma', B \vdash Q}}{\Gamma', B \vdash P \vee Q}\,\vee_r}{\Gamma, \Gamma', A \vee B \vdash P \vee Q}\,\vee_l$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A \vdash P} \quad \overset{\varphi_2}{\Gamma', B \vdash P}}{\Gamma, \Gamma', A \vee B \vdash P}\,\vee_l}{\Gamma, \Gamma', A \vee B \vdash P \vee Q}\,\vee_r \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A \vdash P}}{\Gamma, A \vdash P \vee Q}\,\vee_l \qquad \dfrac{\overset{\varphi_2}{\Gamma', B \vdash P}}{\Gamma', B \vdash P \vee Q}\,\vee_r}{\Gamma, \Gamma', A \vee B \vdash P \vee Q}\,\vee_l$$

$\forall_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P[x/t], A \vdash F} \quad \overset{\varphi_2}{\Gamma', B \vdash F}}{\Gamma, \Gamma', A \vee B, P[x/t] \vdash F}\,\vee_l}{\Gamma, \Gamma', \forall x.Px, A \vee B \vdash F}\,\forall_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A, P[x/t] \vdash F}}{\Gamma, \forall x.Px, A \vdash F}\,\forall_l \qquad \overset{\varphi_2}{\Gamma', B \vdash F}}{\Gamma, \Gamma', \forall x.Px, A \vee B \vdash F}\,\vee_l$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A \vdash F} \quad \overset{\varphi_2}{\Gamma, P[x/t], B \vdash F}}{\Gamma, \Gamma', A \vee B, P[x/t] \vdash F}\,\vee_l}{\Gamma, \Gamma', \forall x.Px, A \vee B \vdash F}\,\forall_l \quad \rightsquigarrow \quad \dfrac{\overset{\varphi_1}{\Gamma, A \vdash F} \qquad \dfrac{\overset{\varphi_2}{\Gamma', B, P[x/t] \vdash F}}{\Gamma', \forall x.Px, B \vdash F}\,\forall_l}{\Gamma, \Gamma, \forall x.Px, A \vee B \vdash F}\,\vee_l$$

$\forall_r$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A \vdash P[x/\alpha]} \quad \overset{\varphi_2}{\Gamma', B \vdash P[x/\alpha]}}{\Gamma, \Gamma', A \lor B \vdash P[x/\alpha]}\,\lor_l}{\Gamma, \Gamma', A \lor B \vdash \forall x.Px}\,\forall_r \quad\rightsquigarrow\quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A \vdash P[x/\alpha]}}{\Gamma, A \vdash \forall x.Px}\,\forall_r \quad \dfrac{\overset{\varphi_2}{\Gamma', B \vdash P[x/\alpha]}}{\Gamma', B \vdash \forall x.Px}\,\forall_r}{\Gamma, \Gamma', A \lor B \vdash \forall x.Px}\,\lor_l$$

$\exists_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P[x/\alpha], A \vdash F} \quad \overset{\varphi_2}{\Gamma', B \vdash F}}{\Gamma, \Gamma', A \lor B, P[x/\alpha] \vdash F}\,\lor_l}{\Gamma, \Gamma', \exists x.Px, A \lor B \vdash F}\,\exists_l \quad\rightsquigarrow\quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A, P[x/\alpha] \vdash F}}{\Gamma, \exists x.Px, A \vdash F}\,\exists_l \quad \overset{\varphi_2}{\Gamma', B \vdash F}}{\Gamma, \Gamma', \exists x.Px, A \lor B \vdash F}\,\lor_l$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A \vdash F} \quad \overset{\varphi_2}{\Gamma', P[x/\alpha], B \vdash F}}{\Gamma, \Gamma', A \lor B, P[x/\alpha] \vdash F}\,\lor_l}{\Gamma, \Gamma', \exists x.Px, A \lor B \vdash F}\,\exists_l \quad\rightsquigarrow\quad \dfrac{\overset{\varphi_1}{\Gamma, A \vdash F} \quad \dfrac{\overset{\varphi_2}{\Gamma', B, P[x/\alpha] \vdash F}}{\Gamma', \exists x.Px, B \vdash F}\,\exists_l}{\Gamma, \Gamma', \exists x.Px, A \lor B \vdash F}\,\lor_l$$

$\exists_r$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma A \vdash P[x/t]} \quad \overset{\varphi_2}{\Gamma', B \vdash P[x/t]}}{\Gamma, \Gamma', A \lor B \vdash P[x/t]}\,\lor_l}{\Gamma, \Gamma', A \lor B \vdash \exists x.Px}\,\exists_r \quad\rightsquigarrow\quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A \vdash P[x/t]}}{\Gamma, A \vdash \exists x.Px}\,\exists_r \quad \dfrac{\overset{\varphi_2}{\Gamma', B \vdash P[x/t]}}{\Gamma', B \vdash \exists x.Px}\,\exists_r}{\Gamma, \Gamma', A \lor B \vdash \exists x.Px}\,\lor_l$$

$\neg_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A \vdash P} \quad \overset{\varphi_2}{\Gamma, B \vdash P}}{\Gamma, \Gamma', A \lor B \vdash P}\,\lor_l}{\Gamma, \Gamma', A \lor B, \neg P \vdash}\,\neg_l \quad\rightsquigarrow\quad \dfrac{\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A \vdash P}}{\Gamma, \neg P, A \vdash}\,\neg_l \quad \dfrac{\overset{\varphi_2}{\Gamma', B \vdash P}}{\Gamma', \neg P, B \vdash}\,\neg_l}{\Gamma, \Gamma', \neg P, \neg P, A \lor B \vdash}\,\lor_l}{\Gamma, \Gamma', \neg P, A \lor B \vdash}\,c_l$$

$\neg_r$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P, A \vdash} \quad \overset{\varphi_2}{\Gamma', B \vdash}}{\Gamma, \Gamma', A \lor B, P \vdash}\,\lor_l}{\Gamma, \Gamma', A \lor B \vdash \neg P}\,\neg_r \quad\rightsquigarrow\quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A, P \vdash}}{\Gamma, A \vdash \neg P}\,\neg_r \quad \dfrac{\overset{\varphi_2}{\Gamma', B \vdash}}{\Gamma', B \vdash \neg P}\,w_r}{\Gamma, \Gamma', A \lor B \vdash \neg P}\,\lor_l$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A \vdash} \quad \overset{\varphi_2}{\Gamma', P, B \vdash}}{\Gamma, \Gamma', A \vee B, P \vdash}\,\vee_l}{\Gamma, \Gamma', A \vee B \vdash \neg P}\,\neg_r \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A \vdash}}{\Gamma, A \vdash \neg P}\,w_l \quad \dfrac{\overset{\varphi_2}{\Gamma', B, P \vdash}}{\Gamma', B \vdash \neg P}\,\neg_r}{\Gamma, \Gamma', A \vee B \vdash \neg P}\,\vee_l$$

$w_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A \vdash F} \quad \overset{\varphi_2}{\Gamma', B \vdash F}}{\Gamma, \Gamma', A \vee B \vdash F}\,\vee_l}{\Gamma, \Gamma', A \vee B, P \vdash F}\,w_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A \vdash F}}{\Gamma, P, A \vdash F}\,w_l \quad \overset{\varphi_2}{\Gamma', B \vdash F}}{\Gamma, \Gamma', P, A \vee B \vdash F}\,\vee_l$$

$w_r$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A \vdash} \quad \overset{\varphi_2}{\Gamma', B \vdash}}{\Gamma, \Gamma', A \vee B \vdash}\,\vee_l}{\Gamma, \Gamma', A \vee B \vdash P}\,w_r \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A \vdash}}{\Gamma, A \vdash P}\,w_r \quad \dfrac{\overset{\varphi_2}{\Gamma', B \vdash}}{\Gamma', B \vdash P}\,w_r}{\Gamma, \Gamma', A \vee B \vdash P}\,\vee_l$$

$c_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P, A \vdash F} \quad \overset{\varphi_2}{\Gamma', P, B \vdash F}}{\Gamma, \Gamma', A \vee B, P, P \vdash F}\,\vee_l}{\Gamma, \Gamma', A \vee B, P \vdash F}\,c_l \quad \rightsquigarrow \quad ?$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P, P, A \vdash F} \quad \overset{\varphi_2}{\Gamma', B \vdash F}}{\Gamma, \Gamma', A \vee B, P, P \vdash F}\,\vee_l}{\Gamma, \Gamma', A \vee B, P \vdash F}\,c_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P, P, A \vdash F}}{\Gamma, P, A \vdash F}\,c_l \quad \overset{\varphi_2}{\Gamma', B \vdash F}}{\Gamma, \Gamma', P, A \vee B \vdash F}\,\vee_l$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A \vdash F} \quad \overset{\varphi_2}{\Gamma', P, P, B \vdash F}}{\Gamma, \Gamma', A \vee B, P, P \vdash F}\,\vee_l}{\Gamma, \Gamma', A \vee B, P \vdash F}\,c_l \quad \rightsquigarrow \quad \dfrac{\overset{\varphi_1}{\Gamma, A \vdash F} \quad \dfrac{\overset{\varphi_2}{\Gamma', P, P, B \vdash F}}{\Gamma', P, B \vdash F}\,c_l}{\Gamma, \Gamma', P, A \vee B \vdash F}\,\vee_l$$

*cut*

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A \vdash P} \quad \overset{\varphi_2}{\Gamma', B \vdash P}}{\Gamma, \Gamma', A \vee B \vdash P}\,\vee_l \quad \overset{\varphi_3}{\Gamma'', P \vdash F}}{\Gamma, \Gamma', \Gamma'', A \vee B \vdash F}\,\mathit{cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A \vdash P} \quad \overset{\varphi_3}{\Gamma'', P \vdash F}}{\Gamma, \Gamma'', A \vdash F}\,\mathit{cut} \quad \dfrac{\overset{\varphi_2}{\Gamma', B \vdash P} \quad \overset{\varphi_3}{\Gamma'', P \vdash F}}{\Gamma', \Gamma'', B \vdash F}\,\mathit{cut}}{\Gamma, \Gamma', \Gamma'', \Gamma'', A \vee B \vdash F}\,\vee_l}{\Gamma, \Gamma', \Gamma'', A \vee B \vdash F}\,c^*$$

$$\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \dfrac{\overset{\varphi_2}{\Gamma', P, A \vdash F} \quad \overset{\varphi_3}{\Gamma'', B \vdash F}}{\Gamma', \Gamma'', A \vee B, P \vdash F}\,\vee_l}{\Gamma, \Gamma', \Gamma'', A \vee B \vdash F}\,\mathit{cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \overset{\varphi_2}{\Gamma', A, P \vdash F}}{\Gamma, \Gamma', A \vdash F}\,\mathit{cut} \quad \overset{\varphi_3}{\Gamma'', B \vdash F}}{\Gamma, \Gamma', \Gamma'', A \vee B \vdash F}\,\vee_l$$

$$\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \dfrac{\overset{\varphi_2}{\Gamma', A \vdash F} \quad \overset{\varphi_3}{\Gamma'', P, B \vdash F}}{\Gamma', \Gamma'', A \vee B, P \vdash F}\,\vee_l}{\Gamma, \Gamma', \Gamma'', A \vee B \vdash F}\,\mathit{cut} \quad \rightsquigarrow \quad \dfrac{\overset{\varphi_2}{\Gamma', A \vdash F} \quad \dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \overset{\varphi_3}{\Gamma'', B, P \vdash F}}{\Gamma, \Gamma'', B \vdash F}\,\mathit{cut}}{\Gamma, \Gamma', \Gamma'', A \vee B \vdash F}\,\vee_l$$

## B.6 Permutation of $\vee_r$ down

$\rightarrow_l$

$$\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \dfrac{\overset{\varphi_2}{\Gamma', Q \vdash B}}{\Gamma', Q \vdash A \vee B}\,\vee_r}{\Gamma, \Gamma', P \rightarrow Q \vdash A \vee B}\,\rightarrow_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \overset{\varphi_2}{\Gamma'', Q \vdash B}}{\Gamma, \Gamma', P \rightarrow Q \vdash B}\,\rightarrow_l}{\Gamma, \Gamma', P \rightarrow Q \vdash A \vee B}\,\vee_r$$

$$\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \dfrac{\overset{\varphi_2}{\Gamma', Q \vdash A}}{\Gamma', Q \vdash A \vee B}\,\vee_r}{\Gamma, \Gamma', P \rightarrow Q \vdash A \vee B}\,\rightarrow_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \overset{\varphi_2}{\Gamma'', Q \vdash A}}{\Gamma, \Gamma', P \rightarrow Q \vdash A}\,\rightarrow_l}{\Gamma, \Gamma', P \rightarrow Q \vdash A \vee B}\,\vee_r$$

$\rightarrow_r$

N/A

$\wedge_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, Q \vdash B}}{\Gamma, Q \vdash A \vee B}\,\vee_r}{\Gamma, P \wedge Q \vdash A \vee B}\,\wedge_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, Q \vdash B}}{\Gamma, P \wedge Q \vdash B}\,\wedge_l}{\Gamma, P \wedge Q \vdash A \vee B}\,\vee_r$$

$$\dfrac{\dfrac{\begin{array}{c}\varphi_1\\ \Gamma, P \vdash B\end{array}}{\Gamma, P \vdash A \vee B}\,\vee_r}{\Gamma, P \wedge Q \vdash A \vee B}\,\wedge_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{array}{c}\varphi_1\\ \Gamma, P \vdash B\end{array}}{\Gamma, P \wedge Q \vdash B}\,\wedge_l}{\Gamma, P \wedge Q \vdash A \vee B}\,\vee_r$$

$$\dfrac{\dfrac{\begin{array}{c}\varphi_1\\ \Gamma, Q \vdash A\end{array}}{\Gamma, Q \vdash A \vee B}\,\vee_r}{\Gamma, P \wedge Q \vdash A \vee B}\,\wedge_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{array}{c}\varphi_1\\ \Gamma, Q \vdash A\end{array}}{\Gamma, P \wedge Q \vdash A}\,\wedge_l}{\Gamma, P \wedge Q \vdash A \vee B}\,\vee_r$$

$$\dfrac{\dfrac{\begin{array}{c}\varphi_1\\ \Gamma, P \vdash A\end{array}}{\Gamma, P \vdash A \vee B}\,\vee_r}{\Gamma, P \wedge Q \vdash A \vee B}\,\wedge_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{array}{c}\varphi_1\\ \Gamma, P \vdash A\end{array}}{\Gamma, P \wedge Q \vdash A}\,\wedge_l}{\Gamma, P \wedge Q \vdash A \vee B}\,\vee_r$$

$\wedge_r$

N/A

$\vee_l$

$$\dfrac{\dfrac{\begin{array}{c}\varphi_1\\ \Gamma, P \vdash B\end{array}}{\Gamma, P \vdash A \vee B}\,\vee_r \quad \dfrac{\begin{array}{c}\varphi_2\\ \Gamma', Q \vdash B\end{array}}{\Gamma', Q \vdash A \vee B}\,\vee_r}{\Gamma, \Gamma', P \vee Q \vdash A \vee B}\,\vee_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{array}{c}\varphi_1\\ \Gamma, P \vdash B\end{array} \quad \begin{array}{c}\varphi_2\\ \Gamma', Q \vdash B\end{array}}{\Gamma, \Gamma', P \vee Q \vdash B}\,\vee_l}{\Gamma, \Gamma', P \vee Q \vdash A \vee B}\,\vee_r$$

$$\dfrac{\dfrac{\begin{array}{c}\varphi_1\\ \Gamma, P \vdash A\end{array}}{\Gamma, P \vdash A \vee B}\,\vee_r \quad \dfrac{\begin{array}{c}\varphi_2\\ \Gamma', Q \vdash A\end{array}}{\Gamma', Q \vdash A \vee B}\,\vee_r}{\Gamma, \Gamma', P \vee Q \vdash A \vee B}\,\vee_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{array}{c}\varphi_1\\ \Gamma, P \vdash A\end{array} \quad \begin{array}{c}\varphi_2\\ \Gamma', Q \vdash A\end{array}}{\Gamma, \Gamma', P \vee Q \vdash A}\,\vee_l}{\Gamma, \Gamma', P \vee Q \vdash A \vee B}\,\vee_r$$

$$\dfrac{\dfrac{\begin{array}{c}\varphi_1\\ \Gamma, P \vdash A\end{array}}{\Gamma, P \vdash A \vee B}\,\vee_r \quad \dfrac{\begin{array}{c}\varphi_2\\ \Gamma', Q \vdash B\end{array}}{\Gamma', Q \vdash A \vee B}\,\vee_r}{\Gamma, \Gamma', P \vee Q \vdash A \vee B}\,\vee_l \quad \rightsquigarrow \quad ?$$

$$\dfrac{\dfrac{\begin{array}{c}\varphi_1\\ \Gamma, P \vdash B\end{array}}{\Gamma, P \vdash A \vee B}\,\vee_r \quad \dfrac{\begin{array}{c}\varphi_2\\ \Gamma', Q \vdash A\end{array}}{\Gamma', Q \vdash A \vee B}\,\vee_r}{\Gamma, \Gamma', P \vee Q \vdash A \vee B}\,\vee_l \quad \rightsquigarrow \quad ?$$

$\vee_r$

N/A

$\forall_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P[x/t] \vdash B}}{\Gamma, P[x/t] \vdash A \vee B}\,\vee_r}{\Gamma, \forall x.Px \vdash A \vee B}\,\forall_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P[x/t] \vdash B}}{\Gamma, \forall x.Ax \vdash B}\,\forall_l}{\Gamma, \forall x.Px \vdash A \vee B}\,\vee_r$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P[x/t] \vdash A}}{\Gamma, P[x/t] \vdash A \vee B}\,\vee_r}{\Gamma, \forall x.Px \vdash A \vee B}\,\forall_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P[x/t] \vdash A}}{\Gamma, \forall x.Ax \vdash A}\,\forall_l}{\Gamma, \forall x.Px \vdash A \vee B}\,\vee_r$$

$\forall_r$

N/A

$\exists_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P[x/\alpha] \vdash B}}{\Gamma, P[x/\alpha] \vdash A \vee B}\,\vee_r}{\Gamma, \exists x.Px \vdash A \vee B}\,\exists_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P[x/\alpha] \vdash B}}{\Gamma, \exists x.Px \vdash B}\,\exists_l}{\Gamma, \exists x.Px \vdash A \vee B}\,\vee_r$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P[x/\alpha] \vdash A}}{\Gamma, P[x/\alpha] \vdash A \vee B}\,\vee_r}{\Gamma, \exists x.Px \vdash A \vee B}\,\exists_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P[x/\alpha] \vdash A}}{\Gamma, \exists x.Px \vdash A}\,\exists_l}{\Gamma, \exists x.Px \vdash A \vee B}\,\vee_r$$

$\exists_r$

N/A

$\neg_l$

N/A

$\neg_r$

N/A

$w_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash A}}{\Gamma \vdash A \vee B}\,\vee_r}{\Gamma, P \vdash A \vee B}\,w_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash A}}{\Gamma, P \vdash A}\,w_l}{\Gamma, P \vdash A \vee B}\,\vee_r$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash B}}{\Gamma \vdash A \vee B}\,\vee_r}{\Gamma, P \vdash A \vee B}\,w_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash B}}{\Gamma, P \vdash B}\,w_l}{\Gamma, P \vdash A \vee B}\,\vee_r$$

$w_r$

N/A

$c_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P, P \vdash A}}{\Gamma, P, P \vdash A \vee B}\,\vee_r}{\Gamma, P \vdash A \vee B}\,c_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P, P \vdash A}}{\Gamma, P \vdash A}\,c_l}{\Gamma, P \vdash A \vee B}\,\vee_r$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P, P \vdash B}}{\Gamma, P, P \vdash A \vee B}\,\vee_r}{\Gamma, P \vdash A \vee B}\,c_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P, P \vdash B}}{\Gamma, P \vdash B}\,c_l}{\Gamma, P \vdash A \vee B}\,\vee_r$$

$cut$

$$\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \dfrac{\overset{\varphi_2}{\Gamma', P \vdash A}}{\Gamma', P \vdash A \vee B}\,\vee_l}{\Gamma, \Gamma' \vdash A \vee B}\,cut \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \overset{\varphi_2}{\Gamma', P \vdash A}}{\Gamma, \Gamma' \vdash A}\,cut}{\Gamma, \Gamma' \vdash A \vee B}\,\vee_r$$

$$\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \dfrac{\overset{\varphi_2}{\Gamma', P \vdash B}}{\Gamma', P \vdash A \vee B}\,\vee_l}{\Gamma, \Gamma' \vdash A \vee B}\,cut \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \overset{\varphi_2}{\Gamma', P \vdash B}}{\Gamma, \Gamma' \vdash B}\,cut}{\Gamma, \Gamma' \vdash A \vee B}\,\vee_r$$

## B.7 Permutation of $\forall_l$ down

$\to_l$

$$\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \dfrac{\overset{\varphi_2}{\Gamma', Q, A[x/t] \vdash F}}{\Gamma', \forall x.Ax, Q \vdash F}\,\forall_l}{\Gamma, \Gamma', \forall x.Ax, P \to Q \vdash F}\,\to_l \quad\rightsquigarrow\quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \overset{\varphi_2}{\Gamma', A[x/t], Q \vdash F}}{\Gamma, \Gamma', P \to Q, A[x/t] \vdash F}\,\to_l}{\Gamma, \Gamma', \forall x.Ax, P \to Q \vdash F}\,\forall_l$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A[x/t] \vdash P}}{\Gamma, \forall x.Ax \vdash P}\,\forall_l \quad \overset{\varphi_2}{\Gamma', Q \vdash F}}{\Gamma, \Gamma', \forall x.Ax, P \to Q \vdash F}\,\to_l \quad\rightsquigarrow\quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A[x/t] \vdash P} \quad \overset{\varphi_2}{\Gamma', Q \vdash F}}{\Gamma, \Gamma', P \to Q, A[x/t] \vdash F}\,\to_l}{\Gamma, \Gamma', \forall x.Ax, P \to Q \vdash F}\,\forall_l$$

$\to_r$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P, A[x/t] \vdash Q}}{\Gamma, \forall x.Ax, P \vdash Q}\,\forall_l}{\Gamma, \forall x.Ax \vdash P \to Q}\,\to_r \quad\rightsquigarrow\quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A[x/t], P \vdash Q}}{\Gamma, A[x/t] \vdash P \to Q}\,\to_r}{\Gamma, \forall x.Ax \vdash P \to Q}\,\forall_l$$

$\wedge_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, Q, A[x/t] \vdash F}}{\Gamma, \forall x.Ax, Q \vdash F}\,\forall_l}{\Gamma, P \wedge Q, \forall x.Ax \vdash F}\,\wedge_l \quad\rightsquigarrow\quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A[x/t], Q \vdash F}}{\Gamma, P \wedge Q, A[x/t] \vdash F}\,\wedge_l}{\Gamma, \forall x.Ax, P \wedge Q \vdash F}\,\forall_l$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P, A[x/t] \vdash F}}{\Gamma, \forall x.Ax, P \vdash F}\,\forall_l}{\Gamma, P \wedge Q, \forall x.Ax \vdash F}\,\wedge_l \quad\rightsquigarrow\quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A[x/t], P \vdash F}}{\Gamma, P \wedge Q, A[x/t] \vdash F}\,\wedge_l}{\Gamma, \forall x.Ax, P \wedge Q \vdash F}\,\forall_l$$

$\wedge_r$

$$\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \dfrac{\overset{\varphi_2}{\Gamma', A[x/t] \vdash Q}}{\Gamma', \forall x.Ax \vdash Q}\,\forall_l}{\Gamma, \Gamma', \forall x.Ax \vdash P \wedge Q}\,\wedge_r \quad\rightsquigarrow\quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \overset{\varphi_2}{\Gamma', A[x/t] \vdash Q}}{\Gamma, \Gamma', A[x/t] \vdash P \wedge Q}\,\wedge_r}{\Gamma, \Gamma', \forall x.Ax \vdash P \wedge Q}\,\forall_l$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A[x/t] \vdash P}}{\Gamma, \forall x.Ax \vdash P}\,\forall_l \qquad \overset{\varphi_2}{\Gamma' \vdash Q}}{\Gamma, \Gamma', \forall x.Ax \vdash P \wedge Q}\,\wedge_r \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A[x/t] \vdash P} \qquad \overset{\varphi_2}{\Gamma' \vdash Q}}{\Gamma, \Gamma', A[x/t] \vdash P \wedge Q}\,\wedge_r}{\Gamma, \Gamma', \forall x.Ax \vdash P \wedge Q}\,\forall_l$$

$\vee_l$

$$\dfrac{\overset{\varphi_1}{\Gamma, P \vdash F} \qquad \dfrac{\overset{\varphi_2}{\Gamma', Q, A[x/t] \vdash F}}{\Gamma', \forall x.Ax, Q \vdash F}\,\forall_l}{\Gamma, \Gamma', \forall x.Ax, P \vee Q \vdash F}\,\vee_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P \vdash F} \qquad \overset{\varphi_2}{\Gamma', A[x/t], Q \vdash F}}{\Gamma, \Gamma', P \vee Q, A[x/t] \vdash F}\,\vee_l}{\Gamma, \Gamma', \forall x.Ax, P \vee Q \vdash F}\,\forall_l$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P, A[x/t] \vdash F}}{\Gamma, \forall x.Ax, P \vdash F}\,\forall_l \qquad \overset{\varphi_2}{\Gamma', Q \vdash F}}{\Gamma, \Gamma', \forall x.Ax, P \vee Q \vdash F}\,\vee_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A[x/t], P \vdash F} \qquad \overset{\varphi_2}{\Gamma', Q \vdash F}}{\Gamma, \Gamma, P \vee Q, A[x/t] \vdash F}\,\vee_l}{\Gamma, \Gamma', \forall x.Ax, P \vee Q \vdash F}\,\forall_l$$

$\vee_r$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A[x/t] \vdash Q}}{\Gamma, \forall x.Ax \vdash Q}\,\forall_l}{\Gamma, \forall x.Ax \vdash P \vee Q}\,\vee_r \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A[x/t] \vdash Q}}{\Gamma, A[x/t] \vdash P \vee Q}\,\vee_r}{\Gamma, \forall x.Ax \vdash P \vee Q}\,\forall_l$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A[x/t] \vdash P}}{\Gamma, \forall x.Ax \vdash P}\,\forall_l}{\Gamma, \forall x.Ax \vdash P \vee Q}\,\vee_r \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A[x/t] \vdash P}}{\Gamma, A[x/t] \vdash P \vee Q}\,\vee_r}{\Gamma, \forall x.Ax \vdash P \vee Q}\,\forall_l$$

$\forall_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P[x/t], A[x/t'] \vdash F}}{\Gamma, \forall x.Ax, P[x/t] \vdash F}\,\forall_l}{\Gamma, \forall x.Ax, \forall x.Px \vdash F}\,\forall_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A[x/t'], P[x/t] \vdash F}}{\Gamma, \forall x.Px, A[x/t'] \vdash F}\,\forall_l}{\Gamma, \forall x.Ax, \forall x.Px \vdash F}\,\forall_l$$

$\forall_r$

Note: watch out for eigenvariable violations (if $t$ contains $\alpha$).

$$\begin{array}{c}\varphi_1\\ \dfrac{\dfrac{\Gamma, A[x/t] \vdash P[x/\alpha]}{\Gamma, \forall x.Ax \vdash P[x/\alpha]}\,\forall_l}{\Gamma, \forall x.Ax \vdash \forall x.Px}\,\forall_r\end{array} \quad \rightsquigarrow \quad \begin{array}{c}\varphi_2\\ \dfrac{\dfrac{\Gamma, A[x/t] \vdash P[x/\alpha]}{\Gamma, A[x/t] \vdash \forall x.Px}\,\forall_r}{\Gamma, \forall x.Ax \vdash \forall x.Px}\,\forall_l\end{array}$$

$\exists_l$

Note: watch out for eigenvariable violations (if $t$ contains $\alpha$)

$$\begin{array}{c}\varphi_1\\ \dfrac{\dfrac{\Gamma, P[x/\alpha], A[x/t] \vdash F}{\Gamma, \forall x.Ax, P[x/\alpha] \vdash F}\,\forall_l}{\Gamma, \exists x.Px, \forall x.Ax \vdash F}\,\exists_l\end{array} \quad \rightsquigarrow \quad \begin{array}{c}\varphi_1\\ \dfrac{\dfrac{\Gamma, A[x/t], P[x/\alpha] \vdash F}{\Gamma, \exists x.Px, A[x/t] \vdash F}\,\exists_l}{\Gamma, \forall x.Ax, \exists x.Px \vdash F}\,\forall_l\end{array}$$

$\exists_r$

$$\begin{array}{c}\varphi_1\\ \dfrac{\dfrac{\Gamma, A[x/t'] \vdash P[x/t]}{\Gamma, \forall x.Ax \vdash P[x/t]}\,\forall_l}{\Gamma, \forall x.Ax \vdash \exists x.Px}\,\exists_r\end{array} \quad \rightsquigarrow \quad \begin{array}{c}\varphi_1\\ \dfrac{\dfrac{\Gamma, A[x/t'] \vdash P[x/t]}{\Gamma, A[x/t'] \vdash \exists x.Px}\,\exists_r}{\Gamma, \forall x.Ax \vdash \exists x.Px}\,\forall_l\end{array}$$

$\neg_l$

$$\begin{array}{c}\varphi_1\\ \dfrac{\dfrac{\Gamma, A[x/t] \vdash P}{\Gamma, \forall x.Ax \vdash P}\,\forall_l}{\Gamma, \forall x.Ax, \neg P \vdash}\,\neg_l\end{array} \quad \rightsquigarrow \quad \begin{array}{c}\varphi_1\\ \dfrac{\dfrac{\Gamma, A[x/t] \vdash P}{\Gamma, \neg P, A[x/t] \vdash}\,\neg_l}{\Gamma, \forall x.Ax, \neg P \vdash}\,\forall_l\end{array}$$

$\neg_r$

$$\begin{array}{c}\varphi_1\\ \dfrac{\dfrac{\Gamma, P, A[x/t] \vdash}{\Gamma, \forall x.Ax, P \vdash}\,\forall_l}{\Gamma, \forall x.Ax \vdash \neg P}\,\neg_r\end{array} \quad \rightsquigarrow \quad \begin{array}{c}\varphi_1\\ \dfrac{\dfrac{\Gamma, A[x/t], P \vdash}{\Gamma, A[x/t] \vdash \neg P}\,\neg_r}{\Gamma, \forall x.Ax \vdash \neg P}\,\forall_l\end{array}$$

$w_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A[x/t] \vdash F}}{\Gamma, \forall x.Ax \vdash F}\;\forall_l}{\Gamma, \forall x.Ax, P \vdash F}\;w_l \quad\rightsquigarrow\quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A[x/t] \vdash F}}{\Gamma, P, A[x/t] \vdash F}\;w_l}{\Gamma, \forall x.Ax, P \vdash F}\;\forall_l$$

$w_r$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A[x/t] \vdash}}{\Gamma, \forall x.Ax \vdash}\;\forall_l}{\Gamma, \forall x.Ax \vdash P}\;w_r \quad\rightsquigarrow\quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A[x/t] \vdash}}{\Gamma, A[x/t] \vdash P}\;w_r}{\Gamma, \forall x.Ax \vdash P}\;\forall_l$$

$c_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P, P, A[x/t] \vdash F}}{\Gamma, \forall x.Ax, P, P \vdash F}\;\forall_l}{\Gamma, \forall x.Ax, P \vdash F}\;c_l \quad\rightsquigarrow\quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A[x/t], P, P \vdash F}}{\Gamma, P, A[x/t] \vdash F}\;c_l}{\Gamma, \forall x.Ax, P \vdash F}\;\forall_l$$

$cut$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A[x/t] \vdash P}}{\Gamma, \forall x.Ax \vdash P}\;\forall_l \qquad \overset{\varphi_2}{\Gamma', P \vdash F}}{\Gamma, \Gamma', \forall x.Ax \vdash F}\;cut \quad\rightsquigarrow\quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A[x/t] \vdash P} \qquad \overset{\varphi_2}{\Gamma', P \vdash F}}{\Gamma, \Gamma', A[x/t] \vdash F}\;cut}{\Gamma, \Gamma', \forall x.Ax \vdash F}\;\forall_l$$

$$\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \qquad \dfrac{\overset{\varphi_2}{\Gamma', P, A[x/t] \vdash F}}{\Gamma', \forall x.Ax, P \vdash F}\;\forall_l}{\Gamma, \Gamma', \forall x.Ax \vdash F}\;cut \quad\rightsquigarrow\quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \qquad \overset{\varphi_2}{\Gamma', A[x/t], P \vdash F}}{\Gamma, \Gamma', A[x/t] \vdash F}\;cut}{\Gamma, \Gamma', \forall x.Ax \vdash F}\;\forall_l$$

## B.8 Permutation of $\forall_r$ down

$\to_l$

$$\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \qquad \dfrac{\overset{\varphi_2}{\Gamma', Q \vdash A[x/\alpha]}}{\Gamma', Q \vdash \forall x.Ax}\;\forall_r}{\Gamma, \Gamma', P \to Q \vdash \forall x.Ax}\;\to_l \quad\rightsquigarrow\quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \qquad \overset{\varphi_2}{\Gamma, Q \vdash A[x/\alpha]}}{\Gamma, \Gamma', P \to Q \vdash A[x/\alpha]}\;\to_l}{\Gamma, \Gamma', P \to Q \vdash \forall x.Ax}\;\forall_r$$

$\rightarrow_r$

N/A

$\wedge_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, Q \vdash A[x/\alpha]}}{\Gamma, Q \vdash \forall x.Ax}\;\forall_r}{\Gamma, P \wedge Q \vdash \forall x.Ax}\;\wedge_l \quad\rightsquigarrow\quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, Q \vdash A[x/\alpha]}}{\Gamma, P \wedge Q \vdash A[x/\alpha]}\;\wedge_l}{\Gamma, P \wedge Q \vdash \forall x.Ax}\;\forall_r$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P \vdash A[x/\alpha]}}{\Gamma, P \vdash \forall x.Ax}\;\forall_r}{\Gamma, P \wedge Q \vdash \forall x.Ax}\;\wedge_l \quad\rightsquigarrow\quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P \vdash A[x/\alpha]}}{\Gamma, P \wedge Q \vdash A[x/\alpha]}\;\wedge_l}{\Gamma, P \wedge Q \vdash \forall x.Ax}\;\forall_r$$

$\wedge_r$

N/A

$\vee_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P \vdash A[x/\alpha]}}{\Gamma, P \vdash \forall x.Ax}\;\forall_r \qquad \dfrac{\overset{\varphi_2}{\Gamma', Q \vdash A[x/\alpha]}}{\Gamma', Q \vdash \forall x.Ax}\;\forall_r}{\Gamma, \Gamma', P \vee Q \vdash \forall x.Ax}\;\vee_l \quad\rightsquigarrow\quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P \vdash A[x/\alpha]} \qquad \overset{\varphi_2}{\Gamma', Q \vdash A[x/\alpha]}}{\Gamma, \Gamma', P \vee Q \vdash A[x/\alpha]}\;\vee_l}{\Gamma, \Gamma', P \vee Q \vdash \forall x.Ax}\;\forall_r$$

$\vee_r$

N/A

$\forall_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P[x/t] \vdash A[x/\alpha]}}{\Gamma, P[x/t] \vdash \forall x.Ax}\;\forall_r}{\Gamma, \forall x.Px \vdash \forall x.Ax}\;\forall_l \quad\rightsquigarrow\quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P[x/t] \vdash A[x/\alpha]}}{\Gamma, \forall x.Px \vdash A[x/\alpha]}\;\forall_l}{\Gamma, \forall x.Px \vdash \forall x.Ax}\;\forall_r$$

$\forall_r$

N/A

$\exists_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P[x/\alpha] \vdash A[x/\alpha']}}{\Gamma, P[x/\alpha] \vdash \forall x.Ax}\ \forall_r}{\Gamma, \exists x.Px \vdash \forall x.Ax}\ \exists_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P[x/\alpha] \vdash A[x/\alpha']}}{\Gamma, \exists x.Px \vdash A[x/\alpha']}\ \exists_l}{\Gamma, \exists x.Px \vdash \forall x.Ax}\ \forall_r$$

$\exists_r$

N/A

$\neg_l$

N/A

$\neg_r$

N/A

$w_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash A[x/\alpha]}}{\Gamma \vdash \forall x.Ax}\ \forall_r}{\Gamma, P \vdash \forall x.Ax}\ w_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash A[x/\alpha]}}{\Gamma, P \vdash A[x/\alpha]}\ w_l}{\Gamma, P \vdash \forall x.Ax}\ \forall_r$$

$w_r$

N/A

$c_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P, P \vdash A[x/\alpha]}}{\Gamma, P, P \vdash \forall x.Ax}\ \forall_r}{\Gamma, P \vdash \forall x.Ax}\ c_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P, P \vdash A[x/\alpha]}}{\Gamma, P \vdash A[x/\alpha]}\ c_l}{\Gamma, P \vdash \forall x.Ax}\ \forall_r$$

$cut$

$$\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \dfrac{\overset{\varphi_2}{\Gamma', P \vdash A[x/\alpha]}}{\Gamma', P \vdash \forall x.Ax}\ \forall_r}{\Gamma, \Gamma' \vdash \forall x.Ax}\ cut \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \overset{\varphi_2}{\Gamma', P \vdash A[x/\alpha]}}{\Gamma, \Gamma' \vdash A[x/\alpha]}\ cut}{\Gamma, \Gamma' \vdash \forall x.Ax}\ \forall_r$$

## B.9 Permutation of $\exists_l$ down

$\to_l$

$$\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \dfrac{\overset{\varphi_2}{\Gamma', Q, A[x/\alpha] \vdash F}}{\Gamma', \exists x.Ax, Q \vdash F}\,\exists_l}{\Gamma, \Gamma', \exists x.Ax, P \to Q \vdash F}\,\to_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \overset{\varphi_2}{\Gamma', A[x/\alpha], Q \vdash F}}{\Gamma, \Gamma', P \to Q, A[x/\alpha] \vdash F}\,\to_l}{\Gamma, \Gamma', \exists x.Ax, P \to Q \vdash F}\,\exists_l$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A[x/\alpha] \vdash P}}{\Gamma, \exists x.Ax \vdash P}\,\exists_l \quad \overset{\varphi_2}{\Gamma', Q \vdash F}}{\Gamma, \Gamma', \exists x.Ax, P \to Q \vdash F}\,\to_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A[x/\alpha] \vdash P} \quad \overset{\varphi_2}{\Gamma', Q \vdash F}}{\Gamma, \Gamma', P \to Q, A[x/\alpha] \vdash F}\,\to_l}{\Gamma, \Gamma', \exists x.Ax, P \to Q \vdash F}\,\exists_l$$

$\to_r$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P, A[x/\alpha] \vdash Q}}{\Gamma, \exists x.Ax, P \vdash Q}\,\exists_l}{\Gamma, \exists x.Ax \vdash P \to Q}\,\to_r \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A[x/\alpha], P \vdash Q}}{\Gamma, A[x/\alpha] \vdash P \to Q}\,\to_r}{\Gamma, \exists x.Ax \vdash P \to Q}\,\exists_l$$

$\wedge_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, Q, A[x/\alpha] \vdash F}}{\Gamma, \exists x.Ax, Q \vdash F}\,\exists_l}{\Gamma, P \wedge Q, \exists x.Ax \vdash F}\,\wedge_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A[x/\alpha], Q \vdash F}}{\Gamma, P \wedge Q, A[x/\alpha] \vdash F}\,\wedge_l}{\Gamma, \exists x.Ax, P \wedge Q \vdash F}\,\exists_l$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P, A[x/\alpha] \vdash F}}{\Gamma, \exists x.Ax, P \vdash F}\,\exists_l}{\Gamma, P \wedge Q, \exists x.Ax \vdash F}\,\wedge_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A[x/\alpha], P \vdash F}}{\Gamma, P \wedge Q, A[x/\alpha] \vdash F}\,\wedge_l}{\Gamma, \exists x.Ax, P \wedge Q \vdash F}\,\exists_l$$

$\wedge_r$

$$\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \dfrac{\overset{\varphi_2}{\Gamma', A[x/\alpha] \vdash Q}}{\Gamma', \exists x.Ax \vdash Q}\,\exists_l}{\Gamma, \Gamma', \exists x.Ax \vdash P \wedge Q}\,\wedge_r \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \overset{\varphi_2}{\Gamma', A[x/\alpha] \vdash Q}}{\Gamma, \Gamma', A[x/\alpha] \vdash P \wedge Q}\,\wedge_r}{\Gamma, \Gamma', \exists x.Ax \vdash P \wedge Q}\,\exists_l$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A[x/\alpha] \vdash P}}{\Gamma, \exists x.Ax \vdash P}\exists_l \quad \overset{\varphi_2}{\Gamma' \vdash Q}}{\Gamma, \Gamma', \exists x.Ax \vdash P \wedge Q}\wedge_r \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A[x/\alpha] \vdash P} \quad \overset{\varphi_2}{\Gamma' \vdash Q}}{\Gamma, \Gamma', A[x/\alpha] \vdash P \wedge Q}\wedge_r}{\Gamma, \Gamma', \exists x.Ax \vdash P \wedge Q}\exists_l$$

$\vee_l$

$$\dfrac{\overset{\varphi_1}{\Gamma, P \vdash F} \quad \dfrac{\overset{\varphi_2}{\Gamma', Q, A[x/\alpha] \vdash F}}{\Gamma', \exists x.Ax, Q \vdash F}\exists_l}{\Gamma, \Gamma', \exists x.Ax, P \vee Q \vdash F}\vee_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P \vdash F} \quad \overset{\varphi_2}{\Gamma', A[x/\alpha], Q \vdash F}}{\Gamma, \Gamma', P \vee Q, A[x/\alpha] \vdash F}\vee_l}{\Gamma, \Gamma', \exists x.Ax, P \vee Q \vdash F}\exists_l$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P, A[x/\alpha] \vdash F}}{\Gamma, \exists x.Ax, P \vdash F}\exists_l \quad \overset{\varphi_2}{\Gamma', Q \vdash F}}{\Gamma, \Gamma', \exists x.Ax, P \vee Q \vdash F}\vee_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A[x/\alpha], P \vdash F} \quad \overset{\varphi_2}{\Gamma', Q \vdash F}}{\Gamma, \Gamma', P \vee Q, A[x/\alpha] \vdash F}\vee_l}{\Gamma, \Gamma', \exists x.Ax, P \vee Q \vdash F}\exists_l$$

$\vee_r$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A[x/\alpha] \vdash Q}}{\Gamma, \exists x.Ax \vdash Q}\exists_l}{\Gamma, \exists x.Ax \vdash P \vee Q}\vee_r \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A[x/\alpha] \vdash Q}}{\Gamma, A[x/\alpha] \vdash P \vee Q}\vee_r}{\Gamma, \exists x.Ax \vdash P \vee Q}\exists_l$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A[x/\alpha] \vdash P}}{\Gamma, \exists x.Ax \vdash P}\exists_l}{\Gamma, \exists x.Ax \vdash P \vee Q}\vee_r \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A[x/\alpha] \vdash P}}{\Gamma, A[x/\alpha] \vdash P \vee Q}\vee_r}{\Gamma, \exists x.Ax \vdash P \vee Q}\exists_l$$

$\forall_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P[x/t], A[x/\alpha] \vdash F}}{\Gamma, \exists x.Ax, P[x/t] \vdash F}\exists_l}{\Gamma, \forall x.Px, \exists x.Ax \vdash F}\forall_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A[x/\alpha], P[x/t] \vdash F}}{\Gamma, \forall x.Px, A[x/\alpha] \vdash F}\forall_l}{\Gamma, \exists x.Ax, \forall x.Px \vdash F}\exists_l$$

$\forall_r$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A[x/\alpha'] \vdash P[x/\alpha]}}{\Gamma, \exists x.Ax \vdash P[x/\alpha]}\ \exists_l}{\Gamma, \exists x.Ax \vdash \forall x.Px}\ \forall_r \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A[x/\alpha'] \vdash P[x/\alpha]}}{\Gamma, A[x/\alpha'] \vdash \forall x.Px}\ \forall_r}{\Gamma, \exists x.Ax \vdash \forall x.Px}\ \exists_l$$

$\exists_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P[x/\alpha], A[x/\alpha'] \vdash F}}{\Gamma, \exists x.Ax, P[x/\alpha] \vdash F}\ \exists_l}{\Gamma, \exists x.Ax, \exists x.Px \vdash F}\ \exists_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A[x/\alpha'], P[x/\alpha] \vdash F}}{\Gamma, \exists x.Px, A[x/\alpha'] \vdash F}\ \exists_l}{\Gamma, \exists x.Ax, \exists x.Px \vdash F}\ \exists_l$$

$\exists_r$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A[x/\alpha] \vdash P[x/t]}}{\Gamma, \exists x.Ax \vdash P[x/t]}\ \exists_l}{\Gamma, \exists x.Ax \vdash \exists x.Px}\ \exists_r \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A[x/\alpha] \vdash P[x/t]}}{\Gamma, A[x/\alpha] \vdash \exists x.Px}\ \exists_r}{\Gamma, \exists x.Ax \vdash \exists x.Px}\ \exists_l$$

$\neg_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A[x/\alpha] \vdash P}}{\Gamma, \exists x.Ax \vdash P}\ \exists_l}{\Gamma, \exists x.Ax, \neg P \vdash}\ \neg_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A[x/\alpha] \vdash P}}{\Gamma, \neg P, A[x/\alpha] \vdash}\ \neg_l}{\Gamma, \exists x.Ax, \neg P \vdash}\ \exists_l$$

$\neg_r$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P, A[x/\alpha] \vdash}}{\Gamma, \exists x.Ax, P \vdash}\ \exists_l}{\Gamma, \exists x.Ax \vdash \neg P}\ \neg_r \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A[x/\alpha], P \vdash}}{\Gamma, A[x/\alpha] \vdash \neg P}\ \neg_r}{\Gamma, \exists x.Ax \vdash \neg P}\ \exists_l$$

$w_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A[x/\alpha] \vdash F}}{\Gamma, \exists x.Ax \vdash F}\ \exists_l}{\Gamma, P, \exists x.Ax \vdash F}\ w_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A[x/\alpha] \vdash F}}{\Gamma, P, A[x/\alpha] \vdash F}\ w_l}{\Gamma, P, \exists x.Ax \vdash F}\ \exists_l$$

$w_r$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A[x/\alpha] \vdash}}{\Gamma, \exists x.Ax \vdash}\ \exists_l}{\Gamma, \exists x.Ax \vdash P}\ w_r \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A[x/\alpha] \vdash}}{\Gamma, A[x/\alpha] \vdash P}\ w_r}{\Gamma, \exists x.Ax \vdash P}\ \exists_l$$

$c_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P, PA[x/\alpha] \vdash F}}{\Gamma, P, P\exists x.Ax \vdash F}\ \exists_l}{\Gamma, P, \exists x.Ax \vdash F}\ c_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P, P, A[x/\alpha] \vdash F}}{\Gamma, P, A[x/\alpha] \vdash F}\ c_l}{\Gamma, P, \exists x.Ax \vdash F}\ \exists_l$$

$cut$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A[x/\alpha] \vdash P}}{\Gamma, \exists x.Ax \vdash P}\ \exists_l \quad \overset{\varphi_2}{\Gamma', P \vdash F}}{\Gamma, \Gamma', \exists x.Ax \vdash F}\ cut \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A[x/\alpha] \vdash P} \quad \overset{\varphi_2}{\Gamma', P \vdash F}}{\Gamma, \Gamma', A[x/\alpha] \vdash F}\ cut}{\Gamma, \Gamma', \exists x.Ax \vdash F}\ \exists_l$$

$$\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \dfrac{\overset{\varphi_2}{\Gamma, P, A[x/\alpha] \vdash F}}{\Gamma', \exists x.Ax, P \vdash F}\ \exists_l}{\Gamma, \Gamma', \exists x.Ax \vdash F}\ cut \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \overset{\varphi_2}{\Gamma', A[x/\alpha], P \vdash F}}{\Gamma, \Gamma', A[x/\alpha] \vdash F}\ cut}{\Gamma, \Gamma', \exists x.Ax \vdash F}\ \exists_l$$

## B.10 Permutation of $\exists_r$ down

$\rightarrow_l$

$$\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \dfrac{\overset{\varphi_2}{\Gamma', Q \vdash A[x/t]}}{\Gamma', Q \vdash \exists x.Ax}\ \exists_r}{\Gamma, \Gamma', P \rightarrow Q \vdash \exists x.Ax}\ \rightarrow_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \overset{\varphi_2}{\Gamma', Q \vdash A[x/t]}}{\Gamma, \Gamma', P \rightarrow Q \vdash A[x/t]}\ \rightarrow_l}{\Gamma, \Gamma', P \rightarrow Q \vdash \exists x.Ax}\ \exists_r$$

$\rightarrow_r$

N/A

$\wedge_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, Q \vdash A[x/t]}}{\Gamma, Q \vdash \exists x.Ax}\ \exists_r}{\Gamma, P \wedge Q \vdash \exists x.Ax}\ \wedge_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, Q \vdash A[x/t]}}{\Gamma, P \wedge Q \vdash A[x/t]}\ \wedge_l}{\Gamma, P \wedge Q \vdash \exists x.Ax}\ \exists_r$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P \vdash A[x/t]}}{\Gamma, P \vdash \exists x.Ax}\ \exists_r}{\Gamma, P \wedge Q \vdash \exists x.Ax}\ \wedge_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P \vdash A[x/t]}}{\Gamma, P \wedge Q \vdash A[x/t]}\ \wedge_l}{\Gamma, P \wedge Q \vdash \exists x.Ax}\ \exists_r$$

$\wedge_r$

N/A

$\vee_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P \vdash A[x/t]}}{\Gamma, P \vdash \exists x.Ax}\ \exists_r \quad \dfrac{\overset{\varphi_2}{\Gamma', Q \vdash A[x/t']}}{\Gamma', Q \vdash \exists x.Ax}\ \exists_r}{\Gamma, \Gamma', P \vee Q \vdash \exists x.Ax}\ \vee_l \quad \rightsquigarrow \quad ?$$

$\vee_r$

N/A

$\forall_l$

$$\dfrac{\dfrac{\Gamma, P[x/t] \vdash A[x/t']}{\Gamma, P[x/t] \vdash \exists x.Ax}\ \exists_r}{\Gamma, \forall x.Px \vdash \exists x.Ax}\ \forall_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\Gamma, P[x/t] \vdash A[x/t']}{\Gamma, \forall x.Px \vdash A[x/t']}\ \forall_l}{\Gamma, \forall x.Px \vdash \exists x.Ax}\ \exists_r$$

$\forall_r$

N/A

$\exists_l$

Note: watch out for eigenvariable violations (if $t$ contains $\alpha$)

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P[x/\alpha] \vdash A[x/t]}}{\Gamma, P[x/\alpha] \vdash \exists x.Ax}\ \exists_r}{\Gamma, \exists x.Px \vdash \exists x.Ax}\ \exists_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P[x/\alpha] \vdash A[x/t]}}{\Gamma, \exists x.Px \vdash A[x/t]}\ \exists_l}{\Gamma, \exists x.Px \vdash \exists x.Ax}\ \exists_r$$

$\exists_r$

N/A

$\neg_l$

N/A

$\neg_r$

N/A

$w_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash A[x/t]}}{\Gamma \vdash \exists x.Ax}\ \exists_r}{\Gamma, P \vdash \exists x.Ax}\ w_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash A[x/t]}}{\Gamma, P \vdash A[x/t]}\ w_l}{\Gamma, P \vdash \exists x.Ax}\ \exists_r$$

$w_r$

N/A

$c_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P, P \vdash A[x/t]}}{\Gamma, P, P \vdash \exists x.Ax}\ \exists_r}{\Gamma, P \vdash \exists x.Ax}\ c_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P, P \vdash A[x/t]}}{\Gamma, P \vdash A[x/t]}\ c_l}{\Gamma, P \vdash \exists x.Ax}\ \exists_r$$

*cut*

$$\dfrac{\stackrel{\varphi_1}{\Gamma \vdash P} \quad \dfrac{\stackrel{\varphi_2}{\Gamma', P \vdash A[x/t]}}{\Gamma', P \vdash \exists x.Ax}\,\exists_r}{\Gamma, \Gamma' \vdash \exists x.Ax}\,cut \quad \to \quad \dfrac{\dfrac{\stackrel{\varphi_1}{\Gamma \vdash P} \quad \stackrel{\varphi_2}{\Gamma', P \vdash A[x/t]}}{\Gamma, \Gamma' \vdash A[x/t]}\,cut}{\Gamma, \Gamma' \vdash \exists x.Ax}\,\exists_r$$

## B.11 Permutation of $\neg_l$ down

$\to_l$

$$\dfrac{\stackrel{\varphi_1}{\Gamma \vdash P} \quad \dfrac{\stackrel{\varphi_2}{\Gamma', Q \vdash A}}{\Gamma', \neg A, Q \vdash}\,\neg_l}{\Gamma, \Gamma', \neg A, P \to Q \vdash}\,\to_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\stackrel{\varphi_1}{\Gamma \vdash P} \quad \stackrel{\varphi_2}{\Gamma', Q \vdash A}}{\Gamma, \Gamma', P \to Q \vdash A}\,\to_l}{\Gamma, \Gamma', P \to Q, \neg A \vdash}\,\neg_l$$

$\to_r$

N/A

$\wedge_l$

$$\dfrac{\dfrac{\stackrel{\varphi_1}{\Gamma, Q \vdash A}}{\Gamma, \neg A, Q \vdash}\,\neg_l}{\Gamma, P \wedge Q, \neg A \vdash}\,\wedge_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\stackrel{\varphi_1}{\Gamma, Q \vdash A}}{\Gamma, P \wedge Q \vdash A}\,\wedge_l}{\Gamma, P \wedge Q, \neg A \vdash}\,\neg_l$$

$$\dfrac{\dfrac{\stackrel{\varphi_1}{\Gamma, P \vdash A}}{\Gamma, \neg A, P \vdash}\,\neg_l}{\Gamma, P \wedge Q, \neg A \vdash}\,\wedge_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\stackrel{\varphi_1}{\Gamma, P \vdash A}}{\Gamma, P \wedge Q \vdash A}\,\wedge_l}{\Gamma, P \wedge Q, \neg A \vdash}\,\neg_l$$

$\wedge_r$

N/A

$\vee_l$

$$\dfrac{\stackrel{\varphi_1}{\Gamma, P \vdash} \quad \dfrac{\stackrel{\varphi_2}{\Gamma', Q \vdash A}}{\Gamma', \neg A, Q \vdash}\,\neg_l}{\Gamma, \Gamma, \neg A, P \vee Q \vdash}\,\vee_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\stackrel{\varphi_1}{\Gamma, P \vdash}}{\Gamma, P \vdash A}\,w_r \quad \stackrel{\varphi_2}{\Gamma', Q \vdash A}}{\Gamma, \Gamma', P \vee Q \vdash A}\,\vee_l}{\Gamma, \Gamma', P \vee Q, \neg A \vdash}\,\neg_l$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P \vdash A}}{\Gamma, \neg A, P \vdash}\ \neg_l \qquad \overset{\varphi_2}{\Gamma', Q \vdash}}{\Gamma, \Gamma', \neg A, P \vee Q \vdash}\ \vee_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P \vdash A} \qquad \dfrac{\overset{\varphi_2}{\Gamma', Q \vdash}}{\Gamma', Q \vdash A}\ w_r}{\Gamma, \Gamma', P \vee Q \vdash A}\ \vee_l}{\Gamma, \Gamma', P \vee Q, \neg A \vdash}\ \neg_l$$

$\vee_r$

N/A

$\forall_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P[x/t] \vdash A}}{\Gamma, \neg A, P[x/t] \vdash}\ \neg_l}{\Gamma, \forall x.Px, \neg A \vdash}\ \forall_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P[x/t] \vdash A}}{\Gamma, \forall x.Px \vdash A}\ \forall_l}{\Gamma, \forall x.Px, \neg A \vdash}\ \neg_l$$

$\forall_r$

N/A

$\exists_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P[x/\alpha] \vdash A}}{\Gamma, \neg A, P[x/\alpha] \vdash}\ \neg_l}{\Gamma, \exists x.Px, \neg A \vdash}\ \exists_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P[x/\alpha] \vdash A}}{\Gamma, \exists x.Px \vdash A}\ \exists_l}{\Gamma, \exists x.Px, \neg A \vdash}\ \neg_l$$

$\exists_r$

N/A

$\neg_l$

N/A

$\neg_r$

$$\dfrac{\dfrac{\Gamma, P \vdash A}{\Gamma, \neg A, P \vdash}\ \neg_l}{\Gamma, \neg A \vdash \neg P}\ \neg_r \quad \rightsquigarrow \quad ?$$

$w_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash A}}{\Gamma, \neg A \vdash}\ \neg_l}{\Gamma, \neg A, P \vdash}\ w_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash A}}{\Gamma, P \vdash A}\ w_l}{\Gamma, \neg A, P \vdash}\ \neg_l$$

$w_r$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash A}}{\Gamma, \neg A \vdash}\ \neg_l}{\Gamma, \neg A \vdash P}\ w_r \quad \rightsquigarrow \quad ?$$

$c_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P, P \vdash A}}{\Gamma, \neg A, P, P \vdash}\ \neg_l}{\Gamma, \neg A, P \vdash}\ c_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P, P \vdash A}}{\Gamma, P \vdash A}\ c_l}{\Gamma, \neg A, P \vdash}\ \neg_l$$

$cut$

$$\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \dfrac{\overset{\varphi_2}{\Gamma', P \vdash A}}{\Gamma', \neg A, P \vdash}\ \neg_l}{\Gamma, \Gamma', \neg A \vdash}\ cut \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \overset{\varphi_2}{\Gamma', P \vdash A}}{\Gamma, \Gamma' \vdash A}\ cut}{\Gamma, \Gamma', \neg A \vdash}\ \neg_l$$

## B.12 Permutation of $\neg_r$ down

$\rightarrow_l$

$$\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \dfrac{\overset{\varphi_2}{\Gamma', Q, A \vdash}}{\Gamma', Q \vdash \neg A}\ \neg_r}{\Gamma, \Gamma', P \rightarrow Q \vdash \neg A}\ \rightarrow_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \overset{\varphi_2}{\Gamma', A, Q \vdash}}{\Gamma, \Gamma', P \rightarrow Q, A \vdash}\ \rightarrow_l}{\Gamma, \Gamma', P \rightarrow Q \vdash \neg A}\ \neg_r$$

$\rightarrow_r$

N/A

$\wedge_l$

$$\cfrac{\cfrac{\overset{\varphi_1}{\Gamma, Q, A \vdash}}{\Gamma, Q \vdash \neg A}\ \neg_r}{\Gamma, P \wedge Q \vdash \neg A}\ \wedge_l \quad \rightsquigarrow \quad \cfrac{\cfrac{\overset{\varphi_1}{\Gamma, A, Q \vdash}}{\Gamma, P \wedge Q, A \vdash}\ \wedge_l}{\Gamma, P \wedge Q \vdash \neg A}\ \neg_r$$

$$\cfrac{\cfrac{\overset{\varphi_1}{\Gamma, P, A \vdash}}{\Gamma, P \vdash \neg A}\ \neg_r}{\Gamma, P \wedge Q \vdash \neg A}\ \wedge_l \quad \rightsquigarrow \quad \cfrac{\cfrac{\overset{\varphi_1}{\Gamma, A, P \vdash}}{\Gamma, P \wedge Q, A \vdash}\ \wedge_l}{\Gamma, P \wedge Q \vdash \neg A}\ \neg_r$$

$\wedge_r$

N/A

$\vee_l$

$$\cfrac{\cfrac{\overset{\varphi_1}{\Gamma, P, A \vdash}}{\Gamma, P \vdash \neg A}\ \neg_r \quad \cfrac{\overset{\varphi_2}{\Gamma', Q, A \vdash}}{\Gamma', Q \vdash \neg A}\ \neg_r}{\Gamma, \Gamma', P \vee Q \vdash \neg A}\ \vee_l \quad \rightsquigarrow \quad \cfrac{\cfrac{\cfrac{\overset{\varphi_1}{\Gamma, A, P \vdash} \quad \overset{\varphi_2}{\Gamma', A, Q \vdash}}{\Gamma, \Gamma', P \vee Q, A, A \vdash}\ \vee_l}{\Gamma, \Gamma', P \vee Q, A \vdash}\ c_l}{\Gamma, \Gamma', P \vee Q \vdash \neg A}\ \neg_r$$

$\vee_r$

N/A

$\forall_l$

$$\cfrac{\cfrac{\overset{\varphi_1}{\Gamma, P[x/t], A \vdash}}{\Gamma, P[x/t] \vdash \neg A}\ \neg_r}{\Gamma, \forall x.Px \vdash \neg A}\ \forall_l \quad \rightsquigarrow \quad \cfrac{\cfrac{\overset{\varphi_1}{\Gamma, A, P[x/t] \vdash}}{\Gamma, \forall x.Px, A \vdash}\ \forall_l}{\Gamma, \forall x.Px \vdash \neg A}\ \neg_r$$

$\forall_r$

N/A

$\exists_l$

$$\dfrac{\dfrac{\begin{array}{c}\varphi_1\\ \Gamma, P[x/\alpha], A \vdash\end{array}}{\Gamma, P[x/\alpha] \vdash \neg A}\,\neg_r}{\Gamma, \exists x.Px \vdash \neg A}\,\exists_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{array}{c}\varphi_1\\ \Gamma, A, P[x/\alpha] \vdash\end{array}}{\Gamma, \exists x.Px, A \vdash}\,\exists_l}{\Gamma, \exists x.Px \vdash \neg A}\,\neg_r$$

$\exists_r$

N/A

$\neg_l$

N/A

$\neg_r$

N/A

$w_l$

$$\dfrac{\dfrac{\begin{array}{c}\varphi_1\\ \Gamma, A \vdash\end{array}}{\Gamma \vdash \neg A}\,\neg_r}{\Gamma, P \vdash \neg A}\,w_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{array}{c}\varphi_1\\ \Gamma, A \vdash\end{array}}{\Gamma, P, A \vdash}\,w_l}{\Gamma, P \vdash \neg A}\,\neg_r$$

$w_r$

N/A

$c_l$

$$\dfrac{\dfrac{\begin{array}{c}\varphi_1\\ \Gamma, P, P, A \vdash\end{array}}{\Gamma, P, P \vdash \neg A}\,\neg_r}{\Gamma, P \vdash \neg A}\,c_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{array}{c}\varphi_1\\ \Gamma, P, P, A \vdash\end{array}}{\Gamma, P, A \vdash}\,c_l}{\Gamma, P \vdash \neg A}\,\neg_r$$

$cut$

$$\dfrac{\begin{array}{c}\varphi_1\\ \Gamma \vdash P\end{array} \quad \dfrac{\begin{array}{c}\varphi_2\\ \Gamma', P, A \vdash\end{array}}{\Gamma', P \vdash \neg A}\,\neg_r}{\Gamma, \Gamma' \vdash \neg A}\,cut \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{array}{c}\varphi_1\\ \Gamma \vdash P\end{array} \quad \begin{array}{c}\varphi_2\\ \Gamma', A, P \vdash\end{array}}{\Gamma, \Gamma', A \vdash}\,cut}{\Gamma, \Gamma' \vdash \neg A}\,\neg_r$$

## B.13 Permutation of $w_l$ down

$\to_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash P}}{\Gamma, A \vdash P}\,w_l \qquad \overset{\varphi_2}{\Gamma', Q \vdash F}}{\Gamma, \Gamma', P \to Q, A \vdash F}\,\to_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \overset{\varphi_2}{\Gamma', Q \vdash F}}{\Gamma, \Gamma', P \to Q \vdash F}\,\to_l}{\Gamma, \Gamma', P \to Q, A \vdash F}\,w_l$$

$$\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \qquad \dfrac{\overset{\varphi_2}{\Gamma', Q \vdash F}}{\Gamma', A, Q \vdash F}\,w_l}{\Gamma, \Gamma', P \to Q, A \vdash F}\,\to_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \overset{\varphi_2}{\Gamma', Q \vdash F}}{\Gamma, \Gamma', P \to Q \vdash F}\,\to_l}{\Gamma, \Gamma', P \to Q, A \vdash F}\,w_l$$

$\to_r$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P \vdash Q}}{\Gamma, A, P \vdash Q}\,w_l}{\Gamma, A \vdash P \to Q}\,\to_r \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P \vdash Q}}{\Gamma \vdash P \to Q}\,\to_r}{\Gamma, A \vdash P \to Q}\,w_l$$

$\wedge_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P \vdash F}}{\Gamma, P, A \vdash F}\,w_l}{\Gamma, P \wedge Q, A \vdash F}\,\wedge_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P \vdash F}}{\Gamma, P \wedge Q \vdash F}\,\wedge_l}{\Gamma, P \wedge Q, A \vdash F}\,w_l$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, Q \vdash F}}{\Gamma, Q, A \vdash F}\,w_l}{\Gamma, P \wedge Q, A \vdash F}\,\wedge_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, Q \vdash F}}{\Gamma, P \wedge Q \vdash F}\,\wedge_l}{\Gamma, P \wedge Q, A \vdash F}\,w_l$$

$\wedge_r$

$$\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \qquad \dfrac{\overset{\varphi_2}{\Gamma' \vdash Q}}{\Gamma', A \vdash Q}\,w_l}{\Gamma, \Gamma', A \vdash P \wedge Q}\,\wedge_r \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \overset{\varphi_2}{\Gamma' \vdash Q}}{\Gamma, \Gamma' \vdash P \wedge Q}\,\wedge_r}{\Gamma, \Gamma', A \vdash P \wedge Q}\,w_l$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash P}}{\Gamma, A \vdash P}\,w_l \qquad \overset{\varphi_2}{\Gamma' \vdash Q}}{\Gamma, \Gamma', A \vdash P \wedge Q}\,\wedge_r \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \qquad \overset{\varphi_2}{\Gamma' \vdash Q}}{\Gamma, \Gamma' \vdash P \wedge Q}\,\wedge_r}{\Gamma, \Gamma', A \vdash P \wedge Q}\,w_l$$

$\vee_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P \vdash F}}{\Gamma, A, P \vdash F}\,w_l \qquad \overset{\varphi_2}{\Gamma', Q \vdash F}}{\Gamma, \Gamma', A, P \vee Q \vdash F}\,\vee_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P \vdash F} \qquad \overset{\varphi_2}{\Gamma', Q \vdash F}}{\Gamma, \Gamma', P \vee Q \vdash F}\,\vee_l}{\Gamma, \Gamma', P \vee Q, A \vdash F}\,w_l$$

$$\dfrac{\overset{\varphi_1}{\Gamma, P \vdash F} \qquad \dfrac{\overset{\varphi_2}{\Gamma', Q \vdash F}}{\Gamma', A, Q \vdash F}\,w_l}{\Gamma, \Gamma', A, P \vee Q \vdash F}\,\vee_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P \vdash F} \qquad \overset{\varphi_2}{\Gamma', Q \vdash F}}{\Gamma, \Gamma', P \vee Q \vdash F}\,\vee_l}{\Gamma, \Gamma', P \vee Q, A \vdash F}\,w_l$$

$\vee_r$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash P}}{\Gamma, A \vdash P}\,w_l}{\Gamma, A \vdash P \vee Q}\,\vee_r \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash P}}{\Gamma \vdash P \vee Q}\,\vee_r}{\Gamma, A \vdash P \vee Q}\,w_l$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash Q}}{\Gamma, A \vdash Q}\,w_l}{\Gamma, A \vdash P \vee Q}\,\vee_r \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash Q}}{\Gamma \vdash P \vee Q}\,\vee_r}{\Gamma, A \vdash P \vee Q}\,w_l$$

$\forall_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P[x/t] \vdash F}}{\Gamma, A, P[x/t] \vdash F}\,w_l}{\Gamma, \forall x.Px, A \vdash F}\,\forall_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P[x/t] \vdash F}}{\Gamma, \forall x.Px \vdash F}\,\forall_l}{\Gamma, \forall x.Px, A \vdash F}\,w_l$$

$\forall_r$

$$\dfrac{\dfrac{\begin{matrix}\varphi_1\\ \Gamma \vdash P[x/\alpha]\end{matrix}}{\Gamma, A \vdash P[x/\alpha]}\, w_l}{\Gamma, A \vdash \forall x.Px}\, \forall_r \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{matrix}\varphi_1\\ \Gamma \vdash P[x/\alpha]\end{matrix}}{\Gamma \vdash \forall x.Px}\, \forall_r}{\Gamma, A \vdash \forall x.Px}\, w_l$$

$\exists_l$

$$\dfrac{\dfrac{\begin{matrix}\varphi_1\\ \Gamma, P[x/\alpha] \vdash F\end{matrix}}{\Gamma, A, P[x/\alpha] \vdash F}\, w_l}{\Gamma, A, \exists x.Px \vdash F}\, \exists_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{matrix}\varphi_1\\ \Gamma, P[x/\alpha] \vdash F\end{matrix}}{\Gamma, \exists x.Px \vdash F}\, \exists_l}{\Gamma, A, \exists x.Px \vdash F}\, w_l$$

$\exists_r$

$$\dfrac{\dfrac{\begin{matrix}\varphi_1\\ \Gamma \vdash P[x/t]\end{matrix}}{\Gamma, A \vdash P[x/t]}\, w_l}{\Gamma, A \vdash \exists x.Px}\, \exists_r \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{matrix}\varphi_1\\ \Gamma \vdash P[x/t]\end{matrix}}{\Gamma \vdash \exists x.Px}\, \exists_r}{\Gamma, A \vdash \exists x.Px}\, w_l$$

$\neg_l$

$$\dfrac{\dfrac{\begin{matrix}\varphi_1\\ \Gamma \vdash P\end{matrix}}{\Gamma, A \vdash P}\, w_l}{\Gamma, \neg P, A \vdash}\, \neg_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{matrix}\varphi_1\\ \Gamma \vdash P\end{matrix}}{\Gamma, \neg P \vdash}\, \neg_l}{\Gamma, \neg P, A \vdash}\, w_l$$

$\neg_r$

$$\dfrac{\dfrac{\begin{matrix}\varphi_1\\ \Gamma, P \vdash\end{matrix}}{\Gamma, A, P \vdash}\, w_l}{\Gamma, A \vdash \neg P}\, \neg_r \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{matrix}\varphi_1\\ \Gamma, P \vdash\end{matrix}}{\Gamma \vdash \neg P}\, \neg_r}{\Gamma, A \vdash \neg P}\, w_l$$

$w_l$

$$\dfrac{\dfrac{\begin{matrix}\varphi_1\\ \Gamma \vdash F\end{matrix}}{\Gamma, A \vdash F}\, w_l}{\Gamma, P, A \vdash F}\, w_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{matrix}\varphi_1\\ \Gamma \vdash F\end{matrix}}{\Gamma, P \vdash F}\, w_l}{\Gamma, P, A \vdash F}\, w_l$$

$w_r$

$$\dfrac{\dfrac{\genfrac{}{}{0pt}{}{\varphi_1}{\Gamma \vdash}}{\Gamma, A \vdash}\,w_l}{\Gamma, A \vdash P}\,w_r \quad\rightsquigarrow\quad \dfrac{\dfrac{\genfrac{}{}{0pt}{}{\varphi_1}{\Gamma \vdash}}{\Gamma \vdash P}\,w_r}{\Gamma, A \vdash P}\,w_l$$

$c_l$

$$\dfrac{\dfrac{\genfrac{}{}{0pt}{}{\varphi_1}{\Gamma, P, P \vdash F}}{\Gamma, A, P, P \vdash F}\,w_l}{\Gamma, P, A \vdash F}\,c_l \quad\rightarrow\quad \dfrac{\dfrac{\genfrac{}{}{0pt}{}{\varphi_1}{\Gamma, P, P \vdash F}}{\Gamma, P \vdash F}\,c_l}{\Gamma, P, A \vdash F}\,w_l$$

$cut$

$$\dfrac{\dfrac{\genfrac{}{}{0pt}{}{\varphi_1}{\Gamma \vdash P}}{\Gamma, A \vdash P}\,w_l \qquad \genfrac{}{}{0pt}{}{\varphi_2}{\Gamma', P \vdash F}}{\Gamma, \Gamma', A \vdash F}\,cut \quad\rightsquigarrow\quad \dfrac{\dfrac{\genfrac{}{}{0pt}{}{\varphi_1}{\Gamma \vdash P} \quad \genfrac{}{}{0pt}{}{\varphi_2}{\Gamma', P \vdash F}}{\Gamma, \Gamma' \vdash F}\,cut}{\Gamma, \Gamma', A \vdash F}\,w_l$$

$$\dfrac{\genfrac{}{}{0pt}{}{\varphi_1}{\Gamma \vdash P} \qquad \dfrac{\genfrac{}{}{0pt}{}{\varphi_2}{\Gamma', P \vdash F}}{\Gamma', A, P \vdash F}\,w_l}{\Gamma, \Gamma', A \vdash F}\,cut \quad\rightsquigarrow\quad \dfrac{\dfrac{\genfrac{}{}{0pt}{}{\varphi_1}{\Gamma \vdash P} \quad \genfrac{}{}{0pt}{}{\varphi_2}{\Gamma', P \vdash F}}{\Gamma, \Gamma' \vdash F}\,cut}{\Gamma, \Gamma', A \vdash F}\,w_l$$

## B.14 Permutation of $w_r$ down

$\rightarrow_l$

$$\dfrac{\genfrac{}{}{0pt}{}{\varphi_1}{\Gamma \vdash P} \qquad \dfrac{\genfrac{}{}{0pt}{}{\varphi_2}{\Gamma', Q \vdash}}{\Gamma', Q \vdash A}\,w_r}{\Gamma, \Gamma', P \rightarrow Q \vdash A}\,\rightarrow_l \quad\rightsquigarrow\quad \dfrac{\dfrac{\genfrac{}{}{0pt}{}{\varphi_1}{\Gamma \vdash P} \quad \genfrac{}{}{0pt}{}{\varphi_2}{\Gamma', Q \vdash}}{\Gamma, \Gamma', P \rightarrow Q \vdash}\,\rightarrow_l}{\Gamma, \Gamma', P \rightarrow Q \vdash A}\,w_r$$

$\rightarrow_r$

N/A

$\wedge_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P \vdash}}{\Gamma, P \vdash A}\,w_r}{\Gamma, P \wedge Q \vdash A}\,\wedge_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P \vdash}}{\Gamma, P \wedge Q \vdash}\,\wedge_l}{\Gamma, P \wedge Q \vdash A}\,w_r$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, Q \vdash}}{\Gamma, Q \vdash A}\,w_r}{\Gamma, P \wedge Q \vdash A}\,\wedge_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, Q \vdash}}{\Gamma, P \wedge Q \vdash}\,\wedge_l}{\Gamma, P \wedge Q \vdash A}\,w_r$$

$\wedge_r$

N/A

$\vee_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P \vdash}}{\Gamma, P \vdash A}\,w_r \quad \dfrac{\overset{\varphi_2}{\Gamma', Q \vdash}}{\Gamma', Q \vdash A}\,w_r}{\Gamma, \Gamma', P \vee Q \vdash A}\,\vee_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P \vdash} \quad \overset{\varphi_2}{\Gamma', Q \vdash}}{\Gamma, \Gamma', P \vee Q \vdash}\,\vee_l}{\Gamma, \Gamma', P \vee Q \vdash A}\,w_r$$

$\vee_r$

N/A

$\forall_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P[x/t] \vdash}}{\Gamma, P[x/t] \vdash A}\,w_r}{\Gamma, \forall x.Px \vdash A}\,\forall_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P[x/t] \vdash}}{\Gamma, \forall x.Px \vdash}\,\forall_l}{\Gamma, \forall x.Px \vdash A}\,w_r$$

$\forall_r$

N/A

$\exists_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P[x/\alpha] \vdash}}{\Gamma, P[x/\alpha] \vdash A}\, w_r}{\Gamma, \exists x.Px \vdash A}\, \exists_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P[x/\alpha] \vdash}}{\Gamma, \exists x.Px \vdash}\, \exists_l}{\Gamma, \exists x.Px \vdash A}\, w_r$$

$\exists_r$

N/A

$\neg_l$

N/A

$\neg_r$

N/A

$w_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash}}{\Gamma \vdash A}\, w_r}{\Gamma, P \vdash A}\, w_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash}}{\Gamma, P \vdash}\, w_l}{\Gamma, P \vdash A}\, w_r$$

$w_r$

N/A

$c_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P, P \vdash}}{\Gamma, P, P \vdash A}\, w_r}{\Gamma, P \vdash A}\, c_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P, P \vdash}}{\Gamma, P \vdash}\, c_l}{\Gamma, P \vdash A}\, w_r$$

$cut$

$$\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \qquad \dfrac{\overset{\varphi_2}{\Gamma', P \vdash}}{\Gamma', P \vdash A}\, w_r}{\Gamma, \Gamma' \vdash A}\, cut \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \qquad \overset{\varphi_2}{\Gamma', P \vdash}}{\Gamma, \Gamma' \vdash}\, cut}{\Gamma, \Gamma' \vdash A}\, w_r$$

## B.15 Permutation of $c_l$ down

$\to_l$

$$\frac{\dfrac{\overset{\varphi_1}{\Gamma, A, A \vdash P}}{\Gamma, A \vdash P}\,c_l \qquad \overset{\varphi_2}{\Gamma', Q \vdash F}}{\Gamma, \Gamma', P \to Q, A \vdash F}\,\to_l \quad \rightsquigarrow \quad \frac{\dfrac{\overset{\varphi_1}{\Gamma, A, A \vdash P} \qquad \overset{\varphi_2}{\Gamma', Q \vdash F}}{\Gamma, \Gamma', P \to Q, A, A \vdash F}\,\to_l}{\Gamma, \Gamma', P \to Q, A \vdash F}\,c_l$$

$$\frac{\overset{\varphi_1}{\Gamma \vdash P} \qquad \dfrac{\overset{\varphi_2}{\Gamma', Q, A, A \vdash F}}{\Gamma', A, Q \vdash F}\,c_l}{\Gamma, \Gamma', P \to Q, A \vdash F}\,\to_l \quad \rightsquigarrow \quad \frac{\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \qquad \overset{\varphi_2}{\Gamma', A, A, Q \vdash F}}{\Gamma, \Gamma', P \to Q, A, A \vdash F}\,\to_l}{\Gamma, \Gamma', P \to Q, A \vdash F}\,c_l$$

$\to_r$

$$\frac{\dfrac{\overset{\varphi_1}{\Gamma, P, A, A \vdash Q}}{\Gamma, A, P \vdash Q}\,c_l}{\Gamma, A \vdash P \to Q}\,\to_r \quad \rightsquigarrow \quad \frac{\dfrac{\overset{\varphi_1}{\Gamma, A, A, P \vdash Q}}{\Gamma, A, A \vdash P \to Q}\,\to_r}{\Gamma, A \vdash P \to Q}\,c_l$$

$\wedge_l$

$$\frac{\dfrac{\overset{\varphi_1}{\Gamma, P, A, A \vdash F}}{\Gamma, P, A \vdash F}\,c_l}{\Gamma, P \wedge Q, A \vdash F}\,\wedge_l \quad \rightsquigarrow \quad \frac{\dfrac{\overset{\varphi_1}{\Gamma, A, A, P \vdash F}}{\Gamma, P \wedge Q, A, A \vdash F}\,\wedge_l}{\Gamma, P \wedge Q, A \vdash F}\,c_l$$

$$\frac{\dfrac{\overset{\varphi_1}{\Gamma, Q, A, A \vdash F}}{\Gamma, Q, A \vdash F}\,c_l}{\Gamma, P \wedge Q, A \vdash F}\,\wedge_l \quad \rightsquigarrow \quad \frac{\dfrac{\overset{\varphi_1}{\Gamma, A, A, Q \vdash F}}{\Gamma, P \wedge Q, A, A \vdash F}\,\wedge_l}{\Gamma, P \wedge Q, A \vdash F}\,c_l$$

$\wedge_r$

$$\frac{\overset{\varphi_1}{\Gamma \vdash P} \qquad \dfrac{\overset{\varphi_2}{\Gamma', A, A \vdash Q}}{\Gamma', A \vdash Q}\,c_l}{\Gamma, \Gamma', A \vdash P \wedge Q}\,\wedge_r \quad \rightsquigarrow \quad \frac{\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \qquad \overset{\varphi_2}{\Gamma', A, A \vdash Q}}{\Gamma, \Gamma', A, A \vdash P \wedge Q}\,\wedge_r}{\Gamma, \Gamma', A \vdash P \wedge Q}\,c_l$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A, A \vdash P}}{\Gamma, A \vdash P}\,c_l \qquad \overset{\varphi_2}{\Gamma' \vdash Q}}{\Gamma, \Gamma', A \vdash P \wedge Q}\,\wedge_r \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A, A \vdash P} \qquad \overset{\varphi_2}{\Gamma' \vdash Q}}{\Gamma, \Gamma', A, A \vdash P \wedge Q}\,\wedge_r}{\Gamma, \Gamma', A \vdash P \wedge Q}\,c_l$$

$\vee_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P, A, A \vdash F}}{\Gamma, A, P \vdash F}\,c_l \qquad \overset{\varphi_2}{\Gamma', Q \vdash F}}{\Gamma, \Gamma', A, P \vee Q \vdash F}\,\vee_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A, A, P \vdash F} \qquad \overset{\varphi_2}{\Gamma', Q \vdash F}}{\Gamma, \Gamma', P \vee Q, A, A \vdash F}\,\vee_l}{\Gamma, \Gamma', P \vee Q, A \vdash F}\,c_l$$

$$\dfrac{\overset{\varphi_1}{\Gamma, P \vdash F} \qquad \dfrac{\overset{\varphi_2}{\Gamma', Q, A, A \vdash F}}{\Gamma', A, Q \vdash F}\,c_l}{\Gamma, \Gamma', A, P \vee Q \vdash F}\,\vee_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P \vdash F} \qquad \overset{\varphi_2}{\Gamma', A, A, Q \vdash F}}{\Gamma, \Gamma', P \vee Q, A, A \vdash F}\,\vee_l}{\Gamma, \Gamma', P \vee Q, A \vdash F}\,c_l$$

$\vee_r$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A, A \vdash P}}{\Gamma, A \vdash P}\,c_l}{\Gamma, A \vdash P \vee Q}\,\vee_r \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A, A \vdash P}}{\Gamma, A, A \vdash P \vee Q}\,\vee_r}{\Gamma, A \vdash P \vee Q}\,c_l$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A, A \vdash Q}}{\Gamma, A \vdash Q}\,c_l}{\Gamma, A \vdash P \vee Q}\,\vee_r \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A, A \vdash Q}}{\Gamma, A, A \vdash P \vee Q}\,\vee_r}{\Gamma, A \vdash P \vee Q}\,c_l$$

$\forall_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P[x/t], A, A \vdash F}}{\Gamma, A, P[x/t] \vdash F}\,c_l}{\Gamma, \forall x.Px, A \vdash F}\,\forall_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, A, A, P[x/t] \vdash F}}{\Gamma, \forall x.Px, A, A \vdash F}\,\forall_l}{\Gamma, \forall x.Px, A \vdash F}\,c_l$$

$\forall_r$

$$\begin{array}{c}\varphi_1\\ \dfrac{\dfrac{\Gamma, A, A \vdash P[x/\alpha]}{\Gamma, A \vdash P[x/\alpha]}\,c_l}{\Gamma, A \vdash \forall x.Px}\,\forall_r\end{array} \quad\rightsquigarrow\quad \begin{array}{c}\varphi_1\\ \dfrac{\dfrac{\Gamma, A, A \vdash P[x/\alpha]}{\Gamma, A, A \vdash \forall x.Px}\,\forall_r}{\Gamma, A \vdash \forall x.Px}\,c_l\end{array}$$

$\exists_l$

$$\begin{array}{c}\varphi_1\\ \dfrac{\dfrac{\Gamma, P[x/\alpha], A, A \vdash F}{\Gamma, A, P[x/\alpha] \vdash F}\,c_l}{\Gamma, A, \exists x.Px \vdash F}\,\exists_l\end{array} \quad\rightsquigarrow\quad \begin{array}{c}\varphi_1\\ \dfrac{\dfrac{\Gamma, A, A, P[x/\alpha] \vdash F}{\Gamma, \exists x.Px, A, A \vdash F}\,\exists_l}{\Gamma, A, \exists x.Px \vdash F}\,c_l\end{array}$$

$\exists_r$

$$\begin{array}{c}\varphi_1\\ \dfrac{\dfrac{\Gamma, A, A \vdash P[x/t]}{\Gamma, A \vdash P[x/t]}\,c_l}{\Gamma, A \vdash \exists x.Px}\,\exists_r\end{array} \quad\rightsquigarrow\quad \begin{array}{c}\varphi_1\\ \dfrac{\dfrac{\Gamma, A, A \vdash P[x/t]}{\Gamma, A, A \vdash \exists x.Px}\,\exists_r}{\Gamma, A \vdash \exists x.Px}\,c_l\end{array}$$

$\neg_l$

$$\begin{array}{c}\varphi_1\\ \dfrac{\dfrac{\Gamma, A, A \vdash P}{\Gamma, A \vdash P}\,c_l}{\Gamma, \neg P, A \vdash}\,\neg_l\end{array} \quad\rightsquigarrow\quad \begin{array}{c}\varphi_1\\ \dfrac{\dfrac{\Gamma, A, A \vdash P}{\Gamma, \neg P, A, A \vdash}\,\neg_l}{\Gamma, \neg P, A \vdash}\,c_l\end{array}$$

$\neg_r$

$$\begin{array}{c}\varphi_1\\ \dfrac{\dfrac{\Gamma, P, A, A \vdash}{\Gamma, A, P \vdash}\,c_l}{\Gamma, A \vdash \neg P}\,\neg_r\end{array} \quad\rightsquigarrow\quad \begin{array}{c}\varphi_1\\ \dfrac{\dfrac{\Gamma, A, A, P \vdash}{\Gamma, A, A \vdash \neg P}\,\neg_r}{\Gamma, A \vdash \neg P}\,c_l\end{array}$$

$w_l$

$$\begin{array}{c}\varphi_1\\ \dfrac{\dfrac{\Gamma, A, A \vdash F}{\Gamma, A \vdash F}\,c_l}{\Gamma, P, A \vdash F}\,w_l\end{array} \quad\rightsquigarrow\quad \begin{array}{c}\varphi_1\\ \dfrac{\dfrac{\Gamma, A, A \vdash F}{\Gamma, P, A, A \vdash F}\,w_l}{\Gamma, P, A \vdash F}\,c_l\end{array}$$

$w_r$

$$\dfrac{\dfrac{\stackrel{\varphi_1}{\Gamma, A, A \vdash}}{\Gamma, A \vdash}\, c_l}{\Gamma, A \vdash P}\, w_r \quad \rightsquigarrow \quad \dfrac{\dfrac{\stackrel{\varphi_1}{\Gamma, A, A \vdash}}{\Gamma, A, A \vdash P}\, w_r}{\Gamma, A \vdash P}\, c_l$$

$c_l$

$$\dfrac{\dfrac{\stackrel{\varphi_1}{\Gamma, P, P, A, A \vdash F}}{\Gamma, A, P, P \vdash F}\, c_l}{\Gamma, P, A \vdash F}\, c_l \quad \rightarrow \quad \dfrac{\dfrac{\stackrel{\varphi_1}{\Gamma, P, P, A, A \vdash F}}{\Gamma, P, A, A \vdash F}\, c_l}{\Gamma, P, A \vdash F}\, c_l$$

$cut$

$$\dfrac{\dfrac{\stackrel{\varphi_1}{\Gamma, A, A \vdash P}}{\Gamma, A \vdash P}\, c_l \quad \stackrel{\varphi_2}{\Gamma', P \vdash F}}{\Gamma, \Gamma', A \vdash F}\, cut \quad \rightsquigarrow \quad \dfrac{\dfrac{\stackrel{\varphi_1}{\Gamma, A, A \vdash P} \quad \stackrel{\varphi_2}{\Gamma', P \vdash F}}{\Gamma, \Gamma', A, A \vdash F}\, cut}{\Gamma, \Gamma', A \vdash F}\, c_l$$

$$\dfrac{\stackrel{\varphi_1}{\Gamma \vdash P} \quad \dfrac{\stackrel{\varphi_2}{\Gamma', P, A, A \vdash F}}{\Gamma', A, P \vdash F}\, c_l}{\Gamma, \Gamma', A \vdash F}\, cut \quad \rightsquigarrow \quad \dfrac{\dfrac{\stackrel{\varphi_1}{\Gamma \vdash P} \quad \stackrel{\varphi_2}{\Gamma', A, A, P \vdash F}}{\Gamma, \Gamma', A, A \vdash F}\, cut}{\Gamma, \Gamma', A \vdash F}\, c_l$$

## B.16 Permutation of $cut$ down

$\to_l$

$$\dfrac{\dfrac{\stackrel{\varphi_1}{\Gamma \vdash A} \quad \stackrel{\varphi_2}{\Gamma', A \vdash P}}{\Gamma, \Gamma' \vdash P}\, cut \quad \stackrel{\varphi_3}{\Gamma'', Q \vdash F}}{\Gamma, \Gamma', \Gamma'', P \to Q \vdash F}\, \to_l \quad \rightsquigarrow \quad \dfrac{\stackrel{\varphi_1}{\Gamma \vdash A} \quad \dfrac{\stackrel{\varphi_2}{\Gamma', A \vdash P} \quad \stackrel{\varphi_3}{\Gamma'', Q \vdash F}}{\Gamma', \Gamma'', P \to Q, A \vdash F}\, \to_l}{\Gamma, \Gamma', \Gamma'', P \to Q \vdash F}\, cut$$

$$\dfrac{\stackrel{\varphi_1}{\Gamma \vdash P} \quad \dfrac{\stackrel{\varphi_2}{\Gamma', Q \vdash A} \quad \stackrel{\varphi_3}{\Gamma'', A \vdash F}}{\Gamma', \Gamma'', Q \vdash F}\, cut}{\Gamma, \Gamma', \Gamma'', P \to Q \vdash F}\, \to_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\stackrel{\varphi_1}{\Gamma \vdash P} \quad \stackrel{\varphi_2}{\Gamma', Q \vdash A}}{\Gamma, \Gamma', P \to Q \vdash A}\, \to_l \quad \stackrel{\varphi_3}{\Gamma'', A \vdash F}}{\Gamma, \Gamma', \Gamma'', P \to Q \vdash F}\, cut$$

$$\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \qquad \dfrac{\overset{\varphi_2}{\Gamma' \vdash A} \quad \overset{\varphi_3}{\Gamma'', Q, A \vdash F}}{\Gamma', \Gamma'', Q \vdash F}\,cut}{\Gamma, \Gamma', \Gamma'', P \to Q \vdash F}\to_l \quad \rightsquigarrow \quad \dfrac{\overset{\varphi_2}{\Gamma' \vdash A} \qquad \dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \overset{\varphi_3}{\Gamma'', Q, A \vdash F}}{\Gamma, \Gamma'', P \to Q, A \vdash F}\to_l}{\Gamma, \Gamma', \Gamma'', P \to Q \vdash F}\,cut$$

$\to_r$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P \vdash A} \quad \overset{\varphi_2}{\Gamma', A \vdash Q}}{\Gamma, \Gamma', P \vdash Q}\,cut}{\Gamma, \Gamma' \vdash P \to Q}\to_r \quad \rightsquigarrow \quad ?$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \overset{\varphi_2}{\Gamma', P, A \vdash Q}}{\Gamma, \Gamma', P \vdash Q}\,cut}{\Gamma, \Gamma' \vdash P \to Q}\to_r \quad \rightsquigarrow \quad \dfrac{\overset{\varphi_1}{\Gamma \vdash A} \qquad \dfrac{\overset{\varphi_2}{\Gamma', P, A \vdash Q}}{\Gamma', A \vdash P \to Q}\to_r}{\Gamma, \Gamma' \vdash P \to Q}\,cut$$

$\wedge_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P \vdash A} \quad \overset{\varphi_2}{\Gamma', A \vdash F}}{\Gamma, \Gamma', P \vdash F}\,cut}{\Gamma, \Gamma', P \wedge Q \vdash F}\wedge_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P \vdash A}}{\Gamma, P \wedge Q \vdash F}\wedge_l \qquad \overset{\varphi_2}{\Gamma', A \vdash F}}{\Gamma, \Gamma', P \wedge Q \vdash F}\,cut$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \overset{\varphi_2}{\Gamma', P, A \vdash F}}{\Gamma, \Gamma', P \vdash F}\,cut}{\Gamma, \Gamma', P \wedge Q \vdash F}\wedge_l \quad \rightsquigarrow \quad \dfrac{\overset{\varphi_1}{\Gamma \vdash A} \qquad \dfrac{\overset{\varphi_2}{\Gamma', A, P \vdash F}}{\Gamma', P \wedge Q, A \vdash F}\wedge_l}{\Gamma, \Gamma', P \wedge Q \vdash F}\,cut$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, Q \vdash A} \quad \overset{\varphi_2}{\Gamma', A \vdash F}}{\Gamma, \Gamma', Q \vdash F}\,cut}{\Gamma, \Gamma', P \wedge Q \vdash F}\wedge_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, Q \vdash A}}{\Gamma, P \wedge Q \vdash F}\wedge_l \qquad \overset{\varphi_2}{\Gamma', A \vdash F}}{\Gamma, \Gamma', P \wedge Q \vdash F}\,cut$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \overset{\varphi_2}{\Gamma', Q, A \vdash F}}{\Gamma, \Gamma', Q \vdash F}\,cut}{\Gamma, \Gamma', P \wedge Q \vdash F}\wedge_l \quad \rightsquigarrow \quad \dfrac{\overset{\varphi_1}{\Gamma \vdash A} \qquad \dfrac{\overset{\varphi_2}{\Gamma', A, Q \vdash F}}{\Gamma', P \wedge Q, A \vdash F}\wedge_l}{\Gamma, \Gamma', P \wedge Q \vdash F}\,cut$$

$\bigwedge_r$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \overset{\varphi_2}{\Gamma', A \vdash P}}{\Gamma, \Gamma' \vdash P}\,\mathit{cut} \quad \overset{\varphi_3}{\Gamma'' \vdash Q}}{\Gamma, \Gamma', \Gamma'' \vdash P \wedge Q}\,\wedge_r \quad \rightsquigarrow \quad \dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \dfrac{\overset{\varphi_2}{\Gamma', A \vdash P} \quad \overset{\varphi_3}{\Gamma'' \vdash Q}}{\Gamma', \Gamma'', A \vdash P \wedge Q}\,\wedge_r}{\Gamma, \Gamma', \Gamma'' \vdash P \wedge Q}\,\mathit{cut}$$

$$\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \dfrac{\overset{\varphi_2}{\Gamma' \vdash A} \quad \overset{\varphi_3}{\Gamma'', A \vdash Q}}{\Gamma', \Gamma'' \vdash Q}\,\mathit{cut}}{\Gamma, \Gamma', \Gamma'' \vdash P \wedge Q}\,\wedge_r \quad \rightsquigarrow \quad \dfrac{\overset{\varphi_2}{\Gamma' \vdash A} \quad \dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \overset{\varphi_3}{\Gamma'', A \vdash Q}}{\Gamma, \Gamma'', A \vdash P \wedge Q}\,\wedge_r}{\Gamma, \Gamma', \Gamma'' \vdash P \wedge Q}\,\mathit{cut}$$

$\bigvee_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P \vdash A} \quad \overset{\varphi_2}{\Gamma', A \vdash F}}{\Gamma, \Gamma', P \vdash F}\,\mathit{cut} \quad \overset{\varphi_3}{\Gamma'', Q \vdash F}}{\Gamma, \Gamma', \Gamma'', P \vee Q \vdash F}\,\vee_l \quad \rightsquigarrow \quad ?$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \overset{\varphi_2}{\Gamma', P, A \vdash F}}{\Gamma, \Gamma', P \vdash F}\,\mathit{cut} \quad \overset{\varphi_3}{\Gamma'', Q \vdash F}}{\Gamma, \Gamma', \Gamma'', P \vee Q \vdash F}\,\vee_l \quad \rightsquigarrow \quad \dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \dfrac{\overset{\varphi_2}{\Gamma', A, P \vdash F} \quad \overset{\varphi_3}{\Gamma'', Q \vdash F}}{\Gamma', \Gamma'', P \vee Q, A \vdash F}\,\vee_l}{\Gamma, \Gamma', \Gamma'', P \vee Q \vdash F}\,\mathit{cut}$$

$$\dfrac{\overset{\varphi_1}{\Gamma, P \vdash F} \quad \dfrac{\overset{\varphi_2}{\Gamma', Q \vdash A} \quad \overset{\varphi_3}{\Gamma'', A \vdash F}}{\Gamma', \Gamma'', Q \vdash F}\,\mathit{cut}}{\Gamma, \Gamma', \Gamma'', P \vee Q \vdash F}\,\vee_l \quad \rightsquigarrow \quad ?$$

$$\dfrac{\overset{\varphi_1}{\Gamma, P \vdash F} \quad \dfrac{\overset{\varphi_2}{\Gamma' \vdash A} \quad \overset{\varphi_3}{\Gamma'', Q, A \vdash F}}{\Gamma', \Gamma'', Q \vdash F}\,\mathit{cut}}{\Gamma, \Gamma', \Gamma'', P \vee Q \vdash F}\,\vee_l \quad \rightsquigarrow \quad \dfrac{\overset{\varphi_2}{\Gamma' \vdash A} \quad \dfrac{\overset{\varphi_1}{\Gamma, P \vdash F} \quad \overset{\varphi_3}{\Gamma'', A, Q \vdash F}}{\Gamma, \Gamma'', P \vee Q, A \vdash F}\,\vee_l}{\Gamma, \Gamma', \Gamma'', P \vee Q \vdash F}\,\mathit{cut}$$

$\bigvee_r$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \overset{\varphi_2}{\Gamma', A \vdash P}}{\Gamma, \Gamma' \vdash P}\,\mathit{cut}}{\Gamma, \Gamma' \vdash P \vee Q}\,\vee_r \quad \rightsquigarrow \quad \dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \dfrac{\overset{\varphi_2}{\Gamma', A \vdash P}}{\Gamma', A \vdash P \vee Q}\,\vee_r}{\Gamma, \Gamma' \vdash P \vee Q}\,\mathit{cut}$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \overset{\varphi_2}{\Gamma', A \vdash Q}}{\Gamma, \Gamma' \vdash Q}\ \mathit{cut}}{\Gamma, \Gamma' \vdash P \vee Q}\ \vee_r \quad \rightsquigarrow \quad \dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \dfrac{\overset{\varphi_2}{\Gamma', A \vdash Q}}{\Gamma', A \vdash P \vee Q}\ \vee_r}{\Gamma, \Gamma' \vdash P \vee Q}\ \mathit{cut}$$

$\forall_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P[x/t] \vdash A} \quad \overset{\varphi_2}{\Gamma', A \vdash F}}{\Gamma, \Gamma', P[x/t] \vdash F}\ \mathit{cut}}{\Gamma, \Gamma', \forall x.Px \vdash F}\ \forall_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P[x/t] \vdash A}}{\Gamma, \forall x.Px \vdash A}\ \forall_l \quad \overset{\varphi_2}{\Gamma', A \vdash F}}{\Gamma, \Gamma', \forall x.Px \vdash F}\ \mathit{cut}$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \overset{\varphi_2}{\Gamma', P[x/t], A \vdash F}}{\Gamma, \Gamma', P[x/t] \vdash F}\ \mathit{cut}}{\Gamma, \Gamma', \forall x.Px \vdash F}\ \forall_l \quad \rightsquigarrow \quad \dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \dfrac{\overset{\varphi_2}{\Gamma', A, P[x/t] \vdash F}}{\Gamma', \forall x.Px, A \vdash F}\ \forall_l}{\Gamma, \Gamma', \forall x.Px \vdash F}\ \mathit{cut}$$

$\forall_r$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \overset{\varphi_2}{\Gamma', A \vdash P[x/\alpha]}}{\Gamma, \Gamma' \vdash P[x/\alpha]}\ \mathit{cut}}{\Gamma, \Gamma' \vdash \forall x.Px}\ \forall_r \quad \rightsquigarrow \quad \dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \dfrac{\overset{\varphi_2}{\Gamma', A \vdash P[x/\alpha]}}{\Gamma', A \vdash \forall x.Px}\ \forall_r}{\Gamma, \Gamma' \vdash \forall x.Px}\ \mathit{cut}$$

$\exists_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P[x/\alpha] \vdash A} \quad \overset{\varphi_2}{\Gamma', A \vdash F}}{\Gamma, \Gamma', P[x/\alpha] \vdash F}\ \mathit{cut}}{\Gamma, \Gamma', \exists x.Px \vdash F}\ \exists_l \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P[x/\alpha] \vdash A}}{\Gamma, \exists x.Px \vdash A}\ \exists_l \quad \overset{\varphi_2}{\Gamma', A \vdash F}}{\Gamma, \Gamma', \exists x.Px \vdash F}\ \mathit{cut}$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \overset{\varphi_2}{\Gamma', P[x/\alpha], A \vdash F}}{\Gamma, \Gamma', P[x/\alpha] \vdash F}\ \mathit{cut}}{\Gamma, \Gamma', \exists x.Px \vdash F}\ \exists_l \quad \rightsquigarrow \quad \dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \dfrac{\overset{\varphi_2}{\Gamma', A, P[x/\alpha] \vdash F}}{\Gamma', \exists x.Px, A \vdash F}\ \exists_l}{\Gamma, \Gamma', \exists x.Px \vdash F}\ \mathit{cut}$$

$\exists_r$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \overset{\varphi_2}{\Gamma', A \vdash P[x/t]}}{\Gamma, \Gamma' \vdash P[x/t]}\ \mathit{cut}}{\Gamma, \Gamma' \vdash \exists x.Px}\ \exists_r \quad \rightsquigarrow \quad \dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \dfrac{\overset{\varphi_2}{\Gamma', A \vdash P[x/t]}}{\Gamma', A \vdash \exists x.Px}\ \exists_r}{\Gamma, \Gamma' \vdash \exists x.Px}\ \mathit{cut}$$

$\neg_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \overset{\varphi_2}{\Gamma', A \vdash P}}{\Gamma, \Gamma' \vdash P}\ \mathit{cut}}{\Gamma, \Gamma', \neg P \vdash}\ \neg_l \quad \rightsquigarrow \quad \dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \dfrac{\overset{\varphi_2}{\Gamma', A \vdash P}}{\Gamma', \neg P, A \vdash}\ \neg_l}{\Gamma, \Gamma', \neg P \vdash}\ \mathit{cut}$$

$\neg_r$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \overset{\varphi_2}{\Gamma', P, A \vdash}}{\Gamma, \Gamma', P \vdash}\ \mathit{cut}}{\Gamma, \Gamma' \vdash \neg P}\ \neg_r \quad \rightsquigarrow \quad \dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \dfrac{\overset{\varphi_2}{\Gamma', P, A \vdash}}{\Gamma', A \vdash \neg P}\ \neg_r}{\Gamma, \Gamma' \vdash \neg P}\ \mathit{cut}$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P \vdash A} \quad \overset{\varphi_2}{\Gamma', A \vdash}}{\Gamma, \Gamma', P \vdash}\ \mathit{cut}}{\Gamma, \Gamma' \vdash \neg P}\ \neg_r \quad \rightsquigarrow \quad ?$$

$w_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \overset{\varphi_2}{\Gamma', A \vdash F}}{\Gamma, \Gamma' \vdash F}\ \mathit{cut}}{\Gamma, \Gamma', P \vdash F}\ w_l \quad \rightsquigarrow \quad \dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \dfrac{\overset{\varphi_2}{\Gamma', A \vdash F}}{\Gamma', P, A \vdash F}\ w_l}{\Gamma, \Gamma', P \vdash F}\ \mathit{cut}$$

$w_r$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \overset{\varphi_2}{\Gamma', A \vdash}}{\Gamma, \Gamma' \vdash}\ \mathit{cut}}{\Gamma, \Gamma' \vdash P}\ w_r \quad \rightsquigarrow \quad \dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \dfrac{\overset{\varphi_2}{\Gamma', A \vdash}}{\Gamma', A \vdash P}\ w_r}{\Gamma, \Gamma' \vdash P}\ \mathit{cut}$$

$c_l$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P \vdash A} \quad \overset{\varphi_2}{\Gamma', P, A \vdash F}}{\Gamma, \Gamma', P, P \vdash F}\;\mathit{cut}}{\Gamma, \Gamma', P \vdash F}\;c_l \qquad \rightsquigarrow \quad ?$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P, P \vdash A} \quad \overset{\varphi_2}{\Gamma', A \vdash F}}{\Gamma, \Gamma', P, P \vdash F}\;\mathit{cut}}{\Gamma, \Gamma', P \vdash F}\;c_l \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma, P, P \vdash A}}{\Gamma, P \vdash A}\;c_l \quad \overset{\varphi_2}{\Gamma', A \vdash F}}{\Gamma, \Gamma', P \vdash F}\;\mathit{cut}$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \overset{\varphi_2}{\Gamma', P, P, A \vdash F}}{\Gamma, \Gamma', P, P \vdash F}\;\mathit{cut}}{\Gamma, \Gamma', P \vdash F}\;c_l \qquad \rightsquigarrow \qquad \dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \dfrac{\overset{\varphi_2}{\Gamma', P, P, A \vdash F}}{\Gamma', P, A \vdash F}\;c_l}{\Gamma, \Gamma', P \vdash F}\;\mathit{cut}$$

$\mathit{cut}$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \overset{\varphi_2}{\Gamma', A \vdash P}}{\Gamma, \Gamma' \vdash P}\;\mathit{cut} \quad \overset{\varphi_3}{\Gamma'', P \vdash F}}{\Gamma, \Gamma', \Gamma'' \vdash F}\;\mathit{cut} \qquad \rightsquigarrow \qquad \dfrac{\overset{\varphi_1}{\Gamma \vdash A} \quad \dfrac{\overset{\varphi_2}{\Gamma', A \vdash P} \quad \overset{\varphi_3}{\Gamma'', P \vdash F}}{\Gamma', \Gamma'', A \vdash F}\;\mathit{cut}}{\Gamma, \Gamma', \Gamma'' \vdash F}\;\mathit{cut}$$

$$\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \dfrac{\overset{\varphi_2}{\Gamma', P \vdash A} \quad \overset{\varphi_3}{\Gamma'', A \vdash F}}{\Gamma', \Gamma'', P \vdash F}\;\mathit{cut}}{\Gamma, \Gamma', \Gamma'' \vdash F}\;\mathit{cut} \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \overset{\varphi_2}{\Gamma', P \vdash A}}{\Gamma, \Gamma' \vdash A}\;\mathit{cut} \quad \overset{\varphi_3}{\Gamma'', A \vdash F}}{\Gamma, \Gamma', \Gamma'' \vdash F}\;\mathit{cut}$$

$$\dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \dfrac{\overset{\varphi_2}{\Gamma' \vdash A} \quad \overset{\varphi_3}{\Gamma'', P, A \vdash F}}{\Gamma', \Gamma'', P \vdash F}\;\mathit{cut}}{\Gamma, \Gamma', \Gamma'' \vdash F}\;\mathit{cut} \qquad \rightsquigarrow \qquad \dfrac{\overset{\varphi_2}{\Gamma' \vdash A} \quad \dfrac{\overset{\varphi_1}{\Gamma \vdash P} \quad \overset{\varphi_3}{\Gamma'', P, A \vdash F}}{\Gamma, \Gamma'', A \vdash F}\;\mathit{cut}}{\Gamma, \Gamma', \Gamma'' \vdash F}\;\mathit{cut}$$

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