Proof search tree and cut elimination

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Abstract

A new cut elimination method is obtained here by "proof mining" (unwinding) from the following non-effective proof that begins with extracting an infinite branch $\mathcal{B}$ when the canonical search tree $\mathcal{T}$ for a given formula E of first order logic is not finite. The branch $\mathcal{B}$ determines a semivaluation so that $\mathcal{B} \models \bar{E}$ and (*) every semivaluation can be extended to a total valuation. Since for every derivation d of E and every model $\mathcal{M}$, $\mathcal{M} \models E$, this provides a contradiction showing that $\mathcal{T}$ is finite, $\exists l(\mathcal{T} < l)$. A primitive recursive function $L(d)$ such that $\mathcal{T} < L(d)$ is obtained using instead of (*) the statement: For every r, if the canonical search tree $\mathcal{T}^{r+1}$ with cuts of complexity $r+1$ is finite, then $\mathcal{T}^r$ is finite.

In our proof the reduction of $(r+1)$-cuts does not introduce new r-cuts but preserves only one of the branches.

1 Introduction

Continuing work done in [5] we obtain a new cut elimination method by "proof mining" (unwinding) from a familiar non-effective proof consisting of four parts.

1. If the canonical search tree $\mathcal{T}$ for a given formula E of first order logic is not finite, then there exists an infinite branch $\mathcal{B}$ of $\mathcal{T}$.

2. The branch $\mathcal{B}$ determines a semivaluation (partial model for subformulas of E) so that $\mathcal{B} \models \bar{E}$.

3. Every semivaluation can be extended to a total valuation (model).

4. For every derivation d of E and every model $\mathcal{M}$, $\mathcal{M} \models E$.

This is a contradiction showing that $\mathcal{T}$ is finite, $\exists l(\mathcal{T} < l)$. A primitive recursive function $L(d)$ such that $\mathcal{T} < L(d)$ is obtained in the present paper after replacing the statement 3 by

(3') For every r if the canonical search tree $\mathcal{T}^{r+1}$ with cuts of quantifier complexity $r+1$ is finite, then $\mathcal{T}^r$ is finite.

By quantifier complexity we mean the number of quantifiers in the cut formula. Our proof of (3') provides a reduction of $(r+1)$-cuts that (unlike familiar Gentzen's reduction) does not introduce new r-cuts but preserves only one of the branches, depending however of the truth-values of subformulas in the "current" node of the tree $\mathcal{T}^r$. This node is changed in the course of reduction.

To save notation we assume that the endformula of an original derivation d is a Σ^0_1 sentence of first order logic,

$$E = \exists \mathbf{x} M(\mathbf{x}), \qquad \mathbf{x} := x_1, \dots x_p,$$

with a quantifier free M, but given derivation d as well as proof search trees may contain formulas of an arbitrary complexity.

$\mathcal{T}^r$ is a canonical proof search tree for S with cuts over formulas of quantifier complexity $\leq r$ beginning with a quantifier. In particular

$\mathcal{T} := \mathcal{T}^0$ is a proof search tree with $\exists$ rules instantiated by terms in H (the Herbrand universe for E) and **cuts over all atomic formulas with terms in** H. We wish to prove:

If d is a derivation of E, then $\mathcal{T} < l$ for some l.

It is assumed that the eigenvariables for the $\forall$ inference (that can occur for $r > 0$) in a proof search tree is uniquely determined by the formula introduced by the rule. In this way the eigenvariables in $\mathcal{T}^r$ for different r but the same principal formula are the same.

The step 3 in the non-effective proof of cut-elimination consists in an *extension of a semivaluation* to all formulas of complexity r and hence to a total valuation. This is proved by induction on the quantifier complexity r of the semivaluation with a trivial base $r = 0$ and the induction step:

$$\exists f Sem(f, r) \to \exists f Sem(f, r+1) \tag{1.1}$$

where $Sem(f, r)$ means that f is a semivaluation defined for all formulas of complexity r.

As noticed in [4], given derivation $d : E$ with cuts of quantifier complexity R provides a bound l such that $\mathcal{T}^R < l$: this l is a level in $\mathcal{T}^R$ where all rules present in d had been already applied. In particular for $R = 0$ any derivation with only atomic cuts provides a bound for $\mathcal{T}$.

As G.Kreisel pointed out [1], semivaluations are closely related to infinite branches of the proof search tree. By König's Lemma,

$$\exists f Sem(f, r) \iff \mathcal{T}^r \text{ is infinite},$$

therefore (1.1) can be converted into the implication

$$\exists l(\mathcal{T}^{r+1} < l) \to \exists l(\mathcal{T}^r < l) \tag{1.2}$$

of Σ^0_1-formulas. A familiar proof of this implication consists of converting it back to (1.1) and using arithmetical comprehension.

There are (at least) two ways of replacing this with a primitive recursive proof of (1.2). One way is to note that a standard Gentzen-style cut reduction applied to all cuts of complexity $r+1$ in $\mathcal{T}^{r+1}$ leads to a derivation d of complexity r that provides a simple bound for $\mathcal{T}^r$.

This argument led to the algorithm for bounding $\mathcal{T}$ discussed in [5] : eliminate all non-atomic cuts, then restrict $\mathcal{T}$ as above, completely bypassing (1.1).

Our present method relies on a non-effective proof of (1.2) which is closer to extension of a semivaluation f of complexity r to a semivaluation, say g of complexity $r+1$. If $\mathcal{T}^{r+1}$, or any other derivation $d : E$ of complexity $r+1$ is finite, and g is given, assumption $g(E) = \bot$ leads to a contradiction by computing the truth values of all formulas and sequent S in $\mathcal{T}^{r+1}$ and proving $g(S) = \top$. The values of g for the formulas of complexity $\leq r$ with parameters from $\mathcal{T}^r$ are the same as the given values of f.

Instead of using comprehension to determine the values under g of remaining formulas (of complexity $r+1$), consider which of these values are needed at the beginning under the "depth first" strategy: in the uppermost leftmost $r+1$-cut try to compute the left hand side premise $\exists x A(x)$ first:

$$\dfrac{\dfrac{\dfrac{\Delta, A(t_i)}{\Delta, \exists x A(x)}}{\vdots \atop \Gamma, \exists x A(x)} \qquad \dfrac{\vdots b \atop \Gamma, \bar{A}(b)}{\Gamma, \forall x \bar{A}(x)}\ \forall}{\Gamma, E} \tag{1.3}$$

The values $f(A(t_i))$ are all defined, and it is possible to act as if

$$g(\exists x A(x)) = \max_i f(A(t_i))$$

for all $A(t_i)$ *in a given derivation*. This allows to prune the cut (1.3) retaining only one of the premises depending on this value of $g(\exists x A(x))$. At the end, all $r+1$-cuts are removed and (1.2) is established.

We present a non-effective proof of (1.2) modified in this way for the case $r = 0$ in Section 3. The same proof works for arbitrary r and by conservativeness of Konig's Lemma over primitive recursive arithmetic PRA its unwinding provides a primitive recursive cut elimination algorithm (as pointed out by U. Kohlenbach). Section 4 presents our unwinding leading to a new cut elimination procedure for first order logic.

Discussions with G. Kreisel, S. Feferman and especially U. Kohlenbach helped to clarify the goal of this work and the statements of results and proofs.

2 Preliminaries

2.1 Tree notation

Let's recall some notation concerning finite sequences of natural numbers. We use $\mathrm{a}, \mathrm{b}, \mathrm{c}$ as variables for *binary finite sequences*

$\mathrm{a} =< a_0, \ldots, a_n >$ where $a_i \in \{0,1\}$, $lth(\mathrm{a}) := n+1$, $(\mathrm{a})_i := a_i$.

Concatenation $*$:

$< a_0, \ldots, a_n > * < b_0, \ldots, b_m > := < a_0, \ldots, a_n, b_0, \ldots, b_m >$.

$<>$ is the empty sequence with $lth(<>) = 0$. $\mathrm{a} \subseteq \mathrm{b} : \iff \exists \mathrm{c} \mathrm{b} = \mathrm{a} * \mathrm{c}$;

$\mathrm{a} < \mathrm{b}$ iff a *lexicographically strictly precedes* b, that is situated strictly to the left in the tree of all finite sequences:

$\mathrm{a} \subset \mathrm{b}$ or for some $j < lth(\mathrm{a})$, $(a)_i = (b)_i$ for all $i < j$, and $(a)_j < (b)_j$.

Consider a primitive recursive tree $\mathcal{T}$ of *binary sequences* with the root $<>$:

$$\mathrm{b} \in \mathcal{T} \& \mathrm{a} \subseteq \mathrm{b} \to \mathrm{a} \in \mathcal{T}; \qquad \mathrm{a} \in \mathcal{T} \to (\forall i < lth(\mathrm{a}))(\mathrm{a})_i \leq 1$$

$\mathcal{T}_{\mathrm{a}}$ is the subtree of $\mathcal{T}$ with the root a: $\{\mathrm{b} \in \mathcal{T} : \mathrm{a} \subseteq \mathrm{b}\}$.

In fact we use labeled trees. $\mathcal{T}(\mathrm{a}) = 0$ means $\mathrm{a} \notin \mathcal{T}$, while $\mathcal{T}(\mathrm{a}) \neq 0$ means that $\mathrm{a} \in \mathcal{T}$ and contains some additional information. A node $\mathrm{a} \in \mathcal{T}$ is a *leaf* if $\mathrm{b} \supset \mathrm{a}$ implies $\mathrm{b} \notin \mathcal{T}$. In this case all branches of $\mathcal{T}$ through a are *closed*.

$$\mathcal{T} < l := \ (\forall \mathrm{a} : lth(\mathrm{a}) = l)(\mathrm{a} \notin \mathcal{T}); \qquad \mathcal{T} > l := \ (\exists \mathrm{a} : lth(\mathrm{a}) = l + 1)(\mathrm{a} \in \mathcal{T})$$

and similar bounded formulas with replacement of $<, >$ by $\leq, \geq$.

2.2 Tait Calculus; Canonical proof trees

We consider first order formulas in positive normal form (negations only at atomic formulas). Negation $\bar{A}$ of a formula A is defined in a standard way by de-Morgan rules. Derivable objects are *sequents*, that is multisets of formulas.

Axioms: $A, \bar{A}, \Gamma$

Inference Rules:

$$\frac{A, \Gamma \quad B, \Gamma}{A \& B, \Gamma} \ \& \qquad \frac{A, B, \Gamma}{A \vee B, \Gamma} \ \vee \qquad \frac{M(t), \exists x B(x), \Gamma}{\exists x B(x), \Gamma} \ \exists$$

$$\frac{B(a), \Gamma}{\forall x B(x), \Gamma} \ \forall \qquad \frac{C, \Gamma \quad \bar{C}, \Gamma}{\Gamma} \ cut$$

The *eigenvariable* a in $\forall$ inference should be fresh. The term t in the rule $\exists$ is called *the term* of that rule.

Definition 2.1 *The* Herbrand Universe H *of a* Σ^0_1*-formula* E *consists of all terms generated from constants and free variables occurring in* M *by function symbols occurring in* M*. If the initial supply is empty, add a new constant.*

For a given formula $E = \exists x_1 \ldots x_p M$ list all p-tuples of terms in of the Herbrand universe H in a sequence

$$\mathbf{t}_1, \ldots, \mathbf{t}_i, \ldots \tag{2.1}$$

We assume also some Godel enumeration $Gn(A)$ of all terms and formulas A. The *canonical proof search tree* $\mathcal{T}$ for a sentence $E = \exists \mathbf{x} M(\mathbf{x})$ is constructed by bottom-up application of the rules $\&, \vee$ (first) and $\exists$ and atomic cut when $\&, \vee$ are not applicable.

$\mathcal{T}$ assigns sequents to nodes a of the tree of finite binary sequences. To express a $\notin \mathcal{T}$ (when a is situated over an axiom or the second premise of a one-premise inference rule) we write $\mathcal{T}(\mathrm{a}) = 0$.

$\mathcal{T}(<>)$ contains sequent E. If $\mathcal{T}(\mathrm{a})$ is already constructed and is not an axiom (:=*closed* ' node or branch), then it is extended preserving all existing formulas. Principal formulas of the propositional rules are preserved (for book-keeping). If all branches of $\mathcal{T}$ are closed, then the whole *tree is closed*.

The following *fairness conditions* are assumed. There exists a primitive recursive function L_0 such that for each $\mathrm{a} \in \mathcal{T}$ and every non-closed $\mathrm{b} \supseteq \mathrm{a}, \mathrm{b} \in \mathcal{T}$ with $lth(\mathrm{b}) \geq lth(\mathrm{a}) + L_0(\mathrm{a})$

1. If $C\&D \in \mathcal{T}(\mathrm{a})$ then $C \in \mathcal{T}(\mathrm{b})$ or $D \in \mathcal{T}(\mathrm{b})$,
2. If $C \vee D \in \mathcal{T}(\mathrm{a})$ then $C \in \mathcal{T}(\mathrm{b})$ and $D \in \mathcal{T}(\mathrm{b})$,
3. $B(\mathbf{t}_i) \in \mathcal{T}(\mathrm{b})$ for every $i \leq lth(\mathrm{a})$,
4. For every atomic formula A over H with $Gn(A) < lth(\mathrm{a})$ either $A \in \mathcal{T}(\mathrm{b})$ or $\bar{A} \in \mathcal{T}(\mathrm{b})$.

For $r > 0$ let H^r be the Herbrand universe for $\mathcal{T}^r$ generated by the functions (including constants) in E from the eigenvariables (chosen in a standard way as above) for all $\forall$-formulas of quantifier complexity $\leq r$.

We say that a formula A *agrees* with a node $\mathrm{a} \in \mathcal{T}^r$ if A contains free only eigenvariables of the $\forall$ rules situated under a.

The canonical proof search tree $\mathcal{T}^r$ of complexity $r > 0$ is defined similarly to $\mathcal{T}$, but now cuts are applied to atomic formulas and formulas of complexity $\leq r$ beginning with quantifier that agree with a given node.

The following *fairness conditions* are assumed. There exists a primitive recursive function L_r such that for each $\mathrm{a} \in \mathcal{T}^r$ and every non-closed $\mathrm{b} \supseteq \mathrm{a}, \mathrm{b} \in \mathcal{T}^r$ with $lth(\mathrm{b}) \geq lth(\mathrm{a}) + L_r(\mathrm{a})$

1. If $C\&D \in \mathcal{T}(\mathrm{a})$ then $C \in \mathcal{T}(\mathrm{b})$ or $D \in \mathcal{T}(\mathrm{b})$,
2. If $C \vee D \in \mathcal{T}(\mathrm{a})$ then $C \in \mathcal{T}(\mathrm{b})$ and $D \in \mathcal{T}(\mathrm{b})$,
3. If $\exists y C(y) \in \mathcal{T}(\mathrm{a})$, then $C(t) \in \mathcal{T}(\mathrm{b})$ for every term t with $Gn(t) \leq lth(\mathrm{a})$,
4. For every every formula A over H^r with $Gn(A) < lth(\mathrm{a})$ that agrees with a and is atomic or begins with a quantifier, either $A \in \mathcal{T}(\mathrm{b})$ or $\bar{A} \in \mathcal{T}(\mathrm{b})$.

3 Cuts of rank 1: modified non-effective proof

Let us modify the proof of (1.2) for $r = 0$ restated as follows:

Lemma 3.1 *$\mathcal{T}$ is infinite $\Rightarrow \mathcal{T}^1$ is infinite.*

Proof. Suppose $\mathcal{T}$ is infinite. Then every non-closed branch $\mathcal{B}$ of $\mathcal{T}$ (existing by Koenig's Lemma) is a *countermodel* for E: $\mathcal{B} \models \bar{E}$.

Write $\mathcal{B} \models^+ A$ to express that A is propositionally implied by the values of formulas present in the branch $\mathcal{B}$: there are $A_1, \ldots A_k \in \mathcal{T}(\mathrm{b})$ for some $\mathrm{b} \in \mathcal{B}$ such that $\bar{A}_1 \& \ldots \& \bar{A}_k \to A$ is a tautology. Note that for every quantifier free formula A with parameters in H (the Herbrand universe of E).

$$\mathcal{B} \models^+ A \text{ or } \qquad \mathcal{B} \models^+ \bar{A},$$

since all atomic formulas in A are decided by cuts in every branch of $\mathcal{T}$.

Consider a new inference rule:

$$\frac{\mathcal{B} \models^+ A \quad \bar{A}, \Gamma}{\Gamma} \ \mathcal{B}\text{-cut}$$

for a quantifier free A with all terms in the Herbrand universe H.

Define *extended derivations* as ones using $\mathcal{B}$-cut and cuts of rank 1 in addition to ordinary cut free rules.

Assume there exists an extended derivation d of E and prove $\mathcal{B} \models E$ using induction on the number of cuts in d. This implies $\mathcal{T}$ is finite by contradiction with $\mathcal{B} \not\models E$.

Induction base. No cuts. Induction on d using the fact that $\mathcal{B}$ is defined for all needed subformulas.

Induction step.

1. For some $\exists$ inference in (1.3) one has $\mathcal{B} \models^+ A(t_i)$. Replace (1.3) by the rule

$$\frac{\mathcal{B} \models^+ A(t_i) \quad \begin{array}{c} \vdots t_i \\ \bar{A}(t_i), \Delta \end{array}}{\Delta} \ \mathcal{B}\text{-cut}$$

   where the right hand side premise is obtained by substitution of t_i for b. Now apply IH.

2. For some paricular cut in (1.3) and for all $\exists$ inferences as above $\mathcal{B} \models \bar{A}(t)$. Replace the cut (1.3) by its left branch, erasing all formulas $\exists x A$ traceable to this cut and replacing corresponding $\exists$ inference by a $\mathcal{B}$-cut with the premise $\mathcal{B} \models \bar{A}(t)$. Now IH is applicable.

Let us assume the only free variables of the terms t_i are the eigenvariables of the $\forall$ inferences situated below. Then if the case 1 does not obtain, the situation in the case 2 always occurs for one of the cuts (1.3), namely for the uppermost cut in the leftmost (with respect to $r+1$-cuts) branch. Indeed, $r+1$-eigenvariables do nor occur in the conclusions of $\exists$ rules in that branch. This concludes the proof. $\dashv$

4 Reduction of quantifier complexity

We present a combinatorial proof of (1.2) obtained by "unwinding" the proof for complexity 1 in the previous subsection.

We use finite objects similar to $\mathcal{B}$-derivations. For a node $\mathrm{a} \in \mathcal{T}^r$ consider a rule

$$\frac{A \in \mathcal{T}^r(\mathrm{a}) \quad A, \Gamma}{\Gamma} \text{ a-cut}$$

for a formula A agreeing with a and use a-*derivations* using this rule. For comparison with $\mathcal{B}$-cut recall that $\mathcal{B} \models \bar{A}$ for $A \in \mathcal{T}(\mathrm{a}), \mathrm{a} \in \mathcal{B}$.

Assume also that every free variable of the term t in $\exists$ rule either occurs free in the conclusion or is an eigenvariable of a rule occurring below. This can be achieved by replacing redundant free variables by a constant 0. (Recall however remarks by G. Kreisel [2] on essential use of such "redundant" variables in unwinding of mathematical proofs).

4.1 A bound for $\mathcal{T}^r$

4.1.1 Eliminating $r+1$-cuts

Till the end of the subsection 4.1.1

$$d : \Gamma, E$$

denotes an a-derivation of complexity $r+1$, with Γ agreeing with a, complexity of $\Gamma \leq r$, $\mathrm{a} \in \mathcal{T}^r$ with eigenvariables for $\forall$-formulas of the complexity $\leq r$ chosen in a standard way.

At the beginning $d = \mathcal{T}^r$, at the end all $r+1$-cuts are replaced by a-rules for a suitable a that changes in the process.

Definition 4.1 *A formula A is decided by* a *if $A \in \mathcal{T}(\mathrm{a})$ (i.e. A is explicitly false in* a*) or $\bar{A} \in \mathcal{T}(\mathrm{a})$ (i.e. A is explicitly true in* a*).*

Consider an $r+1$-cut in d:

$$\frac{\Delta, \exists x A(x) \quad \Delta, \forall x \bar{A}(x)}{\Delta, E} \; cut \tag{4.1}$$

Such a cut is *leftmost*, if there are no $\forall$-premises of $r+1$-cuts below it, that is it is in the left branch of every $r+1$-cut situated below. Choose a leftmost cut (4.1) such that there are no $r+1$-cuts above it. List all side formulas of $\exists$ rules traceable to the formula $\exists x A(x)$.

$$A(t_1), \ldots, A(t_m) \tag{4.2}$$

Note that the terms t_i do not contain eigenvariables of the $\forall$ rules of complexity $r+1$, so $t_i \in H^r$.

Lemma 4.2 *Let* $\mathrm{a} \in \mathcal{T}^r$*,* $d : \Gamma, E$ *be* a*-derivation, and all formulas* $A(t_i)$ *in (4.2) be decided by* a*.*

Then the cut (4.1) can be replaced by a*-cuts, leading to a new* a*-derivation of* Γ, E*.*

Proof. Consider possible cases.

1. At least one of the side formulas $A(t_i)$ is "true" in a: $\bar{A}(t_i) \in \mathcal{T}(\mathrm{a})$. Then we can replace (4.1) by a-cuts. Delete the left branch, replace all formulas traceable to $\forall x \bar{A}(x)$ by $\bar{A}(t_i)$ and substitute the eigenvariable of rules introducing $\forall x \bar{A}(x)$ by t_i. Both (4.1) and all such $\forall$ rules become a-cuts:

$$\frac{\bar{A}(t_i) \in \mathcal{T}^r(\mathrm{a}) \quad \bar{A}(t_i), \Sigma}{\Sigma} \tag{4.3}$$

2. All side formulas $A(t_i)$ are "false" in a: $A(t_i) \in \mathcal{T}^r(\mathrm{a})$. Then the formula $\exists x A$ is redundant. Delete the right branch of the cut (4.1). From the left branch delete all formulas traceable to $\exists x A(x)$. Then $\exists$ rules introducing this formula become a-cuts:

$$\frac{A(t_i) \in \mathcal{T}^r(\mathrm{a}) \quad A(t_i), \Sigma}{\Sigma} \tag{4.4}$$

$\dashv$

We say that the transformation in Lemma 4.2 *reduces* d to a new derivation. We show that d can be (primitive recursively in all parameters) reduced to derivations without $(r+1)$-cuts by climbing up $\mathcal{T}^r$.

Lemma 4.3 *Let* $\mathrm{a} \in \mathcal{T}^r$*,* $d : \Gamma, E$ *be* a*-derivation. Then there exists a level* $l \geq lth(\mathrm{a})$ *such that every node* $\mathrm{b} \supset \mathrm{a},\ \mathrm{b} \in \mathcal{T}^r$ *decides all formulas in (4.2).*

Proof. Formulas $A(t_i)$ contain only eigenvariables of the rules of complexity $\leq r$, hence they contain only terms in H^r. Using the fairness function L_r find a level $l \geq lth(\mathrm{a})$ such that all these formulas appear (possibly negated) by the level l. $\dashv$

For $\mathrm{a} \in \mathcal{T}^r$ let $L^{prop}(\mathrm{a})$ be the first level $\geq lth(\mathrm{a})$ of $\mathcal{T}^r$ saturated with respect to propositional rules appled to all formulas in $\mathcal{T}^r(\mathrm{a})$.

Lemma 4.4 *Let* $\mathrm{a} \in \mathcal{T}^r$*,* $d : \Gamma$ *be an* a*-derivation,* $\Gamma \supseteq \mathcal{T}^r(\mathrm{a})$ *and let* d *consist of* a*-cuts and propositional rules. Then*

$$\mathcal{T}^r_{\mathrm{a}} < L^{prop}(\mathrm{a})$$

that is the restriction of $\mathcal{T}^r_{\mathrm{a}}$ *to this level is a derivation.*

Proof. Let $A_1, \ldots, A_n$ be all a-cut formulas in d. Then

$$\mathcal{T}^r(\mathrm{a}) \supseteq \Gamma, A_1, \ldots A_n \tag{4.5}$$

by the proviso in the a-cut. Now use induction on d. Induction base is obvious, the case of a-cut in the induction step follows from (4.5). Consider a propositional rule, say

$$\frac{A, \Gamma \quad B, \Gamma}{A \& B, \Gamma}$$

There is a level $l \leq L^{prop}(\mathrm{a})$ such that every non-closed node $\mathrm{b} \supseteq \mathrm{a}$, $lth(\mathrm{b}) = l$ contains A or B and $L^{prop}(\mathrm{b}) \leq L^{prop}(\mathrm{a})$. Now apply IH. $\dashv$

For an arbitrary finite sequence $A_1, \ldots, A_m$ of formulas of complexity $\leq r$ with parameters in H^r that agree with a let $L^E(\mathrm{a}, A_1, \ldots, A_m, d)$ be the first level $l \geq lth(\mathrm{a})$ containing $A_1, \ldots, A_m$ up to negation and saturated with respect to all rules appled to all formulas in $\mathcal{T}^r(\mathrm{a})$, $A_1, \ldots A_m$ in d. The latter condition means in particular that if $A_i = \exists x B(x)$ is instantiated by a term t in d and $A_i \in \mathcal{T}^r(\mathrm{b})$ with $\mathrm{b} \supseteq \mathrm{a}$, $lth(\mathrm{b}) \geq l$, then $B(t) \in \mathcal{T}^r(\mathrm{b})$.

Lemma 4.5 *Let* $\mathrm{a} \in \mathcal{T}^r$, $d : \Gamma$ *be an* a*-derivation of complexity* r, $\Gamma \supseteq \mathcal{T}^r(\mathrm{a})$. *Then*

$$\mathcal{T}^r_{\mathrm{a}} < L^E(\mathrm{a}, A_1, \ldots, A_m, d)$$

where $A_1, \ldots, A_m$ *is the complete list of the principal and side formulas of quantifier rules in* d.

Proof. Like in the previous Lemma, with all quantifier rules treated using the new bound. $\dashv$

Lemma 4.6 *There is a primitive recursive function* L_1 *such that* $\mathcal{T}^r < L_1(d)$ *for every* $d : E$ *of complexity* $r + 1$ *with the standard choice of eigenvariables.*

Proof. Combine the previous Lemmata. $\dashv$

The original derivation $d : E$ may fail to satisfy the standardness condition for eigenvariables. This condition can be enforced by renaming eigenvariables and deleting redundant formulas and branches. We use fairness properties of $\mathcal{T}^r$ instead.

Theorem 4.7 *There is a primitive recursive function* L_* *such that* $\mathcal{T}^r < L_*(d)$ *for every* $d : E$ *of complexity* r.

Proof. Rename eigenvariables in d into standard eigenvariables. This invalidates some of the $\forall$ rules in d by violating the proviso for eigenvariables. However Lemma 4.5 is still applicable to the resulting figure. $\dashv$

Theorem 4.8 *There is a primitive recursive function* L *such that* $\mathcal{T} < L(d)$ *for every* $d : E$.

Proof. Iterate the previous theorem. $\dashv$

Let's see what happens if $d : E$ is not a derivation but an infinite figure constructed by inference rules of predicate logic. In this case the proofs we gave in Lemma 3 and Lemma 4.2 do not go through for several reasons. First, it is possible that the given branch $\mathcal{B}$ or the leftmost branch (with respect to $r + 1$-cuts) contains infinitely many $r + 1$-cuts so that every $\exists$-formula is instantiated by a term containing $r + 1$-eigenvariable. Second, even if there is only finite number of $r + 1$-cuts, the search through all instance $A(t_i)$ of $\exists x A(x)$ can be infinite. This agrees with the fact that finding the branch in $\mathcal{T}^{r+1}$ is not in general recursive (in a given branch of $\mathcal{T}^r$).

References

[1] G. Kreisel,G. Mints, S. Simpson, The Use of Abstract Language in Elementary Metamathematics. Lecture Notes in Mathematics, 253, 1975, p.38-131

[2] G. Kreisel on redundant variables in ∃ terms

[3] G. Mints, E-theorems, J. Soviet Math. v. 8,no. 3, 1977, p. 323-329

[4] G. Mints, The Universality of the Canonical Tree.Soviet Math. Dokl. 1976, 14, p.527-532

[5] G. Mints Unwinding a non-effective cut elimination proof, LNCS ,2006,p.