Uniform proofs of standard completeness for extensions of first-order MTL[☆]

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Abstract

We provide general –and automatedly verifiable– sufficient conditions that ensure standard completeness for logics formalized Hilbert-style. Our approach subsumes many existing results and allows for the discovery of new fuzzy logics which extend first-order Monoidal T-norm Logic with propositional axioms.

1. Introduction

This work is part of a project which aims to extract suitable proof systems from Hilbert calculi, and use them to prove interesting properties for large classes of logics in a uniform way. The property we consider in this paper is *standard completeness*, that is completeness of a logic with respect to algebras based on truth values in the real interval $[0, 1]$. Standard completeness has received increased attention in the last few years to formalize the engineering tool of Fuzzy Logic and as an interesting mathematical result in itself, e.g. [22, 16]. In a standard complete logic indeed connectives are interpreted by suitable functions on $[0, 1]$. For example conjunction and implication can be interpreted by a left-continuous t-norm[1] and its residuum. Gödel and Łukasiewicz logic are prominent examples of logics that are based on particular t-norms, while Hajek's Basic Logic [22] and Monoidal T-norm Logic **MTL** [19] are based on the whole class of continuous and left-continuous t-norms, respectively.

Establishing whether a logic is standard complete is often a challenging task which deserves a paper on its own, e.g., [21, 12, 24, 25]. It is traditionally proved by semantic techniques which are inherently logic-specific. Given a logic **L** described as a Hilbert system, a semantic proof usually consists of the following four steps (see, e.g., [21, 12, 29, 18, 23]):

1. The algebraic semantics of the logic is identified (L-algebras).
2. It is shown that if a formula is not valid in an L-algebra, then it is not valid in a countable L-chain (linearly ordered L-algebra).

[☆]Work supported by the Austrian Science Fund (FWF) START project Y544-N23.
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[1]T-norms are the main tool in fuzzy set theory to combine vague information.

3. The crucial step is to show that any countable L-chain can be embedded into a countable *dense* L-chain by adding countably many new elements to the algebra and extending the operations appropriately. This establishes *rational completeness*: a formula is derivable in **L** iff it is valid in all countable dense L-chains.
4. Finally, a countable dense L-chain is embedded into a standard L-algebra, that is an L-algebra with lattice reduct $[0, 1]$, using a Dedekind Mac-Neille style completion.

Rational completeness is the most difficult step to establish as it relies on finding the right embedding, if any. A different approach to prove standard completeness was proposed in [27] by using proof-theoretic techniques. The idea in [27] is that the admissibility of a particular syntactic rule (called *density*) in a logic **L** can lead to a proof of rational completeness for **L**. This is for instance the case when **L** is any axiomatic extension of **MTL**. Introduced by Takeuti and Titani in [32] the density rule formalized Hilbert-style has the following form

$$\frac{(\alpha \to p) \vee (p \to \beta) \vee \gamma}{(\alpha \to \beta) \vee \gamma}$$

where p is a propositional variable not occurring in the formulas α, β, or γ. Ignoring γ, this can be read contrapositively as saying (very roughly) "if $\alpha > \beta$, then $\alpha > p$ and $p > \beta$ for some p"; hence the name "density" and the intuitive connection with rational completeness.

This new approach was used in [27] to prove standard completeness for various logics, including some for which semantic techniques did not appear to work. Following this method, standard completeness for a logic **L** is proved by:

(a) first defining a suitable proof system *HL* for **L** extended with the density rule

(b) checking that this rule is eliminable (or admissible) in *HL*, i.e. that density does not enlarge the set of provable formulas

(c) standard completeness may then be obtained in many cases (but not in general) by means of the Dedekind Mac-Neille completion.

Hilbert systems do not help performing step (b) above (density-elimination), which needs instead cut-free Gentzen-style calculi.

In this paper we consider **MTL**$\forall$ [29] – the first-order version of **MTL** – as the basic system and investigate standard completeness of its axiomatic extensions. **MTL**$\forall$ is defined by adding to **MTL** suitable axioms and rules for the quantifiers $\forall$ and $\exists$, semantically interpreted as infimum and supremum. A convenient proof system for **MTL**$\forall$, in which the density rule is syntactically similar to the cut rule, uses hypersequents that are disjunctions of Gentzen sequents [15, 2, 28].

Density-elimination was shown in [5, 27] for various calculi, including the hypersequent calculus *HMTL* for **MTL**. These proofs are calculi-specific and use heavy combinatorial arguments, in close analogy with Gentzen-style proofs of cut-elimination. A different method to eliminate applications of the density rule from derivations was introduced in [14]. In this approach, similar to normalization for natural deduction systems, applications of the density rule are removed by making suitable substitutions for the newly introduced propositional variables. It is shown in [14] that each hypersequent calculus obtained by extending the first-order version of *HMTL* by certain *sequent rules* admits density-elimination. As shown in [8] sequent rules can only capture Hilbert axioms of a certain syntactic form (i.e., within the class $\mathcal{N}_2$ in the classification of [10]). Though more general than the proofs in [5, 27], the result in [14] does not apply to the axioms in the next level of the classification (that is, the class $\mathcal{P}_3$) in spite of the fact that the corresponding algebraic equations are preserved by Dedekind Mac-Neille completion when applied to subdirectly irreducible algebras [9]. Formalized by *hypersequent rules*, $\mathcal{P}_3$ axioms are actually used to define many interesting extensions of **MTL**; among them weak nilpotent minimum logic **WNM** [19, 10].

In this paper we provide general sufficient conditions for standard completeness that apply to many extensions of **MTL**$\forall$ with $\mathcal{P}_3$ axioms. Our conditions – formulated on the shape of hypersequent rules – can be checked on the base of the Hilbert axioms of the considered logics using the PROLOG-system *AxiomCalc* [6], available at *http://www.logic.at/people/lara/axiomcalc.html*. This automates the steps (a)-(c) above and applies to infinitely many logics.

The paper is organized as follows: given any axiomatic extensions of **MTL**$\forall$ with axioms within the class $\mathcal{P}_3$

- (Section 2) shows that the algorithm in [10] – extracting (hyper)sequent rules out of $\mathcal{P}_3$ axioms – works for logics extending **MTL**$\forall$;
- (Section 3) contains a general sufficient condition[2] on hypersequent rules that ensures density-elimination;
- (Section 4) closes the cycle showing how to use the results in [14, 9] to prove that the corresponding logics are standard complete.

Our approach subsumes many existing results on standard completeness for specific logics and allows for the computerized discovery of new standard complete logics (fuzzy logics, in the sense of [22]). For instance, it applies to first-order **WNM** (already known to be standard complete) and to many new axiomatic extensions of **MTL**$\forall$ (see Section 5 for some examples).

2. Analytic calculi for extensions of *MTL*$\forall$

In this section we show how to define analytic calculi for a large class of axiomatic extensions of first-order Monoidal T-norm Logic **MTL**$\forall$.
We start by recalling **MTL**$\forall$. Propositional **MTL** was introduced in [19] and

[2] An earlier version of this paper introducing (a more complicated version of) this condition for propositional logics appeared as [6].

shown in [21] to be complete with respect to the class of left-continuous t-norms and their residua[3]. A Hilbert system for **MTL** is obtained by adding the prelinearity axiom

$$(lin) \quad (\alpha \to \beta) \vee (\beta \to \alpha)$$

to the Full Lambek calculus with exchange and weakening FL_{ew} (i.e., intuitionistic logic without contraction [31]). Many interesting logics, including Gödel and Łukasiewicz logics, can be obtained by extending **MTL** with suitable axioms. For example, weak nilpotent minimum logic **WNM** arises by adding to **MTL** the axiom (wnm) (see Table 2).

The logic **MTL**$\forall$ is the first-order version of **MTL**. **MTL**$\forall$ is defined over a usual first-order language $\mathcal{L}$, consisting of variables, constants, functions and predicate symbols. Terms and atomic formulas are defined in the usual way. Formulas are built from atomic formulas and the constants 0 and 1, by using the connectives $\to$ (implication), $\wedge$ (additive conjunction), $\cdot$ (multiplicative conjunction), $\vee$ (disjunction) and the quantifiers $\forall$, $\exists$. We use $\neg\alpha$ as an abbreviation for $\alpha \to 0$. Following standard practice, we do not explicitly distinguish between formulas and metavariables for formulas and we use $\alpha, \beta, \ldots$ for both.

A Hilbert calculus for **MTL**$\forall$ extends that for **MTL**, with:

the *generalization rule* $\quad \dfrac{\alpha}{\forall x\alpha}\ (Gen)$

the usual axioms for quantifiers in intuitionistic logic (below t is a term in the language $\mathcal{L}$ and the notation β^x means that x does not occur free in β):

$$\begin{array}{llll} (\forall 1) & \forall x\alpha(x) \to \alpha(t) & (\exists 1) & \alpha(t) \to \exists x\alpha(x) \\ (\forall 2) & \forall x(\beta^x \to \alpha) \to (\beta^x \to \forall x\alpha) & (\exists 2) & \forall x(\alpha \to \beta^x) \to (\exists x\alpha \to \beta^x) \end{array}$$

and the shifting law of the universal quantifier over disjunction

$$(\forall 3) \quad \forall x(\beta^x \vee \alpha) \to (\beta^x \vee \forall x\alpha)$$

The above calculus for **MTL**$\forall$ does not help proving the redundancy of the density rule (below p is a propositional variable not occurring in α, β, or γ):

$$\frac{(\alpha \to p) \vee (p \to \beta) \vee \gamma}{(\alpha \to \beta) \vee \gamma}$$

that is the core of our approach to prove standard completeness. We need instead to work with *analytic calculi*, i.e. calculi in which derivations consist only of subformulas of the formulas to be proved. An analytic calculus for the logic **MTL**$\forall$ has been introduced in [15], and is based on Avron's *hypersequents*.

[3] A t-norm is a commutative, associative, increasing function $* : [0,1]^2 \to [0,1]$ with identity element 1. $*$ is *left continuous* iff whenever $\{x_n\}, \{y_n\}$ $(n \in N)$ are increasing sequences in $[0,1]$ s.t. their suprema are x and y, then $\sup\{x_n * y_n : n \in N\} = x * y$. The residuum of $*$ is a function $\to^*\colon [0,1]^2 \to [0,1]$ where $x \to^* y = \max\{z \mid x * z \leq y\}$. As shown in [19], it is equivalent for a t-norm to be left-continuous and to admit a residuum.

Definition 1. [3] A *hypersequent* is a finite multiset $S_1 \mid \ldots \mid S_n$ where each $S_i, i = 1 \ldots n$ is a sequent, called a *component* of the hypersequent.

The symbol "$\mid$" is intended to denote disjunction at the meta-level.

Notation. Hypersequents will be denoted by G, H and sequents (possibly built from metavariables) by S_i, C_i. Within a sequent $S := \Gamma \Rightarrow \Pi$ we will denote by $L(S)$ the multiset[4] Γ in its left hand side, and by $R(S)$ its right hand side Π. To distinguish between rule applications and rule schemas, we will denote finite (possibly empty) multisets of formulas with $\Gamma, \Delta, \Sigma, \Theta, \Lambda$ and metavariables for multisets of formulas with $\overline{\Gamma}, \overline{\Delta}, \overline{\Sigma}, \overline{\Theta}, \overline{\Lambda}$. Metavariables $\overline{\Pi}, \overline{\Psi}$ will stand for *stoups*, i.e. either a formula or the empty set.

Henceforth we will only consider *single conclusion* calculi: for any component S, $R(S)$ is either a formula or the empty set. As in sequent calculus, a hypersequent calculus consists of initial axioms, cut, logical and structural rules. Structural rules are divided into internal and external rules. Axioms, cut, logical and internal structural rules are as in sequent calculus; the only difference being the presence of a context G representing a (possibly empty) hypersequent. External structural rules, which permit the interaction between various components, increase the expressive power of the hypersequent calculus with respect to sequent calculus.

A *derivation* d of a hypersequent G from $G_1, \ldots, G_n$ is defined in the usual way, i.e. as a tree whose nodes are hypersequents, edges correspond to rule applications, G is the root, and leaves are $G_1, \ldots, G_n$ or axioms of the calculus. Henceforth we will denote such derivation in a hypersequent calculus HL as

$$d, G_1, \ldots, G_n \vdash_{HL} G$$

Given a sequent calculus, its hypersequent version is obtained (i) by adding to all its rules a context G, and (ii) by extending the calculus with the external structural rules of weakening and contraction ((ec) and (ew) in Table 1). Some care is needed for the quantifier rules ($\forall r$) and ($\exists l$), as discussed in Example 2. The hypersequent calculus $HMTL\forall$ for **MTL**$\forall$ is shown in Table 1. Notice that the eigenvariable condition for the rules ($\forall r$) and ($\exists l$) applies to the whole rule conclusion, i.e. the variable a must not occur in (the instantiation of) $G \mid \overline{\Gamma} \Rightarrow \forall x \alpha(x)$ and $G \mid \overline{\Gamma}, \exists x \alpha(x) \Rightarrow \overline{\Pi}$, respectively.

Defining an analytic calculus for an axiomatic extension of a logic is in general a difficult task. A systematic way to extract (hyper)sequent rules from suitable Hilbert axioms extending the full Lambek calculus with exchange FL_e was introduced in [10]. We recall below this result and lift it to first-order logics extending **MTL**$\forall$, thus constructing cut-free hypersequent calculi for a large class of axiomatic extensions of **MTL**$\forall$. A key concept for the method in [10] is the substructural hierarchy, a syntactic classification of Hilbert axioms in the language of FL_e. This classification accounts for the intuitive difficulty

[4] The use of multisets avoids to consider the exchange rule explicitly.

$$\frac{G \mid \overline{\Gamma} \Rightarrow \alpha \quad G \mid \alpha, \overline{\Delta} \Rightarrow \overline{\Pi}}{G \mid \overline{\Gamma}, \overline{\Delta} \Rightarrow \overline{\Pi}} \ (cut) \qquad \frac{}{G \mid \alpha \Rightarrow \alpha} \ (init) \qquad \frac{}{G \mid 0 \Rightarrow} \ (0l)$$

$$\frac{G \mid \overline{\Gamma} \Rightarrow \alpha \quad G \mid \overline{\Delta} \Rightarrow \beta}{G \mid \overline{\Gamma}, \overline{\Delta} \Rightarrow \alpha \cdot \beta} \ (\cdot\, r) \qquad \frac{G \mid \alpha, \beta, \overline{\Gamma} \Rightarrow \overline{\Pi}}{G \mid \alpha \cdot \beta, \overline{\Gamma} \Rightarrow \overline{\Pi}} \ (\cdot\, l) \qquad \frac{G \mid \overline{\Gamma} \Rightarrow \overline{\Pi}}{G \mid 1, \overline{\Gamma} \Rightarrow \overline{\Pi}} \ (1l)$$

$$\frac{G \mid \overline{\Gamma} \Rightarrow \alpha \quad G \mid \beta, \overline{\Delta} \Rightarrow \overline{\Pi}}{G \mid \overline{\Gamma}, \alpha \to \beta, \overline{\Delta} \Rightarrow \overline{\Pi}} \ (\to l) \qquad \frac{G \mid \alpha, \overline{\Gamma} \Rightarrow \beta}{G \mid \overline{\Gamma} \Rightarrow \alpha \to \beta} \ (\to r) \qquad \frac{G \mid \overline{\Gamma} \Rightarrow}{G \mid \overline{\Gamma} \Rightarrow 0} \ (0r)$$

$$\frac{G \mid \overline{\Gamma} \Rightarrow \alpha \quad G \mid \overline{\Gamma} \Rightarrow \beta}{G \mid \overline{\Gamma} \Rightarrow \alpha \wedge \beta} \ (\wedge r) \qquad \frac{G \mid \alpha_i, \overline{\Gamma} \Rightarrow \overline{\Pi}}{G \mid \alpha_1 \wedge \alpha_2, \overline{\Gamma} \Rightarrow \overline{\Pi}} \ (\wedge_i l)_{i=1,2} \qquad \frac{}{G \mid \ \Rightarrow 1} \ (1r)$$

$$\frac{G \mid \alpha, \overline{\Gamma} \Rightarrow \overline{\Pi} \quad G \mid \beta, \overline{\Gamma} \Rightarrow \overline{\Pi}}{G \mid \alpha \vee \beta, \overline{\Gamma} \Rightarrow \overline{\Pi}} \ (\vee l) \qquad \frac{G \mid \overline{\Gamma} \Rightarrow \alpha_i}{G \mid \overline{\Gamma} \Rightarrow \alpha_1 \vee \alpha_2} \ (\vee_i r)_{i=1,2} \qquad \frac{G \mid \overline{\Gamma} \Rightarrow \overline{\Pi}}{G \mid \overline{\Gamma}, \alpha \Rightarrow \overline{\Pi}} \ (wl)$$

$$\frac{G \mid \overline{\Gamma}, \alpha(t) \Rightarrow \overline{\Pi}}{G \mid \overline{\Gamma}, \forall x \alpha(x) \Rightarrow \overline{\Pi}} \ (\forall l) \qquad \frac{G \mid \overline{\Gamma} \Rightarrow \alpha(a)}{G \mid \overline{\Gamma} \Rightarrow \forall x \alpha(x)} \ (\forall r) \quad (a \ eigenvariable)$$

$$\frac{G \mid \overline{\Gamma} \Rightarrow \alpha(t)}{G \mid \overline{\Gamma} \Rightarrow \exists x \alpha(x)} \ (\exists r) \qquad \frac{G \mid \overline{\Gamma}, \alpha(a) \Rightarrow \overline{\Pi}}{G \mid \overline{\Gamma}, \exists x \alpha(x) \Rightarrow \overline{\Pi}} \ (\exists l) \quad (a \ eigenvariable)$$

$$\frac{G \mid \overline{\Gamma} \Rightarrow \overline{\Pi} \mid \overline{\Gamma} \Rightarrow \overline{\Pi}}{G \mid \overline{\Gamma} \Rightarrow \overline{\Pi}} \ (ec) \qquad \frac{G}{G \mid \overline{\Gamma} \Rightarrow \overline{\Pi}} \ (ew) \qquad \frac{G \mid \overline{\Gamma} \Rightarrow}{G \mid \overline{\Gamma} \Rightarrow \overline{\Pi}} \ (wr)$$

$$\frac{G \mid \overline{\Gamma}_1, \overline{\Delta}_1 \Rightarrow \overline{\Pi}_1 \quad G \mid \overline{\Gamma}_2, \overline{\Delta}_2 \Rightarrow \overline{\Pi}_2}{G \mid \overline{\Gamma}_1, \overline{\Gamma}_2 \Rightarrow \overline{\Pi}_1 \mid \overline{\Delta}_1, \overline{\Delta}_2 \Rightarrow \overline{\Pi}_2} \ (com)$$

Table 1: Hypersequent calculus $HMTL\forall$ for $MTL\forall$

to deal with these axioms proof theoretically and at the same time for the corresponding algebraic equations to be preserved under suitable order-theoretic completions over residuated lattices [9]. The substructural hierarchy consists of classes $(\mathcal{N}_n, \mathcal{P}_n)$, with $n \geq 0$; an axiom belongs to a class $\mathcal{N}$ (resp. $\mathcal{P}$) if its most external connective has negative polarity, i.e. its right introduction rule is invertible (resp. positive polarity and left invertible rules). In particular the connectives $\to$ and $\wedge$ of FL_e are negative, while $\vee$ and $\cdot$ are positive. The general grammar for determining the classes $\mathcal{N}_n, \mathcal{P}_n$ has the following structure:

$$\begin{aligned}
&\mathcal{P}_0 ::= \mathcal{N}_0 ::= \text{the set of atomic formulas} \\
&\mathcal{P}_{n+1} ::= \mathcal{N}_n \mid \mathcal{P}_{n+1} \cdot \mathcal{P}_{n+1} \mid \mathcal{P}_{n+1} \vee \mathcal{P}_{n+1} \mid 1 \\
&\mathcal{N}_{n+1} ::= \mathcal{P}_n \mid \mathcal{P}_{n+1} \to \mathcal{N}_{n+1} \mid \mathcal{N}_{n+1} \wedge \mathcal{N}_{n+1} \mid 0
\end{aligned}$$

A graphical representation of the classification is depicted in Figure 1. The arrows $\to$ there stand for inclusions $\subseteq$ of the classes.

The hierarchy provides the basis for exploring the connection between axioms and structural rules preserving cut-elimination when added to a calculus for FL_e. Indeed, [10] contains an algorithm that transforms:

$$\begin{tikzcd}
\mathcal{P}_0 \arrow[r] \arrow[d, leftrightarrow] \arrow[dr] & \mathcal{P}_1 \arrow[r] \arrow[dr] & \mathcal{P}_2 \arrow[r] \arrow[dr] & \mathcal{P}_3 \arrow[r] \arrow[dr] & \mathcal{P}_4 \arrow[r] \arrow[dr] & \cdots \\
\mathcal{N}_0 \arrow[r] \arrow[ur] & \mathcal{N}_1 \arrow[r] \arrow[ur] & \mathcal{N}_2 \arrow[r] \arrow[ur] & \mathcal{N}_3 \arrow[r] \arrow[ur] & \mathcal{N}_4 \arrow[r] \arrow[ur] & \cdots
\end{tikzcd}$$

Figure 1: Substructural hierarchy $(\mathcal{N}_n, \mathcal{P}_n)$ [10]

- Axioms within the class $\mathcal{N}_2$ into equivalent sequent structural rules
- Axioms within the class $\mathcal{P}_3$ into equivalent hypersequent structural rules.

By equivalence between axioms $\alpha_1, \ldots, \alpha_n$ and rules $(r_1), \ldots, (r_m)$ we mean that

$$\beta_1, \ldots, \beta_k \vdash_{FL_e + \{\alpha_1, \ldots, \alpha_n\}} \alpha \text{ iff } \Rightarrow \beta_1, \ldots, \Rightarrow \beta_k \vdash_{FL_e^* + \{(r_1), \ldots, (r_m)\}} \Rightarrow \alpha,$$

where FL_e^* stands for either the sequent or the hypersequent calculus for FL_e. (See the Appendix A for an explicit formulation of the axioms within the classes $\mathcal{N}_2$ and $\mathcal{P}_3$). The rules generated by the algorithm in [10] preserve cut-elimination when added to the (hyper)sequent calculus for FL_e; in general, however, the subformula property (and hence analiticity) is ensured only in presence of the weakening rules ((wl) and (wr) in Table 1), i.e. taking FL_{ew} as the basic system. In this case all the rules generated by the algorithm are *completed*.

Definition 2. Let (R) be a hypersequent structural rule:

$$\frac{G \,|\, S_1 \quad \ldots \quad G \,|\, S_n}{G \,|\, C_1 \,|\, \ldots \,|\, C_q}$$

(R) is said to be *completed* if it satisfies the following conditions:

- *Strong subformula property* : Each metavariable occurring in $L(S_i)$ (respectively in $R(S_i)$), with $i = \{1, \ldots, n\}$, occurs also in $L(C_j)$ (respectively in $R(C_j)$), for some j in $\{1, \ldots, q\}$.
- *Linear conclusion* : Each metavariable occurs at most once in $G|C_1| \ldots |C_q$.
- *Coupling* : Let $R(C_j) = \overline{\Pi}$ with $\overline{\Pi}$ non empty, j in $\{1, \ldots, q\}$. There is a metavariable $\overline{\Sigma}$ in $L(C_j)$ such that, whenever $\overline{\Pi}$ belongs to $R(S_i)$, for a given i in $\{1, \ldots, n\}$, $\overline{\Sigma}$ belongs to $L(S_i)$ as well.

Example 1. The prelinearity axiom (lin) is in the class $\mathcal{P}_2$. The algorithm in [10] constructs the equivalent completed rule (com) (see Table 1); this rule was first introduced in [2] to define a hypersequent calculus for Gödel logic G (i.e. **MTL** with contraction (c), see Table 2).

The algorithm in [10] constructs an analytic calculus for any propositional extension of **MTL** with additional axioms within the class $\mathcal{P}_3$. We show below that this also holds taking **MTL**$\forall$ as base calculus. This is not a trivial observation as the correspondence axioms-structural rules does not hold anymore once we consider first-order logics. Indeed, in the presence of quantifiers, the rules generated by the algorithm might be not anymore equivalent to the starting axioms, as shown by the following example.

Example 2. Let $FL_{ew}\forall$ be the first-order version of the logic FL_{ew}. Consider now $FL_{ew}\forall$ extended with the axiom (lin). The algorithm in [10] extract, out of this axiom, the (com) rule (cf. Example 1). As shown below, (com) is not anymore equivalent to (lin) when considering $FL_{ew}\forall$ as base calculus. Indeed, first notice that the quantifier rules of the hypersequent version of the calculus for $FL_{ew}\forall$ have to be as in Table 1, that is the eigenvariable condition in $(\forall r)$ and $(\exists l)$ has to apply to the whole rule conclusion, as otherwise using (com) we can easily derive $\exists x\alpha(x) \Rightarrow \forall x\alpha(x)$ for each formula α.
However, the addition of (com) to the hypersequent version of $FL_{ew}\forall$ allows the following derivation of axiom $(\forall 3)$, which is *not* a theorem of $FL_{ew}\forall + (lin)$:

$$
\begin{prooftree}
\AxiomC{$\alpha(a) \Rightarrow \alpha(a)$}
\AxiomC{$\alpha(a) \Rightarrow \alpha(a)$}
\AxiomC{$\beta \Rightarrow \beta$}
\RightLabel{(com)}
\BinaryInfC{$\beta \Rightarrow \alpha(a) \mid \alpha(a) \Rightarrow \beta$}
\AxiomC{$\beta \Rightarrow \beta$}
\RightLabel{$2\times(\vee l)+(ew)'s$}
\TrinaryInfC{$\alpha(a) \vee \beta \Rightarrow \alpha(a) \mid \alpha(a) \vee \beta \Rightarrow \beta$}
\RightLabel{$2\times(\forall l)$}
\UnaryInfC{$\forall x(\alpha(x) \vee \beta) \Rightarrow \alpha(a) \mid \forall x(\alpha(x) \vee \beta) \Rightarrow \beta$}
\RightLabel{$(\forall r)$}
\UnaryInfC{$\forall x(\alpha(x) \vee \beta) \Rightarrow \forall x\alpha(x) \mid \forall x(\alpha(x) \vee \beta) \Rightarrow \beta$}
\RightLabel{$2\times(\vee r)$}
\UnaryInfC{$\forall x(\alpha(x) \vee \beta) \Rightarrow \forall x\alpha(x) \vee \beta \mid \forall x(\alpha(x) \vee \beta) \Rightarrow \forall x\alpha(x) \vee \beta$}
\RightLabel{(ec)}
\UnaryInfC{$\forall x(\alpha(x) \vee \beta) \Rightarrow \forall x\alpha(x) \vee \beta$}
\RightLabel{$(\rightarrow r)$}
\UnaryInfC{$\Rightarrow \forall x(\alpha(x) \vee \beta) \rightarrow (\forall x\alpha(x) \vee \beta)$}
\end{prooftree}
$$

Consider the usual formula-interpretation I of a hypersequent $H = \Gamma_1 \Rightarrow \Pi_1 \mid \ldots \mid \Gamma_n \Rightarrow \Pi_n$ (see, e.g., [2, 28, 15, 10]), namely

- $I(\Gamma_1 \Rightarrow \Pi_1 \mid \ldots \mid \Gamma_n \Rightarrow \Pi_n) = I(\Gamma_1 \Rightarrow \Pi_1) \vee \cdots \vee I(\Gamma_n \Rightarrow \Pi_n)$

where the interpretation of a sequent $\Gamma \Rightarrow \Pi$ is:

- $I(\Gamma \Rightarrow \Pi) = \odot\Gamma \rightarrow \beta$, if Π is a formula β
- $I(\Gamma \Rightarrow) = \odot\Gamma \rightarrow 0$, otherwise

$\odot\Gamma$ stands for the multiplicative conjunction $\cdot$ of all the formulas in Γ, and is 1 when Γ is empty.
Let **L**$\forall$ be any logic extending **MTL**$\forall$ with axioms in $\mathcal{P}_3$. Henceforth we denote with $HL\forall$ the hypersequent calculus obtained by adding to $HMTL\forall$ the corresponding completed rules generated by the algorithm in [10].

Theorem 1 (Soundness and completeness). *Let* $\mathbf{L}\forall$ *be any logic extending* $\mathbf{MTL}\forall$ *with axioms in* $\mathcal{P}_3$*. A hypersequent* H *is derivable in* $HL\forall$ *if and only if* $I(H)$ *is derivable in* $\mathbf{L}\forall$*.*

Proof. ($\Rightarrow$) Proceeds by induction on the length of the derivation of H in $HL\forall$. For logical and structural rules we refer to [10]. Among the quantifier rules, the only non-trivial case is $(\forall, r)$. This case is handled by using axiom ($\forall3$), which belongs to $\mathbf{L}\forall$ (as it extends $\mathbf{MTL}\forall$). Indeed, assume that $I(G) \vee I(\Gamma \Rightarrow \alpha(a))$ is derivable in $\mathbf{L}\forall$. By the generalization rule (Gen), $\forall x(I(G) \vee I(\Gamma \Rightarrow \alpha(x)))$ is also derivable in $\mathbf{L}\forall$. Recall that, for the eigenvariable condition, a must not occur in $I(G)$. Then we may assume that x does not occur there either. Hence, using axiom ($\forall3$) we obtain that $I(G) \vee \forall x I(\Gamma \Rightarrow \alpha(x))$ is derivable. The result follows using the fact that $\forall x I(\Gamma \Rightarrow \alpha(x)) \to I(\Gamma \Rightarrow \forall x \alpha(x))$ is derivable in $\mathbf{L}\forall$ (from axiom ($\forall2$)) .
($\Leftarrow$) It is enough to show that all axioms and rules in $\mathbf{L}\forall$ are derivable in $HL\forall$. This is immediate for ($\forall1$), ($\forall2$), ($\exists1$), ($\exists2$) and (Gen). Modus ponens is simulated by (cut). For the propositional axioms of $\mathbf{L}\forall$ extending $\mathbf{MTL}\forall$ we refer to [10]. A derivation of ($\forall3$) proceeds as in Example 2 (recall that $HL\forall$ is an extension of $HMTL\forall$).

Let $d(t)$ and $G(t)$ denote the results of substituting the term t for all free occurrences of a in the derivation $d(a)$ and hypersequent $G(a)$, respectively.

Lemma 1. *If* $d(a), G_1(a), \ldots, G_n(a) \vdash_{HL\forall} G(a)$ *and* t *is a term whose variables are all free and do not occur in* $d(a)$*, then* $d(t), G_1(t), \ldots, G_n(t) \vdash_{HL\forall} G(t)$*.*

The substitution lemma above and the properties of completed rules are the keys for proving the following theorem.

Theorem 2. *Let* $\mathbf{L}\forall$ *be any logic extending* $\mathbf{MTL}\forall$ *with axioms in* $\mathcal{P}_3$*. The hypersequent calculus* $HL\forall$ *admits cut-elimination.*

Proof. Cut-admissibility for (the hypersequent version of) FL_e extended by any set of completed rule was proved in [10]. The algebraic proof there was reformulated syntactically in [11] (cut-elimination) and extended to calculi having more than one formula on the right hand side of their sequents. It is easy to see that the addition of the quantifier rules in Table 1 does not harm this proof, as we can use the substitution lemma to rename variables, when needed.

Recalling that axioms within class $\mathcal{P}_3$ are transformed into equivalent *completed* rules by the algorithm in [10], we get the following.

Corollary 1. *Any logic extending* $\mathbf{MTL}\forall$ *with axioms within* $\mathcal{P}_3$ *has an analytic hypersequent calculus.*

The above corollary can be shown in a constructive way and indeed the analytic calculi for these extensions of $\mathbf{MTL}\forall$ can be automatedly constructed by using the PROLOG-system *AxiomCalc* [6].

Class	Axiom	Rule (cf. Table 3)	**MTL**$\forall$+
$\mathcal{N}_2$	$\alpha \to \alpha \cdot \alpha$	*(c)*	Gödel logic $G\forall$
	$\alpha^{n-1} \to \alpha^n$	*(c_n)*	$C_nMTL\forall$ [12]
$\mathcal{P}_2$	$\alpha \vee \neg\alpha$	*(em)*	Classical Logic
$\mathcal{P}_3$	$\neg\alpha \vee \neg\neg\alpha$	*(lq)*	$SMTL\forall$ [18]
	$\alpha \vee (\alpha \to \beta) \vee (\alpha \wedge \beta \to \gamma)$	*(bc2)*	3-valued $G\forall$ (with (c))
	$\neg(\alpha \cdot \beta) \vee (\alpha \wedge \beta \to \alpha \cdot \beta)$	(wnm)	$WNM\forall$[19]
	$\neg(\alpha \cdot \beta)^n \vee ((\alpha \wedge \beta)^{n-1} \to (\alpha \cdot \beta)^n)$	*(wnm^n)*	investigated in [6]
	$\neg(\alpha^n) \vee (\alpha^{n-1} \to \alpha^n)$	$(wnm1^n)$	(new!)
	$\bigvee_{i<k}(\neg\alpha_i \to \neg\alpha_{i+1})$	*(inv_k)*	investigated in [30]

Table 2: Some axioms and their corresponding logics

In what follows, the notation α^k ($k \geq 0$), stands for $\alpha \cdot \dots \cdot \alpha$, k times, for a formula α, while X^k stands for $X, \dots, X$, k times, where X is either a multiset Γ or a metavariable $\overline{\Gamma}$.

Example 3. The axiom $(wnm^3) := \neg(\alpha \cdot \beta)^3 \vee ((\alpha \wedge \beta)^2 \to (\alpha \cdot \beta)^3)$ is in the class $\mathcal{P}_3$. The equivalent rule generated by the algorithm in [10] is

$$\dfrac{\begin{array}{c}\{G \mid \overline{\Gamma}_1^3, \overline{\Gamma}_i^3, \overline{\Delta} \Rightarrow \overline{\Pi}\}_{1\leq i\leq 8} \qquad \{G \mid \overline{\Gamma}_2^3, \overline{\Gamma}_i^3, \overline{\Delta} \Rightarrow \overline{\Pi}\}_{1\leq i\leq 8} \\ \{G \mid \overline{\Gamma}_i^3, \overline{\Gamma}_{i+1}^3, \overline{\Delta} \Rightarrow \overline{\Pi}\}_{3\leq i\leq 7} \quad \{G \mid \overline{\Gamma}_i^3, \overline{\Gamma}_{i+3}^3, \overline{\Delta} \Rightarrow \overline{\Pi}\}_{3\leq i\leq 5} \quad G \mid \overline{\Gamma}_3^3, \overline{\Gamma}_8^3, \overline{\Delta} \Rightarrow \overline{\Pi}\end{array}}{G \mid \overline{\Gamma}_3, \overline{\Gamma}_4, \overline{\Gamma}_5, \overline{\Gamma}_6, \overline{\Gamma}_7, \overline{\Gamma}_8 \Rightarrow \mid \overline{\Gamma}_1, \overline{\Gamma}_2, \overline{\Delta} \Rightarrow \overline{\Pi}}\ (wnm^3)$$

See Tables 2 and 3 for more examples.

2.1. The density rule

The density rule was introduced by Takeuti and Titani in their axiomatization of first-order Gödel logic [32], called there Intuitionistic Fuzzy Logic. In hypersequent calculi density is a *structural rule* of the form:

$$\dfrac{G \mid \overline{\Sigma}, p \Rightarrow \overline{\Pi} \mid \overline{\Lambda} \Rightarrow p}{G \mid \overline{\Sigma}, \overline{\Lambda} \Rightarrow \overline{\Pi}}\ (D)$$

where p is a propositional variable not occurring in (any instance of) $\overline{\Sigma}, \overline{\Lambda}, \overline{\Pi}$ or G (p is an *eigenvariable*). Notice that adding the density rule to a hypersequent calculus can have a dramatic effect. For instance, the addition of (D) to the calculus $HMTL\forall$ + *(em)* (cf. Table 3) permits to derive the empty sequent. A similar situation arises for $HMTL\forall$ + *(bc2)* (cf. Table 3), where the empty sequent can be derived as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{q \Rightarrow q \qquad \dfrac{p \Rightarrow p \quad q \Rightarrow q}{q \Rightarrow p \mid p \Rightarrow q}\,(com) \qquad p \Rightarrow p}{\Rightarrow p \mid p \Rightarrow q \mid p \Rightarrow q \mid q \Rightarrow}\,(bc2)}{\Rightarrow p \mid p \Rightarrow q \mid q \Rightarrow}\,(ec)}{\Rightarrow p \mid p \Rightarrow}\,(D)}{\Rightarrow}\,(D)$$

$$\frac{G\,|\,\overline{\Gamma},\overline{\Gamma},\overline{\Delta}\Rightarrow\overline{\Pi}}{G\,|\,\overline{\Gamma},\overline{\Delta}\Rightarrow\overline{\Pi}}\;(c)\qquad \frac{G\,|\,\overline{\Gamma},\overline{\Delta}\Rightarrow\overline{\Pi}}{G\,|\,\overline{\Gamma}\Rightarrow\;|\,\overline{\Delta}\Rightarrow\overline{\Pi}}\;(em)\qquad \frac{G\,|\,\overline{\Gamma}_1,\overline{\Gamma}_2\Rightarrow}{G\,|\,\overline{\Gamma}_1\Rightarrow\;|\,\overline{\Gamma}_2\Rightarrow}\;(lq)$$

$$\frac{G\,|\,\overline{\Gamma}_1,\overset{n}{\ldots},\overline{\Gamma}_1,\overline{\Delta}\Rightarrow\overline{\Pi}\quad\ldots\quad G\,|\,\overline{\Gamma}_{n-1},\overset{n}{\ldots},\overline{\Gamma}_{n-1},\overline{\Delta}\Rightarrow\overline{\Pi}}{G\,|\,\overline{\Gamma}_1,\ldots,\overline{\Gamma}_{n-1},\overline{\Delta}\Rightarrow\overline{\Pi}}\;(c_n)$$

$$\frac{G\,|\,\overline{\Gamma}_1,\overline{\Delta}_2\Rightarrow\overline{\Pi}_2\quad G\,|\,\overline{\Gamma}_1,\overline{\Delta}_3\Rightarrow\overline{\Pi}_3\quad G\,|\,\overline{\Gamma}_2,\overline{\Delta}_3\Rightarrow\overline{\Pi}_3}{G\,|\,\overline{\Delta}_3\Rightarrow\overline{\Pi}_3\,|\,\overline{\Gamma}_2,\overline{\Delta}_2\Rightarrow\overline{\Pi}_2\,|\,\overline{\Gamma}_1,\overline{\Delta}_1\Rightarrow\overline{\Pi}_1}\;(bc2)$$

$$\frac{G\,|\,\overline{\Gamma}_1,\overline{\Gamma}_1,\overline{\Delta}_1\Rightarrow\overline{\Pi}_1\quad G\,|\,\overline{\Gamma}_2,\overline{\Gamma}_1,\overline{\Delta}_1\Rightarrow\overline{\Pi}_1\quad G\,|\,\overline{\Gamma}_2,\overline{\Gamma}_3,\overline{\Delta}_1\Rightarrow\overline{\Pi}_1\quad G\,|\,\overline{\Gamma}_1,\overline{\Gamma}_3,\overline{\Delta}_1\Rightarrow\overline{\Pi}_1}{G\,|\,\overline{\Gamma}_2,\overline{\Gamma}_3\Rightarrow\;|\,\overline{\Gamma}_1,\overline{\Delta}_1\Rightarrow\overline{\Pi}_1}\;(wnm)$$

$$\frac{\{G\,|\,\overline{\Gamma}^n_i,\overline{\Gamma}^n_j,\overline{\Delta}\Rightarrow\overline{\Pi}\}_{1\leq i\leq(n-1),1\leq j\leq(3n-1)}\quad\{G\,|\,\overline{\Gamma}^n_i,\overline{\Gamma}^n_{i+(2p-1)},\overline{\Delta}\Rightarrow\overline{\Pi}\}_{1\leq p\leq n,n\leq i\leq(3n-2p)}}{G\,|\,\overline{\Gamma}_n,\cdots,\overline{\Gamma}_{3n-1}\Rightarrow\;|\,\overline{\Gamma}_1,\cdots,\overline{\Gamma}_{n-1},\overline{\Delta}\Rightarrow\overline{\Pi}}\;(wnm^n)$$

$$\frac{\{G\,|\,\overline{\Gamma}_{2i},\overline{\Gamma}_{2i+1}\Rightarrow\;\}_{1\leq i\leq(k-2)}}{G\,|\,\overline{\Gamma}_1,\overline{\Gamma}_2\Rightarrow\;|\ldots|\,\overline{\Gamma}_{2k-3},\overline{\Gamma}_{2k-2}\Rightarrow}\;(inv_k)$$

Table 3: Some completed rules

There is nothing surprising since, as shown in [27], the addition and subsequent elimination of (D) from any extension of $HMTL$ leads to rational completeness for the formalized logic, and the two calculi above formalize logics that are not rational complete: classical and 3-valued Gödel logic (see Table 2).

As shown in the next section, for many extensions of $HMTL\forall$, adding (D) has no effect on which hypersequents are derivable: applications of (D) can be *eliminated* from derivations.

3. Convergent rules and Density Elimination

We identify a (large) class of hypersequent rules – called *convergent* – that allow for density elimination.

Notation: Let $S := X, Y \Rightarrow p$ be a sequent. Henceforth we indicate by $S[{}^{\Lambda}/_{X}]^l[{}^{\Sigma\Rightarrow\Psi}/_{p}]^r$ (where X is either a metavariable for multisets of formulas or a propositional variable) the sequent $\Lambda, Y, \Sigma \Rightarrow \Psi$, obtained by replacing each occurrence of X (resp. p) on the left (right) with a Λ on the left (a Σ on the left and a Ψ on the right).

Let V be a set of metavariables and $\sigma : V \to V$. We will indicate with $S[\{{}^{\sigma(\overline{\Gamma})}/_{\overline{\Gamma}}\}_{\overline{\Gamma}\in V}]^l$ the sequent obtained by substituting in the left hand side of S each occurrence of a metavariable $\overline{\Gamma}$ in V with a metavariable $\sigma(\overline{\Gamma})$.

Definition 3. A *p-axiom* is a hypersequent of the form $G|\Theta, p^k \Rightarrow p$, for any G, Θ, and $k > 0$.

Notice that any p-axiom is derivable from the axiom $p \Rightarrow p$ using weakenings.

Our proof of density elimination uses and refines the method in [14] of *density elimination by substitutions*, which is first presented below in an informal way. Let d be a cut-free subderivation ending in the following uppermost application of density

$$\dfrac{\begin{array}{c}\vdots\, d' \\ G' \mid \Sigma, p \Rightarrow \Pi \mid \Lambda \Rightarrow p\end{array}}{G' \mid \Sigma, \Lambda \Rightarrow \Pi}\,(\mathrm{D})$$

(D) is removed by substituting the occurrences of p in d in an "asymmetric" way, according to whether p occurs in the left or in the right hand side of a sequent. More precisely, each component S of any hypersequent in d is replaced by $S[^{\Lambda}/_{p}]^{l}[^{\Sigma\Rightarrow\Pi}/_{p}]^{r}$. This way, the application of (D) above is simply replaced by an application of (ec).

A problem

Note however that the labeled tree that results by applying the "asymmetric" substitution to d (we denote it by d^*) is in general not a correct derivation anymore. This is due to the possible presence in d of *p-axioms* $G|\Theta, p^k \Rightarrow p$ that become in d^* hypersequents $G|\Theta, \Lambda^k, \Sigma \Rightarrow \Pi$ and are no longer derivable.

To solve the problem and obtain a correct density-free derivation we need to remove all applications of the old *p-axioms*. This is done by suitably restructuring the tree d^*. Looking at the original (cut-free) derivation d bottom-up it is clear that p-axioms can originate only from applications of external structural rules that "mix" the content of various components from the conclusion. We describe below how to restructure d^* in presence of critical applications of (com) (see Table 1) and of a class of rules which we call *convergent* (Definition 4 below).

Communication rule ([14]): Density-elimination was proved in [14] for calculi containing only (com) as external structural rule "mixing" the content of components. The *only* problematic case to handle was when in d one of the premises of (com) led to a p-axiom as, e.g., in the following case

$$\dfrac{\begin{array}{c}\vdots \\ G \mid \Gamma_1, \Gamma_2 \Rightarrow \Psi\end{array} \quad \begin{array}{c}\vdots \\ p \Rightarrow p\end{array}}{G \mid \Gamma_1 \Rightarrow p \mid \Gamma_2, p \Rightarrow \Psi}\,(\mathrm{com})$$

The restructuring of d^* was handled there by *removing* this application of (com) and replacing it with a (sub)derivation starting from the premise $G \mid \Gamma_1, \Gamma_2 \Rightarrow \Psi$ and containing suitable applications of (cut).

Convergent rules: We show below that many external structural rules (called *convergent*), though mixing components, allow for a suitable restructuring of d^*. Indeed convergent rules have the property that, whenever a premise S is a p-axiom, the remaining premises can be used to derive its substituted version $S[^{\Lambda}/_{p}]^{l}[^{\Sigma\Rightarrow\Pi}/_{p}]^{r}$. We illustrate the idea behind convergent rules and the way we restructure the derivation d^* in Theorem 3 first with an example.

Example 4. Consider the rule (wnm) of Table 3

$$\frac{G\,|\,\overline{\Gamma}_1,\overline{\Gamma}_1,\overline{\Delta}_1 \Rightarrow \overline{\Pi}_1 \quad G\,|\,\overline{\Gamma}_2,\overline{\Gamma}_1,\overline{\Delta}_1 \Rightarrow \overline{\Pi}_1 \quad G\,|\,\overline{\Gamma}_2,\overline{\Gamma}_3,\overline{\Delta}_1 \Rightarrow \overline{\Pi}_1 \quad G\,|\,\overline{\Gamma}_1,\overline{\Gamma}_3,\overline{\Delta}_1 \Rightarrow \overline{\Pi}_1}{G\,|\,\overline{\Gamma}_2,\overline{\Gamma}_3 \Rightarrow \;|\,\overline{\Gamma}_1,\overline{\Delta}_1 \Rightarrow \overline{\Pi}_1}$$

Notice that the metavariables in the premise $\overline{\Gamma}_1,\overline{\Gamma}_1,\overline{\Delta}_1 \Rightarrow \overline{\Pi}_1$ are all included in one component of the conclusion, i.e. $\overline{\Gamma}_1,\overline{\Delta}_1 \Rightarrow \overline{\Pi}_1$. Hence this premise is a p-axiom in the original derivation d only if the conclusion component $\overline{\Gamma}_1,\overline{\Delta}_1 \Rightarrow \overline{\Pi}_1$ is already a p-axiom. Therefore we can assume that each application of (wnm) in d always contains at least one premise that is not a p-axiom. This premise will play a crucial role in the transformation of d^* into a correct derivation. As a particular simple case, assume that d contains the following application of (wnm):

$$\dfrac{\begin{array}{c}\vdots\, d_1\\ \Gamma_1,\Gamma_1 \Rightarrow p\end{array} \quad \dfrac{p \Rightarrow p}{p,\Gamma_1 \Rightarrow p}\,(wl) \quad \dfrac{p \Rightarrow p}{p,p \Rightarrow p}\,(wl) \quad \dfrac{p \Rightarrow p}{p,\Gamma_1 \Rightarrow p}\,(wl)}{p,p \Rightarrow \;|\,\Gamma_1 \Rightarrow p}\,(wnm)$$

We want to derive $p,p \Rightarrow \;|\,\Gamma_1 \Rightarrow p\ [{}^{\Lambda}/_{p}]^l[{}^{\Sigma\Rightarrow\Pi}/_{p}]^r$, i.e. $\Lambda,\Lambda \Rightarrow \;|\,\Gamma_1,\Sigma \Rightarrow \Pi$. The idea is to use the non p-axiom premise $\Gamma_1,\Gamma_1 \Rightarrow p$ to derive the substituted version of the other premises, i.e. $p,\Gamma_1 \Rightarrow p\ [{}^{\Lambda}/_{p}]^l[{}^{\Sigma\Rightarrow\Pi}/_{p}]^r$ and $p,p \Rightarrow p\ [{}^{\Lambda}/_{p}]^l[{}^{\Sigma\Rightarrow\Pi}/_{p}]^r$. This way the incorrect (sub)derivation in d^*

$$\dfrac{\begin{array}{c}\vdots\, d_1^*\\ \Gamma_1,\Gamma_1,\Sigma \Rightarrow \Pi\end{array} \quad \dfrac{\begin{array}{c}\vdots\, ??\\ \Lambda,\Sigma \Rightarrow \Pi\end{array}}{\Lambda,\Gamma_1,\Sigma \Rightarrow \Pi}\,(wl) \quad \dfrac{\begin{array}{c}\vdots\, ??\\ \Lambda,\Sigma \Rightarrow \Pi\end{array}}{\Lambda,\Lambda,\Sigma \Rightarrow \Pi}\,(wl) \quad \dfrac{\begin{array}{c}\vdots\, ??\\ \Lambda,\Sigma \Rightarrow \Pi\end{array}}{\Lambda,\Gamma_1,\Sigma \Rightarrow \Pi}\,(wl)}{\Lambda,\Lambda \Rightarrow \;|\,\Gamma_1,\Sigma \Rightarrow \Pi}\,(wnm)$$

(d_1^* is obtained by applying the asymmetric substitution $[{}^{\Lambda}/_{p}]^l[{}^{\Sigma\Rightarrow\Pi}/_{p}]^r$ to all the sequents in d_1) is replaced by

$$\dfrac{\begin{array}{c}\vdots\, d_1^*\\ \Gamma_1,\Gamma_1,\Sigma \Rightarrow \Pi\end{array} \quad \begin{array}{c}\vdots\, d_1^*\\ \Gamma_1,\Gamma_1,\Sigma \Rightarrow \Pi\\ \vdots\, (*)\\ \Lambda,\Gamma_1,\Sigma \Rightarrow \Pi\end{array} \quad \begin{array}{c}\vdots\, d_1^*\\ \Gamma_1,\Gamma_1,\Sigma \Rightarrow \Pi\\ \vdots\, (**)\\ \Lambda,\Lambda,\Sigma \Rightarrow \Pi\end{array} \quad \begin{array}{c}\vdots\, d_1^*\\ \Gamma_1,\Gamma_1,\Sigma \Rightarrow \Pi\\ \vdots\, (*)\\ \Lambda,\Gamma_1,\Sigma \Rightarrow \Pi\end{array}}{\Lambda,\Lambda \Rightarrow \;|\,\Gamma_1,\Sigma \Rightarrow \Pi}\,(wnm)$$

where $(*)$ and $(**)$ contain suitable applications of (cut) using Lemma 2(ii) below.

Note that, in the general case, the premise $\overline{\Gamma}_1,\overline{\Gamma}_1,\overline{\Delta}_1 \Rightarrow \overline{\Pi}_1$ of (wnm) might not suffice to derive the substituted version of all the p-axiom premises. However, the special relation among the premises (see Definition 4 and Example 5 below) assures that we can always find other premises of (wnm) which are not p-axioms and that can do the work.

We are now ready to introduce the notion of convergent rules.

Definition 4. Let (r) be a completed hypersequent structural rule:

$$\frac{G \,|\, S_1 \quad \dots \quad G \,|\, S_n}{G \,|\, C_1 \,|\, \dots \,|\, C_q}\ (r)$$

and V be the set of different metavariables appearing in $L(S_1) \cup \dots \cup L(S_n)$.

- A premise $G|S_i$ of (r) is said to be a *pivot*-premise if there is a component C_j of the conclusion such that $R(S_i) = R(C_j)$ and the (set of) metavariables in $L(S_i)$ are all contained in $L(C_j)$.
- The rule (r) is said to be *convergent* if, for any premise $G|S_i$ of (r), either $R(S_i) = \emptyset$ or there is a map $\sigma : V \to V$ such that:
  (i) $G|S_i[\{^{\sigma(\overline{\Gamma})}/_{\overline{\Gamma}}\}_{\overline{\Gamma} \in V}]^l$ is a premise of (r) which is a pivot.
  (ii) For any $W \subset V$, the hypersequent $G|S_i[\{^{\sigma(\overline{\Gamma})}/_{\overline{\Gamma}}\}_{\overline{\Gamma} \in W}]^l$ is a premise of (r).

  Notice that both conditions (i) and (ii) are trivially satisfied if $G|S_i$ is a pivot premise itself, by letting σ be the identity function.

Example 5. We show that the rule (wnm) (see Example 4)

$$\frac{G \,|\, \overline{\Gamma}_1, \overline{\Gamma}_1, \overline{\Delta}_1 \Rightarrow \overline{\Pi}_1 \quad G \,|\, \overline{\Gamma}_2, \overline{\Gamma}_1, \overline{\Delta}_1 \Rightarrow \overline{\Pi}_1 \quad G \,|\, \overline{\Gamma}_1, \overline{\Gamma}_3, \overline{\Delta}_1 \Rightarrow \overline{\Pi}_1 \quad G \,|\, \overline{\Gamma}_2, \overline{\Gamma}_3, \overline{\Delta}_1 \Rightarrow \overline{\Pi}_1}{G \,|\, \overline{\Gamma}_2, \overline{\Gamma}_3 \Rightarrow \,|\, \overline{\Gamma}_1, \overline{\Delta}_1 \Rightarrow \overline{\Pi}_1}$$

is convergent. Let $V = \{\overline{\Gamma}_1, \overline{\Gamma}_2, \overline{\Gamma}_3, \overline{\Delta}_1\}$ be the set of all metavariables appearing on the left hand side of its premises. For each premise of (wnm), we verify that conditions (i) and (ii) of Definition 4 are satisfied.

- $G|\overline{\Gamma}_1, \overline{\Gamma}_1, \overline{\Delta}_1 \Rightarrow \overline{\Pi}_1$ is a pivot premise. Conditions (i) and (ii) are satisfied, letting σ be the identity function.
- $G|\overline{\Gamma}_2, \overline{\Gamma}_1, \overline{\Delta}_1 \Rightarrow \overline{\Pi}_1$. Take the map σ that acts as the identity on V except for $\sigma(\overline{\Gamma}_2) = \overline{\Gamma}_1$. Condition (i) is satisfied, as $G|\overline{\Gamma}_2, \overline{\Gamma}_1, \overline{\Delta}_1 \Rightarrow \overline{\Pi}_1[\{^{\sigma(\overline{\Gamma})}/_{\overline{\Gamma}}\}_{\overline{\Gamma} \in V}]^l$ is the pivot premise $G|\overline{\Gamma}_1, \overline{\Gamma}_1, \overline{\Delta}_1 \Rightarrow \overline{\Pi}_1$. Condition (ii) is satisfied as well, as for any $W \subset V$, the hypersequent $G|\overline{\Gamma}_2, \overline{\Gamma}_1, \overline{\Delta}_1 \Rightarrow \overline{\Pi}_1[\{^{\sigma(\overline{\Gamma})}/_{\overline{\Gamma}}\}_{\overline{\Gamma} \in W}]^l$ would be either $G|\overline{\Gamma}_2, \overline{\Gamma}_1, \overline{\Delta}_1 \Rightarrow \overline{\Pi}_1$ itself or $G|\overline{\Gamma}_1, \overline{\Gamma}_1, \overline{\Delta}_1 \Rightarrow \overline{\Pi}_1$.
- $G|\overline{\Gamma}_3, \overline{\Gamma}_1, \overline{\Delta}_1 \Rightarrow \overline{\Pi}_1$. Similar to the previous case, taking σ to be the identity function, except for $\sigma(\overline{\Gamma}_3) = \overline{\Gamma}_1$.
- $G|\overline{\Gamma}_2, \overline{\Gamma}_3, \overline{\Delta}_1 \Rightarrow \overline{\Pi}_1$. Take σ to be the identity, except for $\sigma(\overline{\Gamma}_2) = \overline{\Gamma}_1$ and $\sigma(\overline{\Gamma}_3) = \overline{\Gamma}_1$. Condition (i) is satisfied, as $G|\overline{\Gamma}_2, \overline{\Gamma}_3, \overline{\Delta}_1 \Rightarrow \overline{\Pi}_1[\{^{\sigma(\overline{\Gamma})}/_{\overline{\Gamma}}\}_{\overline{\Gamma} \in V}]^l$ is the pivot premise $G|\overline{\Gamma}_1, \overline{\Gamma}_1, \overline{\Delta}_1 \Rightarrow \overline{\Pi}_1$. For condition (ii), let $W = V \backslash \{\overline{\Gamma}_3\}$ or $W = V \backslash \{\overline{\Gamma}_2\}$. We obtain that $G|\overline{\Gamma}_2, \overline{\Gamma}_3, \overline{\Delta}_1 \Rightarrow \overline{\Pi}_1[\{^{\sigma(\overline{\Gamma})}/_{\overline{\Gamma}}\}_{\overline{\Gamma} \in W}]^l$ corresponds to the premises $G \,|\, \overline{\Gamma}_3, \overline{\Gamma}_1, \overline{\Delta}_1 \Rightarrow \overline{\Pi}_1$ and $G \,|\, \overline{\Gamma}_2, \overline{\Gamma}_1, \overline{\Delta}_1 \Rightarrow \overline{\Pi}_1$, respectively. For other choices of $W \subset V$, the verification of the condition is straightforward.

Example 6. All internal structural rules are convergent and so are the rules *(wnm)*, *(lq)*, *(inv$_k$)* and *(wnm^n)* in Table 3.

Completed rules that are not convergent are *(em)* and *(bc2)*.

Remark 1. *The program* AxiomCalc *[6] verifies whether a rule is convergent. Notice that the program actually works on the basis of the definition of convergent rules given in [6]. Though syntactically different, this definition turns out to be equivalent to the one presented here (Definition 4).*

The technical lemma below, which allows us to suitably "move" multisets of formulas between components, is the key for our proof of density elimination. In what follows, the *length* $|d|$ *of a derivation* d is, as usual, the (maximal number of inference rules) + 1, occurring on any branch.

Lemma 2. *Let $HL\forall$ be any calculus extending $HMTL\forall$ with convergent rules.*

(i) Any derivation d of a hypersequent H can be transformed into a derivation of $H[^{\alpha}/_{p}]^{l}[^{\Rightarrow\alpha}/_{p}]^{r}$, for any formula α and propositional variable p.

(ii) Let d' and d_1 be respectively derivations of $G'|\Sigma, p \Rightarrow \Pi|\Lambda \Rightarrow p$ ($p \notin G', \Sigma, \Pi, \Lambda$) and $G|\Theta, \Delta \Rightarrow \Psi$. Then there exists a derivation of $G|G'|\Theta, \Lambda \Rightarrow \Psi|\Sigma, \Delta \Rightarrow \Pi$.

Proof.

(i) Replace p in d everywhere with α. The claim is proved by induction on the length of the resulting derivation, as convergent rules are completed (and hence substitutive, cf. the definition and the analogous lemma in [14]), possibly using Lemma 1 to rename variables.

(ii) By (i) and d' we have a derivation d_2 of $G'|\Sigma, \odot\Delta \Rightarrow \Pi|\Lambda \Rightarrow \odot\Delta$ where $\odot\Delta$ stands for the multiplicative conjunction $\cdot$ of the formulas in Δ (note that $p \notin G', \Sigma, \Pi, \Lambda$). The desired derivation follows by applying (cut) between $G|G'|\Theta, \Lambda \Rightarrow \Psi|\Delta \Rightarrow \odot\Delta$ and the end hypersequent of

$$\dfrac{\dfrac{\dfrac{\vdots\, d_2}{G'|\Sigma, \odot\Delta \Rightarrow \Pi|\Lambda \Rightarrow \odot\Delta}}{G|G'|\Sigma, \odot\Delta \Rightarrow \Pi|\Lambda \Rightarrow \odot\Delta}\,(ew) \qquad \dfrac{\dfrac{\vdots\, d_1}{G|\Theta, \Delta \Rightarrow \Psi}}{G|G'|\Sigma, \odot\Delta \Rightarrow \Pi|\Theta, \odot\Delta \Rightarrow \Psi}\,(\cdot l) + (ew)}{G|G'|\Theta, \Lambda \Rightarrow \Psi|\Sigma, \odot\Delta \Rightarrow \Pi}\,(cut)$$

We are now ready to present the main theorem of this section, namely density elimination. Henceforth, we denote by S_i^* the sequent $S_i[^{\Lambda}/_{p}]^{l}[^{\Sigma\Rightarrow\Pi}/_{p}]^{r}$, and by G^*,H^*, the hypersequents G,H, where the same substitution is applied to each one of their components. A (D)-free derivation is a derivation not containing the (D) rule.

Theorem 3 (Density Elimination). *Any calculus $HL\forall$ extending $HMTL\forall$ with convergent rules admits density elimination.*

Proof. It is enough to consider topmost applications of (D). Take the derivation d in $HL\forall + (D)$, ending in an application of (D)

$$\dfrac{\begin{array}{c}\vdots\, d' \\ G' \,|\, \Sigma, p \Rightarrow \Pi \,|\, \Lambda \Rightarrow p\end{array}}{G' \,|\, \Sigma, \Lambda \Rightarrow \Pi}\ (D)$$

where d' is a (D)-free derivation.

Claim: *For each hypersequent H in d' that is not a p-axiom, one can find a (D)-free derivation of $G'|H^*$.*

The result on density elimination follows from this claim. Just let H be $G'|\Lambda \Rightarrow p|\Sigma, p \Rightarrow \Pi$. We get that $G' \,|\, G' \,|\, \Lambda, \Sigma \Rightarrow \Pi \,|\, \Lambda, \Sigma \Rightarrow \Pi$ is derivable (note that $(G')^* = G'$ by the eigenvariable condition on p). The desired (D)-free proof of $G'|\Lambda, \Sigma \Rightarrow \Pi$ follows by multiple applications of (ec).

The proof of the claim proceeds by induction on the length of the cut-free subderivation d_H of H in $HL\forall$. We distinguish cases according to the last rule (r) applied in d_H. All cases but those involving convergent rules are handled as in the analogous proof in [14]. To make the proof self-contained we discuss here also these cases. The cases $|d_H| = 1$, or when (r) is (ec) or (ew) are easy. When (r) is a logical, a quantifier or an internal structural rule, if its conclusion is not a p-axiom then so is(are) its premise(s). Hence the claim follows, by using the induction hypothesis and a subsequent application of the rule (r), using Lemma 1 to rename variables, when needed.

For (com), we show how to handle the case mentioned in the introduction of the section, i.e. when d' contains

$$\dfrac{\begin{array}{c}\vdots\, d_1 \\ G \,|\, \Gamma_1, \Gamma_2 \Rightarrow \Psi\end{array} \quad \begin{array}{c}\vdots \\ G \,|\, p \Rightarrow p\end{array}}{G \,|\, \Gamma_1, p \Rightarrow \Psi \,|\, \Gamma_2 \Rightarrow p}\ (com)$$

By the inductive hypothesis we get a derivation $d_1^* \vdash G' \,|\, G^* \,|\, \Gamma_1, \Gamma_2 \Rightarrow \Psi$. The desired (D)-free derivation of

$$G' \,|\, G^* \,|\, \Gamma_1, \Lambda \Rightarrow \Psi \,|\, \Gamma_2, \Sigma \Rightarrow \Pi$$

follows by Lemma 2 (ii) (applied to d_1^* and $d' \vdash G' \,|\, \Sigma, p \Rightarrow \Pi \,|\, \Lambda \Rightarrow p$). See the proof in [14] for all possible cases involving (com).

Convergent rules: Assume that (r) is a convergent rule of the form

$$\frac{G \,|\, S_1 \quad \ldots \quad G \,|\, S_n}{G \,|\, C_1 \,|\, \ldots \,|\, C_q}\ (r)$$

and that its conclusion is not a p-axiom. We show how to find a (D)-free derivation of

$$G'|G^*|C_1^*|\ldots|C_q^*$$

Take a premise $G|S_i$ of (r). If $G|S_i$ is not a p-axiom, the inductive hypothesis gives us a derivation of $G'|G^*|S_i^*$. Note that this is always the case when $R(S_i) = \emptyset$, and when $G|S_i$ is a pivot, as in the latter case the metavariables instantiated to obtain S_i are all included in one component of the conclusion (see Definition 4). Thus, if $G|S_i$ was a p-axiom, the conclusion would be a p-axiom as well, contradicting the assumption. Assume now that $G|S_i$ is a p-axiom (and it is not a pivot). We show below that we can always obtain a (D)-free derivation of

$$G' \,|\, G^* \,|\, S_i^* \,|\, C_s^*$$

for some $s \in \{1, \dots, q\}$. Let $G|S_i$ be an instantiation of a premise of the kind $G|\overline{\Gamma}_1, \dots, \overline{\Gamma}_m \Rightarrow \overline{\Pi}_1$. Being $G|S_i$ a p-axiom, $\overline{\Pi}_1$ is instantiated with p and at least one of the instantiations of $\overline{\Gamma}_1, \dots, \overline{\Gamma}_m$ contains some p's. W.l.o.g., let $\overline{\Gamma}_1, \dots \overline{\Gamma}_r$, with $r \leq m$ be the metavariables in $L(S_i)$ whose instantiations contain at least one p. Being (r) a convergent rule, by Definition 4(i) there exists a map σ acting on the set V of metavariables in $L(S_1) \cup \ldots \cup L(S_m)$, such that $G|S_i[\{{}^{\sigma(\overline{\Gamma})}/_{\overline{\Gamma}}\}_{\overline{\Gamma} \in V}]^l$ is a pivot premise for (r), that we call $G|S_{piv}$. Assume that $G|S_{piv}$ is an instantiation of a premise $\overline{\Delta}_1, \dots, \overline{\Delta}_m \Rightarrow \overline{\Pi}_1$ and that $\sigma(\overline{\Gamma}_i) = \overline{\Delta}_i$ for any $i \in \{1, \dots, m\}$. Notice that, being $G \,|\, S_{piv}$ a pivot, none of the instances of the metavariables $\overline{\Delta}_1, \dots \overline{\Delta}_m$ contains a p, as otherwise $G \,|\, S_{piv}$ would be a p-axiom. Consider now the restriction of σ to the set of metavariables $W = \{\overline{\Gamma}_1, \dots, \overline{\Gamma}_r\} \subset V$. Definition 4(ii) ensures that $G \,|\, S_i[\{{}^{\sigma(\overline{\Gamma}_i)}/_{\overline{\Gamma}_i}\}_{\overline{\Gamma}_i \in W}]^l$ is a premise of (r), say $G \,|\, S_k$. Notice that, in case $r = m$ (i.e. all metavariables $\overline{\Gamma}_1, \dots \overline{\Gamma}_m$ in $G \,|\, S_i$ were instantiated with at least one p), $G \,|\, S_k$ would simply coincide with the pivot premise $G|S_{piv}$. In the most general case, $G \,|\, S_k$ is an instantiation of a premise of the kind $G \,|\, \overline{\Delta}_1, \dots, \overline{\Delta}_r, \overline{\Gamma}_{r+1}, \dots, \overline{\Gamma}_m \Rightarrow \overline{\Pi}_1$. From our assumptions, the instances Δ_i of $\overline{\Delta}_i$, for $i \in \{1, \dots, r\}$ and the instances Γ_j of $\overline{\Gamma}_j$, for $j \in \{r+1, \dots m\}$ do not contain any p, while $\overline{\Pi}_1$ is instantiated with p. Hence $G \,|\, S_k$ is not a p-axiom and by the i.h. we can get a (D)-free derivation of $G' \,|\, G^* \,|\, S_k^* = G' \,|\, G^* \,|\, \Delta_1, \dots, \Delta_r, \Gamma_{r+1}, \dots \Gamma_m, \Sigma \Rightarrow \Pi$. By r iterative applications of Lemma 2(ii), starting from $G' \,|\, G^* \,|\, S_k^*$ and the derivation d' of $G' \,|\, \Sigma, p \Rightarrow \Pi \,|\, \Lambda \Rightarrow p$, we eventually obtain a derivation of

$$(*)\; G' \,|\, G^* \,|\, \Lambda^r, \Gamma_{r+1}, \dots \Gamma_m, \Sigma \Rightarrow \Pi \,|\, \Delta_1, \Sigma \Rightarrow \Pi \,|\, \ldots \,|\, \Delta_r, \Sigma \Rightarrow \Pi$$

Notice that all the metavariables $\overline{\Delta}_1, \dots \overline{\Delta}_r$ belong to the pivot premise $G|S_{piv}$. Hence, by Definition 4 these metavariables also belong to one component of the conclusion, say C_s. Thus, by repeated applications of (wl) to all sequents in $(*)$ of the form $\Delta_i, \Sigma \Rightarrow \Pi$, followed by (ec), we obtain

$$(**)\; G' \,|\, G^* \,|\, \Lambda^r, \Gamma_{r+1}, \dots \Gamma_m, \Sigma \Rightarrow \Pi \,|\, C_s^*$$

The desired hypersequent $G' \,|\, G^* \,|\, S_i^* \,|\, C_s^*$ is obtained by final applications of (wl) to the component $\Lambda^r, \Gamma_{r+1}, \dots \Gamma_m, \Sigma \Rightarrow \Pi$ in $(**)$. Notice that S_i contains at least r times p, as we assumed that the instantiations of the metavariables $\overline{\Gamma}_1, \dots \overline{\Gamma}_r$ contained at least one p. Hence S_i^* can be obtained by (wl) from $\Lambda^r, \Gamma_{r+1}, \dots, \Gamma_m, \Sigma \Rightarrow \Pi$, as S_i^* should contain at least r times Λ.

By summarizing, when the last rule (r) in d_H is convergent, for each premise $G \mid S_i$ we have:

- If $G \mid S_i$ is not a p-axiom, $G' \mid G^* \mid S_i^*$ is (D)-free derivable.
- If $G \mid S_i$ is a p-axiom, then $G' \mid G^* \mid S_i^* \mid C_s^*$ is (D)-free derivable.

The required derivation of $G' \mid G^* \mid C_1^* \mid \ldots \mid C_q^*$ follows by (ew), (r) and possibly subsequent applications of (ec). This completes the proof of the main claim.

4. From Density Elimination to Standard Completeness

Let us take stock of what we have achieved so far. Let $\mathbf{L}\forall$ be a logic extending $\mathbf{MTL}\forall$ with axioms within the class $\mathcal{P}_3$ in the substructural hierarchy of [10] (see Section 2 and the Appendix):

- We introduced a hypersequent calculus $HL\forall$ for $\mathbf{L}\forall$, by extending $HMTL\forall$ with structural rules equivalent to the additional axioms (Section 2).
- We showed that, if these rules are convergent, density elimination holds for the calculus $HL\forall$ (Section 3).

In this section we combine these results with those in [13, 9] to prove that $\mathbf{L}\forall$ is standard complete. The two missing steps towards this proof amount to showing (in fact adapting from [13, 9]) that:

- If a hypersequent calculus $HL\forall$ admits density elimination, then the logic $\mathbf{L}\forall$ is *rational complete*, i.e. $\mathbf{L}\forall$ is complete with respect to countable dense algebras (Section 4.1).
- Any countable dense algebra can be regularly embedded into a *standard algebra*, i.e. an algebra whose lattice reduct is the real unit interval $[0, 1]$ (Section 4.2).

4.1. Density Elimination ⇒ Rational Completeness

We briefly recall the usual structures that provide algebraic semantics for our logics (see, e.g. [4, 23, 19] for more details).

Definition 5. An *FL_{ew}-algebra* is a structure $\mathcal{A} = (A, \wedge, \vee, \cdot, \rightarrow, 0, 1)$ where $(A, \wedge, \vee, 0, 1)$ is a bounded lattice, $(A, \cdot, 1)$ is a commutative monoid and for each $x, y, z \in A$ the *residuation* property holds, i.e. $x \cdot z \leq y \Leftrightarrow z \leq x \rightarrow y$.
An *MTL-algebra* is an FL_{ew}-algebra, satisfying the *prelinearity* equation, i.e., for every $x, y \in A$, $1 = x \rightarrow y \vee y \rightarrow x$.

FL_{ew}-algebras can be defined equationally, hence they form algebraic *varieties*, see, e.g., [7, 20]. An FL_{ew}-algebra $\mathcal{A} = (A, \wedge, \vee, \cdot, \rightarrow, 0, 1)$ is

- a *chain* if, for every $x, y \in A$, either $x \leq y$ or $y \leq x$ holds.

- a *dense algebra* if, for every $x, y \in A$, whenever $x \not\leq y$, there is a $z \in A$ such that $x \not\leq z$ and $z \not\leq y$.
- a *complete algebra* if, for every $X \subseteq A$, $inf X \in A$ and $sup X \in A$, where inf and sup are related to the lattice ordering $\leq$.

Notice that, with a slight abuse of notation, we used the same symbols to denote algebraic operations and logical connectives. Similarly, with α we will denote both a quantifier-free axiom and the corresponding algebraic term, where each connective symbol stands for the corresponding algebraic operation.

Definition 6. A *convergent* axiom is an axiom whose transformation by the algorithm from [10] gives a convergent rule.

Henceforth **L**$\forall$ indicates any logic extending **MTL**$\forall$ with any set of convergent axioms.

Definition 7. An *L-algebra* is an MTL-algebra satisfying the algebraic equation $1 = \alpha$, for each convergent axiom α in **L**$\forall$.

We will call an L-algebra that is a chain, an *L-chain.* The notion of interpretation for **L**$\forall$ is as usual and it is sketched below (see, e.g. [13, 4] for more details). An *$\mathcal{A}$-valuation* on a (non-empty) domain D is defined as a function v, with parameters in D, from closed formulas (sentences) to an L-algebra $\mathcal{A}$. v is extended from atomic to complex formulas, in the usual truth-functional way, interpreting connectives with corresponding algebraic operations in $\mathcal{A}$. In particular, quantifiers $\forall$ and $\exists$ are interpreted as inf and sup, with respect to the lattice ordering in $\mathcal{A}$. We restrict here to *safe* valuations, i.e. $\mathcal{A}$-valuations v, such that $v(\alpha)$ is defined for every sentence α.
Let $T \cup \{\alpha\}$ be a set of sentences. We say that α is a *semantical consequence* of T, and write $T \models_{\mathcal{A}} \alpha$, if $1 = v(\beta)$ for each $\beta \in T$ implies that $1 = v(\alpha)$, for any safe $\mathcal{A}$-valuation v, with $\mathcal{A}$ an L-algebra. The logic **L**$\forall$ is sound with respect to valuations on L-algebras. More precisely, the following holds, see e.g. [4].

Theorem 4. *Let $T \cup \{\alpha\}$ be a set of sentences. $T \vdash_{L\forall} \alpha$ implies $T \models_{\mathcal{A}} \alpha$, for any L-algebra $\mathcal{A}$.*

The converse direction (*completeness*) also holds, see e.g. [4]. However, as shown in [13], density elimination allows us to prove something stronger: the *rational completeness* of **L**$\forall$, i.e. the completeness of **L**$\forall$ with respect to valuations on dense countable L-chains.

Theorem 5 (Rational Completeness). *Let $T \cup \{\alpha\}$ be a set of sentences and* **L**$\forall$ *any logic extending $MTL\forall$ with convergent axioms. Then $T \vdash_{L\forall} \alpha$ iff $T \models_{\mathcal{A}} \alpha$ for $\mathcal{A}$ any countable dense L-chain.*

Proof. By Theorem 3, the hypersequent calculus $HL\forall$ for **L**$\forall$ admits density-elimination. The claim follows by Theorem 4 in [13].

4.2. Rational Completeness ⇒ Standard Completeness

The last step is achieved by means of a *Dedekind Mac-Neille completion* (DM-completion for short). This is a well-known construction generalizing to various ordered algebraic structures, Dedekind's embedding of the rational numbers into the extended real field (i.e. $\mathbb{R}$ with $\pm\infty$).

Definition 8 (Dedekind Mac-Neille completion). (e.g. [20, 27])
Let $\mathcal{A} = (A, \wedge, \vee, \cdot, \rightarrow, 1, 0)$ be an FL_{ew}-algebra and $X \subseteq A$. The sets of upper and lower bounds of X are defined as follows:
$X^u = \{y \in A \text{ s.t. } x \leq y \text{ for all } x \in X\}$ $\quad X^l = \{y \in A \text{ s.t. } y \leq x \text{ for all } x \in X\}$
Moreover, let $DM(A) = \{X \subseteq A : (X^u)^l = X\}$ and

- $X \vee_{DM} Y = ((X \cup Y)^u)^l \qquad X \wedge_{DM} Y = X \cap Y$
- $X \cdot_{DM} Y = (\{x \cdot y : x \in X, y \in Y\}^u)^l$
- $X \rightarrow_{DM} Y = \{z \in A | z \cdot x \in Y \text{ for all } x \in X\}$
- $1_{DM} = A \quad 0_{DM} = \{0\}$

The structure $\mathcal{A}^+ = (DM(A), \wedge_{DM}, \vee_{DM}, \rightarrow_{DM}, \cdot_{DM}, 1_{DM}, 0_{DM})$ is a complete FL_{ew}-algebra, called the *Dedekind Mac-Neille completion* of $\mathcal{A}$.

The following is a well known fact about DM-completions, e.g. [27, 20, 7].

Lemma 3. *Let $\mathcal{A}$ be an FL_{ew}-algebra and $\mathcal{A}^+$ its DM-completion. The map $e : A \rightarrow DM(A)$, associating to any $x \in A$ the set $\{x\}^l$ in $DM(A)$, is a* regular embedding*, i.e. an injective map, preserving all operations and all existing inf and sup in $\mathcal{A}$.*

We recall now (a slightly modified form of) a result in [9], which will be crucial for proving standard completeness. First notice that the syntactic classification of Hilbert axioms in [10] (i.e. the substructural hierarchy, see Section 2) applies to algebraic equations as well [9, 8]. In particular, an equation $\alpha = 1$ belongs to the same class of the hierarchy as the corresponding formula α.

Lemma 4. *Let $\mathcal{A}$ be an FL_{ew}-algebra satisfying a given set of equations within the class $\mathcal{P}_3$ in the substructural hierarchy. Then its DM-completion $\mathcal{A}^+$ satisfies the same equations as $\mathcal{A}$, provided that the following holds:*
$(*)$ *Whenever $\mathcal{A}$ satisfies $1 = \beta \vee \gamma$, then $\mathcal{A}$ satisfies either $1 = \beta$ or $1 = \gamma$.*

Proof. The claim follows by Theorem 4.3 in [9], where it is shown that equations within the class $\mathcal{P}_3$ are preserved under DM-completion for FL_{ew}-algebras that are *subdirectly irreducible* (see, e.g., [7]). An easy inspection of the proofs in [9] reveals that the restriction to subdirectly irreducible algebras is needed only in Lemma 3.1 of [9], to prove $(*)$.

We then obtain the following.

Theorem 6. *Dense L-chains are preserved under DM-completion.*

Proof. The fact that dense FL_{ew}-chains are preserved under DM-completion is shown e.g. in Theorem 27 of [27]. Recall that the equations defining L-algebras are within the class $\mathcal{P}_3$. The claim follows by Lemma 4, as for chains either $\beta \vee \gamma = \beta$ or $\beta \vee \gamma = \gamma$ holds.

Theorem 3 and Theorem 6 ensure the existence of a *regular* embedding from a dense L-chain to a complete dense L-chain (its DM-completion). This, together with rational completeness (Theorem 5), leads to standard completeness for the first-order logics we consider, using standard arguments (see e.g. [17, 27, 13, 23]).

Theorem 7 (Standard Completeness). *Let $T \cup \{\alpha\}$ be a set of sentences and* $\mathbf{L}\forall$ *any logic extending* $\mathbf{MTL}\forall$ *with convergent axioms. Then $T \vdash_{L\forall} \alpha$ iff $T \models_{\mathcal{A}} \alpha$ for $\mathcal{A}$ any L-algebra with lattice reduct $[0,1]$ (*standard *L-algebra).*

5. Conclusion

We have shown that every logic extending $\mathbf{MTL}\forall$ with (any set of) convergent axioms is standard complete. Our approach subsumes existing results and allows for the discovery of new fuzzy logics. As an example consider the $\mathcal{P}_3$ axioms ($n > 1$)

$$(wnm^n):\ \neg(\alpha \cdot \beta)^n \vee ((\alpha \wedge \beta)^{n-1} \to (\alpha \cdot \beta)^n).$$

These axioms, that generalize the peculiar axiom (wnm) of weak nilpotent minimum logic **WNM**, were first introduced in [6]. The corresponding rules are convergent (see Table 3 and Example 6) and hence the (infinitely many) logics $\mathbf{MTL}\forall + (wnm^n)$ are standard complete. The $\mathcal{P}_3$ axiom

$$(wnm1):\ \neg(\alpha^2) \vee (\alpha \to \alpha^2)$$

which is the single-variable variant of (wnm) (see [1]) is also convergent (easy check using the program *AxiomCalc*). Hence the logic $\mathbf{MTL}\forall + (wnm1)$ is standard complete. The same holds for the single-variable variant of (wnm^n), i.e. :

$$(wnm1^n):\ \neg(\alpha^n) \vee (\alpha^{n-1} \to \alpha^n)$$

Hence $\mathbf{MTL}\forall + (wnm1^n)$ is standard complete. Notice that different n's in $(wnm1^n)$ determine different logics. This follows by the facts that for any n, (1) as shown in [12] $\mathbf{MTL}\forall + (c_n)$ and $\mathbf{MTL}\forall + (c_{n+1})$ are distinct (see Table 3) and (2) $\mathbf{MTL}\forall + (wnm1^n)$ is intermediate between $\mathbf{MTL}\forall + (c_{n+1})$ and $\mathbf{MTL}\forall + (c_n)$.

Remark 2. *Despite of its generality, convergency is not a necessary condition for standard completeness. E.g. as shown in [26], the logic $\Omega(S_nMTL)$, obtained by extending* **MTL** *with both the n-contraction axiom (c_n) (see Table 2) and $(\alpha^{n-1} \to \beta) \vee (\beta \to \beta \cdot \alpha)$, is standard complete. The latter axiom is in $\mathcal{P}_3$, but is not convergent. (this can be easily checked by using the system* AxiomCalc*).*

Acknowledgement

We would like to thank Kazushige Terui and the anonymous reviewers for their helpful comments and for suggesting an important simplification in the definition of convergent rules.

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Appendix A.

The normal form of axioms within the class $\mathcal{N}_2$ is the following:

$\mathcal{N}_2$: Axioms have the form $\bigwedge_{1\leq i\leq n}\delta_i$, in which every δ_i is a $\alpha_1\cdots\alpha_m \to \beta$ where:
- $\beta = 0$ or $\beta_1 \vee \cdots \vee \beta_k$ and each β_l is a multiplicative conjunction of propositional variables and
- each α_i is of the form $\bigwedge_{1\leq j\leq p}\gamma_i^j \to \beta_i^j$ where
  - ∘ $\beta_i^j = 0$ or a propositional variable, and
  - ∘ γ_i^j is a multiplicative conjunction or a disjunction of propositional variables (or 1).

The normal form of axioms within the classes $\mathcal{P}_{n+1}$ is the following:

$\mathcal{P}_{n+1}$: Axioms have the form $\bigvee\bigodot\mathcal{N}_n$ i.e. any axiom in $\mathcal{P}_{n+1}$ is a disjunction of multiplicative conjunctions of axioms in $\mathcal{N}_n$.