A Calculus for Rational Łukasiewicz Logic and Related Systems

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Abstract. We introduce hypersequent calculi for Rational Łukasiewicz logic and for the logic $KZ(\pi)$, an extension of Kleene-Zadeh logic, motivated by game semantic investigations.

1 Introduction

Fuzzy Logic is nowadays a vast research area, which offers many different methods and tools to handle vagueness for computational purposes. In particular, in the area of so-called Fuzzy Logic in narrow sense or Mathematical Fuzzy Logic [4] many axiomatic systems have been so far introduced and investigated, in order to characterize valid inferences under vagueness. One of the most prominent such system is *Łukasiewicz logic* Ł. This logic is an important example of a t-norm based logic, together with Gödel and Product logic, see e.g. [4, 9]. In its intended or *standard* semantics, truth values are taken over the real interval $[0, 1]$ and the (strong) conjunction and implication connectives are interpreted by the well known Łukasiewicz t-norm $x * y = \max(0, x + y - 1)$ and its residuum $x \to y = min(1, 1 - x + y)$, respectively.

In this paper we focus on *Rational Łukasiewicz logic* RŁ, an expansion of Ł with a family of unary connectives $\{\delta_n\}_{n\in\mathbb{N}}$, standing for *division* operators. In other words, the intended evaluation v over the real interval $[0, 1]$ of a formula $\delta_n\alpha$ is defined by $v(\delta_n\alpha) = v(\alpha)/n$ where / stands for the usual division.

The name of the logic hints at the fact that constants corresponding to all the rational numbers in $[0, 1]$ are definable in RŁ. Not surprisingly, therefore, RŁ has been shown in [3] to be a conservative extension of the so-called Rational Pavelka logic [9]. The logic has also a nice functional representation, in analogy to the famous McNaughton theorem for Łukasiewicz [11]: formulas in RŁ correspond to continuous piecewise linear functions with rational coefficients over $[0, 1]$, see [2, 3].

RŁ has been systematically investigated in [3], where a Hilbert system and a corresponding algebraic semantics DMV (divisible MV algebras) have been introduced.

In this work we present a hypersequent calculus HRŁ for RŁ, which extends the calculus for Łukasiewicz logic introduced in [12]. In Section 2 we introduce the calculus and show its soundness and completeness with respect to the standard semantics. In Section 3 we then move to consider the logic $KZ(\pi)$, an expansion of Kleene-Zadeh logic (KZ in the following), introduced in [7] by considerations of game-semantic nature. A hypersequent calculus for $KZ(\pi)$ is easily obtained from suitable restrictions on HRŁ. Indeed, the logic $KZ(\pi)$ can be seen as a proper fragment of RŁ. In Section 4 we conclude by pointing to future work. In particular, we suggest that our calculi may provide a useful framework for a proof-theoretic investigation of fuzzy logics extended with so-called fuzzy quantifiers [8], such as *many, few, about half*, etc.

2 The hypersequent calculus *HR*Ł

In this section we introduce a calculus for the logic RŁ, i.e. the expansion of Łukasiewicz logic with division operators. Recall that in a language for propositional Łukasiewicz logic only the constant (or 0-ary connective) $\bot$ and the connective $\to$ are needed, other connectives $\neg, \cdot, \oplus, \wedge, \vee, \top$ being definable in terms of $\to, \bot$. In the following, we call *atomic formula* a propositional variable or the constant $\bot$. As usual, formulas are built recursively from atomic formulas. Any evaluation v on the standard semantics assigns truth values in $[0,1]$ to propositional variables, the value 0 to $\bot$, and is extended truth functionally by letting $v(\alpha \to \beta) = \min(1, 1 - v(\alpha) + v(\beta))$. For the remaining connectives, evaluations v are determined as follows:

$$\begin{array}{ll} v(\top) = 1, & v(\neg\alpha) = 1 - v(\alpha) \\ v(\alpha \cdot \beta) = \min(0, v(\alpha) + v(\beta) - 1), & v(\alpha \oplus \beta) = \min(1, v(\alpha) + v(\beta)) \\ v(\alpha \wedge \beta) = \min(v(\alpha), v(\beta)), & v(\alpha \vee \beta) = \max(v(\alpha), v(\beta)) \end{array}$$

Notation. In what follows, given an integer n, we denote by α^n a multiset of α's and by $n\alpha$ the formula $\alpha \oplus \cdots \oplus \alpha$. More precisely, we let

$$\alpha^1 = \alpha \quad \alpha^n = \alpha, \alpha^{n-1} \quad \text{and} \quad 1\alpha = \alpha \quad n\alpha = \alpha \oplus (n-1)\alpha.$$

The language of RŁ is obtained extending that of Ł with the set of unary connectives $\{\delta_n\}_{n \in \mathbb{N}}$. (Standard) Evaluations for RŁ are defined extending those for Ł with the condition:

$$v(\delta_n \alpha) = \frac{v(\alpha)}{n}$$

for any δ_n. Clearly $v(\delta_m(\delta_n\alpha)) = v(\delta_{mn}\alpha)$ and $v(\delta_1\alpha) = v(\alpha)$. Hence we will identify in the following any formula of the kind $\delta_m(\delta_n\alpha)$ with $\delta_{mn}\alpha$ and $\delta_1\alpha$ with α. Note that for any rational number n/m in $[0,1]$, a corresponding constant $n(\delta_m\top)$ is definable in RŁ and clearly satisfies $v(n(\delta_m\top)) = n/m$ for any evaluation v.

A Hilbert-style axiomatization of the logic RŁ has been introduced in [3]. It is obtained by adding the following axioms to the Hilbert system for Łukasiewicz logic

$$\begin{array}{cc} (\delta 1a) & n(\delta_n\varphi) \to \varphi \\ (\delta 1b) & \varphi \to n(\delta_n\varphi) \\ (\delta 2) & \neg\delta_n\varphi \oplus \neg(n-1)(\delta_n\varphi) \end{array}$$

The axiomatic system is shown in [3] to be complete w.r.t. the standard semantics over $[0,1]$, via algebraic methods. More precisely, a corresponding general algebraic semantics, the variety of *divisible MV algebras* (DMV algebras) is introduced and shown to be generated by its members on the real interval $[0,1]$. In what follows we introduce a Gentzen-style calculus for the logic RŁ that is based on *hypersequents.* We exhibit a direct proof of the completeness of the calculus w.r.t the standard semantics. First, we recall the notion of sequent and hypersequent (see e.g. [1, 13]).

Definition 1. *A* hypersequent *is a non-empty finite multiset* $S_1 \mid \dots \mid S_n$ *where each* $S_i, i = 1, \dots, n$ *is a sequent, called a* component *of the hypersequent. A* (multiple-conclusioned) sequent *is in turn an object of the form* $\Gamma \Rightarrow \Pi$*, where* Γ, Π *are multisets of formulas.*

Our hypersequent calculus for RŁ is an extension of the hypersequent calculus for Łukasiewiz logic introduced in [12]. In Table 1 we recall the calculus HŁ for Ł , with some unessential modifications. We include also rules for the connectives $\oplus, \neg, \top$, although they are not necessary, being derivable from the rules for $\rightarrow, \bot$.

$$
\begin{array}{ccc}
\overline{\Rightarrow}\ (emp) & \dfrac{}{\alpha \Rightarrow \alpha}\ (id) & \dfrac{}{\bot \Rightarrow \alpha}\ (\bot) \\[2ex]
\dfrac{G \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}{G \mid \Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2}\ (split) & \dfrac{G \mid \Gamma_1 \Rightarrow \Delta_1 \quad G \mid \Gamma_2 \Rightarrow \Delta_2}{G \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\ (mix) & \dfrac{}{\Rightarrow \top}\ (\top) \\[2ex]
\dfrac{G \mid H \mid H}{G \mid H}\ (ec) & \dfrac{G}{G \mid H}\ (ew) & \dfrac{G \mid \Gamma \Rightarrow \Delta}{G \mid \Gamma, \Pi \Rightarrow \Delta}\ (wl) \\[2ex]
\dfrac{G \mid \Gamma, \beta \Rightarrow \alpha, \Delta \mid \Gamma \Rightarrow \Delta}{G \mid \Gamma, \alpha \rightarrow \beta \Rightarrow \Delta}\ (\rightarrow l) & \dfrac{G \mid \Gamma \Rightarrow \Delta \quad G \mid \Gamma, \alpha \Rightarrow \beta, \Delta}{G \mid \Gamma \Rightarrow \alpha \rightarrow \beta, \Delta}\ (\rightarrow r) & \\[2ex]
\dfrac{G \mid \Gamma, \alpha, \beta \Rightarrow \bot, \Delta}{G \mid \Gamma, \alpha \oplus \beta \Rightarrow \Delta}\ (\oplus l) & \dfrac{G \mid \Gamma, \bot \Rightarrow \alpha, \beta, \Delta \quad G \mid \Gamma \Rightarrow \Delta}{G \mid \Gamma \Rightarrow \alpha \oplus \beta, \Delta}\ (\oplus r) & \\[2ex]
\dfrac{G \mid \Gamma, \bot \Rightarrow \alpha, \Delta}{G \mid \Gamma, \neg\alpha \Rightarrow \Delta}\ (\neg l) & \dfrac{G \mid \Gamma, \alpha \Rightarrow \bot, \Delta}{G \mid \Gamma \Rightarrow \neg\alpha, \Delta}\ (\neg r) &
\end{array}
$$

Table 1. Hypersequent calculus HŁ for Łukasiewicz logic

We are now ready to introduce the calculus for RŁ.

Definition 2. *The calculus HRŁ is obtained by adding to the calculus for Łukasiewicz logic in Table 1 the rules in Table 2.*

Hypersequents are usually interpreted as particular formulas in a logic: for instance, the symbol $\mid$ is generally interpreted as a disjunction $\vee$ and $\Rightarrow$ as an implication $\rightarrow$. This is not the case of HŁ, where hypersequents are directly interpreted over the standard semantics of the logic. The evaluation of a multiset

$$\frac{G \mid \Gamma, (\delta_n\alpha)^n \Rightarrow \Delta, \bot^{n-1}}{G \mid \Gamma, \alpha \Rightarrow \Delta}\ (\delta\uparrow l) \qquad \frac{G \mid \Gamma, \bot^{n-1} \Rightarrow (\delta_n\alpha)^n, \Delta}{G \mid \Gamma \Rightarrow \alpha, \Delta}\ (\delta\uparrow r)$$

$$\frac{G \mid \Gamma, \alpha, \bot^{n-1} \Rightarrow \Delta}{G \mid \Gamma, (\delta_n\alpha)^n \Rightarrow \Delta}\ (\delta\downarrow l) \qquad \frac{G \mid \Gamma \Rightarrow \alpha, \bot^{n-1}, \Delta}{G \mid \Gamma \Rightarrow (\delta_n\alpha)^n, \Delta}\ (\delta\downarrow r)$$

$$\frac{G \mid \Gamma, \delta_n\beta, \delta_n\top \Rightarrow \delta_n\alpha, \Delta \mid \Gamma, \delta_n\top \Rightarrow \Delta}{G \mid \Gamma, \delta_n(\alpha \to \beta) \Rightarrow \Delta}\ (\delta\to l) \qquad \frac{G \mid \Gamma, \delta_n\alpha \Rightarrow \delta_n\top, \delta_n\beta, \Delta \quad \Gamma \Rightarrow \delta_n\top, \Delta}{G \mid \Gamma \Rightarrow \delta_n(\alpha\to\beta), \Delta}\ (\delta\to r)$$

Table 2. Additional Rules for HRŁ

Γ of formulas is defined in [12, 13] for Ł as:

$$v(\Gamma) = 1 + \sum_{\alpha\in\Gamma}(v(\alpha) - 1).$$

We will adopt the same notion for the evaluation of a multiset of formulas in the logic RŁ . The validity of a hypersequent is then defined as follows.

Definition 3. *Let $G = \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$ be a hypersequent in HRŁ . We say that G is valid and denote it by $\models_{RL} G$ iff for any valuation v there is a component $\Gamma_i \Rightarrow \Delta_i$ such that $v(\Gamma_i) \leq v(\Delta_i)$ ($i \in \{1, \ldots, n\}$).*

As usual, we denote by $\vdash_{HR\text{Ł}} G$ the fact that a hypersequent G is derivable in HRŁ. Note that the rules for $(\delta \to)$ allow for a form of deep inference: they do not necessarily operate on the most external connective, i.e. on δ_n, but inside the formula. As an example to illustrate the functioning of the calculus, we show how to derive the axiom $(\delta 2)$ in page 3:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\bot^n \Rightarrow \bot^n}{\varphi, \bot^n \Rightarrow \bot^n}\,(wl)}{\bot, (\delta_n\varphi)^n \Rightarrow \bot^n}\,(\delta\downarrow l)}{\bot, \delta_n\varphi, (n-1)(\delta_n\varphi) \Rightarrow \bot, \bot}\,(\oplus l)\times(n-2)}{\bot \Rightarrow \neg\delta_n\varphi, \neg(n-1)(\delta_n\varphi)}\,(\neg r)\times 2 \qquad \dfrac{}{\Rightarrow}\,(emp)}{\Rightarrow \neg\delta_n\varphi \oplus \neg(n-1)(\delta_n\varphi)}\,(\oplus r)$$

where $\bot^n \Rightarrow \bot^n$ is clearly derivable by repeated applications of (mix) with the axiom $\bot \Rightarrow \bot$.

Lemma 1. *The rules for HRŁ in Table 2 are sound and invertible.*

Proof. We consider only two rules, the others being similar. First, we consider the rule $(\delta \to r)$. Assume that the premises hold. The case where the context G is valid is trivial. W.l.o.g. let us assume thus:

$$(*) \qquad v(\Gamma) + (\frac{v(\alpha)}{n} - 1) \leq (\frac{1}{n} - 1) + (\frac{v(\beta)}{n} - 1) + v(\Delta)$$

and

$$(**) \qquad v(\Gamma) \leq v(\Delta) + (\frac{1}{n} - 1).$$

In case $v(\alpha) \leq v(\beta)$, we have $v(\alpha \to \beta) = 1$, hence the conclusion just amounts to $(**)$. In case $v(\beta) \leq v(\alpha)$, we have $v(\alpha \to \beta) = 1 - v(\alpha) + v(\beta)$, hence the conclusion holds iff

$$v(\Gamma) \leq (\frac{1 - v(\alpha) + v(\beta)}{n} - 1) + v(\Delta)$$

which follows from $(*)$ by easy computations. For invertibility, we assume that

$$(***) \qquad v(\Gamma) \leq (\frac{v(\alpha \to \beta)}{n} - 1) + v(\Delta).$$

In case $v(\alpha) \leq v(\beta)$ this amounts to $(**)$ i.e. the right premise. Combining $v(\alpha) \leq v(\beta)$ and $(**)$ we can easily obtain $(*)$, i.e. the left premise. In case $v(\beta) \leq v(\alpha)$ we obtain the left premise $(*)$ by easy computations. From $(***)$ we easily get

$$v(\Gamma) \leq \frac{1}{n} + \frac{v(\beta) - v(\alpha)}{n} - 1 + v(\Delta) \leq \frac{1}{n} - 1 + v(\Delta)$$

i.e. the right premise $(**)$.

Let us consider now the rule $(\delta \uparrow l)$ and assume its premises hold, i.e.

$$v(\Gamma) + n(\frac{v(\alpha)}{n} - 1) \leq v(\Delta) - (n - 1).$$

This is clearly equivalent to

$$v(\Gamma) + v(\alpha) - 1 \leq v(\Delta).$$

that is, the conclusion of $(\delta \uparrow l)$. The same reasoning gives also the invertibility of $(\delta \uparrow l)$.

In what follows we call δ-*atomic* any formula of kind $\delta_n\alpha$, with α atomic formula.[1]. We call a hypersequent δ-atomic if it only contains δ-atomic formulas. Towards the completeness theorem, we show first two useful technical lemmas.

Lemma 2. *(i) The following rules are derivable in HRŁ and invertible.*

$$\frac{G \mid \Gamma^m, \bot^{(m-1)n} \Rightarrow \Delta^m}{G \mid \Gamma, (\delta_m \top)^n \Rightarrow \Delta}\ (div\ l) \qquad \frac{G \mid \Gamma^m \Rightarrow \bot^{(m-1)n}, \Delta^m}{G \mid \Gamma \Rightarrow (\delta_m \top)^n, \Delta}\ (div\ r)$$

(ii) The sequent $\delta_m\alpha \Rightarrow \delta_m\top$ is derivable in HRŁ.

[1] A formula of the kind $\delta_n(\delta_m\alpha)$ with α atomic is considered δ-atomic as well. Recall that we identify such formulas with $\delta_{mn}\alpha$.

Proof. (i). The rule $(div\ l)$ is derivable as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{G \mid \Gamma^m, \bot^{(m-1)n} \Rightarrow \Delta^m}{G \mid \Gamma^m, \top^n, \bot^{(m-1)n} \Rightarrow \Delta^m}\ (wl)}{G \mid \Gamma^m, ((\delta_m \top)^m)^n \Rightarrow \Delta^m}\ (\delta \downarrow l) \times n}{G \mid \Gamma, (\delta_m \top)^n \Rightarrow \Delta \mid \ldots \mid \Gamma, (\delta_m \top)^n \Rightarrow \Delta}\ (split) \times m}{G \mid \Gamma, (\delta_m \top)^n \Rightarrow \Delta}\ (ec) \times m$$

For invertibility, note that $(\delta \downarrow l)$ and (ec) are invertible in general. The applications of $(split)$ and (wl) above can be easily shown to be invertible as well. The rule $(div\ r)$ is derivable in a similar way, using repeated (mix) with the sequent $\Rightarrow \top$ instead of (wl) and $(\delta \downarrow r)$ instead of $(\delta \downarrow l)$.

(ii). A derivation of $\delta_m \alpha \Rightarrow \delta_m \top$ is obtained as follows:

$$\dfrac{\dfrac{\dfrac{\bot^{m-1} \Rightarrow \bot^{m-1}}{\alpha, \bot^{m-1} \Rightarrow \bot^{m-1}}\ (wl)}{(\delta_m \alpha)^m \Rightarrow \bot^{m-1}}\ (\delta \downarrow l)}{\delta_m \alpha \Rightarrow \delta_m \top}\ (div\ r)$$

Where $\bot^{m-1} \Rightarrow \bot^{m-1}$ is derivable by repeated applications of (mix) with the axiom $\bot \Rightarrow \bot$.

Lemma 3. *If* $\vdash_{HRL} G \mid \Gamma, \delta_n \top \Rightarrow \Delta$ *then* $\vdash_{HRL} G \mid \Gamma, \delta_n \alpha \Rightarrow \Delta$.

Proof. We reason by induction on the length of the derivation of $G \mid \Gamma, \delta_n \top \Rightarrow \Delta$. For the base case, if we have an axiom of the form $\delta_n \top \Rightarrow \delta_n \top$, we replace it by the derivable sequent $\delta_n \alpha \Rightarrow \delta_n \top$ (see Lemma 2). In the remaining cases, the lemma just follows by a suitable application of the induction hypothesis on the last applied rule in a derivation of $G \mid \Gamma, \delta_n \top \Rightarrow \Delta$.

We are now ready for the completeness theorem, which follows the basic structure of the argument for Łukasiewicz logic as presented e.g. in [5, 12, 13].

Theorem 1. *[Completeness] Let G be a hypersequent in HRŁ. If* $\models_{RL} G$*, then* $\vdash_{HRL} G$

Proof. By the invertibility of logical rules (see Lemma 1), it is sufficient to show the claim only for δ-atomic hypersequents. We reason by induction on the number k of different propositional variables occurring on the left hand side of the components of G. In case $k = 0$, there can only be $\bot, \delta_n \top$ on the left hand side of any component. By applying (mix) backwards with $\bot \Rightarrow \bot$, $\delta_n \top \Rightarrow \delta_n \top$, we remove any simultaneous occurrence of $\bot$ and $\delta_n \top$ on both sides of a sequent. We then apply the rules $(div\ l)$ and $(div\ r)$ backwards (see Lemma 2) to obtain a hypersequent G_1 where no occurrences of $\delta_n \top$ appear. It is clear that G_1 is valid iff G is valid and moreover, if G_1 is derivable, G is derivable as well. To conclude the base case, we are now left to show that if G_1 is valid, it is provable. Note that any component of G_1 can only be of the form $(\bot)^n \Rightarrow \Delta$ for a certain n. If there is a component such that $n \geq |\Delta|$, then the whole hypersequent is derivable by $(\bot)$, (mix),(wl) and (ew). Assume this is not the case and consider an evaluation which assigns the value 0 to any propositional variable. It can be easily shown that this evaluation would falsify the whole hypersequent G_1, thus contradicting our assumption that G_1 is valid.

We consider now the case where $k > 0$ and we pick an atomic variable q occurring on the left of at least one sequent in G. By suitable backwards application of the rules $(\delta \uparrow r)$ and $(\delta \uparrow l)$, we can obtain a hypersequent where all occurrences of q are of the form $\delta_n q$, for the same integer n. W.l.o.g. we assume $n \geq 2$ (in case $n = 1$ the proof proceeds as the one in [13]). We remove any occurrence of $\delta_n q$ on both sides of each sequent, applying backwards repeatedly (mix) with the axiom $\delta_n q \Rightarrow \delta_n q$. We obtain thus a valid hypersequent, whose components contain $\delta_n q$ either only on the right or on the left. We multiply the components of this hypersequent applying (ec) and $(split)$ backwards, so to obtain

$$G' = G_0 \mid \{\Gamma_i, (\delta_n q)^\lambda \Rightarrow \Delta_i \mid \Pi_j \Rightarrow (\delta_n q)^\lambda, \Sigma_j\}_{i \in I, j \in J}$$

where I, J are finite sets of indices and $\lambda \in \mathbb{N}$. Clearly we still have $\models_{R\text{Ł}} G'$. Let us consider now the hypersequent

$$H = G_0 \mid \{\Gamma_i, \Pi_j \Rightarrow \Sigma_j, \Delta_i \mid \Gamma_i, (\delta_n \top)^\lambda \Rightarrow \Delta_i \mid \Pi_j \Rightarrow (\delta_n q)^\lambda, \Sigma_j\}_{i \in I, j \in J}$$

which contains fewer distinct variables on the left than G'. We claim that, if H is derivable, G' is derivable as well. Indeed, from a derivation of H, by suitable applications of (mix) with $\delta_n q \Rightarrow \delta_n q$ and $(split)$, we can obtain a derivation of

$$G_0 \mid \{\Gamma_i, (\delta_n q)^\lambda \Rightarrow \Delta_i \mid \Gamma_i, (\delta_n \top)^\lambda \Rightarrow \Delta_i \mid \Pi_j \Rightarrow (\delta_n q)^\lambda, \Sigma_j\}_{i \in I, j \in J}$$

Applying Lemma 3 to the latter hypersequent and (ec), we obtain our desired derivation of G'. It suffices now to show that H is valid, as in this case we obtain $\vdash_{HR\text{Ł}} H$ by the induction hypothesis. For a contradiction, let us suppose that there exists a valuation v such that $v(\Gamma) > v(\Delta)$ for all components $\Gamma \Rightarrow \Delta \in H$. We let

$$x = \max(\{v(\Delta_i) - v(\Gamma_i)\}_{i \in I} \cup \{-\lambda\})$$
$$y = \min(\{v(\Pi_j) - v(\Sigma_j)\}_{j \in J} \cup \{0\}).$$

Assume $x \geq y$. We would have either $v(\Gamma_i) + v(\Pi_j) \leq v(\Sigma_j) + v(\Delta_i)$ or $-\lambda \geq v(\Pi_j) - v(\Sigma_j)$ or $v(\Delta_i) - v(\Gamma_i) \geq 0$. In any of these cases, we can easily obtain a contradiction with the assumption that the valuation v does not satisfy H. Hence we have $x < y$. We claim that there is a $w \in [0, 1]$ such that $x < \lambda(\frac{w}{n} - 1) < y$. Towards this aim, let us first show the two following facts:

$$(a) \quad x < \lambda(\frac{1}{n} - 1) \quad (b) \quad \lambda(\frac{v(q)}{n} - 1) < y$$

Let us start from (a). In case $x = -\lambda$ we get $-\lambda < \lambda(\frac{1}{n} - 1)$ which clearly holds. Assume instead that $x = v(\Delta_i) - v(\Gamma_i)$ for some $i \in I$. We have $v(\Delta_i) - v(\Gamma_i) <$

$\lambda(\frac{1}{n}-1)$ as otherwise v would satisfy the component $\Gamma_i, (\delta_n\top)^\lambda \Rightarrow \Delta_i$ of H and this would contradict our assumption. Let us now consider the inequation (b) and assume it does not hold. In case $y = 0$, we would have $\lambda(\frac{v(q)}{n}-1) \geq 0$, which is clearly a contradiction. Otherwise, there would be an index $j \in J$ such that $y = v(\Pi_j) - v(\Sigma_j) \leq \lambda(\frac{v(q)}{n}-1)$. Hence we would have $v(\Pi_j) \leq v(\Sigma_j, (\delta_n q)^\lambda)$, which contradicts the assumption that v does not satisfy the hypersequent H. Recall now that $x < y$. If either $x < \lambda(\frac{v(q)}{n}-1) < y$ or $x < \lambda(\frac{1}{n}-1) < y$ we are done. Otherwise we have

$$\lambda(\frac{v(q)}{n}-1) < x < y < \lambda(\frac{1}{n}-1)$$

Also in this latter case we can find a $w \in [0,1]$ (actually in $(v(q),1)$) such that $x < \lambda(\frac{w}{n}-1) < y$. We define now a new valuation $v'(q)$ which differs from v only for letting $v'(q) = w$. We have thus $x < \lambda(\frac{v'(q)}{n}-1) < y$. Hence $v'(\Delta_i) - v'(\Gamma_i) < \lambda(\frac{v'(q)}{n}-1)$ and $\lambda(\frac{v'(q)}{n}-1) < v'(\Pi_j) - v'(\Sigma_j)$, i.e.

$$v'(\Gamma_i, (\delta_n q)^\lambda) > v'(\Delta_i) \qquad v'(\Pi_j) > v'(\Sigma_j, (\delta_n q)^\lambda)$$

for any $i \in I, j \in J$. This means that G' is not valid, which contradicts our initial assumption.

In Theorem 1 we have directly shown the completeness of the hypersequent calculus with respect to the *standard* semantics over the real interval $[0,1]$. Notice that our calculus does not include the (cut) rule

$$\frac{G \mid \Gamma, \alpha \Rightarrow \Delta \quad \Sigma \Rightarrow \alpha, \Pi}{G \mid \Gamma, \Sigma \Rightarrow \Pi, \Delta}\ (cut)$$

which can be easily proved to be sound with respect to the standard semantics. The completeness of our (cut)-free calculus shows thus that the (cut) rule is actually *admissible* for HRŁ. This means that the addition of (cut) to the calculus HRŁ would not change the set of derivable formulas.

3 A hypersequent calculus for the logic $KZ(\pi)$

In this section we introduce a calculus for the logic $KZ(\pi)$, which we obtain as a restriction of the calculus HRŁ. $KZ(\pi)$ was introduced in [7] in the context of an investigation into Hintikkas's game semantics for fuzzy logic. Hintikka-games [10] are essentially two-person zero-sum games. The players, say Myself and You, in each move stepwise reduce a complex logic formula, until atomic formulas are reached. A state of the game is fully determined by the formula at stake and by an attribution of roles (attacker and defender) to the two players. For propositional (classical) logic, the rules for decomposing complex into atomic formulas are as follows:

($R_\wedge$) If I assert (i.e. defend) $\alpha \wedge \beta$ then You attack by pointing either to the left or to the right subformula. As corresponding defense, I then have to assert either α or β, according to Your choice.
($R_\vee$) If I assert $\alpha \vee \beta$ then I have to assert either α or β at My own choice.
($R_\neg$) If I assert $\neg\alpha$ then You have to assert α. In other words, our roles are switched: the game continues with You as defender and Me as attacker (of α).

Once the players reach an atomic formula, the game ends. We say that I win if in a final state I assert an atomic formula α and $v(\alpha) = 1$ (my payoff is 1). Similarly, I lose if $v(\alpha) = 0$ (my payoff is 0). In case in a final state You assert an atomic formula α, the winning conditions and related payoffs are reverted.

The game-theoretical framework just sketched provides an alternative characterization for truth in classical logic: the satisfiability of a formula corresponds to the existence of a winning strategy (i.e. ending with payoff 1) for Myself in the corresponding game.

Hintikka games were not originally meant to deal with many-valued logic: atomic formulas are indeed interpreted only as either true or false. Nevertheless, it is possible to drop this requirement and admit evaluations over $[0, 1]$, while retaining the basic game-theoretical framework. As shown in [7], this results in a game-theoretic semantics for the $\{\wedge, \vee, \neg, \bot\}$-fragment of Łukasiewicz logic, i.e. the so-called Kleene Zadeh logic (KZ). More precisely, a formula α in KZ evaluates to $w \in [0, 1]$ under a certain evaluation iff the corresponding Hintikka game has a payoff value w for Myself. Moreover, it is shown in [7] that any additional rule in a Hintikka game, involving only choices between two players and role switches, (such as $(R_\wedge), (R_\vee), (R_\neg)$) always corresponds to a definable connective in KZ. Hence a different kind of game rule is needed to go beyond the logic KZ. The logic $KZ(\pi)$ is obtained in [7] expanding KZ with a new binary connective π, characterized by the following *random choice* rule:

(R_π) If the current formula is $\alpha\pi\beta$ then a uniformly random choice determines whether the game continues with α or with β.

The corresponding truth function for this connective is obtained as

$$v(\alpha\pi\beta) = (v(\alpha) + v(\beta))/2.$$

This truth function matches the corresponding game semantics, provided that we consider *expected* payoff instead of payoff. More precisely, it is shown in [7] that a formula of $KZ(\pi)$ has a value w under a given evaluation iff the expected payoff in the corresponding Hintikka game for Myself is w. Note that the logic $KZ(\pi)$ is a proper extension of KZ, but it is incomparable with Ł: indeed, the connective π is not definable from the connectives of Ł, nor can $\cdot, \oplus, \to$ be defined from $\pi, \wedge, \vee, \neg, \bot$. The addition to $KZ(\pi)$ of a further unary connective, standing for a doubling of the truth values, is needed to capture the whole Ł while retaining the Hintikka-style game-semantics.

We can see $KZ(\pi)$, however, as a fragment of the logic RŁ, which we considered in Section 2. The fragment is generated by the atomic formulas and the

$$\frac{G \mid \Gamma \Rightarrow \Delta}{G \mid \Gamma, \Pi \Rightarrow \Delta}\ (wl)$$

$$\frac{}{\Rightarrow}\ (emp) \qquad \frac{}{\alpha \Rightarrow \alpha}\ (id)$$

$$\frac{}{(\delta_{2^m})\bot \Rightarrow \alpha}\ (\bot l) \qquad \frac{}{\Rightarrow \top}\ (\top)$$

$$\frac{G \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}{G \mid \Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2}\ (split) \qquad \frac{G \mid \Gamma_1 \Rightarrow \Delta_1 \quad G \mid \Gamma_2 \Rightarrow \Delta_2}{G \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\ (mix)$$

$$\frac{G \mid H \mid H}{G \mid H}\ (ec) \qquad \frac{G}{G \mid H}\ (ew)$$

$$\frac{G \mid \Gamma, (\delta_{2^m})\ \alpha \Rightarrow \Delta \quad G \mid \Gamma, (\delta_{2^m})\ \beta \Rightarrow \Delta}{G \mid \Gamma, (\delta_{2^m})\ \alpha \vee \beta \Rightarrow \Delta}\ (\vee l) \qquad \frac{G \mid \Gamma \Rightarrow (\delta_{2^m})\ \alpha, \Delta \mid \Gamma \Rightarrow (\delta_{2^m})\ \beta, \Delta}{G \mid \Gamma \Rightarrow (\delta_{2^m})\ \alpha \vee \beta, \Delta}\ (\vee r)$$

$$\frac{G \mid \Gamma, (\delta_{2^m})\ \alpha \Rightarrow \Delta \mid \Gamma, (\delta_{2^m})\ \beta \Rightarrow \Delta}{G \mid \Gamma, (\delta_{2^m})\ \alpha \wedge \beta \Rightarrow \Delta}\ (\wedge l) \qquad \frac{G \mid \Gamma \Rightarrow (\delta_{2^m})\ \alpha, \Delta \quad G \mid \Gamma \Rightarrow (\delta_{2^m})\ \beta, \Delta}{G \mid \Gamma \Rightarrow (\delta_{2^m})\ \alpha \wedge \beta, \Delta}\ (\wedge r)$$

$$\frac{G \mid \Gamma, (\delta_{2^{m+1}})\ \alpha, (\delta_{2^{m+1}})\ \beta \Rightarrow \bot, \Delta}{G \mid \Gamma, (\delta_{2^m})\ \alpha\pi\beta \Rightarrow \Delta}\ (\pi l) \qquad \frac{G \mid \Gamma, \bot \Rightarrow (\delta_{2^{m+1}})\ \alpha, (\delta_{2^{m+1}})\ \beta, \Delta}{G \mid \Gamma \Rightarrow (\delta_{2^m})\ \alpha\pi\beta, \Delta}\ (\pi r)$$

$$\frac{G \mid \Gamma, \bot, (\delta_{2^m})\ \top \Rightarrow (\delta_{2^m})\ \alpha, \Delta}{G \mid \Gamma, (\delta_{2^m})\ (\neg\alpha) \Rightarrow \Delta}\ (\neg l) \qquad \frac{G \mid \Gamma, (\delta_{2^m})\ \alpha \Rightarrow \bot, (\delta_{2^m})\ \top, \Delta}{G \mid \Gamma \Rightarrow (\delta_{2^m})\ (\neg\alpha), \Delta}\ (\neg r)$$

$$\frac{G \mid \Gamma, \delta_2\alpha, \delta_2\alpha \Rightarrow \Delta, \bot}{G \mid \Gamma, \alpha \Rightarrow \Delta}\ (\delta \uparrow l) \qquad \frac{G \mid \Gamma, \bot \Rightarrow \delta_2\alpha, \delta_2\alpha, \Delta}{G \mid \Gamma \Rightarrow \alpha, \Delta}\ (\delta \uparrow r)$$

$$\frac{G \mid \Gamma, \alpha, \bot \Rightarrow \Delta}{G \mid \Gamma, \delta_2\alpha, \delta_2\alpha \Rightarrow \Delta}\ (\delta \downarrow l) \qquad \frac{G \mid \Gamma \Rightarrow \alpha, \bot, \Delta}{G \mid \Gamma \Rightarrow \delta_2\alpha, \delta_2\alpha, \Delta}\ (\delta \downarrow r)$$

Table 3. Calculus $HKZ(\pi)$ for $KZ(\pi)$

connectives $\wedge, \vee, \neg, \pi$ where $x\pi y := \delta_2 x \oplus \delta_2 y$. Note that, in turn the unary connective δ_2 is definable in $KZ(\pi)$ by letting $\delta_2\alpha = \alpha\pi\bot$. By these simple observations we can thus obtain a hypersequent calculus for $KZ(\pi)$ as a fragment of that for RŁ. We present the calculus explicitly in Table 3. Note that only δ-formulas of the kind $\delta_{2^m}\alpha$ can occur in a proof of a hypersequent in the language of $KZ(\pi)$[2].

Lemma 4. *The logical rules and $(\delta \uparrow), (\delta \downarrow)$ in Table 3 are sound and invertible for RŁ.*

Proof. By simple arithmetic computation, as for the proof of Lemma 1. Notice that the rules $(\delta \uparrow)$ and $(\delta \downarrow)$ are just particular cases of the corresponding ones in Table 1. Similarly, the rules for $(\neg)$ and (π) are just special cases of the rules $(\delta \to)$ and $(\oplus)$, respectively.

Being the logical rules invertible, the completeness proof in Theorem 1 can be adapted to the fragment $KZ(\pi)$.

[2] As for RŁ, we identify $\delta_{2^m}\delta_{2^n}\alpha$ with $\delta_{2^{m+n}}\alpha$ and $\delta_{2^0}\alpha$, i.e. $\delta_1\alpha$, with α.

Theorem 2. *Let G be a hypersequent in the language of $KZ(\pi)$. If $\models_{KZ(\pi)} G$, then $\vdash_{HKZ(\pi)} G$*

Proof. Using the invertibility of the logical rules, we can reduce G to an atomic hypersequent. Applying the rules $(\delta \uparrow l)$ and $(\delta \uparrow r)$ backwards we obtain a valid hypersequent which contains only δ-atomic formulas of the kind $(\delta_{2^m})\ \alpha$, for a given m. The rest of the proof proceeds as in Theorem 1 (note that Lemmas 2 and 3 apply to $HKZ(\pi)$ as well).

4 Conclusions and future work

Variants of $KZ(\pi)$, with similar game theoretical motivations, can also be defined as fragments of RŁ. First, in the definition of the game rule (R_π) in page 9 we can drop the requirement that the formula is chosen according to a random uniform distribution. A generalized connective π_r for any rational number r in $[0, 1]$ can be introduced via the game rule:

(R_{π_r}) If the current formula is $\alpha\pi_r\beta$ then the game continues with α with probability r and with β with probability $1 - r$.

Let $r = m/n$, for m, n natural numbers. The corresponding truth function for π_r is $v(\alpha\pi_r\beta) = (m/n)v(\alpha) + (1 - m/n)v(\beta)$. The connective π_r is clearly definable in RŁ as $\alpha\pi_r\beta := m(\delta_n\alpha) \oplus (n - m)\delta_n\beta$.

In a different direction, we can also consider π-like connectives of arbitrary arity[3], i.e. connectives of kind π^n, arising from the following game rule:

(R_{π^n}) If the current formula is $\pi^n(\alpha_1, \dots, \alpha_n)$ then a uniform random choice determines whether the game continues with one of the $\alpha_1, \dots, \alpha_n$.

The corresponding truth function is clearly the average of the truth values $v(\alpha_1), \dots, v(\alpha_n)$, i.e.

$$v(\pi_n(\alpha_1, \dots, \alpha_n)) = \sum_{i=1,\dots,n} \frac{v(\alpha_i)}{n}.$$

The connective is definable in RŁ by letting $\pi^n(\alpha_1, \dots, \alpha_n) = \delta_n\alpha_1 \oplus \dots \oplus \delta_n\alpha_n$. The connective π^n is strictly related to the *random witness quantifier*, introduced by game semantics means in [6], as an extension of first-order Łukasiewicz logic. The random witness quantifier is determined by the following game rule

(R_Π) If the current formula is $\Pi x F(x)$ then an element c from the domain D is chosen randomly and the game continues with $F(c)$.

As we might expect, for a finite domain D, the corresponding truth function for Πx is defined as:

$$v(\Pi x F(x)) = \sum_{d \in D} \frac{v(F(d))}{|D|}$$

[3] Note that π is not associative in general.

This coincides with the truth function of the connective π^n for $n = |D|$. It is thus possible to investigate the properties of the quantifier $\Pi(x)$ in a finite domain by means of the corresponding connective π^n, which is in turn definable in RŁ. As shown in [6, 7], the mechanism of random choice provides a guiding principle for the characterization and systematic introduction of families of so-called *fuzzy quantifiers*, i.e. expressions such as "few", "many", "about half". Many such quantifiers are indeed definable over an extension of first-order Łukasiewicz logic with $\Pi(x)$. Our calculus HRŁ can thus provide a natural framework where a proof-theoretical study of these quantifiers can be further developed.

We leave also as a topic of future research the closer investigation of the connection between the calculi HRŁ, $HKZ(\pi)$ and the game semantics of the corresponding logic, along the lines of [5].

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