From Semi-Fuzzy to Fuzzy Quantifiers via Łukasiewicz Logic and Games*

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Abstract. Various challenges for lifting semi-fuzzy quantifier models to fully fuzzy ones are discussed. The aim is to embed such models into Łukasiewicz logic in a systematic manner. Corresponding extensions of Giles' game with random choices of constants as well as precisifications of fuzzy models are introduced for this purpose.

1 Introduction

Fuzzy logic provides formal models of vague quantifier expressions like many, few, almost all, about half, etc. Following Zadeh [14], the literature on corresponding *fuzzy quantifiers* is huge: we refer to the monograph [12] and to the more recent survey article [2] for an overview of relevant literature. Following a useful and well argued suggestion by Glöckner [12], a truth function for a fuzzy quantifier should be determined in two separate steps: (1) define a suitable *semi-fuzzy* quantifier, where the (scope and range) predicates are crisp (i.e. classical 0/1-valued) and (2) lift the semi-fuzzy quantifier to a (fully) fuzzy quantifier in some systematic and uniform manner. Regarding step (1) we will refer to an approach based on extensions of Giles's game for Łukasiewicz logic [10] that involve random choices of witness elements. But in this paper we will focus on step (2). After reviewing various shortcomings of existing approaches, Glöckner proposed an axiomatic approach for this second step, arriving at a corresponding quantifier fuzzification mechanism (QFM). However, Glöckner's QFM is still unsatisfying in some respects. In particular it is incompatible with the paradigm of mathematical fuzzy logic [1], where implication is understood as the residuum of (strong) conjunction.

After reviewing some basic notions regarding quantifiers, Łukasiewicz logic Ł, and Giles's game for Ł, we will explain some problems that may arise for lifting semi-fuzzy quantifier models to fully fuzzy ones. We then discuss in a systematic manner various quantifier fuzzification methods that arise from considering *precisifications* of fuzzy interpretations. A central aim in this endeavor is to embed the quantifier models into (suitable extensions of) Łukasiewicz logic. Moreover, we want to avoid *ad hoc* definitions of truth functions. For this reason, our main tools are certain extensions of Giles game, where one considers random choices

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of domain elements (constants) as well as choices of precisifications, in addition to moves by the two strategic players of the game. We conclude with a brief summary and some hints on further topics for related research.

2 Types of Quantification

We are interested in models of vague quantifier expressions like almost all, about half, at least about a quarter. We focus on *unary* (also known as *monadic* or *type* $\langle 1 \rangle$) quantification, where the *scope* of the quantified statement consists of a single formula and where the quantifier binds a single object variable. Vagueness will be modeled by *fuzziness*. A fuzzy set $\widetilde{S}_D$ is a function of type $D \to [0,1]$, where the (crisp) set D is the underlying *domain* or *universe*. Similarly, an n-ary fuzzy relation is a function of type $D^n \to [0,1]$. Every interpretation $\mathcal{M}$ with domain D assigns an n-ary fuzzy relation over D to each n-ary predicate symbol. Any unary *fuzzy quantifier* $\widetilde{Q}$ is interpreted by a *truth function* which assigns a *truth degree* (*truth value*) in $[0,1]$ to each fuzzy set over the domain. As a special case of fuzzy quantification, we obtain *semi-fuzzy quantifiers* by restricting the scope to classical predicates (corresponding to crisp sets).

Throughout this paper we will assume that the domain D is *finite*; an assumption that is justified by the intended application of modeling natural language expressions. We will focus on a specific, but very common type of quantifiers, namely *proportionality quantifiers*, where, in the (unary) semi-fuzzy case, the degree of truth of the quantified sentence depends only on the fraction of domain elements that satisfy the scope predicate. Given an interpretation $\mathcal{M}$ with domain D and a formula F we define[1]

$$Prop_{\mathcal{M}}(F) = \sum_{d \in D} \frac{v_{\mathcal{M}}(F(d))}{|D|}.$$

When F is a classical formula, then $Prop_{\mathcal{M}}(F)$ is $|\{d \in D : v_{\mathcal{M}}(F(d)) = 1\}|/|D|$ and denotes the proportion of elements of the domain satisfying F under $\mathcal{M}$. Hence, if Q is a semi-fuzzy proportionality quantifier, $v_{\mathcal{M}}(\mathsf{Q}xF(x))$ is uniquely determined by $Prop_{\mathcal{M}}(F)$. In the general fuzzy case, we can read $Prop_{\mathcal{M}}(F)$ as the *average truth value* of F under $\mathcal{M}$. It is much more straightforward to judge the linguistic adequateness of semi-fuzzy quantifiers as models of vague (proportional) quantification, than to deal directly with the general case, where the scope predicate may be vague as well. For this reason, as already mentioned in the introduction, Glöckner [12] suggested to split the design of adequate fuzzy models of vague quantifiers into two separate steps:

(1) specify the truth function for a semi-fuzzy quantifier,
(2) lift the function obtained in (1) to the fully fuzzy case.

For step (2) Glöckner introduced the notion of a *quantifier fuzzification mechanism (QFM)* and presented a range of axioms that should be satisfied by a QFM

[1] For convenience, we identify constant symbols with domain elements.

that lifts a wide class of semi-fuzzy to fuzzy quantifiers in a uniform manner. While we definitely agree with the usefulness of splitting the task of designing fuzzy logic based quantifier models as indicated, there remains a number of challenges. In particular it is left unclear how task (1) can be accomplished without resorting to *ad hoc* decisions for selecting appropriate truth functions.

Of particular importance for the current paper, we moreover argue that the corresponding quantifiers should be embeddable into (full) Łukasiewicz logic or at least into some other t-norm based fuzzy logic, as suggested by the paradigm of Hájek [13], which provides the basis for contemporary Mathematical Fuzzy Logic [1]. The approach of Glöckner [12] as well as that of many others (see [2]) leaves much to be desired in this respect. Here, we will not deal directly with step (1), but rather rely on a framework for the systematic design of semi-fuzzy proportionality quantifiers, based on Giles's game for Łukasiewicz logic (see Section 4).

3 Łukasiewicz Logic

As already indicated, we do not want to consider fuzzy quantifiers in isolation, but rather suggest that such quantifiers should lead to natural generalizations of well understood deductive fuzzy logics, as investigated under the heading of contemporary Mathematical Fuzzy Logic [1]. Among the corresponding t-norm based logics, Łukasiewicz logic Ł can be singled out as particularly important, since it has the unique property that the truth functions of *all* logical connectives are continuous[2] functions [1]. The semantics of the propositional connectives of (full) Łukasiewicz logic is given by the following truth functions:

$$\begin{array}{lll} v_{\mathcal{M}}(F \wedge G) = \min(v_{\mathcal{M}}(F), v_{\mathcal{M}}(G)) & & v_{\mathcal{M}}(F \odot G) = \max(0, v_{\mathcal{M}}(F) + v_{\mathcal{M}}(G) - 1) \\ v_{\mathcal{M}}(F \vee G) = \max(v_{\mathcal{M}}(F), v_{\mathcal{M}}(G)) & & v_{\mathcal{M}}(F \oplus G) = \min(1, v_{\mathcal{M}}(F) + v_{\mathcal{M}}(G)) \\ v_{\mathcal{M}}(F \to G) = \min(1, 1 - v_{\mathcal{M}}(F) + v_{\mathcal{M}}(G)) & & \\ v_{\mathcal{M}}(\bot) = 0 & v_{\mathcal{M}}(\top) = 1 & v_{\mathcal{M}}(\neg F) = 1 - v_{\mathcal{M}}(F) \end{array}$$

Universal and existential quantification is specified as follows:

$$v_{\mathcal{M}}(\forall x F(x)) = \inf_{c \in D}(v_{\mathcal{M}}(F(c))) \qquad v_{\mathcal{M}}(\exists x F(x)) = \sup_{c \in D}(v_{\mathcal{M}}(F(c)))$$

There is a further reason for choosing Łukasiewicz logic as a frame for designing formal models of vague language: already in the 1970s Robin Giles [10, 11] provided a game based semantics for Ł, that allows one to justify the particular choice of truth functions with respect to first principles about approximate reasoning. As we will see in the next section, Giles's game provides a suitable base for extending Ł with further quantifiers in a principled manner.

[2] In rival candidates, like Gödel logic or Product logic the truth function for implication is not continuous.

4 Giles's Game and Semi-Fuzzy Quantifiers

In Giles's game for Ł, two players (You and Myself) stepwise reduce logically complex assertions (formulas) to their atomic components via systematic attack and corresponding defense moves. A state of the game is given by two multisets (*tenets*) of formulas, written as

$$[F_1, \ldots, F_m \mid G_1, \ldots, G_n]\,,$$

where $F_1, \ldots, F_m$ denotes the multiset of formulas currently asserted by You (your tenet), whereas $G_1, \ldots, G_n$ denotes the multiset of formulas currently asserted by Myself (my tenet). The rules of the game specify how the player in role **P** ('proponent') may react to an attack by the player in role **O** ('opponent') on an occurrence of one the formulas asserted by **P**. For example, an attack (by **O**) on $\forall x F(x)$ has to be answered by **P** with the assertion of $F(c)$, where the constant c is chosen by **O**. Whereas in replying to an attack on $\exists x F(x)$, **P** chooses the instance $F(c)$ that replaces the attacked formula occurrence in the multiset of formulas currently asserted by her. Similar rules apply to propositional connectives: if a disjunctive formula $A \vee B$ is attacked, then it is replaced by either A or B, according to a choice by **P**, etc. In particular, implication and strong conjunction are specified by the following rules:

($R_{\to}$) If **P** asserts $F \to G$ then, if **O** chooses to attack this formula occurrence, it is replaced by G in **P**'s tenet and F is added to **O**'s tenet; otherwise, if **O** chooses not to attack this occurrence of $F \to G$, it is removed from **P**'s tenet.

($R_{\odot}$) If **P** asserts $F \odot G$ then **P** has to reply to **O**'s attack by either asserting F as well as G or else $\perp$ instead of $F \odot G$.

In any case, the successor state of the game is obtained by removing the attacked formula occurrence and adding zero or more immediate subformulas, or the logical constant $\perp$ to my or your tenet. This is repeated until a state is reached, where all asserted formulas are atomic. At such a final state the payoff for Myself is given by

$$m - n + 1 + \sum_{1 \le i \le n} v_{\mathcal{M}}(G_i) - \sum_{1 \le i \le m} v_{\mathcal{M}}(F_i),$$

where $v_{\mathcal{M}}(A)$ denotes the truth value[3] assigned to the atomic formula A by the given interpretation $\mathcal{M}$. Giles [10] (essentially) proved that for every formula F

[3] The payoff scheme may look arbitrary at a first glimpse. However it results from Giles's interpretation of the truth value of a given atom A in terms of the expected loss for a player, who has to pay a fixed amount of money (say 1 Euro) to the opposing player, if a certain experiment E_A associated with A fails. Such (binary) experiments may show dispersion, i.e. repeated executions of the same experiment E_A may show different results. However for each A a fixed *failure probability* (risk) is associated to E_A.

of Łukasiewicz logic, there is a strategy for Myself that guarantees a final payoff of $v_{\mathcal{M}}(F)$ if both players play rationally according to the rules of the outlined (finite, two person, perfect information) game. When this is the case, we say that the truth functions used in $v_{\mathcal{M}}(F)$ for interpreting the connectives and quantifiers match the corresponding game rules. Here we are interested in game rules—and the resulting truth functions—for proportional semi-fuzzy quantifiers. To obtain such rules Fermüller and Roschger [7,8] considered *uniformly random choices* of witnessing constants, in addition to the choices made by the two strategic players in roles **P** and **O**, as indicated above for the classical quantifiers $\forall$ and $\exists$. The most basic of such rules introduces a new *random choice quantifier* Π as follows.

(R_Π) If **P** asserts $\Pi x F(x)$ then this formula occurrence is replaced by $F(c)$, where c is a (uniformly) randomly chosen constant.

We will call $\text{Ł}(\Pi)$ the expansion of Ł with the random choice quantifier Π. More generally, rules for a quantified formula $\mathsf{Q}xF(x)$ feature *bets for* and *bets against* instances $F(c)$ of its scope formula, where c is a randomly chosen constant. A bet for $F(c)$ is simply an assertion of $F(c)$ by the corresponding player, whereas a bet against $F(c)$ means that $\perp$ has to be asserted, while the opposing player asserts $F(c)$. Following [7, 8], these notions allow us to formulate, e.g., the following families of rules for so-called *blind choice quantifiers.*

($R_{\mathsf{L}^k_m}$) If **P** asserts $\mathsf{L}^k_m xF(x)$ then **O** may attack by betting for k random instances of $F(x)$, while **P** bets against m random instances of $F(x)$.

($R_{\mathsf{G}^k_m}$) If **P** assert $\mathsf{G}^k_m xF(x)$ then **O** may attack by betting against m random instances of $F(x)$, while **P** bets for k random instances of $F(x)$.

Some clarifications are needed to render these rules intelligible:

1. *'Blind choice'* signifies that the identity of the randomly picked constants $c_1, \dots, c_n$ used for the relevant random instances $F(c_1), \dots, F(c_n)$ is revealed to the players only after they have placed their bets.
2. The choices of constants are *uniformly random* and *independent* of each other. In particular, the same constant may be picked more than once. Therefore the random instances form multisets, rather than sets of formulas.
3. Attacks are always optional in a Giles style game, which means that (the player in role) **O** can always decide that the attacked formula is simply removed from the current state. Giles speaks of a 'principle of limited liability' for attack (LLA) in such a situation.
4. A 'principle of limited liability' for defense (LLD) is also in place: if attacked by **O** then **P** may always decide to replace the attacked formula occurrence by $\perp$, rather than to continue the game as indicated in the above rules.

As shown in [8] the above rules, together with the just mentioned principles of limited liability, allow one to extract the following corresponding truth functions:

$$v_{\mathcal{M}}(\mathsf{L}^k_m xF(x)) = \min\{1, \max\{0, 1 + k - (m+k)Prop_{\mathcal{M}}(F)\}\} \tag{1}$$

$$v_{\mathcal{M}}(\mathsf{G}^k_m xF(x)) = \min\{1, \max\{0, 1 - k + (m+k)Prop_{\mathcal{M}}(F)\}\}. \tag{2}$$

These quantifiers are definable in $\text{Ł}(\Pi)$ using additional truth constants [5].

5 Problems with Lifting

It is tempting to extend the above framework for semi-fuzzy quantifiers to fully fuzzy quantifiers by just applying the same functions and game rules to fuzzy predicates. From a purely mathematical point of view, no problem arises: for any formula F in the scope of a quantifier we can just compute $Prop_{\mathcal{M}}(F)$ and plug the obtained value into the corresponding truth functions. However, this leads to results that run counter to expectations on the behavior of vague quantifiers in natural language, as illustrated by the following example.

Example 1. Let F a predicate standing for "is tall". We want to evaluate the sentence About half (of the elements of the domain) are tall. For modeling About half we use the quantifier $\mathsf{H}_0^1 x$, introduced in [8] and shown there to be equivalent to $\mathsf{G}_1^1 x \wedge \mathsf{L}_1^1 x$. It is straightforward to see that

$$v_{\mathcal{M}}(\mathsf{H}_0^1 x(F(x))) = \max\{0, \min\{2Prop_{\mathcal{M}}(F), 2 - 2Prop_{\mathcal{M}}(F)\}\}.$$

We now consider the following two interpretations $\mathcal{M}_1$ and $\mathcal{M}_2$ under the same domain $D = \{d_1, d_2, d_3, d_4\}$. Under the interpretation $\mathcal{M}_1$ we let $v_{\mathcal{M}_1}(F(d_1)) = v_{\mathcal{M}_1}(F(d_2)) = 0.1$ and $v_{\mathcal{M}_1}(F(d_3)) = v_{\mathcal{M}_1}(F(d_4)) = 0.9$. Under the interpretation $\mathcal{M}_2$ we let instead $v_{\mathcal{M}_2}(F(d)) = 0.5$ for any $d \in D$. Note that $Prop_{\mathcal{M}_1}(F) = Prop_{\mathcal{M}_2}(F) = 0.5$, hence $v_{\mathcal{M}_1}(\mathsf{H}_0^1 xF(x)) = v_{\mathcal{M}_2}(\mathsf{H}_0^1 xF(x)) = 1$.

In the first interpretation we have two almost clear cases of tall people and two almost clear cases of not tall people, and we correctly obtain a high value for $v_{\mathcal{M}_1}(\mathsf{H}_0^1 xF(x))$. In the second interpretation instead, all individuals of the domain are meant to be of perfectly average height. Of course there is no clear fact in this situation which would determine the "correct" truth value, but we would expect it to be smaller than in the first interpretation. As we saw above, however, $v_{\mathcal{M}_1}(\mathsf{H}_0^1 xF(x)) = v_{\mathcal{M}_2}(\mathsf{H}_0^1 xF(x))$. Informally, the approach is not sensitive to the difference between About half (of the people) are tall and All (of the people) are about half tall. Note that using any other truth function for the quantifier About half defined only in terms of the average truth value $Prop_{\mathcal{M}}(F)$ would not help. The example shows that, when evaluating a fuzzy quantifier over fuzzy predicates, one should also keep track of how the truth values are distributed over the elements of the domain.

6 Fuzzification via Random Precisification

Assume that we have a sentence $\tilde{\mathsf{Q}} xF(x)$, where $\tilde{\mathsf{Q}}$ is a fuzzy quantifier corresponding to a semi-fuzzy quantifier Q, and let $\mathcal{M}$ be an interpretation evaluating F over $[0, 1]$. As we saw before, we cannot interpret $\tilde{\mathsf{Q}}$ just in the same way as Q. To obtain more satisfactory models, we first need to associate to the interpretation $\mathcal{M}$ a set of interpretations evaluating F as a classical formula, so that the corresponding semi-fuzzy quantifier Q can be evaluated properly. Following the terminology used in supervaluationist accounts of vagueness [6, 9], we call such a set of interpretations the *admissible precisifications*[4] of $\mathcal{M}$ and denote it by $C_{\mathcal{M}}$. Informally $C_{\mathcal{M}}$ collects the "reasonable" ways of making $\mathcal{M}$ precise (i.e. classical) over atomic formulas. A simple idea for the evaluation of fuzzy quantifiers via reduction to precisifications has been introduced in [4]. It can be formulated via the following *random-precisification* based rule, which extends Giles' game for $\mathsf{Ł}(\mathsf{Q})$, where Q is a semi-fuzzy quantifier, to a corresponding fuzzy quantifier $\tilde{\mathsf{Q}}$:

($R^{RP}_{\tilde{\mathsf{Q}}}$) If **P** asserts $\tilde{\mathsf{Q}}xF(x)$ and **O** attacks the formula, a precisification $\mathcal{M}'$ is chosen randomly from $C_{\mathcal{M}}$ and **P** has to assert $\mathsf{Q}x(F(x))$, where this formula occurrence is evaluated over $\mathcal{M}'$.

A truth function for $\tilde{\mathsf{Q}}$ matching this rule is obtained as the *expected value* of $v_{\mathcal{M}'}(\mathsf{Q}xF(x))$, where $\mathcal{M}'$ ranges over the set of admissible precisifications $C_{\mathcal{M}}$. Even though not explicitly required by the general framework, we can assume that the random choice of a precisification from $C_{\mathcal{M}}$ follows a uniform distribution. A natural way to instantiate ($R^{RP}_{\tilde{\mathsf{Q}}}$) is by letting

$$C_{\mathcal{M}} = \{\mathcal{M}^{\geq\alpha} \mid \alpha \in [0,1]\},$$

where $\mathcal{M}^{\geq\alpha}$ denotes the interpretation such that, for any atomic formula A, $v_{\mathcal{M}^{\geq\alpha}}(A) = 1$ if $v_{\mathcal{M}}(A) \geq \alpha$ and $v_{\mathcal{M}^{\geq\alpha}}(A) = 0$ otherwise. Hence we can think of the random choice of a precisification as coinciding with the random choice of a value α acting as a threshold. The truth function matching ($R^{RP}_{\tilde{\mathsf{Q}}}$) is then obtained as:

$$v_{\mathcal{M}}(\widetilde{\mathsf{Q}}xF(x)) = \int_0^1 v_{\mathcal{M}^{\geq\alpha}}(\mathsf{Q}xF(x))d\alpha.$$

The same evaluation function and corresponding lifting mechanism for fuzzy quantifier is also obtained in [3], though motivated by a different semantics, based on voting models. In [3] it is also shown that the model satisfies many, though not all of Glöckner's desiderata for a quantifier fuzzification mechanism. Let us look now how this approach deals with the Example 1.

Example 1 (continued). Let $b_0 = 0, b_1 = v_{\mathcal{M}_1}(F(d_1)) = v_{\mathcal{M}_1}(F(d_2)) = 0.1$, $b_2 = v_{\mathcal{M}_1}(F(d_3)) = v_{\mathcal{M}_1}(F(d_4)) = 0.9$, $b_3 = 1$. Clearly, for any $b_{i-1} < \alpha \leq b_i$ we have $v_{\mathcal{M}_1^{\geq\alpha}}(\mathsf{H}^1_0xF(x)) = v_{\mathcal{M}_1^{\geq b_i}}(\mathsf{H}^1_0xF(x))$. As the domain D is finite, we get

$$v_{\mathcal{M}_1}(\widetilde{\mathsf{H}^1_0}xF(x)) = \sum_{i=1}^{3}(b_i - b_{i-1}) \cdot v_{\mathcal{M}_1^{\geq b_i}}(\mathsf{H}^1_0xF(x)) =$$

$$= 0.1 \cdot v_{\mathcal{M}_1^{\geq 0.1}}(\mathsf{H}^1_0xF(x)) + 0.8 \cdot v_{\mathcal{M}_1^{\geq 0.9}}(\mathsf{H}^1_0xF(x)) + 0.1 \cdot v_{\mathcal{M}_1^{\geq 1}}(\mathsf{H}^1_0xF(x)) = 0.8.$$

For the interpretation $\mathcal{M}_2$ we instead obtain

$$v_{\mathcal{M}_2}(\widetilde{\mathsf{H}^1_0}xF(x)) = 0.5 \cdot v_{\mathcal{M}_2^{\geq 0.5}}(\mathsf{H}^1_0xF(x)) + 0.5 \cdot v_{\mathcal{M}_1^{\geq 1}}(\mathsf{H}^1_0xF(x)) = 0.$$

[4] Note that, despite the fact that a precisification evaluates atomic formulas classically, the valuation under a precisification of a formula involving a semi-fuzzy quantifier might be an intermediate value in $[0,1]$.

This computation of the expected value of $\mathsf{H}_0^1 x F(x)$ over the admissible precisifications delivers more adequate results than the naive application of the function for H_0^1. From our perspective though, the fuzzification mechanism recalled here still poses a problem: we lose the possibility of expressing fuzzy quantifiers in the language of the logic $\text{Ł}(\Pi)$ – in contrast to the case for the semi-fuzzy quantifiers introduced in [8]. To embed fuzzy quantifiers in a Łukasiewicz logic based framework, we have to consider appropriate expansions of $\text{Ł}(\Pi)$. One way of doing so consists in introducing a random choice quantifier Πp operating over propositional variables rather than over elements of the domain. Game semantically, given a formula F containing occurrences of a propositional variable p, we can interpret $\Pi p F(p)$ via the following rule.

(R_{Π_p}) If **O** attacks $\Pi p F(p)$, a propositional variable p' is randomly chosen and **P** has to assert $F(p')$.

The random choice of a propositional variable can be seen as a syntactic counterpart of the random choice of a threshold truth value. In addition, we need to expand $\text{Ł}(\Pi)$ with the the well-known unary connective Δ (see e.g. [1]), given by $v_{\mathcal{M}}(\Delta F) = 1$ if $v_{\mathcal{M}}(F) = 1$ and 0 otherwise. In what follows, $\text{Ł}^{\Delta}(\Pi)$ denotes the corresponding expansion of $\text{Ł}(\Pi)$. It is easy to see that, if $v_{\mathcal{M}}(p) = \alpha$, then $v_{\mathcal{M}^{\geq\alpha}}(F(x)) = v_{\mathcal{M}}(\Delta(p \to F(x))$. Hence in $\text{Ł}^{\Delta}(\Pi)$ extended with Πp we can express $\tilde{\mathsf{Q}}$ by $\tilde{\mathsf{Q}} x F(x) \equiv \Pi p \mathsf{Q} x(\Delta(p \to F(x)))$ for any fuzzy quantifier $\tilde{\mathsf{Q}}$.

7 A Closeness-Based Approach to Fuzzy Quantifiers

We will now introduce a different approach to lift semi-fuzzy to fuzzy quantifiers. As in the previous case the idea is rooted in Giles' game semantics setting, but has an important advantage: the resulting fuzzy quantifiers are already definable over the logic $\text{Ł}^{\Delta}(\Pi)$. We start by presenting a more abstract framework.

Assume that, at a certain stage of a Giles' game, the player acting as **P** has in its tenet a fuzzy quantified sentence, say $\tilde{\mathsf{Q}} x F(x)$. If this assertion is attacked by **O**, the following two-step defense ensues:

(i) **P** adds $\mathsf{Q}x(F(x))$ to his tenet and evaluates this formula occurrence under a precisification $\mathcal{M}'$ of $\mathcal{M}$ of his choice.
(ii) **P** has to state that $v_{\mathcal{M}}(F(x))$ is "close" to $v_{\mathcal{M}'}(F(x))$.

Playing rationally, the proponent **P** will choose a precisification which maximizes the truth value of the semi-fuzzy quantified sentence, while staying as close as possible to the original truth values of the (fuzzy) predicate (a "reasonable" precisification). Let us consider some possible ways to instantiate the above abstract scheme in Giles' game in such a way as to obtain the expressibility of fuzzy quantifiers in $\text{Ł}^{\Delta}(\Pi)$.

First, in addressing step (i) above, we may reduce the choice of a precisification to the choice of a certain element of the domain, say c, acting as a threshold. In other words, we take precisifications $\mathcal{M}^c$ of $\mathcal{M}$ such that $v_{\mathcal{M}^c}(F(d)) = 1$ if $v_{\mathcal{M}}(F(d)) \geq v_{\mathcal{M}}(F(c))$, and $v_{\mathcal{M}^c}(F(d)) = 0$ otherwise. In this setting, we could actually even remove any explicit reference to $\mathcal{M}^c$: letting $F^c(x) \equiv \Delta(F(c) \to F(x))$, we easily see that $v_{\mathcal{M}^c}(F(x)) = v_{\mathcal{M}}(F^c(x))$. For ease of reference, in the following we also let $F^\top(x) \equiv \Delta F(x)$, where $F^\top$ stands for the choice of $\top$ as a threshold, instead of an element of the domain.

Let us turn now to step (ii). Given two formulas A and B in Łukasiewicz logic, the most obvious way of measuring the closeness of their truth values under a given interpretation is by evaluating $A \leftrightarrow B$, i.e. $(A \to B) \wedge (B \to A)$. Thus, a natural way to evaluate how close the formulas $F^c(x)$ and $F(x)$ are under the interpretation $\mathcal{M}$ is by computing $\mathit{Prop}_{\mathcal{M}}(F^c \leftrightarrow F)$. In the setting of Giles' game, the above ideas for (i) and (ii) result in the following *closeness-based* game rule for $\tilde{\mathsf{Q}}$.

($R^{Cl}_{\tilde{\mathsf{Q}}}$) If **P** asserts $\tilde{\mathsf{Q}}xF(x)$ and **O** attacks the formula, **P** can either invoke LLD (i.e. dismiss this formula occurrence) or add $\mathsf{Q}xF^c(x)$ to his tenet, where c is either an element of the domain of his choice or $\top$. An element d is then randomly chosen and **O** can then choose between the following:
1. **O** adds $F^c(d)$ to his tenet, thereby forcing **P** to add $F(d)$ to his tenet.
2. **O** adds $F(d)$ to his tenet, thereby forcing **P** to add $F^c(d)$ to his tenet.

The states of the game corresponding to **O**'s choices when **P** does not invoke LLD can be depicted as follows (Γ and Σ stand for arbitrary multisets of formulas):

$$\left[\Gamma \mid \Sigma, \tilde{\mathsf{Q}}xF(x)\right]$$

$$[\Gamma, F^c(d) \mid \Sigma, \mathsf{Q}xF^c(x), F(d)] \qquad [\Gamma, F(d) \mid \Sigma, \mathsf{Q}xF^c(x), F^c(d)]$$

Proposition 1. *Let us define the formula $Cl(\mathsf{Q}xF(x))$ as*

$$\exists z(\mathsf{Q}xF^z(x) \odot \Pi y(F^z(y) \leftrightarrow F(y))) \vee (\mathsf{Q}x(\Delta F(x)) \odot \Pi y(F(y) \leftrightarrow \Delta F(y))).$$

The game rules for $Cl(\mathsf{Q}xF(x))$ in Giles' game for Łukasiewicz logic are essentially reducible to ($R^{Cl}_{\tilde{\mathsf{Q}}}$), modulo some irrelevant change of order: letting $v_{\mathcal{M}}(\tilde{\mathsf{Q}}xF(x)) = v_{\mathcal{M}}(Cl(\mathsf{Q}xF(x)))$, where

$$v_{\mathcal{M}}(Cl(\mathsf{Q}xF(x))) = \sup_{c \in D \cup \{\top\}} \big(\max\{0, \mathit{Prop}_{\mathcal{M}}(F^c \leftrightarrow F) + v_{\mathcal{M}}(\mathsf{Q}xF^c(x)) - 1\}\big),$$

we obtain an evaluation of $\tilde{\mathsf{Q}}xF(x)$ matching the game rule ($R^{Cl}_{\tilde{\mathsf{Q}}}$).

Note that the occurrence of $\Pi y(F^z(y) \leftrightarrow F(y))$ in $Cl(\mathsf{Q}xF(x))$ corresponds to asserting that $F^z(y)$ is "close" to $F(y)$, whereas the subformula $\mathsf{Q}x(\Delta F(x)) \odot \Pi y(F(y) \leftrightarrow \Delta F(y))$ reflects the choice of $\top$ as a threshold instead of an element of the domain.

Motivated by game semantics, we have thus obtained an abstract characterization of fuzzy quantifiers in terms of semi-fuzzy ones. We remark that simple changes in the choice of connectives and quantifiers in $Cl(\mathsf{Q}xF(x))$, lead to different fuzzification mechanisms, still expressible in $\text{Ł}^{\Delta}(\Pi)$. Let us check how the approach sketched here fares with respect to Example 1 of Section 5.

Example 1 (continued). Under the interpretation $\mathcal{M}_1$, the supremum in $v_{\mathcal{M}}(Cl(H_0^1 xF(x))$ is obtained equivalently by choosing d_3 or d_4 as a threshold element. Hence $v_{\mathcal{M}_1}(\widetilde{\mathsf{H}_0^1} xF(x)) = v_{\mathcal{M}_1}(\mathsf{H}_0^1 xF^{d_4}(x) \odot \mathit{\Pi} y(F^{d_4}(y) \leftrightarrow F(y)))$. We have $Prop_{\mathcal{M}_1}(F^{d_4}) = 0.5$ and $Prop_{\mathcal{M}_1}\ (F^{d_4} \leftrightarrow F) = \frac{0.9+0.9+0.9+0.9}{4} = 0.9$. Hence $v_{\mathcal{M}_1}(\widetilde{\mathsf{H}_0^1} xF(x)) = 1 \odot 0.9 = 0.9$.

For the interpretation $\mathcal{M}_2$ it does not matter which element of the domain is taken as a threshold (the choice of $\top$ can only make the value of $v_{\mathcal{M}}(Cl(H_0^1 x))$ smaller). In any case $v_{\mathcal{M}_2}(\mathsf{H}_0^1 xF^{d_i}(x)) = 0$, hence $v_{\mathcal{M}_2}(\widetilde{\mathsf{H}_0^1} xF(x)) = 0$.

The method correctly determines two different truth values for $\widetilde{\mathsf{H}_0^1} xF(x)$ for the two interpretations $\mathcal{M}_1$ and $\mathcal{M}_2$. Note that a different value was obtained for $v_{\mathcal{M}_1}(\widetilde{\mathsf{H}_0^1} xF(x))$ under the random precisification mechanism in Section 6. The value 0.8 obtained for $\mathcal{M}_1$ in that approach points to a probabilistic interpretation of the truth values of $F(d_i)$. Indeed 0.8 stands for the probability that one picks the precisifications accepting half of the elements of the domain (d_3, d_4 but not d_1, d_2) as instances of F. What we present in this section instead follows a "metric" intuition: it determines how close the interpretation $\mathcal{M}_1$ is from one where exactly half of the elements of the domain fully satisfy $F(x)$; hence the value 0.9. We contend that both results are plausible and justifiable under the respective (different) underlying intuitions.

Some problems persist, due to our choice of the closeness measure: evaluating how close the truth values of all the elements of the domain are to a precisification can lead indeed to counterintuitive results, as illustrated in the following.

Example 2. Let us consider the quantifier G_1^1, which can be thought of as modeling $\mathsf{At\ least\ about\ half}$. Recall that $v_{\mathcal{M}}(\mathsf{G}_1^1 xF(x)) = \min\{1, 2Prop_{\mathcal{M}}(F)\}$. We compare the truth values of $\mathsf{G}_1^1 xF(x)$ under the interpretation $\mathcal{M}_1$ in Example 1 and under a new interpretation $\mathcal{M}_3$ over the same domain $D = \{d_1, \ldots, d_4\}$. As for the case of $\widetilde{\mathsf{H}_0^1} xF(x)$, we obtain $v_{\mathcal{M}_1}(\widetilde{\mathsf{G}_1^1} xF(x)) = 0.9$. Now let $v_{\mathcal{M}_3}(F(d_1)) = v_{\mathcal{M}_3}(F(d_2)) = 0.4$ and $v_{\mathcal{M}_3}(F(d_3)) = v_{\mathcal{M}_3}(F(d_4)) = 0.9$. For $\mathcal{M}_3$, the supremum of $v_{\mathcal{M}}(Cl(\mathsf{G}_1^1 xF(x)))$ is obtained by choosing the precisification determined by F^{d_4}. Again, we have $v_{\mathcal{M}_3}(\mathsf{G}_1^1 xF^{d_4}(x)) = 1$, but F^{d_4} is less close to F than in $\mathcal{M}_1$. Indeed, we have $v_{\mathcal{M}_3}(\mathit{\Pi} y(F^{d_4}(y) \leftrightarrow F(y))) = \frac{0.6+0.6+0.9+0.9}{4} = 0.75$ hence $v_{\mathcal{M}_3}(\widetilde{\mathsf{G}_1^1} xF(x)) = 1 \odot 0.75 = 0.75$.

Note that the semi-fuzzy quantifier G_1^1 is monotone increasing, which means that $v_{\mathcal{M}_1}(F(d)) \leq v_{\mathcal{M}_3}(F(d))$ for any $d \in D$, implies that $v_{\mathcal{M}_1}(\mathsf{G}_1^1 xF(x)) \leq v_{\mathcal{M}_3}(\mathsf{G}_1^1 xF(x))$. One would expect the same to happen also for the corresponding fuzzy quantifier, i.e. to have $v_{\mathcal{M}_1}(\widetilde{\mathsf{G}_1^1} xF(x)) \leq v_{\mathcal{M}_3}(\widetilde{\mathsf{G}_1^1} xF(x))$. But this is not the case, as shown above. The problem is that, when evaluating in $\mathcal{M}_3$ how close F is to the precisification F^{d_4}, we take into account also those elements (d_1 and d_2) for which $v_{\mathcal{M}_3}(F^{d_4}(d)) = 0$. These values should be indifferent for an increasing quantifier. A simple solution to address this problem is to replace in the subformula $\mathit{\Pi} y(F^z(y) \leftrightarrow F(y))$ of $Cl(\mathsf{Q}xF(x))$ (see Proposition 1) the quantifier occurrence $\mathit{\Pi} y$ by $\forall y$ or $\exists y$, thus obtaining a stricter or looser measure of closeness, respectively. A more general – and hence more satisfactory – solution consists in replacing the rule $(R^{Cl}_{\tilde{\mathsf{Q}}})$ above by a simplified version, which reduces the choices available to **O**. In case we consider a monotone increasing semi-fuzzy quantifier, we allow the opponent **O** only the first choice, which is matched by the truth function $Prop_{\mathcal{M}}(F^c \to F)$. Similarly, for decreasing quantifiers we allow only the second option, corresponding to $Prop_{\mathcal{M}}(F \to F^c)$, while for other quantifiers we leave both options available to **O**. Let $Cl'(\mathsf{Q}xF(x))$ stand for

$$\exists z(\mathsf{Q}xF^z(x) \odot \Pi y(\circ(F^z(y), F(y))) \vee (\mathsf{Q}x(\Delta F(x) \odot \Pi y(\circ(\Delta F(y), F(y))))$$

where $\circ(F^z(y), F(y)) \equiv F^z(y) \to F(y)$ if Q is increasing, $F(y) \to F^z(y)$ if Q is decreasing, and $F(y) \leftrightarrow F^z(y)$ otherwise. The game rules for $Cl'(\mathsf{Q}xF(x))$ correspond to the refinement of $(R^{Cl}_{\tilde{\mathsf{Q}}})$ just discussed; i.e., we obtain the analogue of Proposition 1 for $v_{\mathcal{M}}(\tilde{\mathsf{Q}}xF(x)) = v_{\mathcal{M}}(Cl'(\mathsf{Q}xF(x)))$. We can now solve the problems with monotonicity in Example 2, while still retaining the important property of allowing to define fuzzy quantifiers over $Ł^{\Delta}(\Pi)$.

Example 2 (continued). For $\mathcal{M}_1$ the supremum in $v_{\mathcal{M}_1}(Cl'(\mathsf{Q}xF(x)))$ is obtained considering F^{d_4}. Since G^1_1 is a monotone increasing quantifier, the closeness of F to F^{d_4} is measured by $v_{\mathcal{M}_1}(\Pi y(F^{d_4}(y) \to F(y))) = \frac{1+1+0.9+0.9}{4} = 0.95$ and consequently we obtain $v_{\mathcal{M}_1}(\widetilde{\mathsf{G}^1_1}xF(x)) = 0.95$. The same value is obtained for $v_{\mathcal{M}_3}(\Pi y(F^{d_4}(y) \to F(y)))$, hence $v_{\mathcal{M}_3}(\widetilde{\mathsf{G}^1_1}xF(x)) = v_{\mathcal{M}_3}(\widetilde{\mathsf{G}^1_1}xF(x)) = 0.95$.

More generally, we obtain the following restricted preservation of monotonicity.

Proposition 2. *For any semi-fuzzy quantifier Q, let us interpret $\tilde{\mathsf{Q}}xF(x)$ as $Cl'(\mathsf{Q}xF(x))$; and let $\mathcal{M}$ and $\mathcal{M}'$ be two interpretations such that $v_{\mathcal{M}}(F(d_i)) \lhd v_{\mathcal{M}}(F(d_j))$ iff $v_{\mathcal{M}'}(F(d_i)) \lhd v_{\mathcal{M}'}(F(d_j))$ for arbitrary elements d_i, d_j of a finite domain D, where $\lhd$ is either $=$ or $<$. If Q is a monotone increasing quantifier and $v_{\mathcal{M}}(F(d)) \leq v_{\mathcal{M}'}(F(d))$ for any $d \in D$, then $v_{\mathcal{M}}(\widetilde{\mathsf{Q}}xF(x)) \leq v_{\mathcal{M}'}(\widetilde{\mathsf{Q}}xF(x))$. Analogously, for decreasing quantifiers.*

8 Conclusion

We have investigated different ways of lifting semi-fuzzy to fuzzy quantifiers. The two main approaches presented in Section 6 and 7 have the following advantages: (1) they have a clear semantic foundation, based on Giles' game and (2) they provide models of fuzzy quantifiers compatible with Łukasiewicz logic. The closeness-based method introduced in Section 7 fulfills (2) in an even stronger sense, by allowing for the definition of fuzzy quantifiers over $Ł^{\Delta}(\Pi)$. We partially departed from Glöckner's [12], as the set of axioms presented there for fuzzification mechanisms forces an interpretation of the connectives different from that of Łukasiewicz logic. Nevertheless we maintain that some of the properties listed by Glöckner are relevant for our purposes as well. Among them, we stress the preservation of monotonicity. In the closeness based approach we obtained only a restricted form of this property. Full preservation of monotonicity can be easily achieved if we drop the requirement that a precisification should be identified with the choice of a threshold element. This, however, results in losing the immediate expressibility of quantifiers in $Ł^{\Delta}(\Pi)$. Further refinements of the method yet to be explored can be obtained by changes to the closeness measure.

Another natural research direction is to extend the closeness based approach to binary, or more generally to n-ary vague quantifiers: linguistically adequate models of such quantifiers should also take into account general concerns regarding truth functionality, as already suggested in [4] for the random precisification-based approach.

Finally, we suggest to further explore the advantages of embedding fuzzy quantifiers models into logical calculi, in particular for t-norm based logics. An axiomatization and a proof-theoretic study of semi-fuzzy and fuzzy quantifiers is still lacking, even for the "basic" logic $Ł(\Pi)$. Promising steps in this direction consider modal counterparts of quantifiers, e.g. along the lines suggested in Chapter 8 of Hajek's ground breaking monograph [13].

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