First-order satisfiability in Gödel logics: an NP-complete fragment

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Abstract

Defined over sets of truth values V which are closed subsets of $[0, 1]$ containing both 0 and 1, Gödel logics $\mathbf{G}_V$ are prominent examples of many-valued logics. We investigate a first-order fragment of $\mathbf{G}_V$ extended with Δ that is powerful enough to formalize important properties of fuzzy rule-based systems. The satisfiability problem in this fragment is shown to be NP-complete for all $\mathbf{G}_V$, also in presence of an additional, involutive, negation. In contrast to the one-variable case, in the considered fragment only two infinite-valued Gödel logics extended with Δ differ w.r.t. satisfiability. Only one of them enjoys the finite model property.

Keywords: First-order Gödel logics, satisfiability, monadic logic, one-variable fragment, involutive negation

1. Introduction

Many-valued logics provide a foundation for reasoning in presence of vagueness. The idea behind them is to extend the scope of classical logic by considering sets of truth values larger than the usual $\{0, 1\}$. To this aim, various many-valued systems have been defined. Among them Gödel logics $\mathbf{G}_V$ are the only ones that are completely specified by the order structure of the underlying set V of truth values. This fact characterizes $\mathbf{G}_V$ as logics of comparative truth and make them important formalizations of Fuzzy Logic, see [14].

The addition of the projection operator $\triangle$ or of the classical (involutive) negation $\sim$ enhances the expressive power of Gödel logics and their applicability. For instance, Gödel logic with truth value set $[0, 1]$ extended with $\sim$ is used in [9] to formalize the rules of the fuzzy medical expert system CADIAG-2 [1].

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In contrast with the propositional case, where there is only one infinite-valued Gödel logic w.r.t. tautologies and only one set of satisfiable formulas [5], different infinite sets of truth values determine different first-order Gödel logics. Their number has been settled to countable in [7], when considering the sets of tautologies. Nothing is known about the number of different sets of satisfiable formulas, henceforth denoted by $\mathbf{SAT\text{-}G}_V$.

In this paper we investigate the satisfiability problem for a fragment of $\mathbf{G}_V$ extended with $\triangle$, which is a subset of both the monadic and the one-variable fragment $\mathsf{FO}^1(\mathrm{V})$. We call this fragment $\mathsf{FO}^1_{mon}(\mathrm{V})$. For formulas without $\triangle$, satisfiability in $\mathsf{FO}^1_{mon}(\mathrm{V})$ is equivalent to satisfiability in classical propositional logic.

To appreciate the usefulness of this fragment, notice that the formulas in [9] formalizing the system CADIAG-2 belong to $\mathsf{FO}^1_{mon}(\mathrm{V})$. The considered fragment is also interesting from the mathematical point of view. Indeed, as shown in [4] the presence of the modality $\triangle$ (or of the negation $\sim$) renders the satisfiability problem for infinite-valued Gödel logics undecidable already in the monadic case. In contrast with this result here we prove that the satisfiability problem for $\mathsf{FO}^1_{mon}(\mathrm{V})$ is decidable, and in fact NP-complete, for all Gödel logics. The proof distinguishes two cases determined by a simple topological property of the set of truth values V: 1 isolated and 1 non isolated in V. Prominent examples for the former case being finite-valued Gödel logics (witnessed Gödel logics [15] can be treated in the same way), while for the latter case, Gödel logic with set of truth values $[0,1]$. Despite its decidability, $\mathsf{FO}^1_{mon}(\mathrm{V})$ with 1 non isolated in V does not enjoy the finite model property. Our results still hold when extending Gödel logics with the involutive negation $\sim$. An algorithm to actually check satisfiability in $\mathsf{FO}^1_{mon}(\mathrm{V})$ for Gödel logics with and without $\sim$ is presented. The algorithm is based on a reduction of the problem to suitable propositional finite-valued Gödel logics.

Our decidability proof also shows that for V infinite, in contrast with monadic $\mathsf{FO}^1(\mathrm{V})$ for which countably many distinct sets of satisfiable formulas do exist, $\mathsf{FO}^1_{mon}(\mathrm{V})$ only exhibits two different sets of satisfiable formulas.

2. Preliminaries on Gödel logics

Introduced by Gödel in 1932 to show that intuitionistic logic does not have a characteristic finite matrix, Gödel logics naturally turn up in a number of different contexts; among them fuzzy logic [14], Kripke frames [8], relevance logics [10], the provability logic of Heyting arithmetic [19] and strong equivalence in logic programming [17].

To present their semantics, we consider below a standard first-order language $\mathcal{L}$ with finitely or countably many predicate symbols P and finitely or countably many function symbols f for every finite arity k. In addition to the two quantifiers $\forall$ and $\exists$ we use the connectives $\vee$, $\wedge$, $\to$ and the constant $\bot$ (for 'false'); other connectives are introduced as abbreviations, in particular we let $\neg A := (A \to \bot)$, $\top := \neg\bot$ and $A \leftrightarrow B := (A \to B) \wedge (B \to A)$.

Definition 2.1 (Gödel set)**.** *A* Gödel set *is a closed set* $V \subseteq [0,1]$ *which contains* 0 *and* 1*.*

Let V be any Gödel set. The semantics of Gödel logic, with respect to V as truth value set and to a fixed language $\mathcal{L}$ of predicate logic, is defined using the extended language $\mathcal{L}^U$, that is $\mathcal{L}$ extended with constant symbols for each element of the universe U.

Definition 2.2 (Semantics of Gödel logic)**.** *A V-*interpretation *(or simply* interpretation*) φ into V consists of*

1. *a nonempty set $U = U^\varphi$, the 'universe' of φ,*
2. *for each k-ary predicate symbol P, a function $P^\varphi : U^k \to V$,*
3. *for each k-ary function symbol f, a function $f^\varphi : U^k \to U$.*
4. *for each variable v, a value $v^\varphi \in U$.*

Given an interpretation φ, we can naturally define a value $t^\varphi \in U$ for any term t and a truth value $\varphi(A) \in V$ for any formula A of $\mathcal{L}^U$. For a term $t = f(u_1, \dots, u_k)$ we define $\varphi(t) = f^\varphi(u_1^\varphi, \dots, u_k^\varphi)$ ($d^\varphi = d$, for all $d \in U$). For atomic formulas $A \equiv P(t_1, \dots, t_n)$, we define $\varphi(A) = P^\varphi(t_1^\varphi, \dots, t_n^\varphi)$. For composite formulas A, $\varphi(A)$ is inductively defined by:

$$
\begin{aligned}
\varphi(\bot) &= 0 \\
\varphi(A \wedge B) &= \min(\varphi(A), \varphi(B)) \\
\varphi(A \vee B) &= \max(\varphi(A), \varphi(B)) \\
\varphi(A \to B) &= \begin{cases} 1 & \varphi(A) \le \varphi(B) \\ \varphi(B) & \text{otherwise} \end{cases} \\
\varphi(\forall x\, A(x)) &= \inf\{\varphi(A(u)) : u \in U\} \\
\varphi(\exists x\, A(x)) &= \sup\{\varphi(A(u)) : u \in U\}
\end{aligned}
$$

(Here we use the fact that V is a closed *subset of $[0,1]$ in order to be able to interpret $\forall$ and $\exists$ as* inf *and* sup *in V.)*

Remark 2.3. When $V = \{0, 1\}$, φ is an interpretation of classical logic.

In Gödel logics, the validity of a formula depends only on the *relative ordering* and the *topological type* of the truth values of atomic formulas, and not on their specific values. We recall the following definition from the theory of polish spaces (see, e.g., [16])

Definition 2.4 (Non isolated point)**.** *A* non isolated point *of a topological space is a point x such that for every open neighborhood U of x there is a point $y \in U$ with $y \neq x$.*

For each Gödel set we associate two sets of formulas: the set of tautologies and the set of satisfiable formulas. We refer to the first set as *logic* in the traditional sense (closed under substitution, generalization and modus ponens).

Definition 2.5 ($\mathbf{G}_V$ and $\mathbf{SAT\text{-}G}_V$). *For a Gödel set V we define the* set of valid formulas $\mathbf{G}_V$ *(referred to as* Gödel logic $\mathbf{G}_V$*) and the* set of satisfiable formulas $\mathbf{SAT\text{-}G}_V$ *as the set of formulas A of $\mathcal{L}$ such that $\varphi(A) = 1$ for all, respectively at least one, V-interpretations φ. Each such interpretation is called a* model *of A.*

Notice that in contrast with classical logic (that is Gödel logic with truth value set $V = \{0, 1\}$), in Gödel logics validity and satisfiability are not dual concepts[4].

Equivalence between formulas of Gödel logics are defined in the usual way, i.e., two formulas A and B are equivalent in $\mathbf{G}_V$ ($A \equiv_{\mathbf{G}_V} B$, in symbols) if for all V-interpretations φ, $\varphi(A) = \varphi(B)$. The expression $A \overset{\text{SAT}}{\equiv}_{\mathbf{G}_V} B$ indicates that A is satisfiable in $\mathbf{G}_V$ if and only if so is B. (Henceforth we use $\equiv$ and $\overset{\text{SAT}}{\equiv}$ when the considered logic is clear from the context.)

Proposition 2.6. *Let V be any Gödel set. The following properties hold in each $\mathbf{G}_V$:*

1. $(A \wedge (B \vee C)) \equiv ((A \wedge B) \vee (A \wedge C))$
2. $\exists x(A(x) \wedge B^x) \equiv (\exists x A(x) \wedge B^x)$, *where x does not occur free in B^x*
3. $\forall x(A(x) \wedge B(x)) \equiv (\forall x A(x) \wedge \forall x B(x))$

2.1. Some extensions

Interesting extensions of Gödel logics $\mathbf{G}_V$ are obtained by adding to $\mathcal{L}$ the unary operator $\triangle$ of [2] or a classical, involutive negation $\sim$, see e.g., [11, 12]. We denote these extensions by $\mathbf{G}_V^{\triangle}$ and $\mathbf{G}_V^{\sim}$, respectively. The semantics of $\mathbf{G}_V^{\triangle}$ and $\mathbf{G}_V^{\sim}$ extend that of $\mathbf{G}_V$ as follows (notice that the Gödel set V in $\mathbf{G}_V^{\sim}$ has to be symmetric with respect to $1/2$).

$$\varphi(\triangle A) = \begin{cases} 1 & \text{if } \varphi(A) = 1 \\ 0 & \text{otherwise.} \end{cases}$$
$$\varphi(\sim A) = 1 - \varphi(A)$$

Definitions and terminology for $\mathbf{G}_V$ also apply to $\mathbf{G}_V^{\triangle}$ and $\mathbf{G}_V^{\sim}$.

Remark 2.7. In $\mathbf{G}_V^{\sim}$ the operator $\triangle$ is derivable ($\triangle A \equiv \neg\sim A$).

$\mathbf{G}_V^{\triangle}$ and $\mathbf{G}_V^{\sim}$ are strictly more expressive than $\mathbf{G}_V$. E.g., unlike $\mathbf{G}_V$, in $\mathbf{G}_V^{\triangle}$ we can express 'strict linear order' as

$$\neg\triangle(B \to A) \tag{1}$$

Henceforth we denote by $A \prec B$ the formula above. It is easy to see that for every interpretation φ of $\mathbf{G}_V^{\triangle}$ one has $\varphi(A \prec B) = 1$ if and only if $\varphi(A) < \varphi(B)$.

Proposition 2.8. *Let V be any Gödel set, then $\forall x B(x) \overset{SAT}{\equiv}_{\mathbf{G}_V} \forall x \triangle B(x)$.*

Notice that the corresponding statement does not hold for the existential quantifier. E.g., let $B(x)$ be the formula $A(x) \wedge \forall x \neg\triangle A(x)$; $\exists x B(x)$ is satisfiable in any $\mathbf{G}_V^{\triangle}$ in which 1 is non isolated in V while $\exists x \triangle B(x)$ is not.

[4]The duality holds instead when considering the notion of *positive satisfiability*: a formula A is positive satisfiable if there exists a V-interpretation φ such that $\varphi(A) > 0$.

2.2. (Un)decidability of the monadic fragment

Monadic logic is the fragment of first-order logic with no function symbol and in which all predicates are unary. A general investigation of the (un)decidability status for the validity and the satisfiability problem in monadic Gödel logics was carried out in [3, 4], respectively.

[3] proved that with the possible exception of Gödel logic with truth values set $V_\uparrow = \{1\} \cup \{1 - \frac{1}{k} \mid k \geq 1\}$ (whose (un)decidability status is left open), validity is undecidable when V is infinite.

[4] identified suitable conditions on the topological type of V which determine the decidability or undecidability of $\textbf{SAT-G}_V$ in monadic Gödel logics. $\textbf{SAT-G}_V$ is decidable when 0 is an isolated point in V (i.e., 0 has Cantor-Bendixon rank $|0|_{\text{CB}} = 0$, see e.g., [16]). In the remaining monadic Gödel logics the presence of at least three predicate symbols, one of which is a constant different from 0 or 1, makes $\textbf{SAT-G}_V$ undecidable. Moreover without this constant predicate, the problem remains undecidable for all monadic Gödel logics in which 0 is a limit point of limit points in V (i.e., $|0|_{\text{CB}} \geq 2$). Gödel logic $\mathbf{G}_{[0,1]}$, with $V = [0, 1]$ (also known as Intuitionistic Fuzzy Logic [18]) being a prominent example. Only one Gödel logic is left out from the classification; this is the logic with truth value set $V_\downarrow = \{0\} \cup \{\frac{1}{k} \mid k \geq 1\}$ for which the (un)decidability status of $\textbf{SAT-G}_{V_\downarrow}$ is left open.

The addition of $\triangle$ renders both the validity and the satisfiability problem undecidable for all monadic $\mathbf{G}_V^\triangle$ (and therefore $\mathbf{G}_V^\sim$) with V infinite, in presence of at least two predicate symbols. The problems remain undecidable even when we restrict to prenex formulas[5] [3, 4].

3. The fragment $\mathsf{FO}^1_{mon}(\mathrm{V})$

We introduce the class $\mathsf{FO}^1_{mon}(\mathrm{V})$ of formulas of Gödel logics and provide a suitable normal form for them. The defined normal form will be crucial for the decidability proof in Section 4.2.

Definition 3.1 ($\mathsf{FO}^1_{mon}(\mathrm{V})$)**.** *The class $\mathsf{FO}^1_{mon}(V)$ consists of all closed formulas in the first-order language $\mathcal{L}$ extended with $\triangle$, of the form*

$$\bigvee_{i=1}^{n}(\exists x A^i_1(x) \wedge \ldots \wedge \exists x A^i_{n_i}(x) \wedge \forall x B^i_1(x) \wedge \ldots \wedge \forall x B^i_{m_i}(x))$$

where each A^i_k, B^i_k is a monadic and quantifier-free formula containing no function and constant symbol.

Notice that $\mathsf{FO}^1_{mon}(\mathrm{V})$ is contained in the one-variable fragment ($\mathsf{FO}^1(\mathrm{V})$). The satisfiability problem for formulas in $\mathsf{FO}^1_{mon}(\mathrm{V})$ without $\triangle$ is classically decidable. The proof proceeds as in the case of monadic $\mathbf{G}_V$ in which 0 is isolated in V (see [4]). Indeed

Proposition 3.2. *Let V be any Gödel set. Formulas in $\mathsf{FO}^1_{mon}(V)$ without $\triangle$ are satisfiable in $\mathbf{G}_V$ if and only if they are satisfiable in classical logic.*

[5] In general Gödel logics do not admit equivalent prenex formulas, see e.g. [6].

Proof. Let $Q = \bigvee_{i=1}^{n} (\exists x A_1^i(x) \wedge \ldots \wedge \exists x A_{n_i}^i(x) \wedge \forall x B_1^i(x) \wedge \ldots \wedge \forall x B_{m_i}^i(x))$ be any formula in $\mathsf{FO}^1_{mon}(\mathrm{V})$ without $\triangle$. If Q is satisfiable in classical logic then clearly Q is satisfiable in $\mathbf{G}_V$. For the converse direction, consider any V-interpretation such that $\varphi_{\mathbf{G}}(Q) = 1$. An interpretation $\varphi_{\mathbf{CL}}$ of classical logic such that $\varphi_{\mathbf{CL}}(Q) = 1$ is defined as follows: for any atomic formula P

$$\varphi_{\mathbf{CL}}(P) = \begin{cases} 1 & \text{if } \varphi_{\mathbf{G}}(P) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, let Q_i be any quantifier-free (sub)formula of Q. By induction on the complexity of Q_i we can prove that $\varphi_{\mathbf{G}}(Q_i) = 0$ if and only if $\varphi_{\mathbf{CL}}(Q_i) = 0$ and $\varphi_{\mathbf{G}}(Q_i) > 0$ if and only if $\varphi_{\mathbf{CL}}(Q_i) = 1$. The claim easily follows. □

Notice that each formula in $\mathsf{FO}^1_{mon}(\mathrm{V})$ is equivalent in $\mathbf{G}_V^{\triangle}$ to a prenex formula with prefix $\exists^*\forall^*$. Therefore, by Proposition 3.2, $\mathsf{FO}^1_{mon}(\mathrm{V})$ without $\triangle$ is contained in the Bernays-Schönfinkel class[6] that, for classical logic is known to be *effectively propositional*, i.e., its formulas can be effectively translated into propositional logic formulas by replacing all existing variables by Skolem constants and then grounding the universally quantified variables.

3.1. Chain Normal Form

A normal form similar to the disjunctive normal form of classical logic was introduced in [2] for formulas of propositional Gödel logic. This normal form (called *chain normal form*) is extended below to formulas of $\mathsf{FO}^1_{mon}(\mathrm{V})$. The idea behind it is to enumerate all the orderings of unary predicates over the same variable in a way similar to how one constructs a disjunctive normal form by enumerating all possible truth value assignments of propositional atoms. Notice that, unlike $\mathbf{G}_V$, these orderings are expressible by formulas of $\mathbf{G}_V^{\triangle}$. We use below the following abbreviations (cf. Equation 1)

$$\begin{array}{lll} A \prec B & \text{for} & \neg\triangle(B \to A), \text{ and} \\ A \equiv_\Delta B & \text{for} & \triangle(A \to B) \wedge \triangle(B \to A). \end{array}$$

Definition 3.3 ($\triangle$-chain)**.** *Let F be any formula in $\mathsf{FO}^1_{mon}(V)$ and $P_1, \ldots, P_n$ be the predicates occurring in F. A* $\triangle$-chain *over F is any formula of the form*

$$(\bot \bowtie_0 P_{i_1}(x)) \wedge (P_{i_1}(x) \bowtie_1 P_{i_2}(x)) \wedge (P_{i_2}(x) \bowtie_3 P_{i_3}(x)) \wedge \cdots \wedge (P_{i_n}(x) \bowtie_{n+1} \top)$$

where $\{P_{i_1}, \ldots, P_{i_n}\} = \{P_1, \ldots, P_n\}$, i.e., every predicate symbol occurs exactly twice in the above formula, $\bowtie_i$ is either $\prec$ or $\equiv_\Delta$, and at least one of the $\bowtie_i$'s is $\prec$.

Every $\triangle$-chain describes a possible ordering of the values of predicates of F. Let $\mathcal{C}_F$ be the set of all $\triangle$-chains over F, i.e., $\mathcal{C}_F = \{C_1, \ldots, C_N\}$ for $N = (2^{n+1} - 1)n!$ ($n!$ from the number of permutations of the predicate symbols, and $2^{n+1} - 1$ from the number of possible combinations of $\prec$ and $\equiv_\Delta$).

[6]The Bernays-Schönfinkel class consists of formulas of the form $\exists^*\forall^* A$ where A is quantifier-free and no function symbol occurs.

Every $\triangle$-chain C_i induces equivalence classes over the predicates of F. These are ordered as

$$[\bot] = \alpha_0 \prec \alpha_1 \prec \ldots \prec \alpha_n = [\top] \quad \text{with} \quad \alpha_i = \{P_1^i(x), \ldots, P_{k_i}^i(x)\}$$

where $P_n^i(x) \equiv_\Delta P_m^i(x)$ for all $n, m \in \{1, \ldots k_i\}$ in C_i and at least one element in α_i is related to at least one element in α_{i+1} with $\prec$. Notice that the union of all α_i is the set of all predicate symbols (plus $\top$ and $\bot$) occurring in F, and the intersection of any two α_i is empty.

Furthermore, every interpretation uniquely defines a $\triangle$-chain induced by the natural order of the valuations in the reals.

Lemma 3.4. *$\bigvee_{C \in \mathcal{C}_F} C$ is a tautology in $\mathbf{G}_V^\Delta$.*

Every $\triangle$-chain over F in FO^1_{mon}(V) induces a 'syntactic evaluation' of the (quantifier-free) formulas in F.

Definition 3.5 (Syntactic evaluation)**.** *Let F be a formula in FO^1_{mon}(V) and $A(x)$ any quantifier-free subformula of F. Its* evaluation *$\Phi^C_{A(x)}$ with respect to a $\triangle$-chain C over F is defined inductively as follows:*
Base case:

- *for predicate symbols, $\top$ and $\bot$ define $\Phi^C_{P(x)}$ as*

$$\Phi^C_{P(x)} = \begin{cases} \top & \text{if } P(x) \in [\top] \\ \bot & \text{if } P(x) \in [\bot] \\ P(x) & \text{otherwise} \end{cases}$$

Compound formulas:

- *$\Phi^C_{\triangle A(x)}$ is $\top$ if $\Phi^C_{A(x)} = \top$ and $\Phi^C_{\triangle A(x)}$ is $\bot$, otherwise.*
- *$\Phi^C_{A_k(x) \wedge A_l(x)}$ is either $\Phi^C_{A_k(x)}$ or $\Phi^C_{A_l(x)}$ depending on which of the two occurs earlier in the chain.*
- *$\Phi^C_{A_k(x) \vee A_l(x)}$ is either $\Phi^C_{A_k(x)}$ or $\Phi^C_{A_l(x)}$ depending on which of the two occurs later in the chain.*
- *$\Phi^C_{A_k(x) \to A_l(x)}$ is $\top$ if $\Phi^C_{A_k(x)}$ occurs earlier than $\Phi^C_{A_l(x)}$ in the chain, otherwise it is $\Phi^C_{A_l(x)}$*

The syntactic evaluation of $A(x)$ is a predicate symbol, $\top$ or $\bot$ (when $A(x)$ is a formula prefixed by $\triangle$, then $\Phi^C_{A(x)}$ is either $\top$ or $\bot$). $\Phi^C_{A(x)}$ is a syntactic evaluation of $A(x)$ in the following sense

Proposition 3.6. *Let F be a formula in FO^1_{mon}(V) and $A(x)$ be any quantifier-free subformula of F. For each interpretation φ of $\mathbf{G}_V^\Delta$ and $\triangle$-chain C over F*

$$\varphi(C \wedge A(x)) = \varphi(C \wedge \Phi^C_{A(x)})$$

Proof. Let C be $(\bot \bowtie_0 P_{i_1}(x)) \wedge (P_{i_1}(x) \bowtie_1 P_{i_2}(x)) \wedge \cdots \wedge (P_{i_n}(x) \bowtie_n \top)$ where $\{P_{i_1}, \ldots, P_{i_n}\} = \{P_1, \ldots, P_n\}$. If for some conjunct $R \bowtie S$ in the chain C, $\varphi(R) \bowtie^* \varphi(S)$ where $\bowtie^*$ is $=$ when $\bowtie = \prec$ or $\bowtie^*$ is $<$ when $\bowtie = \equiv_\Delta$, then $\varphi(C) = 0$. The rest follows by Definition 3.5. $\square$

4. Decidability Results

We show that the satisfiability problem for $\mathsf{FO}^1_{mon}(\mathrm{V})$ is decidable for all Gödel logics. We consider two cases, distinguished by the property that 1 is isolated in the truth value set V or it is not. All finite V being prominent examples of the first case, while $V = [0, 1]$ belongs to the second case. Though both cases are decidable, only $\mathsf{FO}^1_{mon}(\mathrm{V})$ in which 1 is isolated in V enjoys the finite model property. These results also hold in presence of the additional negation $\sim$.

4.1. 1 *isolated in* V

In presence of $\triangle$, the satisfiability problem for $\mathsf{FO}^1_{mon}(\mathrm{V})$ is not anymore equivalent to SAT in classical logic.

Example 4.1. *The formula* $\exists x(\neg\neg A(x) \wedge \neg\triangle A(x))$ *is not satisfiable in classical logic while it is satisfiable in* $\mathbf{G}^{\triangle}_V$*, for any Gödel set* $V \neq \{0, 1\}$.

We show below that for any Gödel set V in which 1 is isolated (i.e., 1 has Cantor-Bendixon rank $|1|_{\mathrm{CB}} = 0$) the decidability proof of the satisfiability problem for $\mathsf{FO}^1_{mon}(\mathrm{V})$ proceeds similarly to that for classical formulas: by a process of grounding and instantiation.

Lemma 4.2. *Let* V *be any Gödel set in which* 1 *is isolated and* A_i *and* B_j *be quantifier free formulas of* $\mathbf{G}^{\triangle}_V$*. Then*

$$\bigwedge_{i=1}^{n} \exists x A_i(x) \wedge \forall x B(x) \in \boldsymbol{SAT\text{-}G}^{\triangle}_{\mathrm{V}} \quad \Leftrightarrow \quad \bigwedge_{i=1}^{n} A_i(d_i) \wedge \bigwedge_{i=1}^{n} B(d_i) \in \boldsymbol{SAT\text{-}G}^{\triangle}_{\mathrm{V}}$$

where the d_i *are new constant symbols (Skolem constants).*

Proof. ($\Rightarrow$) Assume that there is an interpretation φ such that $\varphi(\bigwedge_{i=1}^{n} \exists x A_i(x) \wedge \forall x B(x)) = 1$. Then all the instances of $B(u)$ for $u \in U$ will be evaluated to 1 under φ. Furthermore, due to $\varphi(\exists x A_i(x)) = 1$ for all i, and the isolation of 1 in V, for every i there exists an object $u_i \in U$ such that $\varphi(A_i(u_i)) = 1$. Thus, the interpretation that evaluates the Skolem constants d_i to the elements u_i, respectively, is a model for $\bigwedge_{i=1}^{n} A_i(d_i) \wedge \bigwedge_{i=1}^{n} B(d_i)$.

($\Leftarrow$) The n-element universe together with the interpretation satisfying $\bigwedge_{i=1}^{n} A_i(d_i) \wedge \bigwedge_{i=1}^{n} B(d_i)$ is a model of $\bigwedge_{i=1}^{n} \exists x A_i(x) \wedge \forall x B(x)$. □

Theorem 4.3. *The satisfiability problem for* $\mathsf{FO}^1_{mon}(V)$ *in which* 1 *is isolated in* V *is decidable.*

Proof. Let V be any Gödel set in which 1 is isolated and P be any formula in the class $\mathsf{FO}^1_{mon}(\mathrm{V})$. P is satisfiable in $\mathbf{G}^{\triangle}_V$ if and only if so is one of its disjuncts. This has the general form

$$\exists x A_1(x) \wedge \ldots \wedge \exists x A_{n_i}(x) \wedge \forall x B_1(x) \wedge \ldots \wedge \forall x B_{m_i}(x). \tag{2}$$

and it is equivalent to $\exists x A_1(x) \wedge \ldots \wedge \exists x A_{n_i}(x) \wedge \forall x(B_1(x) \wedge \ldots \wedge B_{m_i}(x))$ by Proposition 2.6. The claim follows by Proposition 2.8 and Lemma 4.2. □

4.2. 1 non isolated in V

When 1 is non isolated in V, the satisfiability of an existential formula $\exists x D(x)$ under any interpretation φ does not imply anymore that there exists an element u of the domain such that $\varphi(D(u)) = 1$. Therefore the grounding process in Lemma 4.2 does not work and the decidability proof is more involved. This is also to be expected, as formulas in $\mathsf{FO}^1_{mon}(\mathrm{V})$, with 1 non isolated in V, are not *finitely controllable*, that is not all satisfiable formulas have a finite model.

Example 4.4. *The formula F in $\mathsf{FO}^1_{mon}(V)$*

$$\exists x A(x) \wedge \forall x \neg \triangle A(x)$$

is satisfiable in $\mathbf{G}^{\triangle}_{V}$ where 1 is non isolated in V but it has no finite model (see Example 6.5).

Theorem 4.5. *The satisfiability problem for $\mathsf{FO}^1_{mon}(V)$ in which 1 is non isolated in V is decidable.*

Proof. Let V be any Gödel set in which 1 is non isolated and $\overline{F}$ be any formula in the class $\mathsf{FO}^1_{mon}(\mathrm{V})$. First recall that $\overline{F}$ is satisfiable if and only if so is one of its disjuncts, which has the general form of Equation 2 (cf. Theorem 4.3). Consider, to fix ideas, the case $n_i = m_i = 1$. The general case follows by easy adaptations. Let F be $\exists x A(x) \wedge \forall x B(x)$. We first transform F into a suitable equivalent formula using the chain-normal form. Consider $\exists x A(x)$. By Lemma 3.4:

$$\exists x A(x) \equiv \exists x((\bigvee_{C \in \mathcal{C}_F} C) \wedge A(x))$$

now we push in the existential quantifier (cf. Proposition 2.6)

$$\equiv \bigvee_{C \in \mathcal{C}_F} \exists x(C \wedge A(x))$$

and evaluate the formula $A(x)$ with respect to the chain C (cf. Proposition 3.6), syntactic evaluation

$$\equiv \bigvee_{C \in \mathcal{C}_F} \exists x(C \wedge \Phi^C_{A(x)})$$

Some of the $\Phi^C_{A(x)}$ might be $\perp$. We delete these disjuncts (keeping equivalence of satisfiability). The chains C_i leading to 'syntactic evaluations' $\Phi^{C_i}_{A(x)}$ different from $\perp$ are collected into the set

$$\Gamma = \{C_i : i \in I\} \tag{3}$$

Consider now the universal conjunct $\forall x B(x)$ of F. By Proposition 2.8:

$$\forall x B(x) \overset{\text{SAT}}{\equiv} \forall x \triangle B(x)$$

Similarly as above, by using Lemma 3.4, Proposition 2.6 and Proposition 3.6 we obtain:

$$\equiv \forall x((\bigvee_{C \in \mathcal{C}_F} C) \wedge \triangle B(x)) \equiv \forall x(\bigvee_{C \in \mathcal{C}_F} (C \wedge \triangle B(x))) \equiv \forall x(\bigvee_{C \in \mathcal{C}_F} (C \wedge \Phi^C_{\triangle B(x)}))$$

As a formula with leading $\triangle$ can only evaluate syntactically to $\bot$ or $\top$ (cf. Definition 3.5), we remove the disjuncts with $\bot$ and arrive at

$$\overset{\text{SAT}}{\equiv} \forall x(\bigvee_{C \in \Sigma} C) \tag{4}$$

where $\Sigma \subseteq \mathcal{C}_F$ is the set of chains for which $\Phi^C_{\triangle B(x)} = \top$.

The original formula $\exists x A(x) \wedge \forall x B(x)$ has a model if and only if this holds for its SAT-equivalent formula (where Γ arises from $\exists x A(x)$, cf. Equation 3)

$$F' := \bigvee_{C \in \Gamma} \exists x(C \wedge \Phi^C_{A(x)}) \wedge \forall x \bigvee_{C \in \Sigma} C \tag{5}$$

Claim: $\exists x A(x) \wedge \forall x B(x)$ is satisfiable if and only if there is a $\triangle$-chain $C \in \Gamma \cap \Sigma$.
We will refer to this condition as *satisfiability condition.*

($\Longrightarrow$) Let φ be an evaluation satisfying the original formula. The $\triangle$-chain 'induced' by this evaluation naturally satisfies the condition above.

($\Longleftarrow$) We show below that if the satisfiability condition holds, we can construct an interpretation φ of $\mathbf{G}^\triangle_V$ that is a model for F' (and hence for the original formula F).

Indeed, let C be a $\triangle$-chain matching the satisfiability condition above. Consider the equivalence classes over the predicates of F induced by C. Assume that they are ordered as

$$[\bot] = \alpha_0 \prec \alpha_1 \prec \ldots \prec \alpha_n = [\top] \quad \text{with} \quad \alpha_i = \{P^i_1(x), \ldots, P^i_{k_i}(x)\}$$

Furthermore assume that the equivalence class of $\Phi^C_{A(x)}$ is α_k, i.e., $\Phi^C_{A(x)} \in \alpha_k$.

By the property that 1 is non isolated in the truth value set, we can define the evaluation of atomic formulas on the universe of natural numbers in a way that the following properties are fulfilled (for simplicity below we omit the subscripts in $P^i_j(x)$, and indicate only the respective equivalence class by the superscript):

(1) $\varphi(P^0(c)) = 0$, for all c, which is necessary as all the $P^0(x)$ are in $\alpha_0 = [\bot]$.

(2) if $i < j$ then for all $P^i \in \alpha_i$ and $P^j \in \alpha_j$ and for all c, $\varphi(P^i(c)) < \varphi(P^j(c))$.

(3) if $i \geq k$ and $P^i \in \alpha_i$, then $\lim_{c \to \infty} \varphi(P^i(c)) = 1$

As an example, for $V = [0, 1]$ a satisfying evaluation φ can be defined as

$$\varphi(P^i(c)) = 1 - \frac{1}{(c+2)^i} \tag{6}$$

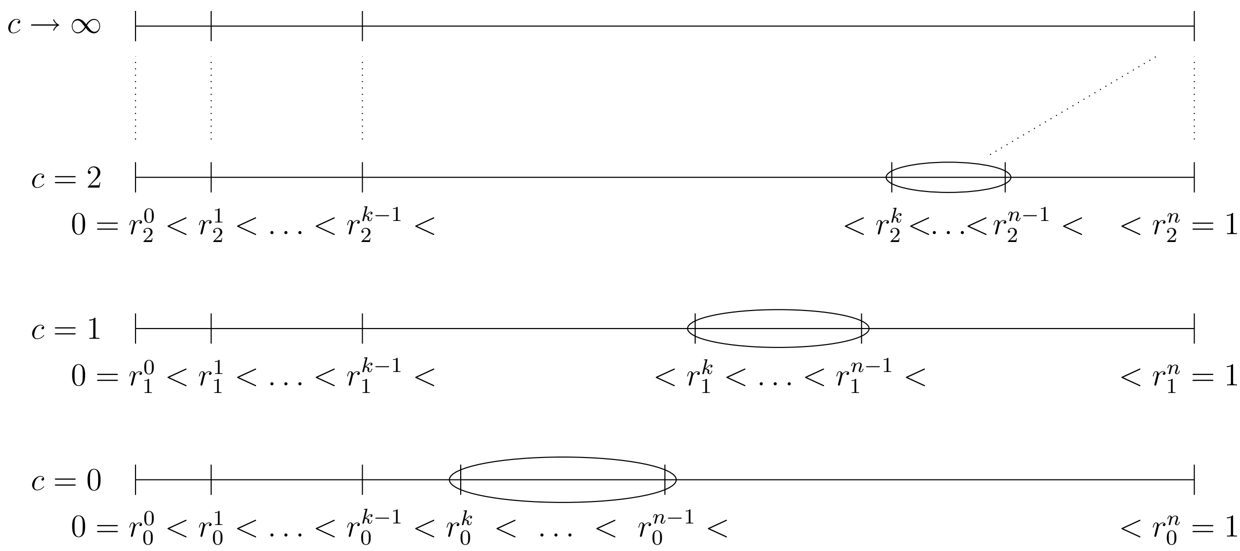

Figure 1: Model construction, $r_c^k = \varphi(P^k(c))$ for $P^k \in \alpha_k$

For truth value sets other than $[0,1]$ (with 1 non isolated) we define the evaluation φ iteratively (cf. Figure 1): for $c = 0$ select the evaluations in the truth value set such that conditions (1) and (2) are satisfied. Having defined the evaluations of $P^i(c)$ select the evaluations of $P^i(c+1)$ (for $i \geq k$) above all the evaluations $P^i(c)$, i.e., above $\max\{\varphi(P^i(c)) : 0 \leq i \leq n\}$. This is possible due to the fact that 1 is non isolated. Furthermore, we can again ensure conditions (1) and (2). Continuing this way we only have to make sure that for all predicates P^i with $i \geq k$ the maximums of the evaluations for c are actually having 1 as the limit (condition (3)). This is again possible being 1 non isolated in V.

We now evaluate in φ the formula F' that is SAT-equivalent to F. Due to conditions (1) and (2) the $\triangle$-chain C is satisfied, that is for all c, $\varphi(C(c)) = 1$. From this and the fact that $C \in \Sigma$ (satisfiability condition) it follows that

$$\varphi(\forall x \bigvee_{C \in \Sigma} C) = 1,$$

Considering that $\Phi^C_{A(x)}$ cannot be member of $\alpha_0 = [\bot]$ (otherwise the $\triangle$-chain C would not be part of Γ) and by (3) we have $\lim_{c \to \infty} \varphi(\Phi^C_{A(x)}(c)) = 1$, thus

$$\varphi(\exists x (C \wedge \Phi^C_{A(x)}(x))) = 1.$$

Hence φ is a model for F' and therefore F is satisfiable.

The extension of the proof to the general case, i.e., to formulas of the form $\exists x A_1(x) \wedge \ldots \wedge \exists x A_n \wedge \forall x B_1(x) \wedge \ldots \wedge \forall x B_m$ (cf. Equation 2) is easy. As in the restricted case we obtain sets $\Gamma_1, \ldots, \Gamma_n$ from each of the A_k, and sets $\Sigma_1, \ldots, \Sigma_m$ from the B_j. The *satisfiability condition* for the general case is therefore

$$\exists C \in \bigcap_{1 \leq i \leq n} \Gamma_i \cap \bigcap_{1 \leq j \leq m} \Sigma_j$$

i.e., there has to be a common chain in all the solutions. The proof in the forward direction is trivial as a satisfying evaluation provides *one* chain that fulfills this condition. For the reverse direction we proceed exactly as in the basic case.

The decidability of the satisfiability problem for $\mathsf{FO}^1_{mon}(\mathrm{V})$ follows from the fact that the satisfiability condition is a finite check over finite objects (i.e. $\triangle$-chains). $\square$

Example 4.6. *Consider the formulas*

$$F := \exists x A(x) \wedge \forall x \neg \triangle A(x) \quad \textit{and} \quad \overline{F} := \exists x \triangle A(x) \wedge \forall x \neg \triangle A(x)$$

F is satisfiable in $G_{[0,1]}$. Indeed, $\Gamma_F = \{A(x) \equiv_\Delta \top,\ \bot \prec A(x) \prec \top\}$ while $\Sigma_F = \{\bot \prec A(x) \prec \top,\ A(x) \equiv_\Delta \bot\}$. The chain $C := \bot \prec A(x) \prec \top$ meets the satisfiability condition.

$\overline{F}$ is not satisfiable in $G_{[0,1]}$. Indeed, $\Gamma_{\overline{F}} = \{A(x) \equiv_\Delta \top\}$ while $\Sigma_{\overline{F}} = \Sigma_F$.

Remark 4.7. In constructing the satisfying evaluation φ, the interpretations of the formulas $P^i(c)$ in *all* equivalence classes (but $[\bot]$) could have been shifted closer and closer to 1 with increasing c. We instead did that only for formulas in the equivalence classes α_i with $i \geq k$ (cf. condition (3)), as this is used in the next section to make a similar proof working in presence of $\sim$.

4.3. Adding the involutive negation

The presence of the involutive negation does not change the decidability results for the satisfiability problem for $\mathsf{FO}^1_{mon}(\mathrm{V})$. The proof for the case 1 isolated in V proceeds exactly as that in Section 4.1. We show below how to modify the decidability proof in Section 4.2 to deal with $\sim$.

To define a chain normal form for formulas in $\mathsf{FO}^1_{mon}(\mathrm{V})$ with $\sim$ we allow the constant $\frac{1}{2}$ as predicate constant in the language and fix its evaluation under every interpretation to be $1/2$.

Definition 4.8 (Literal)**.** *Let $P(x)$ be an atomic formula. Both $P(x)$ and $\sim P(x)$ are called* literals. *We denote with* Atom(L) *the atomic formula for the literal L, i.e., if $L = P(x)$ or $L = \sim P(x)$, then in both cases* Atom$(L) = P(x)$.

Recall that when considering $\sim$ we require that the Gödel set V is symmetric with respect to the rational number $1/2$. The notion of $\triangle$-chains is extended to $\sim$-$\triangle$-chains as follows:

Definition 4.9 ($\sim$-$\triangle$-chain)**.** *Let F be any formula in $\mathsf{FO}^1_{mon}(V)$ with $\sim$, let $P_1, \ldots, P_n$ be the predicates occurring in F. A $\sim$-$\triangle$-*chain *over F is any formula of the form*

$$(\bot \bowtie_n L_n) \wedge (L_n \bowtie_{n-1} L_{n-1}) \wedge \ldots \wedge (L_1 \bowtie_1 \frac{1}{2}) \wedge (\frac{1}{2} \bowtie_1 M_1) \wedge (M_1 \bowtie_2 M_2) \wedge \ldots \wedge (M_n \bowtie_n \top)$$

such that

- *for all $1 \leq i \leq n$,* Atom(L_i) *and* Atom(M_i) *are in $\{P_1, \ldots, P_n\}$, i.e., all the L_i and M_i are literals made from the $P_1, \ldots, P_n$,*

- *for all i, L_i and M_i are dual literals, i.e., if $L_i = {\sim}P(x)$, then $M_i = {\sim}P(x)$, and if $L_i = {\sim}P(x)$, then $M_i = P(x)$, with $P \in \{P_1, \dots, P_n\}$,*
- $\{\text{Atom}(L_1), \dots, \text{Atom}(L_n)\} = \{P_1, \dots, P_n\}$ *(and thus also* $\{\text{Atom}(M_1), \dots, \text{Atom}(M_n)\} = \{P_1, \dots, P_n\}$*),*
- *$\bowtie_i$ is either $\prec$ or $\equiv_\Delta$, and at least one of the $\bowtie_i$'s is $\prec$.*

Note that in $\sim$-$\triangle$-chains each $\bowtie_i$ is mirrored on the left and right side w.r.t. $\frac{1}{2}$. This reflects the relation between dual literals.

Furthermore, the definition of syntactic evaluation (Def. 3.5) has to be extended for $\sim$ by letting

- $\Phi^C_{\sim A(x)} = {\sim}\Phi^C_{A(x)}$

As a consequence the structure of the equivalence classes induced by a $\sim$-$\triangle$-chain changes as follows

$$[\bot] = \beta_n \prec \dots \prec \beta_1 \prec [\frac{1}{2}] \prec \alpha_1 \prec \dots \prec \alpha_n = [\top]$$

where for all $1 \le i \le n$, if $\alpha_i = [L_k]$, then $\beta_i = [M_k]$, i.e., dual literals are representatives of equivalence classes with the same index, but on different side of $[\frac{1}{2}]$.

Proposition 4.10. *Let F be any formula in $\mathsf{FO}^1_{mon}(V)$ with $\sim$ and $A(x)$ any quantifier-free subformula of F. For each interpretation φ of $\mathbf{G}^{\sim}_V$ and each $\sim$-$\triangle$-chain C over F*

$$\varphi(C \wedge A(x)) = \varphi(C \wedge \Phi^C_{A(x)})$$

Proof. Similar to the proof of Proposition 3.6. □

Theorem 4.11. *The satisfiability problem for $\mathsf{FO}^1_{mon}(V)$ in $\mathbf{G}^{\Delta}_V$ extended with $\sim$ is decidable for all V.*

Proof. The case 1 isolated in V proceeds as in the proof of Theorem 4.3.

Assume that 1 is non isolated in V and F is the formula $\exists x A(x) \wedge \forall x B(x)$ of FO^1_{mon}(V) in $\mathbf{G}^{\sim}_V$. As in the proof of Theorem 4.5 (cf. Equation 5) F is satisfiable if and only if so is

$$F' = \bigvee_{C \in \Gamma} \exists x (C \wedge \Phi^C_{A(x)}) \wedge \forall x \bigvee_{C \in \Sigma} C$$

where Γ and Σ contain all $\sim$-$\triangle$-chains leading to 'syntactic evaluations' $\Phi^{C_i}_{A(x)}$ and $\Phi^{C_i}_{\triangle B(x)}$, respectively, different from $\bot$.

Claim: (*satisfiability condition*) F' is satisfiable if and only if

(sat$_a$) there is a $\sim$-$\triangle$-chain $C \in \Gamma \cap \Sigma$.

Moreover let

$$[\bot] = \beta_n \prec \dots \prec \beta_1 \prec [\frac{1}{2}] \prec \alpha_1 \prec \dots \prec \alpha_n = [\top]$$

be the equivalence classes induced by this $\sim$-$\triangle$-chain C, (sat$_b$) the equivalence class of $\Phi^C_{A(x)}$ is α_k for some $1 \leq k \leq n$, i.e., $[\Phi^C_{A(x)}] > [\frac{1}{2}]$.

($\Longleftarrow$) Condition (sat$_a$) is proved as in Theorem 4.5. Due to the fact that the valuations satisfies the existential quantifier, the equivalence class of the syntactic evaluation of $A(x)$ needs to be between (the equivalence class of) $\frac{1}{2}$ and (the equivalence class of) $\top$, which gives (sat$_b$).

($\Longrightarrow$) When conditions (sat$_a$) and (sat$_b$) hold we can define an interpretation $\varphi^{\sim}$ that is a model for F' (and hence for F) similarly to the interpretation φ in the proof of Theorem 4.5. Indeed let $[\Phi^C_{A(x)}] = \alpha_k$. As in the case without $\sim$, the idea is to push to 1 all the equivalences classes α_i greater or equal to α_k, i.e., for which $i \geq k$ (see Remark 4.7). Due to the presence of the constant $\frac{1}{2}$ this can be achieved only if $[\Phi^C_{A(x)}]$ is strictly greater than $[\frac{1}{2}]$. This is guaranteed by condition (sat$_b$).

For the case $V = [0, 1]$ we present below an explicit definition of the interpretation $\varphi^{\sim}(P(c))$, whose domain is the set of natural numbers plus $\frac{1}{2}$ (and n is the number of equivalence classes on each side of $\frac{1}{2}$ arising from the satisfiability condition above)

$$
\begin{aligned}
\varphi^{\sim}(\tfrac{1}{2}) &= 1/2 \\
\varphi^{\sim}(P(c)) &= 1/2(1 - i/n) && \text{if } P(x) \in \beta_i, i < k \\
\varphi^{\sim}(P(c)) &= 1/c^i \cdot 1/2(1 - i/n) && \text{if } P(x) \in \beta_i, i \geq k \\
\varphi^{\sim}(P(c)) &= 1 - \varphi^{\sim}(\sim P(c)) && \text{if } P(x) \in \alpha_i
\end{aligned}
$$

It is easy to see that the following properties hold:

(1)' $\varphi^{\sim}(P(c)) = 0$, for all c, if $P(x) \in \beta_n (= [\perp])$.

(2)' if $i < j$ then for all $P^i \in \alpha_i$ and $P^j \in \alpha_j$ and for all c, $\varphi^{\sim}(P^i(c)) < \varphi^{\sim}(P^j(c))$ (and, symmetrically if $i > j$ then for all $P^i \in \beta_i$ and $P^j \in \beta_j$, $\varphi^{\sim}(P^i(c)) < \varphi^{\sim}(P^j(c))$).

(3)' if $P(x) \in \alpha_i$ with $i \geq k$, then $\lim_{c \to \infty} \varphi^{\sim}(P(c)) = 1$

From the above properties easily follows that $\varphi^{\sim}(F') = 1$.

In the case of arbitrary (but symmetric) truth value sets we use the construction given in the proof of Theorem 4.5 for the α_i's (with $i \geq k$) and define the evaluations for all predicates in the symmetric equivalence classes β_i by 1 minus the evaluation of those in α_i. Finally, the evaluations of the predicates in the remaining equivalence classes are chosen to satisfy conditions $(1)'$ and $(2)'$.

The extension of the proof to the general case proceeds as in Theorem 4.5.

□

Remark 4.12. The results in this section also hold when 1/2 is not in the Gödel set V. In this case we still require that V is symmetric with respect to 1/2 and we define a $\sim$-$\triangle$-chain as $(\perp \bowtie_n L_n) \wedge \ldots (L_2 \bowtie_2 L_1) \wedge (L_1 < M_1) \wedge (M_1 \bowtie_2 M_2) \wedge \ldots \wedge (M_n \bowtie_n \top)$.

5. On the number of SAT-$\mathbf{G}_{\mathrm{V}}^{\triangle}$

Consider the two sets of formulas associated to each truth value set V (cf. Definition 2.5). In propositional logic the choice of any infinite subset of $[0,1]$ leads to the same set of valid formulas. The same holds for the set of satisfiable formulas, see [5]. At the first-order level different infinite Gödel sets V induce instead different sets of valid and of satisfiable formulas. For (validity) Gödel logics their number has been settled to countable in [7]. Nothing is known about the number of sets **SAT-**$\mathbf{G}_{\mathrm{V}}^{\triangle}$ of satisfiable formulas with infinite Gödel set V and $\triangle$. We show below that these are at least countable, as this is already the case when we restrict to monadic formulas of $\mathbf{G}_{\mathrm{V}}^{\triangle}$ only containing one variable. In contrast with this result, the decidability proofs in Section 4 also reveals that in $\mathsf{FO}^1_{mon}(\mathrm{V})$ only two infinite-valued Gödel logics extended with $\triangle$ (or $\sim$) differ w.r.t. satisfiability.

Proposition 5.1. *There are countably many distinct monadic and one-variable* ***SAT-G***$_{\mathrm{V}}^{\triangle}$*, with infinite Gödel set* V*.*

Proof. For $n \geq 1$ let V_n be the truth value set:

$$V_n = \{0,1\} \cup \left\{ \frac{k}{n+1} : 1 \leq k \leq n \right\} \cup \left\{ \frac{k}{n+1} + \frac{1}{l(n+1)} : 1 \leq k \leq n, l \geq 2 \right\}$$

V_n has exactly n accumulation points $\frac{k}{n+1}$ strictly between 0 and 1. Each of them is the infimum of the points $\frac{k}{n+1} + \frac{1}{l(n+1)}$. We show that **SAT-**$\mathbf{G}_{\mathrm{V_n}}^{\triangle}$, for $n \geq 1$, are all different. Consider indeed the following formula

$$\mathrm{INF}_k := \triangle(C_k \leftrightarrow \forall x P_k(x)) \wedge \forall x(C_k \prec P_k(x))$$

(where $\leftrightarrow$ and $\prec$ are defined as in Section 2). INF_k expresses that C_k is a proper infimum in the sense that under a given evaluations φ, INF_k evaluates to 1 if and only if the truth value of C_k is the infimum but not a minimum of the truth values of $P_k(c)$.

Using INF_k we can now define the formulas

$$F_n := \bot \prec C_1 \wedge C_1 \prec C_2 \wedge \cdots \wedge C_{n-1} \prec C_n \wedge C_n \prec \top \wedge \bigwedge_{k=1}^{n} \mathrm{INF}_k$$

that distinguish **SAT-**$\mathbf{G}_{\mathrm{V_m}}^{\triangle}$. For each $n \geq 1$, F_n expresses indeed the fact that there are at least n proper infimum in the open interval $(0,1)$. It is easy to see that $F_m \in$ **SAT-**$\mathbf{G}_{\mathrm{V_n}}^{\triangle}$, i.e., there is a V_n-interpretation satisfying F_m, if and only if $n \geq m$. The if part follows by the existence of more than m accumulation points in V_n while for the only if part notice that every V_n-interpretation with $n < m$ assigns to one of the INF_k a value less than 1. $\square$

In contrast with the above result, as an immediate consequence of Proposition 3.2 and the fact that there is only one set of classically satisfiable formulas, the following corollary holds

Corollary 5.2. *There is only one* ***SAT-G***$_{\mathrm{V}}$ *in* $\mathsf{FO}^1_{mon}(V)$ *without* $\triangle$*.*

The decision methods given in Theorems 4.3 and 4.5 show that the only difference between the sets of formulas $\mathbf{SAT\text{-}G}_{\mathrm{V}}^{\Delta}$ in $\mathsf{FO}^1_{mon}(\mathrm{V})$ is the isolation of 1, which gives the following corollary:

Corollary 5.3. *There are only two different* $\boldsymbol{SAT\text{-}G}_{\mathrm{V}}^{\Delta}$ *in* $\mathsf{FO}^1_{mon}(V)$.

6. Reduction to propositional satisfiability

We reduce the satisfiability problem for $\mathsf{FO}^1_{mon}(\mathrm{V})$ to satisfiability in suitable propositional finite-valued Gödel logics. As a corollary it follows that the satisfiability problem for $\mathsf{FO}^1_{mon}(\mathrm{V})$ is NP-complete for all Gödel sets V , with or without the involutive negation $\sim$.

Definition 6.1 (Propositional reduct)**.** *Let A be any formula in $\mathsf{FO}^1_{mon}(V)$ with or without $\sim$. The* propositional reduct A^p *of A is inductively defined as follows:*

$$\begin{aligned} P_i(x)^p &= P_i \qquad (*)^p = * \quad \textit{for } * \in \{0, 1, \frac{1}{2}\} \\ (\forall x A)^p &= A^p \qquad (\exists x A)^p = A^p \\ (A * B)^p &= A^p * B^p \quad \textit{for } * \in \{\wedge, \vee, \rightarrow\} \\ (*A)^p &= *A^p \qquad \textit{for } * \in \{\neg, \sim, \Delta\} \end{aligned}$$

Henceforth we denote by $\mathbf{G}^{\star}_{\infty}$ the propositional infinite-valued Gödel logic[7] extended with $\star \in \{\triangle, \sim\}$. The following theorem reduces SAT for (first-order) formulas in $\mathsf{FO}^1_{mon}(V)$ to SAT for propositional formulas.

Theorem 6.2. *Let V be any infinite Gödel set,*

$$F = \forall x A_1(x) \wedge \ldots \wedge \forall x A_m(x) \wedge \exists x B_1(x) \wedge \ldots \wedge \exists x B_n(x)$$

be any formula in $\mathsf{FO}^1_{mon}(V)$ (with or without $\sim$) and $A = \forall x \triangle (A_1(x) \wedge \ldots \wedge A_m(x))$.

1. *If* 1 *is isolated in V, we have*

$$F \in \boldsymbol{SAT\text{-}G}^{\star}_{\mathrm{V}} \quad \textit{if and only if} \quad \begin{array}{l} A^p \wedge (\exists x B_1(x))^p \in \boldsymbol{SAT\text{-}G}^{\star}_{\infty} \\ \textit{AND} \ \ldots \ \textit{AND} \\ A^p \wedge (\exists x B_n(x))^p \in \boldsymbol{SAT\text{-}G}^{\star}_{\infty} \end{array}$$

2. *If* 1 *is non isolated in V, we have*

$$F \in \boldsymbol{SAT\text{-}G}^{\star}_{\mathrm{V}} \quad \textit{if and only if} \quad \begin{array}{l} A^p \wedge X_1 \in \boldsymbol{SAT\text{-}G}^{\star}_{\infty} \\ \textit{AND} \ \ldots \ \textit{AND} \\ A^p \wedge X_n \in \boldsymbol{SAT\text{-}G}^{\star}_{\infty} \end{array}$$

   where $X_i = \neg\neg(\exists x B_i(x))^p$*, if* $\star = \triangle$ *and* $X_i = \neg{\sim}((\exists x B_i(x) \rightarrow \frac{1}{2}) \rightarrow \exists x B_i(x))^p$*, when* $\star = \sim$.

[7] Recall that for propositional formulas all infinite-valued Gödel logics coincide.

3. *If* 1 *is non isolated in* V *and* $F \in \boldsymbol{SAT\text{-}G}^{\star}_{\mathrm{V}}$, *then*

$$F \text{ is satisfiable in a finite model} \leftrightarrow A^p \wedge (\exists x B_i(x))^p \in \boldsymbol{SAT\text{-}G}^{\star}_{\infty} \text{ for all } i = 1, \dots, n.$$

Proof. 1. Immediately follows by Lemma 4.2.

2. The satisfiability conditions in the proof of Theorem 4.5 and 4.11 correspond, on the propositional side, to the satisfiability of $A^p \wedge \neg\neg(B_i(x))^p$ (i.e., the syntactic evaluation $\Phi^C_{B_i(x)}$ evaluates to an atom not in the equivalence class of $\bot$), for the case $\star = \triangle$, and to $X_i = \neg{\sim}((\exists x B_i(x) \to \frac{1}{2}) \to \exists x B_i(x))^p$ (i.e. the syntactic evaluation $\Phi^C_{B_i(x)}$ evaluates to an atom in an equivalence class bigger than $[\frac{1}{2}]$), for the case $\star = {\sim}$.

3. By Proposition 2.8, F is satisfiable in $\mathbf{G}^{\sim}_V$ if and only if so is $A \wedge \exists x B_1(x) \wedge \dots \wedge \exists x B_n(x)$. ($\Longrightarrow$) Let φ^f be a finite interpretation that is a model for F. Then for each $j = 1, \dots n$ there exists c_j in its domain such that $\varphi^f(B_j(c_j)) = 1$. Therefore all $\triangle(A_1(c_i) \wedge \dots \wedge A_m(c_i)) \wedge B_i(c_i)$ for $1 \le i \le n$ are satisfiable in φ^f which induces the propositional evaluations satisfying each $A^p \wedge (\exists x B_i(x))^p$ in $\mathbf{G}^{\star}_{\infty}$. ($\Longleftarrow$) Assume that each $A^p \wedge (\exists x B_i(x))^p \in \mathbf{SAT\text{-}G}^{\star}_{\infty}$ and φ^{∞}_i is a model in $\mathbf{G}^{\star}_{\infty}$. Let $P_1, \dots, P_l$ be the atomic formulas in $A^p \wedge B^p_i$, for all $i = 1, \dots, n$. A (finite) model for F in $\mathbf{G}^{\star}_V$ is simply defined by taking $c_1, \dots, c_n$ as domain elements and assigning to each $P_i(c_j)$ a value in V which respects the ordering of the values $\varphi^{\infty}_j(P_i)$ for all $i = 1, \dots, l$. ☐

The above theorem together with the proposition below allow us to reduce the satisfiability problem for $\mathsf{FO}^1_{mon}(\mathrm{V})$ to a check on propositional finite-valued Gödel logics. Henceforth G_k will stand for propositional Gödel logic with k truth values.

Lemma 6.3. *Let* A *be a propositional formula containing* n *distinct variables.*

$$A \in \boldsymbol{SAT\text{-}G}^{\star}_{\infty} \quad \textit{iff} \quad A \in \boldsymbol{SAT\text{-}G}^{\triangle}_{n+2}$$

If A *contains* $\sim$ *then*

$$A \in \boldsymbol{SAT\text{-}G}^{\star}_{\infty} \quad \textit{iff} \quad A \in \boldsymbol{SAT\text{-}G}^{\triangle}_{2n+3}$$

Proof. In presence of $\sim$ we have to consider, for each of the $(n+2)$ values, their negation w.r.t. $\sim$, i.e. for each x, also $1 - x$. These are $2n + 2$ (notice that 0 is the negation of 1, and vice versa) with in addition the value 1/2. ☐

Corollary 6.4. *Let* F *be a formula in* $\mathsf{FO}^1_{mon}(V)$ *containing* n *different predicates.* F *is satisfiable in* $\mathbf{G}^{\triangle}_V$ *(*$\mathbf{G}^{\sim}_V$*, respectively) if and only if the corresponding propositional formulas in Theorem 6.2 are satisfiable in* $\mathbf{G}^{\triangle}_{n+2}$ *(*$\mathbf{G}^{\sim}_{2n+3}$*, respectively).*

Example 6.5. *Consider the formula* $F = \exists x A(x) \wedge \forall x \neg \triangle A(x)$ *of Example 4.4. If* 1 *is non isolated in* V*, then* F *is satisfiable in* $\mathbf{G}_V$ *iff* $(\triangle\neg\triangle A) \wedge \neg\neg A$ *is satisfiable in* $\mathbf{G}^{\triangle}_{\infty}$*. The satisfiability of the propositional formula can be checked in* $\mathbf{G}^{\triangle}_3$*. Note that* F *has no finite models being* $(\triangle\neg\triangle A) \wedge A$ *not satisfiable in* $\mathbf{G}^{\triangle}_{\infty}$*.*

Corollary 6.6. *The satisfiability problem for* FO^1_{mon}*(V) with and without* $\sim$ *is NP-complete.*

Proof. The inclusion in the class NP follows by Corollary 6.4 and e.g., [13]. For the NP-completeness note that SAT in propositional classical logic can be expressed as SAT in $\mathbf{G}^{\triangle}_{\infty}$ by prefixing with $\triangle$ each variable in the classical formula. ☐

Final Remark

CADIAG-2 (Computer Assisted DIAGnosis) is a 'MYCIN-like' expert system assisting in the differential diagnosis in internal medicine, developed at the Medical University of Vienna. Its knowledge base contains more than 20.000 IF-THEN rules expressing relationships between medical entities, e.g., patient's symptoms and diagnoses. In most cases, the relationships and the involved entities are not boolean (yes/no). To check the representation of the medical knowledge in the system, CADIAG-2's rules were formalized in [9] as suitable formulas of $\mathbf{G}^{\sim}_{[0,1]}$ belonging to the class $\mathsf{FO}^1_{mon}(\mathrm{V})$. The resulting formalization is consistency preserving, that is the unsatisfiability in $\mathbf{G}^{\sim}_{[0,1]}$ of the logical formulas implies the existence of errors in the system's rules. The (un)decidability status of the satisfiability problem for these formulas was left open. Theorem 4.11 provides an answer to this question. Furthermore, Corollary 6.4 can be used to actually check the rules of CADIAG-2. This calls first for the development of suitable provers and SAT solvers for propositional finite-valued Gödel logics extended with $\sim$, capable of handling the large set of logical formulas representing the system' rules.

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