A Schütte-Tait style cut-elimination proof for first-order Gödel logic

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Abstract. We present a Schütte-Tait style cut-elimination proof for the hypersequent calculus **HIF** for first-order Gödel logic. This proof allows to bound the depth of the resulting cut-free derivation by $4^{|d|}_{\rho(d)}$, where $|d|$ is the depth of the original derivation and $\rho(d)$ the maximal complexity of cut-formulas in it. We compare this Schütte-Tait style cut-elimination proof to a Gentzen style proof.

1 Introduction

The most important cut-elimination methods in first-order proof theory are the *Gentzen style* procedure [10] (and its variants in the context of natural deduction calculi) and the *Schütte-Tait style* procedure [13, 14]. The latter has been originally introduced to deal with infinitary calculi. From a procedural point of view, these methods differ by their *cut selection rule*: the Gentzen style method selects a highest cut, while the Schütte-Tait style method a largest one (w.r.t. the number of connectives and quantifiers). Consequently, e.g., Gentzen style procedures, generally, will not terminate on calculi with ω rules.

In this paper we formulate cut-elimination proofs, according to both methods, for the hypersequent calculus **HIF** for first-order Gödel logic $\mathbf{G}_\infty$. This logic, also known as intuitionistic fuzzy logic [16], can be axiomatized extending intuitionistic logic IL by the linearity axiom $(A \supset B) \vee (B \supset A)$ and the shifting law of universal quantifier $\forall x(A(x) \vee B) \supset \forall x A(x) \vee B$, where x does not occur free in B. **HIF** has been defined in [8] by incorporating Gentzen's original calculus **LJ** for IL as a sub-calculus and adding to it an additional layer of information by allowing **LJ**-sequents to live in the context of finite multisets of sequents (called *hypersequents*). This opens the possibility to define new rules that, "exchanging information" between different sequents, allow to prove both the linearity axiom and the shifting law of universal quantifier.

The Schütte-Tait style cut-elimination proof introduced in this paper establishes non-elementary primitive recursive bounds for the length of the cut-free proofs in **HIF** in terms of the length and the maximal complexity of cut-formulas in the original proof. Consequently, corresponding bounds apply to the length of Herbrand disjunctions (mid-hypersequents) as well as the length of derivations in the chaining calculus described in [5] for the prenex fragment of $\mathbf{G}_\infty$.

* Research supported by EC Marie Curie fellowship HPMF–CT–1999–00301

Finally, this paper allows to compare both the Gentzen and the Schütte-Tait style procedures in the more general context of the hypersequent notation.

2 Syntax and Semantic of first-order Gödel logic

Propositional finite-valued Gödel logics have been introduced by Gödel in 1933 [11] to show that intuitionistic logic does not have a characteristic finite matrix. Dummett [9] later generalized these to an infinite set of truth-values, and showed that the set of its tautologies – **LC** – is axiomatized extending intuitionistic logic by the linearity axiom $(A \supset B) \vee (B \supset A)$.

The language of Gödel logics is identical to that of classical logic (or intuitionistic logic, for that matter). More precisely, we use the binary *connectives* $\wedge, \vee$, and $\supset$ and the *truth constant* $\bot$. $\neg A$ is defined as $A \supset \bot$. *Object variables* are denoted by x, y, ...; the usual existential and universal *quantifiers*, $\forall$ and $\exists$, refer to these variables. *Bound* and *free* occurrences of variables are defined as usual. Moreover, for every $n \geq 0$, there is an infinite supply of n-ary *predicate symbols* and *function symbols*. Constants are considered as 0-ary function symbols. *Terms* and *formulas* are inductively defined in the usual way. Propositional variables are identified with predicate symbols of arity 0.

In this work we consider the first-order Gödel logic $\mathbf{G}_\infty$ defined over the real unit interval $[0,1]$ [1], also known as *intuitionistic fuzzy logic* [16].

An *interpretation* $\mathcal{I}$ in $\mathbf{G}_\infty$ consists of a non-empty *domain* D and a *valuation function* $v_\mathcal{I}$ that maps constants and object variables to elements of D and n-ary function symbols to functions from D^n into D. $v_\mathcal{I}$ extends in the usual way to function mapping all terms of the language to an element of the domain. Moreover, $v_\mathcal{I}$ maps every n-ary predicate symbol P to a function from D^n into $[0,1]$. The truth-value of an atomic formula $A \equiv P(t_1, \ldots, t_n)$ is thus defined as

$$v_\mathcal{I}(A) = v_\mathcal{I}(P)(v_\mathcal{I}(t_1), \ldots, v_\mathcal{I}(t_n)).$$

For the truth constant $\bot$ we have $v_\mathcal{I}(\bot) = 0$.

The semantics of propositional connectives is given by

$$v_\mathcal{I}(A \supset B) = \begin{cases} 1 & \text{if } v_\mathcal{I}(A) \leq v_\mathcal{I}(B) \\ v_\mathcal{I}(B) & \text{otherwise,} \end{cases}$$

$$v_\mathcal{I}(A \wedge B) = \min(v_\mathcal{I}(A), v_\mathcal{I}(B)) \qquad v_\mathcal{I}(A \vee B) = \max(v_\mathcal{I}(A), v_\mathcal{I}(B)).$$

To assist a concise formulation of the semantics of quantifiers we define the *distribution* of a formula A and a free variable x with respect to an interpretation $\mathcal{I}$ as $\mathrm{Distr}_\mathcal{I}(A(x)) = \{\mathrm{val}_{\mathcal{I}'}(A(x)) \mid \mathcal{I}' \sim_x \mathcal{I}\}$, where $\mathcal{I}' \sim_x \mathcal{I}$ means that $\mathcal{I}'$ is exactly as $\mathcal{I}$ with the possible exception of the domain element assigned to x. The semantics of quantifiers is given by the infimum and supremum of the corresponding distribution:

$$v_\mathcal{I}((\forall x)A(x)) = \inf \mathrm{Distr}_\mathcal{I}(A(x)) \quad v_\mathcal{I}((\exists x)A(x)) = \sup \mathrm{Distr}_\mathcal{I}(A(x)).$$

[1] Different topologies on the set of truth values induce different first-order Gödel logics.

A formula A is a *tautology* iff for all $v_{\mathcal{I}}$, $v_{\mathcal{I}}(A) = 1$. Moreover A is a *logical consequence* of a set of formulas Γ (in symbols $\Gamma \models_{\mathbf{G}_\infty} A$) iff, for all $v_{\mathcal{I}}$, $\min\{v_{\mathcal{I}}(\gamma) \mid \gamma \in \Gamma\} \leq v_{\mathcal{I}}(A)$.

A Hilbert style calculus for $\mathbf{G}_\infty$ is obtained by extending **LC** with the shifting law of universal quantifier $\forall x(A(x) \vee B) \supset \forall x A(x) \vee B$, where x does not occur free in B, see, e.g., [12].

3 Hypersequent Calculi for $\mathbf{G}_\infty$

In [8] an analytic calculus for $\mathbf{G}_\infty$ has been introduced. This calculus —called **HIF**[2]— uses hypersequents, a natural generalization of Gentzen sequents, see [3]. **HIF** is based on Avron's hypersequent calculus **GLC** for **LC** [2].

The most significant feature of **HIF** is its close relation to Gentzen's sequent calculus **LJ** for intuitionistic logic [10]. Indeed, **HIF** contains **LJ** as a sub-calculus and simply adds it an additional layer of information by allowing **LJ**-sequents to live in the context of finite multisets of sequents, as well as suitable (external) structural rules to manipulate sequents with respect to their contexts. In particular, the crucial rule of the calculus **HIF**, added to **LJ**, is the so called communication rule (*com*). It is this rule which increases the expressive power of **HIF** compared to **LJ**.

Recall that a *sequent* is an expression of the form $\Gamma \Rightarrow A$, where Γ is a multiset of formulas and A may be empty.

Definition 1. *A* hypersequent *is a multiset*

$$\Gamma_1 \Rightarrow A_1 \mid \ \ldots \ \mid \Gamma_n \Rightarrow A_n$$

where for every $i = 1, \ldots, n$, $\Gamma_i \Rightarrow A_i$ is a sequent, called component *of the hypersequent.*

The interpretation of the symbol " | " is disjunctive.

In **HIF** the rules for connectives and quantifiers, as well as the internal structural rules, are those of **LJ**. The only difference is the presence of a context G representing a (possibly empty) hypersequent. The structural rules are divided into *internal* and *external rules*. The former deal with formulas within components. These are weakening and contraction. The external rules manipulate whole components within a hypersequent. These are external weakening (EW), contraction (EC), as well as the (*com*) rule. More precisely, **HIF** consists of

Axioms and Cut Rule

$$A \Rightarrow A \ \ (id) \qquad \bot \Rightarrow \ \ (\bot) \qquad \frac{G \mid \Gamma \Rightarrow A \quad G \mid A, \Gamma \Rightarrow C}{G \mid \Gamma \Rightarrow C}\ (cut)$$

[2] **HIF** stands for Hypersequent calculus for Intuitionistic Fuzzy logic.

Internal Structural Rules

$$\frac{G \mid \Gamma \Rightarrow C}{G \mid \Gamma, A \Rightarrow C}\ (w,l) \qquad \frac{G \mid \Gamma \Rightarrow}{G \mid \Gamma \Rightarrow A}\ (w,r) \qquad \frac{G \mid \Gamma, A, A \Rightarrow C}{G \mid \Gamma, A \Rightarrow C}\ (c,l)$$

External Structural Rules

$$\frac{G}{G \mid \Gamma \Rightarrow A}\ (EW) \qquad \frac{G \mid \Gamma \Rightarrow A \mid \Gamma \Rightarrow A}{G \mid \Gamma \Rightarrow A}\ (EC)$$

$$\frac{G \mid \Gamma_1, \Gamma_2 \Rightarrow A \quad G \mid \Gamma_1, \Gamma_2 \Rightarrow B}{G \mid \Gamma_1 \Rightarrow A \mid \Gamma_2 \Rightarrow B}\ (com)$$

Logical Rules

$$\frac{G \mid \Gamma, A \Rightarrow B}{G \mid \Gamma \Rightarrow A \supset B}\ (\supset, r) \qquad \frac{G \mid \Gamma \Rightarrow A \quad G \mid B, \Gamma \Rightarrow C}{G \mid \Gamma, A \supset B \Rightarrow C}\ (\supset, l)$$

$$\frac{G \mid \Gamma \Rightarrow A \quad G \mid \Gamma \Rightarrow B}{G \mid \Gamma \Rightarrow A \wedge B}\ (\wedge, r) \qquad \frac{G \mid \Gamma, A_i \Rightarrow C}{G \mid \Gamma, A_1 \wedge A_2 \Rightarrow C}\ (\wedge_i, l)_{i=1,2}$$

$$\frac{G \mid \Gamma \Rightarrow A_i}{G \mid \Gamma \Rightarrow A_1 \vee A_2}\ (\vee_i, r)_{i=1,2} \qquad \frac{G \mid \Gamma, A \Rightarrow C \quad G \mid \Gamma, B \Rightarrow C}{G \mid \Gamma, A \vee B \Rightarrow C}\ (\vee, l)$$

$$\frac{G \mid A(t), \Gamma \Rightarrow C}{G \mid (\forall x)A(x), \Gamma \Rightarrow C}\ (\forall, l) \qquad \frac{G \mid \Gamma \Rightarrow A(a)}{G \mid \Gamma \Rightarrow (\forall x)A(x)}\ (\forall, r)$$

$$\frac{G \mid A(a), \Gamma \Rightarrow C}{G \mid (\exists x)A(x), \Gamma \Rightarrow C}\ (\exists, l) \qquad \frac{G \mid \Gamma \Rightarrow A(t)}{G \mid \Gamma \Rightarrow (\exists x)A(x)}\ (\exists, r)$$

where $(\forall, r)$ and $(\exists, l)$ must obey the eigenvariable condition: the free variable a must not occur in the lower *hypersequent*.

Definition 2. *In the above rules, Γ and C are called* internal contexts *while G,* external context*. For each rule, the components not in the external context are called* active components*. In the conclusion of each logical rule, the formula in the active component that does not belong to the internal context is called* principal formula*.*

Remark 1. By the presence of (c, l) and (w, l) (resp. (EW) and (EC)), one can derive equivalent versions of the above rules with *multiplicative* internal (resp. external) contexts (see, e.g., [17] for this terminology).

In fact, **HIF** has been originally defined in [8] using a different version of the communication rule, namely

$$\frac{G \mid \Pi_1, \Gamma_1 \Rightarrow A \quad G \mid \Pi_2, \Gamma_2 \Rightarrow B}{G \mid \Pi_1, \Pi_2 \Rightarrow A \mid \Gamma_1, \Gamma_2 \Rightarrow B}\ (com')$$

However, using (w, l) and (c, l), (com) and (com') are interderivable (see [3]).

Definition 3. *The* complexity $|A|$ *of a formula* A *is inductively defined as follows:*

- $|A| = 0$ *if* A *is atomic*
- $|A \wedge B| = |A \vee B| = |A \supset B| = \max(|A|, |B|) + 1$
- $|\forall x A(x)| = |\exists x A(x)| = |A| + 1$

The right (left) rank *of a cut is the number of consecutive hypersequents containing the cut formula, counting upward from the right (left) upper sequent of the cut.*

For the cut-elimination proof in the next section, following Tait [14], we shall consider an equivalent version of **HIF** without explicit (internal and external) contraction rules. In this calculus, we call it $\mathbf{HIF}^{\mathbf{set}}$, hypersequents are considered as *sets* of components, each one of them is a sequent $\Gamma \Rightarrow A$, where Γ is a *set* of formulas. Henceforth, we denote with $\{S_1\} \cup ... \cup \{S_n\}$ a hypersequent in $\mathbf{HIF}^{\mathbf{set}}$ whose components are $S_1, \ldots, S_n$. Rules, are then changed accordingly. Moreover, we only consider atomic axioms, that is of the form

$$A \Rightarrow A \quad \text{and} \quad \bot \Rightarrow \quad \text{where } A \text{ is an atomic formula}$$

Lemma 1. *In* $\mathbf{HIF}^{\mathbf{set}}$ *non atomic axioms can be derived from atomic axioms.*

A derivation d in **HIF** (or $\mathbf{HIF}^{\mathbf{set}}$) is considered, as usual, as an upward rooted tree of hypersequents generated from subtrees by applying the inference rules. This allows for the following definitions:

Definition 4. *The* length $|d|$ *of* d *is the maximal number of inference rules (but weakenings) + 1 occurring on any branch of* d*.*

Remark 2. A different way to avoid counting the number of applications of (internal and external) weakening rules while counting the length of a derivation, is to internalize these rules into axioms. This is done by considering axioms of the form $G \mid \Gamma, A \Rightarrow A$. Then $|d|$ can be simply defined as the maximal number of hypersequents occurring on any branch of d. However, in this case one has to use the multiplicative version of (com), namely (com'), that affects both Lemma 2.2 (see Remark 3) and Definition 7 below.

Definition 5. *Let* d_i*, with* $i < k$*, be the direct subderivations of* d*. The* cut-rank $\rho(d)$ *of* d *is defined by induction as:*

- $\rho(d) = 0$ *if* d *is cut-free*
- $\rho(d) = \max_{i<k} \rho(d_i)$ *if the last inference of* d *is not a cut;*
- $\rho(d) = \max(|A| + 1, \max_{i<k} \rho(d_i))$*, where* A *is the cut formula, otherwise.*

Henceforth we write $d \vdash' \mathrm{H}$ (resp. $d \vdash \mathrm{H}$) if d is a derivation in **HIF** (resp. $\mathbf{HIF}^{\mathbf{set}}$) of H.

Definition 6. *We say that a sequent is* n-reduced *if every formula in the antecedent occurs at most* n *times. A hypersequent is said to be* n-m-reduced *if it is* n*-reduced and every component in it occurs at most* m *times.*

Note that a derivation in $\mathbf{HIF^{set}}$ only contains 1-1-reduced hypersequents.

Let d be a derivation in $\mathbf{HIF^{set}}$. Henceforth we will indicate with $w(d)$ (resp. W(d)) the maximal number of applications of internal weakening (resp. external weakening) occurring on any branch of d.

Lemma 2. *Let H be a 1-1-reduced hypersequent.*

1. *If $d \vdash'$ H, one can find a proof $d' \vdash$ H such that $|d'| \leq |d|$.*
2. *If $d' \vdash$ H one can find a proof $d \vdash'$ H such that $|d| \leq 2|d'| + w(d')$.*

Proof. 1. Straightforward.

2. We show that d does not contain more than two applications of (c, l) and/or (EC) after each inference step in d' (but weakenings). The proof proceeds by induction on $|d'|$. The claim is trivial if H is an axiom. Suppose that the last rule applied in d' is $(\supset, l)$ and d' ends as follows

$$\dfrac{\begin{array}{cc} \vdots\, d'_1 & \vdots\, d'_2 \\ G \,\cup\, \{\Gamma \Rightarrow A\} & G \,\cup\, \{\Gamma, B \Rightarrow C\} \end{array}}{G \,\cup\, \{\Gamma, A \supset B \Rightarrow C\}}\ (\supset,\mathrm{l})$$

By induction hypothesis one can find two proofs d''_1 and d''_2 in **HIF** with the required properties of the 1-1-reduced hypersequents $(G \mid \Gamma \Rightarrow A)^{\#}$ and $(G \mid \Gamma, B \Rightarrow C)^{\#}$. Applying to them the $(\supset, l)$ rule one obtains the hypersequent $G^{\#} \,\cup\, \Gamma^{\#}, A \supset B \Rightarrow C$ that can have at most two equal formulas (if $A \supset B \in \Gamma^{\#}$) and two equal components (if the component $\Gamma^{\#}, A \supset B \Rightarrow C$ is in $G^{\#}$). With at most one application of (c, l) and of (EC), one obtains a 1-1-reduced contraction of $G \mid \Gamma, A \supset B \Rightarrow C$. The cases involving the remaining logical rules as well as the cut rule are analogous.

Suppose the last rule applied in d' is (*com*) and d' ends as follows

$$\dfrac{\begin{array}{cc} \vdots\, d'_1 & \vdots\, d'_2 \\ G \,\cup\, \{\Gamma, \Sigma \Rightarrow A\} & G \,\cup\, \{\Gamma, \Sigma \Rightarrow B\} \end{array}}{G \,\cup\, \{\Gamma \Rightarrow A\} \,\cup\, \{\Sigma \Rightarrow B\}}\ (\mathrm{com})$$

By induction hypothesis one can find two proofs d''_1 and d''_2 in **HIF** with the required properties of the 1-1-reduced hypersequents $(G \mid \Gamma, \Sigma \Rightarrow A)^{\#}$ and $(G \mid \Gamma, \Sigma \Rightarrow B)^{\#}$. Applying the (com) rule to them one obtains the hypersequent $G^{\#} \mid \Gamma^{\#} \Rightarrow A \mid \Sigma^{\#} \Rightarrow B$ in which there can be at most two pairwise equal components (if both the components $\Gamma^{\#} \Rightarrow A$ and $\Sigma^{\#} \Rightarrow B$ are in $G^{\#}$). Applying (EC) at most twice, one obtains a 1-1-reduced contraction of $G \mid \Gamma \Rightarrow A \mid \Sigma \Rightarrow B$.

If the last rule applied in d' is (EW) then the corresponding proof in **HIF** does not contain any additional application of (EC) or (c, l). While in the case of internal weakening one can need an additional application of (EC).

Remark 3. Using multiplicative rules in defining **HIF** and $\mathbf{HIF^{set}}$, the bound on $|d|$ in Lemma 2.2 does not hold anymore.

3.1 A Schütte-Tait style cut-elimination proof

Let $d(s)$ and $H(s)$ denote the result of substituting the term s for all free occurrences of x in the proof $d(x)$ and in the hypersequent $H(x)$, respectively.

Lemma 3 (Substitution). *If $d(x) \vdash \mathrm{H(x)}$, then $d(s) \vdash \mathrm{H(s)}$, with $|d(s)| = |d(x)|$ and $\rho(d(s)) = \rho(d(x))$, where s only contains variables that do not occur in $d(x)$.*

We introduce the notion of *decorated* formulas in a derivation d of $\mathbf{HIF^{set}}$. This notion is intended to trace the cut-formula through d.

Definition 7. *Let $d \vdash \mathrm{H}$ and A be a formula in H that is not the cut-formula of any cut in d. The* decoration *of A (in d) is inductively defined as follows: we denote by A^* a decorated occurrence of A. Given a hypersequent H' in d with some (not necessarily all) decorated A. Let R be the rule introducing H'. We distinguish some cases according to R.*

1. *R is a logical rule, e.g.,*
$$\frac{G \ \cup\ \{\Gamma' \Rightarrow C'\}}{G \ \cup\ \{\Gamma \Rightarrow C\}}$$
   (a) *A is principal in R. Suppose $A^* \in \Gamma$. In the active component, $A^* \in \Gamma'$ if and only if A is a side formula of the inference. Moreover, the decoration in the not-active components of the premise of R is as in the conclusion. That is, for each such a component $\{\Sigma \Rightarrow B\} \in G$, $A^* \in \Sigma$ if and only if $A^* \in \Sigma$ of the corresponding component belonging to the conclusion of R.*
   Suppose C is A^*. The decoration of the not-active components of the premise of R is as in the conclusion.
   (b) *A is not principal in R. If $A^* \in \Gamma$ (resp. C) then $A^* \in \Gamma'$ (resp. C') in the active component. Moreover, the decoration of the not-active components of the premise of R is as in the conclusion.*
   If R is a two premises rule, the definition is analogous.
2. *R is (EW). The decoration of the components in the premise of R is as in the conclusion.*
3. *R is (w, l) or (w, r). Analogous to case 1.*
4. *R is (com).*
$$\frac{G \ \cup\ \{\Gamma, \Sigma \Rightarrow C\} \quad G \ \cup\ \{\Gamma, \Sigma \Rightarrow C'\}}{G \ \cup\ \{\Gamma \Rightarrow C\} \ \cup\ \{\Sigma \Rightarrow C'\}}$$
   Suppose $A^* \in \Gamma$. If $A \notin \Sigma$ (or $A^* \in \Sigma$), then $A^* \in \Gamma, \Sigma$ of both the active components in the premises of R. If A occurs in Σ, then $A^* \in \Gamma, \Sigma$ of only one active component in the premises of R. Suppose $A^* \notin \Gamma$. If $A \notin \Gamma$ and $A^* \in \Sigma$, then $A^* \in \Gamma, \Sigma$ of both the active components in the premises of R. If $A \in \Gamma$ and $A^* \in \Sigma$, then $A^* \in \Gamma, \Sigma$ of only one active component in the premises of R. The decoration in the not-active components in the premises of R is as in the conclusion.
   If C (and/or C') is A^*, then so is in the active component $\{\Gamma, \Sigma \Rightarrow C\}$ (and/or $\{\Gamma, \Sigma \Rightarrow C'\}$). The decoration in the not-active components of the premises of R is as in the conclusion.

5. *R is (cut). Analogous to case 1(b).*

Remark 4. Due to the (com) rule, the decoration of A (in d) is not unique.

Lemma 4 (Inversion).

(i) *If* $d \vdash \mathrm{G} \ \cup \ \{\Gamma, \mathrm{A} \vee \mathrm{B} \Rightarrow \mathrm{C}\}$ *then one can find* $d_1 \vdash \mathrm{G} \ \cup \ \{\Gamma, \mathrm{A} \Rightarrow \mathrm{C}\}$ *and* $d_2 \vdash \mathrm{G} \ \cup \ \{\Gamma, \mathrm{B} \Rightarrow \mathrm{C}\}$
(ii) *If* $d \vdash \mathrm{G} \ \cup \ \{\Gamma, \mathrm{A} \wedge \mathrm{B} \Rightarrow \mathrm{C}\}$ *then one can find* $d_1 \vdash \mathrm{G} \ \cup \ \{\Gamma, \mathrm{A}, \mathrm{B} \Rightarrow \mathrm{C}\}$
(iii) *If* $d \vdash \mathrm{G} \ \cup \ \{\Gamma \Rightarrow \mathrm{A} \wedge \mathrm{B}\}$ *then one can find* $d_1 \vdash \mathrm{G} \ \cup \ \{\Gamma \Rightarrow \mathrm{A}\}$ *and* $d_2 \vdash \mathrm{G} \ \cup \ \{\Gamma \Rightarrow \mathrm{B}\}$
(iv) *If* $d \vdash \mathrm{G} \ \cup \ \{\Gamma \Rightarrow \mathrm{A} \supset \mathrm{B}\}$ *then one can find* $d_1 \vdash \mathrm{G} \ \cup \ \{\Gamma, \mathrm{A} \Rightarrow \mathrm{B}\}$
(v) *If* $d \vdash \mathrm{G} \ \cup \ \{\Gamma, \exists \mathrm{x}\mathrm{A}(\mathrm{x}) \Rightarrow \mathrm{C}\}$ *then one can find* $d_1 \vdash \mathrm{G} \ \cup \ \{\Gamma, \mathrm{A}(\mathrm{a}) \Rightarrow \mathrm{C}\}$
(vi) *If* $d \vdash \mathrm{G} \ \cup \ \{\Gamma \Rightarrow \forall \mathrm{x}\mathrm{A}(\mathrm{x})\}$ *then one can find* $d_1 \vdash \mathrm{G} \ \cup \ \{\Gamma \Rightarrow \mathrm{A}(\mathrm{a})\}$

such that $\rho(d_i) \leq \rho(d)$ *and* $|d_i| \leq |d|$, *for* $i = 1, 2$.

Proof. (i) Let us consider the decoration of $A \vee B$ in d starting from $G \cup \{\Gamma, (A \vee B)^* \Rightarrow C\}$. To obtain the required derivation $d_1 \vdash \mathrm{G} \ \cup \ \{\Gamma, \mathrm{A} \Rightarrow \mathrm{C}\}$ (resp. $d_2 \vdash \mathrm{G} \ \cup \ \{\Gamma, \mathrm{B} \Rightarrow \mathrm{C}\}$), we delete all the right (resp. left) subderivations above any application of $(\vee, l)$ in which the decorated formula $(A \vee B)^*$ is principal and we replace every component $\psi, (A \vee B)^* \Rightarrow D$ in the derivation with $\psi, A \Rightarrow D$ (resp. $\psi, B \Rightarrow D$). (Recall that all axioms are atomic). Clearly $|d_i| \leq |d|$ and $\rho(d_i) \leq \rho(d)$, for $i = 1, 2$.

The remaining cases are analogous.

Remark 5. $(\supset, l)$, $(\forall, l)$ and $(\exists, r)$ are not invertible. Concerning $(\vee, r)$, one has $G \ \cup \ \{\Gamma \Rightarrow A \vee B\}$ can be inverted to $G \ \cup \ \{\Gamma \Rightarrow A\} \ \cup \ \{\Gamma \Rightarrow B\}$ (slightly changing the above bounds). The Schütte-Tait style cut-elimination procedure presuppose that at least one of the two premises of the cut rule is invertible. As we shall see, we will use (i), (iii), (iv), (v) and (vi). Of course we could choose (ii) instead of (iii) or the inversion of $(\vee, r)$ instead of (i). However, the latter choice will transform **LJ**-derivations into derivations containing hypersequents with more than one component.

In the following we write $d, H \vdash \mathrm{G}$ if d is a proof in $\mathbf{HIF^{set}}$ of G from the assumption H. Moreover, $H[{}^B/_A]$ will indicate the hypersequent H in which we uniformly replace A by B.

Lemma 5. *Let* $d \vdash \mathrm{G} \cup \{\Gamma, \mathrm{A}^* \Rightarrow \mathrm{B}\}$, *where* A^* *is an atomic formula decorated in* d *that is not the cut formula of any cut in* d. *One can find a proof* $d', \{\Sigma \Rightarrow A\} \vdash \mathrm{G} \ \cup \ \{\Gamma, \Sigma \Rightarrow \mathrm{B}\}$ *such that* $|d'| \leq |d|$ *and* $\rho(d') = \rho(d)$.

Proof. We replace A^* everywhere in d with Σ. The decorated formula originates

1. in an axiom. Then the axiom is transformed into $\Sigma \Rightarrow A$,
2. by an internal weakening. The weakening on A^* is replaced by stepwise weakenings of formulas B, where $B \in \Sigma$. Note that this does not affect the length of the resulting proof,

3. by an external weakening. The weakening in the component C is replaced by a weakening on $C[{}^{\Sigma}/_{A^*}]$.

The resulting proof d' is correct as it can be shown by induction on $|d'|+w(d')+W(d')$.

Lemma 6 (Reduction). *Let* $d_0 \vdash \mathrm{G} \ \cup \ \{\Gamma \Rightarrow \mathrm{A}\}$ *and* $d_1 \vdash \mathrm{G} \ \cup \ \{\Gamma, \mathrm{A} \Rightarrow \mathrm{C}\}$ *both with cut-rank* $\rho(d_i) \leq |A|$. *Then we can find a derivation* $d \vdash \mathrm{G} \cup \{\Gamma \Rightarrow \mathrm{C}\}$ *with* $\rho(d) \leq |A|$ *and* $|d| \leq 2(|d_0| + |d_1|)$.

Proof. If A is $\bot$ the proof is trivial. Suppose A atomic ($\neq \bot$). The claim follows by Lemma 5 and subsequent concatenation with the proof d_0. Suppose A not atomic.

– $A = B \supset D$. Let us consider the decoration of A in d_1 starting from $G \ \cup \ \{\Gamma, (B \supset D)^* \Rightarrow C\}$. We first replace in d_1 all the components $\{\Psi, (B \supset D)^* \Rightarrow C'\}$ by $\{\Psi, \Gamma \Rightarrow C'\}$. Note that this does not result in a correct proof anymore. We have then to consider the following "correction steps" according to the cases in which the decorated formula originates:
  (i) as principal formula of a logical inference,
  (ii) by an internal weakening,
  (iii) by an external weakening.
  (i) We replace every original inference step of the kind

$$\dfrac{\begin{array}{cc} \vdots & \vdots \\ G' \ \cup \ \{\Psi \Rightarrow B\} & G' \ \cup \ \{\Psi, D \Rightarrow C'\} \end{array}}{G' \ \cup \ \{\Psi, (B \supset D)^* \Rightarrow C'\}} \ (\supset, l)$$

  by (let $G'' = G' \ \cup \ G$)

$$\dfrac{\dfrac{\begin{array}{cc} \vdots & \vdots \, d_0' \\ (G'' \ \cup \ \{\Psi, \Gamma \Rightarrow B\})[{}^{\Gamma}/_{A^*}] & G'' \ \cup \ \{\Gamma, \Psi, B \Rightarrow D\} \end{array}}{G'' \ \cup \ \{\Psi, \Gamma \Rightarrow D\}} \qquad \begin{array}{c} \vdots \\ (G'' \ \cup \ \{\Psi, \Gamma, D \Rightarrow C'\})[{}^{\Gamma}/_{A^*}] \end{array}}{G'' \ \cup \ \{\Psi, \Gamma \Rightarrow C'\}}$$

  (adding some internal and external weakenings) where $d_0' \vdash \mathrm{G} \cup \{\Gamma, \mathrm{B} \Rightarrow \mathrm{D}\}$ is obtained by the Inversion Lemma.
  (ii)The weakening on $(B \supset D)^*$ is replaced by stepwise weakenings of formulas X, where $X \in \Gamma$. Note that this does not affect the length of the resulting proof.
  (iii) The weakening on the component C is replaced by a weakening on $C[{}^{\Gamma}/_{(B \supset D)^*}]$.
  The replacement of components containing decorated formulas $B \supset D$ does not change the length of the proof tree wich remains $\leq |d_1|$ and the cut-rank, which remains $\leq \rho(A)$. This holds also for the correction steps (ii) and (iii), since only weakenings are added. Correction step (i) uses d_0' (with suitable weakenings) as subproof deriving the missing premise of the cut rules replacing $(\supset, l)$ inferences of $(B \supset D)^*$. Therefore $|d| \leq |d_0'|+|d_1|+2 \leq |d_0| + |d_1| + 2 \leq 2(|d_0| + |d_1|)$.

– Cases $A = B \wedge D$ and $A = \forall x A(x)$ are treated analogously.
– $A = \exists x A(x)$. Let us consider the decoration of A in d_0 starting from $G \cup \{\Gamma \Rightarrow (\exists x A(x))^*\}$. We first replace in d_0 all the components $\{\Psi \Rightarrow (\exists x A(x))^*\}$ by $\{\Psi \Rightarrow C\}$. As in the previous case, this does not result in a correct proof anymore. Correction steps (ii) and (iii) are as above. While if $\exists x A(x)$ originates as principal formula of a logical inference, we replace every original inference step of the kind

$$\dfrac{\begin{array}{c}\vdots \\ G' \cup \{\Psi \Rightarrow A(t)\}\end{array}}{G' \cup \{\Psi \Rightarrow (\exists x A(x))^*\}}\ {\scriptstyle(\exists,1)}$$

by

$$\dfrac{\begin{array}{c}\vdots \\ G \cup (G' \cup \{\Psi, \Gamma \Rightarrow A(t)\})[^C/_{(\exists x A(x))^*}]\end{array} \qquad \begin{array}{c}\vdots\, d_1'(t) \\ G \cup G' \cup \{\Gamma, \Psi, A(t) \Rightarrow C\}\end{array}}{G' \cup G \cup \{\Psi, \Gamma \Rightarrow C\}}\ {\scriptstyle(\mathrm{cut})}$$

(adding some external weakenings) where $d_1'(t) \vdash \mathrm{G} \cup \{\Gamma, \mathrm{A(t)} \Rightarrow \mathrm{C}\}$ is obtained by the Inversion Lemma (and Substitution Lemma).
Correction step (i) uses $d_1'(t)$ (with suitable weakenings) as subproof deriving the missing premise of the cut rules replacing $(\exists, r)$ inferences of $(\exists x A(x))^*$.
Therefore $|d| \leq |d_0| + |d_1'(t)| + 1 \leq |d_0| + |d_1| + 1 \leq 2(|d_0| + |d_1|)$.
– Case $A = B \vee D$ is treated analogously.

Theorem 1 (Cut-elimination). *If $d \vdash \mathrm{H}$ and $\rho(d) > 0$, then we can find a derivation $d' \vdash \mathrm{H}$ with $\rho(d') < \rho(d)$ and $|d'| \leq 4^{|d|}$.*

Proof. Proceeds by induction on $|d|$. We may assume that the last inference of d is a cut

$$\dfrac{\begin{array}{c}\vdots\, d_0 \\ G \cup \{\Gamma \Rightarrow A\}\end{array} \quad \begin{array}{c}\vdots\, d_1 \\ G \cup \{\Gamma, A \Rightarrow C\}\end{array}}{G \cup \{\Gamma \Rightarrow C\}}$$

(eventually with subsequent weakenings) with $\rho(d) = |A| + 1$. For otherwise the result follows by the induction hypothesis (making use of the fact that our rules all have finitely many premises).

By the induction hypothesis we have $d_0' \vdash \mathrm{G} \cup \{\Gamma \Rightarrow \mathrm{A}\}$ and $d_1' \vdash \mathrm{G} \cup \{\Gamma, \mathrm{A} \Rightarrow \mathrm{C}\}$ both with cut rank $\rho(d_i') \leq |A|$ and $|d_i'| \leq 4^{|d_i|}$, with $i = 1, 2$. The Reduction Lemma gives a derivation d' with $\rho(d') \leq |A| < \rho(d)$ and $|d'| \leq 2(|d_0'| + |d_1'|) \leq 2(4^{|d_0|} + 4^{|d_1|}) \leq 4^{\max(d_0, d_1)+1} = 4^{|d|}$.

Let $4_0^n = n, 4_{k+1}^n = 4^{4_k^n}$.

Corollary 1. *If $d \vdash \mathrm{H}$, one can find a cut-free proof $d' \vdash \mathrm{H}$ with $|d'| \leq 4_{\rho(d)}^{|d|}$.*

Corollary 2. *If $d' \vdash \mathrm{H}$, one can find a cut-free derivation d of H in* **HIF** *such that $|d| \leq 2 \cdot 4_{\rho(d')}^{|d'|} + w(d'')$, where d'' is the corresponding cut-free derivation in* **HIF$^{\mathbf{set}}$**.

Proof. Immediately follows by Corollary 1 together with Lemma 2.

Note that $w(d'')$ can be easily bounded, e.g., by the total number of occurrences of formulas in d''.

Remark 6. Substitution, Inversion and Reduction Lemma, as well as Lemma 5 transform proofs in $\mathbf{HIF}^{\mathbf{set}}$ without applications of (com) and only containing singleton hypersequent, into proofs with the same properties. Therefore the above Schütte-Tait style cut-elimination proof, with the given bound, also holds for **LJ** in the set theoretic notation.

3.2 A Gentzen style cut-elimination proof

In this section we describe, for comparison, a Gentzen style cut-elimination proof for **HIF**.

Recall that the cut-elimination method of Gentzen proceeds by eliminating the uppermost cut by a double induction on the complexity c of the cut formula and on the sum r of its left and right ranks. In fact, in **LJ**, by the presence of the internal contraction rule one has to consider a derivable generalization of the cut rule, namely, the multi-cut rule

$$\frac{\Gamma \Rightarrow A \quad \Gamma', A^n \Rightarrow B}{\Gamma, \Gamma' \Rightarrow B}\ (mcut)$$

where A^n stands for $A, \ldots, A$ (n times), see, e.g., [15].

Due to the presence of (EC), in hypersequent calculi (and, in particular, in **HIF**) one cannot directly apply Gentzen's argument to show that $(*)$ if $G \mid \Gamma \Rightarrow A$ and $G \mid \Gamma, A^n \Rightarrow B$ are cut-free provable in **HIF**, so is $G \mid \Gamma \Rightarrow B$. A simple way to overcome this problem, is to modify Gentzen's original *Hauptsatz* allowing to reduce certain cuts *in parallel*. E.g., in [2], Avron has used the following induction hypothesis:

$(**)$ If both the hypersequents $G := G' \mid \Gamma_1 \Rightarrow A \mid \ldots \mid \Gamma_n \Rightarrow A$ and $H := H' \mid \Sigma_1, A^{n_1} \Rightarrow B_1 \mid \ldots \mid \Sigma_k, A^{n_k} \Rightarrow B_k$ are cut-free provable in **GLC**, then so is $H' \mid G' \mid \Gamma, \Sigma_1 \Rightarrow B_1 \mid \ldots \mid \Gamma, \Sigma_k \Rightarrow B_k$ where $\Gamma = \Gamma_1, \ldots, \Gamma_n$. It is not hard to see that this formulation is, in fact, equivalent to $(*)$. As we shall see, Avron's induction hypothesis also works for **HIF**.

In analogy with Lemma 2.10 of [15], one can show

Lemma 7. *Let $d(a)$ be a proof in* **HIF** *of a hypersequent S containing the variable a. If throughout the proof, we replace a by a term t, containing only variables that do not occur in $d(a)$, we then obtain a proof $d(t)$ ending with the hypersequent S' obtained by replacing a by t in S.*

Theorem 2 (Cut-elimination). *If a hypersequent H is derivable in* **HIF** *then it is derivable in* **HIF** *without using the cut rule.*

Proof. We show $(**)$ by induction on the pair (c, r). In addition to Avron's proof in [2], we have to consider the cases involving quantifiers. More precisely, let γ and δ be the proofs of G and H, respectively. We consider the following cases:

1. both γ and δ end in some rules for quantifiers such that the principal formula of both rules is just the cut formula;
2. either γ or δ ends in a rule for quantifiers whose principal formula is not the cut formula.

1. Suppose that both γ and δ end in a rule for $\forall$ and the principal formulas of both rules is the cut formula. For instance, δ is

$$\dfrac{\begin{array}{c}\vdots\, \delta_1\\ H' \mid \Sigma_1, A(a), (\forall x A(x))^{n_1-1} \Rightarrow B_1 \mid \ldots \mid \Sigma_k, (\forall x A(x))^{n_k} \Rightarrow B_k\end{array}}{H' \mid \Sigma_1, (\forall x A(x))^{n_1} \Rightarrow B_1 \mid \ldots \mid \Sigma_k, (\forall x A(x))^{n_k} \Rightarrow B_k}\ (\forall,\mathrm{l})$$

and γ is

$$\dfrac{\begin{array}{c}\vdots\, \gamma_1\\ G' \mid \Gamma_1 \Rightarrow A(a) \mid \ldots \mid \Gamma_n \Rightarrow \forall x A(x)\end{array}}{G' \mid \Gamma_1 \Rightarrow \forall x A(x) \mid \ldots \mid \Gamma_n \Rightarrow \forall x A(x)}\ (\forall,\mathrm{r})$$

Applying the induction hypothesis to both γ and δ_1 one gets a proof δ' of $H' \mid G' \mid \Sigma_1, \Gamma, A(a) \Rightarrow B_1 \mid \ldots \mid \Sigma_k, \Gamma \Rightarrow B_k$, where $\Gamma = \Gamma_1, \ldots, \Gamma_n$, while applying the induction hypothesis to γ_1 and δ one gets a proof γ' of $H' \mid G' \mid \Gamma_1 \Rightarrow A(a) \mid \Sigma_1, \Gamma_2, \ldots, \Gamma_n \Rightarrow B_1 \mid \ldots \mid \Sigma_k, \Gamma_2, \ldots, \Gamma_n \Rightarrow B_k$. We now apply again the induction hypothesis, based on the reduced complexity of the cut formula, to γ' and δ'. The desired result is obtained by several applications of $(c, l), (w, l)$ and (EC).

2. Suppose that δ ends as follows

$$\dfrac{\begin{array}{c}\vdots\, \delta_1\\ G' \mid \Gamma_1, A(a) \Rightarrow C \mid \ldots \mid \Gamma_n \Rightarrow C\end{array}}{G' \mid \Gamma_1, \exists x A(x) \Rightarrow C \mid \ldots \mid \Gamma_n \Rightarrow C}\ (\exists,\mathrm{l})$$

Applying the induction hypothesis to the proof γ of the hypersequent $H' \mid \Sigma_1, C^{n_1} \Rightarrow B_1 \mid \ldots \mid \Sigma_k, C^{n_k} \Rightarrow B_k$ and to δ_1 one gets a proof γ' of (a) $H' \mid G' \mid \Sigma_1, \Gamma, A(a) \Rightarrow B_1 \mid \ldots \mid \Sigma_k, \Gamma, A(a) \Rightarrow B_k$, where $\Gamma = \Gamma_1, \ldots, \Gamma_n$. Due to the eigenvariable condition, one cannot directly apply the $(\exists, l)$ rule to (a) in order to obtain the desired result, namely,

$$H' \mid G' \mid \Sigma_1, \Gamma, \exists x A(x) \Rightarrow B_1 \mid \ldots \mid \Sigma_k, \Gamma, \exists x A(x) \Rightarrow B_k.$$

However, the above hypersequent can be obtained from γ' by several applications of $(\exists, l)$, (com') (i.e. (com) + (w, l)) and (EC). The proof of it proceeds by induction on k. Base case: $k = 1$, the claim follows applying the $(\exists, l)$ rule (and Lemma 7). Let $k > 1$. From γ', using only $(\exists, l)$, (com) and (EC), one can derive $H' \mid G' \mid \Sigma_1, \Gamma, \exists x A(x) \Rightarrow B_1 \mid H$, where H stands for $\Sigma_2, \Gamma, A(a) \Rightarrow B_2 \mid \ldots \mid \Sigma_k, \Gamma, A(a) \Rightarrow B_k$. Indeed, by Lemma 7 one can find a proof $\gamma'[b]$ of (b) $H' \mid G' \mid \Sigma_1, \Gamma, A(b) \Rightarrow B_1 \mid \ldots \mid \Sigma_k, \Gamma, A(b) \Rightarrow B_k$, where b is a new variable not occurring in γ'. The derivation of $H' \mid G' \mid \Sigma_1, \Gamma, \exists x A(x) \Rightarrow B_1 \mid H$ is then as follows (we omit contexts that are not involved in the derivation)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(a) \qquad\qquad (b)}{\Sigma_1, \Gamma, A(b) \Rightarrow B_1 \mid \Sigma_3, \Gamma, A(b) \Rightarrow B_3 \mid \cdots \mid \Sigma_k, \Gamma, A(b) \Rightarrow B_k \mid H}{\scriptstyle (\star)} \quad (a)}{\vdots \qquad\qquad \vdots}{\scriptstyle (\star)}}{\Sigma_1, \Gamma, A(b) \Rightarrow B_1 \mid H}{\scriptstyle (\star)}}{\Sigma_1, \Gamma, \exists x A(x) \Rightarrow B_1 \mid H}{\scriptstyle (\exists, l)}$$

where $(\star)$ stands for (com') and (EC).

Remark 7. In [8] the proof of the cut-elimination theorem in Gentzen style has been formulated without using the "extended multi-cut rule" $(**)$. However, as pointed out by Avron, in hypersequent calculi Gentzen's argument works only (as in the case of **LJ** or **LK** without the multi-cut rule) if a suitable notion of decoration is formulated (see, e.g., the "history technique" in [1]).

3.3 Final Remarks

Schütte-Tait style cut-elimination methods make use of the partial (at least one side) invertibility of all logical rules: one side of the cut is reduced immediately. It is easy to see that these methods generally lead to smaller cut-free proofs than the ones obtained with Gentzen style procedures, especially if we admit deletion of subproofs ending with hypersequents containing axioms, e.g.,

$$\dfrac{\dfrac{\begin{matrix} \vdots\, d_1 \\ G \mid \Pi \Rightarrow A \supset B \end{matrix}}{G \mid B, A, \Pi \Rightarrow A \supset B}{\scriptstyle (w,l)} \qquad \dfrac{\dfrac{\dfrac{A \Rightarrow A \quad B \Rightarrow B}{G \mid A \supset B, A \Rightarrow B}{\scriptstyle (\supset,l)(EW)} \quad \dfrac{A \Rightarrow A \quad B \Rightarrow B}{G \mid A \supset B, A \Rightarrow B}{\scriptstyle (\supset,l)(EW)}}{G \mid A \supset B, A \Rightarrow B \wedge B}{\scriptstyle (\wedge,r)}}{G \mid A \supset B, B, A, \Pi \Rightarrow B \wedge B}{\scriptstyle (w,l)'s}}{G \mid B, A, \Pi \Rightarrow B \wedge B}{\scriptstyle (cut)}$$

Indeed, here the Inversion Lemma yields $G \mid B, A, \Pi \Rightarrow B$ and the subproof d_1 is deleted, while a Gentzen style procedure inevitably shifts the cut inside d_1. (See [7] for a comparison on the complexity of Gentzen and Schütte-Tait style procedures in classical first-order logic).

On the other hand, Schütte-Tait style procedures are more arbitrary than Gentzen style procedures (see, e.g., Remark 4), which use the exact properties of the calculus under consideration. In addition, Gentzen style procedures are local and they work even in the case of deductions from arbitrary atomic assumptions closed under cut.

Finally, note that both Gentzen and Schütte-Tait style procedures transform intuitionistic proofs into intuitionistic proofs (within **HIF**): new applications of the (com) rule are not introduced in eliminating cuts. Therefore cut-free derivations even for propositional formulas might lead to long cut-free proofs (recall that the validity problem in intuitionistic logic is P-space complete while the same problem in **LC** is in co-NP). This is not the case, when eliminating cuts from derivations in the sequent-of-relations calculus for **LC** defined in [4] (see also [6]). In this latter calculus, all the rules are invertible. However, it cannot be modified in a simple way in order to include quantifiers.

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