On Completability of Partial Combinatory Algebras

Andrea Asperti

Dipartimento di Matematica
Università di Bologna
P.zza di Porta S.Donato 5, Bologna, Italy
asperti@cs.unibo.it

Agata Ciabattoni

C.d.L in Scienze dell'Informazione
Università di Bologna
Via Sacchi 3, Cesena, Italy
agata@csr.unibo.it

Abstract

A Partial Combinatory Algebra is *completable* if it can be extended to a total one. Klop [11, 12] gave a sufficient condition for completability of a PCA $\mathcal{M} = (M, \cdot, K, S)$ in the form of ten axioms (inequalities) on terms of M. We prove that Klop's sufficient condition is equivalent to the existence of an *injective* s-m-n function over M (that in turns is equivalent to the Padding Lemma). This is proved by working with an alternative characterization of PCA's, recently introduced by the authors (Effective Applicative Structures). As a corollary, we show that nine of Klop's ten axioms are actually redundant (the so called Barendregt's axiom is enough to guarantee completability). Moreover, we prove that any Uniformly Reflexive Structure [17, 18, 16] is completable.

1 Introduction

A *Partial Applicative Structure* is a pair $(A, \cdot)$, where A is an arbitrary set and $\cdot$ is a partial binary operation over A, called application.

Notation

- Instead of $a \cdot b$ we shall often write ab; moreover, we conventionally suppose that application is left associative.
- $ab\downarrow$ means "ab is defined"
  $ab\uparrow$ means "ab is not defined"

- If t_1, t_2 are applicative expressions, $t_1 \simeq t_2$ abbreviates $t_1\downarrow \vee t_2\downarrow \Rightarrow t_1 = t_2$.

Definition 1.1 *A* Partial Combinatory Algebra *(PCA) is a structure $\mathcal{A} = (A, \cdot, K, S)$, where $(A, \cdot)$ is a partial applicative structure and S and K are two* distinguished *elements of A that satisfy the following conditions, for all $a, b, c \in A$:*

1. $Ka\downarrow,\ Sa\downarrow,\ Sab\downarrow$
2. $Kab \simeq a$
3. $Sabc \simeq ac(bc)$

A PCA $\mathcal{Q} = (Q, \cdot, K, S)$ is completable iff there is a total PCA $\mathcal{Q}^1 = (Q^1, \cdot, K^1, S^1)$ and an injection $\phi : Q \to Q^1$ such that $\phi(K) = K^1$, $\phi(S) = S^1$ and

$$\forall a, a_1 \in Q \quad (\mathcal{Q} \models aa_1\downarrow \ \to\ \mathcal{Q}^1 \models \phi(aa_1) = \phi(a)\dot{\phi}(a_1)).$$

Not every Partial Combinatory Algebra can be completed [11, 12]. In the same papers, Klop proved that a sufficient condition for the completability of a PCA, is that of having *unique head normal forms.*

Definition 1.2 *A PCA $\mathcal{Q}$ has unique head normal forms (hnf) iff $\forall a, b \in Q$, the elements K, S, Ka, Sa, Sab are pairwise distinct and Barendregt's axiom holds in $\mathcal{Q}$, that is*

$$\mathcal{Q} \models \forall a, a_1, b, b_1 \in Q \quad (Sab = Sa_1b_1 \to a = a_1 \wedge b = b_1)$$

More precisely, we have the following ten axioms: $K \neq Ka,\ K \neq Sa,\ S \neq Ka,\ S \neq Sa,\ Ka \neq Sa_1,\ S \neq Sab,\ Sa \neq Sa_1b,\ K \neq Sa_1b,\ Ka \neq Sa_1b,\ Sab = Sa_1b_1 \to a = a_1 \wedge b = b_1$.

In this paper we shall prove that Klop's sufficient condition can be equivalently expressed by the representability of an injective s-m-n function in Q. The precise formalization of this property relies on an alternative characterization of PCA's as suitable collections of partial functions, recently introduced by the authors under the name of Effective Applicative Stucture [1, 7]. We also prove that the injectivity of the s-m-n function is equivalent to the Padding Lemma. Since Barendregt's axiom is enough to prove the Padding Lemma we get, as a corollary, that this axiom is enough to ensure completability and, moreover, that any Uniformly Reflexive Structure is completable.

2 Effective Applicative Structures

An Effective Applicative Structure is a collection of indexed partial functions that is closed under composition, contains all projections and an interpreter, and satisfies

the s-m-n theorem of Recursion Theory.
Formally, an Effective Applicative Structure (EAS) is a family Φ^n of functions

$$\Phi^n : M \to \underbrace{(M^n \rightharpoonup M)}_{partial\ functions} \qquad n \in \mathcal{N}$$

over an arbitrary set M, which satisfies the following axioms[1]:

1. it is closed under composition, i.e.
   $\forall s, a_1, \ldots a_r \in M$ and $\forall r, i \in \mathcal{N}$, $\exists f \in M$ such that

   $$\Phi^r_s(\Phi^i_{a_1}(x_1, \cdots, x_i), \cdots \Phi^i_{a_r}(x_1, \cdots, x_i)) \simeq \Phi^i_f(x_1, \cdots, x_i)$$

   $\forall (x_1, \ldots, x_i) \in M^i$

2. it contains all projection functions I^n_i, i.e.
   $\forall i, n \in \mathcal{N}$, $\exists k \in M$ such that
   $$I^n_i \simeq \Phi^n_k$$
   with $I^2_1 \neq I^2_2$

3. it contains the S^m_n functions of the s-m-n theorem, i.e.
   $\forall m, n \in \mathcal{N}$ $\exists j \in M$ such that Φ^{n+1}_j is total[2] and

   $$\Phi^m_{\Phi^{n+1}_j(i, x_1, \cdots, x_n)}(y_1, \cdots, y_m) \simeq \Phi^{m+n}_i(x_1, \cdots, x_n, y_1, \cdots, y_m)$$

   with $(x_1, \cdots, x_n) \in M^n$
   and $(y_1, \cdots, y_m) \in M^m$

4. it contains interpreters, i.e.
   $\forall r \in \mathcal{N}$, $\exists i \in M$ such that

   $$\Phi^{r+1}_i(x, y_1, \cdots, y_r) \simeq \Phi^r_x(y_1, \cdots, y_r)$$

   $\forall x \in M, \forall (y_1, \ldots, y_r) \in M^r$

The definition of EAS looks very natural: closure under composition and existence of projections are obvious properties of effective functions, while the s-m-n theorem and the existence of universal functions are basic results of the theory of effective computability. Many interesting results (such as Kleene's fixed point theorem) can

[1] according to the standard notation of Recusion Theory, we write Φ^n_i instead of $\Phi^n(i)$.
[2] $\Phi^{n+1}_j = S^m_n$

be proved by the only use of the previous assumptions.
There exist many other similar "axiomatic" approaches to Recursion Theory in the literature [17, 18, 16, 9, 6, 8, 13], but all of them make stronger assumptions than ours. The most relevant (and closest) approach is the Basic Recursive Function Theory (BRFT) introduced by Strong [16] (a BRFT characterises the families of functions which form Uniformly Reflexive Structures [17, 18]).

Definition 2.1 *A BRFT (Basic Recursive Function Theory) is a structure* $(D,\ F,\ (\phi^n)_{n \in \mathcal{N}})$ *satisfying:*

- *D is an infinite set*

- *F is a collection of partial functions on D, such that:*

  1. *it is closed under composition*
  2. *it contains all projection functions*
  3. *it contains all constant functions on D*
  4. *it contains the function for definition by cases*

$$f(x, a, b, c) = \begin{cases} b & \textit{if } x = a \\ \\ c & \textit{otherwise} \end{cases}$$

  5. *it contains the S^m_n function of the s-m-n theorem*
  6. *it contains an interpreter*

Essentially, in EAS we drop constants and definition by cases (they are inessential to prove that all partial recursive functions are *representable* in the structure). On the other side, an important consequence of having definition by cases is that application is *essentially* a partial operation (in the sense that it *cannot* be total). This can be proved by a simple diagonal argument. Take two distinct elements a and b and consider the function

$$f(x) = \begin{cases} a & \text{if } \phi_x(x) = b \\ \\ b & \text{otherwise} \end{cases}$$

Since $f \in F$ there exists an index c such that $f = \phi_c$. Supposing $\phi_c(c)$ defined we would get a contradiction.

In [1] we proved that Effective Applicative Stuctures are completely equivalent to Partial Combinatory Algebras. We shall recall the proof in the next section, since we need it for our discussion of completability.

3 Equivalence of EAS and PCA

In this section we shall prove that any Partial Combinatory Algebra provides a model of Effective Applicative Structure and, conversely, for any Effective Applicative Structure its domain can be naturally equipped with a PCA-structure.

3.1 From PCA to EAS

Recall that λ-abstraction can be simulated inside Combinatory Algebras by means of the following rules:

1. $\lambda^* x.x \equiv SKK$
2. $\lambda^* x.M \equiv KM$ if M is a variable $y \neq x$, S or K.
3. $\lambda^* x.(MN) \equiv S(\lambda^* x.M)(\lambda^* x.N)$

Let $\mathcal{M} = (M, \cdot, K, S)$ be a PCA. Let us define a family of (partial) functions $\phi^n : M \to (M^n \to M) \quad n \in \mathcal{N}$ as follows:

$$\phi^n_p(b_1, \cdots, b_n) \simeq pb_1 \cdots b_n$$

(we shall often omit the superscript n when it is clear from the context.)
We have that

1. $\{\phi^n\}$ satisfies closure under composition. In fact $\forall M_1, M_2, \cdots, M_l \in M, \quad \forall n \in \mathcal{N}, \exists F \in M$ such that

$$\phi_{M_1}(\phi_{M_2}(N_1, \cdots, N_n) \cdots \phi_{M_l}(N_1, \cdots, N_n)) = \phi_F(N_1, \cdots, N_n)$$

   with $F \equiv \lambda^* x_1 \cdots x_n.M_1(M_2 x_1 \cdots x_n) \cdots (M_l x_1 \cdots x_n)$

2. $\{\phi^n\}$ contains projection functions, i.e.

$$\phi^k_R(N_1, \cdots, N_k) = N_i$$

   with $R \equiv \lambda^* x_1 \cdots x_k.x_i$

3. $\{\phi^n\}$ contains the S^m_n function of the s-m-n theorem, i.e. $\forall m, n \in \mathcal{N}\ \exists Q \in M$ such that $\phi_Q(P, N_1 \cdots N_n)$ is defined and

$$\phi^m_{\phi_Q(P,N_1,\cdots,N_n)}(M_1, \cdots, M_m) = \phi^{m+n}_P(N_1, \cdots, N_n, M_1, \cdots, M_m)$$

   with $Q \equiv \lambda^* x_1, \cdots x_{n+1} y_1 \cdots y_m.x_1 x_2 \cdots x_{n+1} y_1 \cdots y_m$

4. $\{\phi^n\}$ contains interpreters, i.e. $\forall n \in \mathcal{N} \quad \exists U \in M$ such that

$$\phi_U(M, N_1, \cdots, N_n) = \phi_M(N_1, \cdots, N_n)$$

with $U \equiv \lambda^* x y_1 \cdots y_n . x y_1 \cdots y_n$

Then $\phi^n : M \to (M^n \to M) \quad n \in \mathcal{N}$ is an Effective Applicative Structure.

3.2 From EAS to PCA

Given an Effective Applicative Structure

$$\Phi^n : M \to (M^n \to M) \quad n \in \mathcal{N}$$

we use the interpreter of condition 4. to define a partial binary operation $\cdot : M \times M \to M$:

$$a \cdot b \equiv U^2(a, b) \equiv \Phi^2_u(a, b) \simeq \Phi_a(b)$$

(u is the index of the interpreter with 2 arguments.)
S and K are defined as follows.

- **Existence of** K
  We prove the existence of an index K in M such that, for all $i, j \in M$

  $$(K \cdot i) \cdot j \simeq i$$

  In fact

  $$\begin{array}{lll} i & \simeq I^2_1(i, j) & \\ & \simeq \Phi^2_l(i, j) & \text{by hp. 2. of EAS} \\ & \simeq \Phi^1_{S^1_1(l,i)}(j) & \text{by def. of } S^1_1 \\ & \simeq \Phi^1_{\Phi^2_m(l,i)}(j) & \text{by hp. 3. of EAS} \\ & \simeq \Phi^1_{\Phi^1_{S^1_1(m,l)}(i)}(j) & \text{by def. of } S^1_1 \end{array}$$

  Since S^m_n is total, letting $K \equiv S^1_1(m, l)$ we have:

  $$i \simeq \Phi^1_{\Phi^1_K(i)}(j) \simeq (K \cdot i) \cdot j$$

- **Existence of** S
  We prove the existence of an index $S \in M$ such that $\forall a, b, c \in M$

  $$(((S \cdot a) \cdot b) \cdot c) \simeq (a \cdot c) \cdot (b \cdot c)$$

  In fact

$$
\begin{array}{lll}
(a \cdot c) \cdot (b \cdot c) & \simeq \Phi^2_z(\Phi^2_z(a,c), \Phi^2_z(b,c)) & \text{by def. of application} \\
& \simeq \Phi^2_z(\Phi^2_z(I^3_1(a,b,c), I^3_3(a,b,c)), \Phi^2_z(I^3_2(a,b,c), I^3_3(a,b,c))) & \\
& \simeq \Phi^3_h(a,b,c) & \text{by hp 1. and 2. of EAS} \\
& \simeq \Phi^1_{S^1_2(h,a,b)}(c) & \\
& \simeq \Phi^1_{\Phi^3_j(h,a,b)}(c) & \text{by hp. 3. of EAS} \\
& \simeq \Phi^1_{\Phi^1_{S^1_2(j,h,a)}(b)}(c) & \\
& \simeq \Phi^1_{\Phi^1_{\Phi^3_j(j,h,a)}(b)}(c) & \text{by hp. 3. of EAS} \\
& \simeq \Phi^1_{\Phi^1_{\Phi^1_{S^1_2(j,j,h)}(a)}(b)}(c) &
\end{array}
$$

since S^m_n is total, letting $S \equiv S^1_2(j,j,h)$, we have:

$$(a \cdot c) \cdot (b \cdot c) \simeq \Phi^1_{\Phi^1_{\Phi_S{}^1(a)}(b)}(c) \simeq (((S \cdot a) \cdot b) \cdot c)$$

- $Kx\downarrow, Sx\downarrow$ and $(S(x))y\downarrow$ for all $x, y \in M$
  since S^m_n is a total function.
- It is easy to prove that if $K = S$ then $|M| = 1$.

Then $\mathcal{M} = (M, \cdot, K, S)$ is a PCA.

4 Completability of Partial Combinatory Algebras

In this section we prove that unicity of hnf in PCA's corresponds to the existence of an *injective* S^m_n function in EAS's.
We say that the function S^m_n is injective if

$$S^m_n(\overline{a}) = S^{m_1}_{n_1}(\overline{b}) \Rightarrow m = m_1 \wedge n = n_1 \wedge \overline{a} = \overline{b}$$

(that is, injectivity in n, m, and all its arguments) [3].
First of all we notice that in order to prove the unicity of hnf in a PCA, it is not necessary to check the 10 cases that are possible, i.e. $K \neq Ka$, $K \neq Sa$, $S \neq Ka$, $S \neq Sa$, $Ka \neq Sa_1$, $S \neq Sab$, $Sa \neq Sa_1b$, $K \neq Sa_1b$, $Ka \neq Sa_1b$, $Sab = Sa_1b_1 \to a = a_1 \wedge b = b_1$.
As a matter of fact, the two conditions $S \neq Ka$ and $K \neq Ka$
($\forall a$) are true in every PCA, while a PCA that satisfies $S = Sa$ or $K = Sa$ is already complete.

[3] It is interesting to note that in many texts of Recursion Theory, such as [14], the S^m_n function is explicitly introduced with the additional hypothesis to be injective.

Proposition 4.1 *In any PCA $\mathcal{M}$, $S \neq Ka$ and $K \neq Ka$.*

Proof: Suppose $S = Ka$. Then $\forall b, c$, $Sb = Kab = a = Kac = Sc$. So, $\forall b, c, x, y$, $Sbxy \simeq Scxy$. Let $I \equiv SKK$. Taking $b \equiv K, c \equiv I, y \equiv I, x \equiv K$, we have:

$$I = SKKI = SIKI = KI$$

This is absurd, since for any x we would have $x = Ix = KIx = I$.
Similarly, $K = Ka$ would imply that all terms are equal to a. $\square$

Proposition 4.2 *If a PCA $\mathcal{M}$ satisfies $S = Sa$ or $K = Sa$, then it is total.*

Proof: Suppose $S = Sa$. Then $\forall b, c \in \mathcal{M}$, $Sabc \simeq ac(bc) = Sbc$. Thus $\forall b, c\ (bc)\downarrow$.
Similarly if $K = Sa$, $\forall b, c \in \mathcal{M}$, $Sabc \simeq ac(bc) \simeq Kbc = b$. Thus $\forall b, c\ (bc)\downarrow$. $\square$

These propositions allow us to restrict the sufficient conditions for completability to the remaining 6 cases.

4.1 From injectivity of S^m_n to completability

Let us consider the Partial Combinatory Algebra $\mathcal{M} = (M, \cdot, K, S)$ as defined in 3.2, and suppose that the S^m_n function of s-m-n theorem is injective. We proceed in proving the completability of $\mathcal{M}$ by considering all remaining cases of Klop's sufficient conditions, namely $\forall a, a_1, b, b_1$:

1. $Ka \neq Sa_1$
2. $S \neq Sab$
3. $Sa \neq Sa_1 b$
4. $K \neq Sab$
5. $Ka \neq Sa_1 b$
6. $Sab = Sa_1 b_1 \Rightarrow a = a_1 \wedge b = b_1$

Let us recall first that, by definition,

$$K \equiv S^1_1(m, l)$$

where m is the index in EAS of S^1_1 and l of I^2_1, and

$$S \equiv S^1_2(j, j, h)$$

where j is the index in EAS of S^1_2.

1. $Ka \neq Sa_1$

$$\begin{aligned}
Ka &= U^2(S_1^1(m,l),a) && \text{by def. of application}\\
&= \Phi_{S_1^1(m,l)}(a)\\
&= \Phi_m(l,a) && \text{by def. of } S_1^1\\
&= S_1^1(l,a). && \text{since } m \text{ is the index of } S_1^1
\end{aligned}$$

Conversely,

$$\begin{aligned}
Sa_1 &= U^2(S_2^1(j,j,h),a_1) && \text{by def. of application}\\
&= \Phi_{S_2^1(j,j,h)}(a_1)\\
&= \Phi_j(j,h,a_1) && \text{by def. of } S_2^1\\
&= S_2^1(j,h,a_1) && \text{since } j \text{ is the index of } S_2^1
\end{aligned}$$

Since S_n^m is injective,

$$Ka = S_1^1(l,a) \neq S_2^1(j,h,a_1) = Sa_1 \quad \forall a, a_1$$

2. $S \neq Sab$

$$\begin{aligned}
Sab &= U^2(U^2(S_2^1(j,j,h),a),b) && \text{by def. of application}\\
&= \Phi_j(h,a,b)\\
&= S_2^1(h,a,b) && \text{since } j \text{ is the index of } S_2^1
\end{aligned}$$

If $S = Sab$, from injectivity of S_n^m it follows that $j = h = a = b$. Since $Sa = S_2^1(j,h,a)$, we would have $S = Sa$, and by Proposition 5, $\mathcal{M}$ is already complete.

3. $Sa_1 \neq Sab$
We already proved that

$$Sa_1 = S_2^1(j,h,a_1)$$
$$Sab = S_2^1(h,a,b)$$

If $Sa_1 = Sab$, the injectivity of S_n^m implies that $j = h$, so

$$S = S_2^1(j,j,h) = S_2^1(j,h,h) = Sh$$

and by Proposition 5, $\mathcal{M}$ is already complete.

4. $K \neq Sab$
By the injectivity of S_n^m,

$$K \equiv S_1^1(m,l) \neq S_2^1(h,a,b) \equiv Sab$$

5. $Ka_1 \neq Sab$
Similar to 4.

6. $Sab = Sa_1b_1 \Rightarrow a = a_1 \wedge b = b_1$
Direct consequence of the injectivity of S_n^m.

4.2 From unique head normal form to injectivity of S_n^m

Let us prove that Klop's conditions for completability are enough to ensure the existence of injective S_n^m functions.
Recall that, according to our definition,

$$S_n^m = \lambda^* x_0 x_1 \dots x_n y_1 \dots y_m . x_0 x_1 \dots x_n y_1 \dots y_m$$

Then, by definition of ϕ

$$\phi_{S_n^m}^{n+1}(b, a_1, \dots, a_n) =$$

$$(\lambda^* x_0 x_1 \dots x_n y_1 \dots y_m . x_0 x_1 \dots x_n y_1 \dots y_m) b \; a_1 \; \dots \; a_n =$$

$$(\lambda^* y_1 \dots y_m . x_0 x_1 \dots x_n y_1 \dots y_m)[^b/_{x_0}, ^{a_1}/_{x_1}, \dots, ^{a_n}/_{x_n}]$$

We must prove that, if

$$(*) \quad \phi_{S_n^m}^{n+1}(b, a_1, \dots, a_n) = \phi_{S_q^p}^{q+1}(b', a'_1, \dots, a'_q)$$

then $m = p$, $n = q, b = b'$ and for all i, $a_i = a'_i$.
Let us prove first that (*) implies $m = p$.
The term

$$\lambda^* y_1 \dots y_m . x_0 x_1 \dots x_n y_1 \dots y_m$$

(and all its instances) has the following shape, for any m (trivial induction):

$$SM\lambda^* y_1 \dots y_m . y_m$$

where $M = \lambda^* y_1 \dots y_m . x_0 x_1 \dots x_n y_1 \dots y_{m-1}$.
So, by Barendregt's axiom, (*) would imply

$$\lambda^* y_1 \dots y_m . y_m = \lambda^* y_1 \dots y_p . y_p$$

Suppose that $m \neq p$, and assume $p < m$. If we apply p terms $M_1, \dots, M_p$ to both arguments of the previous equation, we get

$$\lambda^* y_{p+1} \dots y_m . y_m = M_p$$

that is absurd, since every term would be equal to $\lambda^* y_{p+1} \dots y_m . y_m$.
So we can assume $m = p$, and the hypothesis (*) becomes

$$(**) \quad \phi_{S_n^m}^{n+1}(b, a_1, \dots, a_n) = \phi_{S_q^m}^{q+1}(b', a'_1, \dots, a'_q)$$

Let us prove that $n = q$. Let us apply to both arguments of the above equation $m - 1$ terms $c_1, \dots, c_{m-1}$. In particular, (**) becomes:

$$\lambda^* y_m . x_0 x_1 \dots x_n y_1 \dots y_m [^b/_{x_0}, ^{a_1}/_{x_1}, \dots, ^{a_n}/_{x_n}, ^{c_1}/_{y_1}, \dots, ^{c_{m-1}}/_{y_{m-1}}] =$$

$$\lambda^* y_m . x_0 x_1 \dots x_q y_1 \dots y_m [^{b'}/_{x_0}, ^{a'_1}/_{x_1}, \dots, ^{a'_q}/_{x_q}, ^{c_1}/_{y_1}, \dots, ^{c_{m-1}}/_{y_{m-1}}]$$

The term $\lambda^* y_m . x_0 x_1 \dots x_n y_1 \dots y_m$ has the shape

$$S(S \dots \ (S \ (\dots (S(K \ x_0)(K \ x_1)) \dots)(K \ x_n)) \dots (K \ y_{m-1}))I$$

and thus

$$\lambda^* y_m . x_0 x_1 \dots x_n y_1 \dots y_m [^{b}/_{x_0}, ^{a_1}/_{x_1}, \dots, ^{a_n}/_{x_n}, ^{c_1}/_{y_1}, \dots, ^{c_{m-1}}/_{y_{m-1}}] =$$

$$S(S \dots \ (S \ (\dots (S(K \ b)(K \ a_1)) \dots)(K \ a_n)) \dots (K \ c_{m-1}))I$$

Suppose that $n \neq q$, and assume $q < n$. By a repeated use of Barendregt's axiom we would obtain

$$K \ a'_q = SP(K \ a_n)$$

for some term P. But this is impossible due to axiom 5. So $n = q$.
Finally, by a similar argument (a repeated use of Barendregt's axiom) we easily prove that $b = b'$ and for all i, $a_i = a'_i$.

Remark 4.3 *In the previous proof, we just used the axiom $Ka \neq Sa_1 b$ and Barendregt's axiom. So, these two axioms are enough to guarantee completability. It is possible to give a more direct proof of this fact: if we define, $K^1 \equiv \lambda^* xy.Kxy$ and $S^1 \equiv \lambda^* xyz.Sxyz$, it is easy to show that the two axioms above for K and S imply all other axioms for K^1 and S^1.*

5 On the Padding Lemma

An interesting property of the theory of effectively computable functions is the Padding Lemma (see for instance [15]). Essentially, it states that each recursive function f has a recursive infinite set of indexes.
In the framework of applicative structures, such as PCA or URS (BRFT), the Padding Lemma is usually expressed in the following form:

Definition 5.1 *Let $\mathcal{M} = (M, \cdot)$ be an applicative structure. It satisfies the Padding Lemma if and only if $\exists P \in M$ such that, $\forall a, b, x \in M$*

- $Pab\downarrow$
- $Pabx \simeq ax$
- $Pab = Pa_1 b_1 \Rightarrow a = a_1 \wedge b = b_1$.

In general, PCA's (EAS's) do not satisfy the Padding Lemma. A simple counterexample is provided by *extensional* PCA's.

Definition 5.2 *A PCA $\mathcal{Q}$ is extensional iff*

$$\mathcal{Q} \models \forall x, y\ (\forall z\ (xz \simeq yz) \to x = y)$$

Obviously, in extensional Partial Combinatory Algebras each function is represented by just one element. The existence of extensional PCA's has been proved by Bethke in [5]; in the same paper, it is proved that Barendregt's axiom is incompatible with extensionality in nontotal PCA's.
The interesting fact is that, in PCA's, Barendregt's axiom implies the Padding Lemma, and conversely the Padding Lemma implies both Barendregt's axiom and $Ka \neq Sa_1b$ (and thus completability).

Proposition 5.3 *If $\mathcal{M} = (M, \cdot, K, S)$ satisfies Barendregt's axiom, then it satisfies the Padding Lemma.*

Proof: Take $P = \lambda^* yzx.K(yx)z$. The rest is easy. □

Proposition 5.4 *If a PCA $\mathcal{M}$ satisfies the Padding Lemma, then there exist S' and $K' \in M$ that satisfy both Barendregt's axiom and $K'a \neq S'bc\ \forall a, b, c \in M$.*

Proof: Let $K' \equiv \lambda^* x.P(Kx)(PKx)$ and $S' = \lambda^* xy.P(P(Sxy)S)(Pxy)$.
Let us prove that S' satisfies Barendregt's axiom. Note that $S'ab = P(P(Sab)S)(Pab)$. If $S'ab = S'cd$ then $P(P(Sab)S)(Pab) = P(P(Scd)S)(Pcd)$. Since P is injective, $Pab = Pcd$ and by the same reason, $a = c$ and $b = d$.
Suppose now that there are a, b, c such that $K'a = S'bc$. Then $P(Ka)(PKa) = P(P(Sbc)S)(Pbc)$. By the injectivity of P, $Ka = P(Sbc)S$, $K = b$, $a = c$. In particular, $Ka = P(SKa)S$. So, $\forall x \in M$,

$$a = Kax = P(SKa)Sx = SKax = Kx(ax) = x$$

that is absurd. □

Corollary 5.5 *Barendregt's axiom implies completability.*

Proposition 5.6 *Any Uniformly Reflexive Structure satisfies the Padding Lemma.*

Proof: See [18], th.4.3. □

Corollary 5.7 *Any URS is completable.*

Note that we can merely extend the PCA-structure of a URS (and not its BRFT-structure), since the axioms of BRFT imply partiality (in particular, we have no way to extend definition by cases to the completed domain).

6 Conclusions

In this paper we proved that a sufficient condition for the completability of a Partial Combinatory Algebra $\mathcal{M}$ is the representability of an *injective* s-m-n function of Recursion Theory. This fact has some interesting corollaries: we reduce Klop's sufficient conditions for completability from ten to just one axiom; morevoer we prove that all Uniformely Reflexive Structures (a subclass of PCA's that contains Kleene's applicative structure) are completable. The proof uses an alternative characterisation of PCA's, recently proposed by the authors (Effective Applicative Structures). An Effective Applicative Structure is a collection of indexed partial functions that is closed under composition, contains all projections and an interpreter, and satisfies the s-m-n theorem of Recursion Theory. Due to their close relation with the Theory of Recursive Functions and Effective Computability, EAS's seem to provide a more natural and friendly framework than Partial Combinatory Algebras. For this reason, it looks very interesting to rephrase open problems and results from PCA's to EAS's; our work on completability can be seen as a first step in this direction.

References

[1] A.Asperti, A.Ciabattoni. *Effective Applicative Structures.* Proceedings of the sixth biennial conference on Category Theory in Computer Science (CTCS-6), Cambridge, august 1995. LNCS 953, pp.81-95, G.Goos, J.Hartmanis and J.van Leeuwen eds.

[2] H.Barendregt. *The lambda calculus* North-Holland. 1984.

[3] H.Barendregt. *Normed uniformly reflexive structures.* In *λ calculus and Computer Science Theory.* C.Bohm ed., LNCS 37, pp.272-286, Springer Verlag. 1975.

[4] M.J.Beeson. *Foundation of constructive mathematics.* Springer-Verlag. 1985.

[5] I.Bethke. *On the existence of extensional partial combinatory algebras.* J. of Symbolic Logic vol.52. n.3. pp.819-833. 1987

[6] R.E.Byerly. *Mathematical aspects of recursive function theory.* In *Harvey Friedman's research on the foundations of mathematics*, L.A.Harrington ed., pp.339-352, Elsevier Science. 1985.

[7] A.Ciabattoni. *Sull'importanza dei teoremi di enumerazione ed s-m-n nella teoria della calcolabilità.* Tesi di Laurea, Dipartimento di Matematica, Università di Bologna. 1994.

[8] J.E.Fenstad. *On axiomatizing recursion theory.* In *Generalized recursion theory*, North-Holland, pp.385-404. 1974.

[9] H.Friedman. *Axiomatic recursive function theory.* In *Logic Colloquium '69* , R.Gandy e M.Yates eds., pp.113-141, North-Holland. 1969.

[10] J.R.Hindley, J.P.Seldin. *Introduction to combinators and λ-calculus* London Math. Soc. 1990.

[11] J.W.Klop. *Extending partial combinatory algebras.* Bulletin of the European Association for Theoret. Comp.Sci., vol.16, pp.472-482. 1982.

[12] J.W.Klop. Personal Communication.

[13] Y.Moschovakis. *Axioms for computation theories.* In *Logic Colloquium '69* , R.Gandy e M.Yates eds., pp.199-255, North-Holland. 1969.

[14] P.Odifreddi. *Classical recursion theory.* North-Holland. 1989.

[15] H.Rogers. *Theory of recursive functions and effective computability.* McGraw Hill. 1967.

[16] H.Strong. *Algebraically generalized recursive function theory.* IBM J. of Research and Developement, vol.12, pp.465-475. 1968.

[17] E.Wagner. *Uniformly reflexive structures: an axiomatic approach to computability.* Information Sciences vol.1, pp.343-362. 1969.

[18] E.Wagner. *Uniformly reflexive structures: on the nature of gödelizations and relative computability.* Trans. Amer. Math. Soc., vol.144, pp.1-41. 1969.