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# Analytic Calculi for Monoidal T-norm Based Logic

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**Abstract.** Monoidal t-norm based logic $\mathbf{MTL}$ is the logic of left continuous $t$-norms. We introduce two analytic calculi for first-order $\mathbf{MTL}$. These are obtained by lifting two sequent calculi for different fragments of this logic to the hypersequent level with subsequent addition of Avron's communication rule. Our calculi enable to prove the mid(hyper)sequent theorem. As corollaries follow Herbrand's theorem for first-order $\mathbf{MTL}$, the decidability of its $\forall\exists$-fragment and admissibility of Skolemization.

**Keywords:** Basic fuzzy logics, monoidal t-norm based logic, hypersequent calculi, mid(hyper)sequent theorem

## 1. Introduction

Fuzzy logic is the base for reasoning with vague knowledge [22]. The most important formal representations of fuzzy logic are infinite-valued Łukasiewicz, Gödel, and Product logic (see [12]). These correspond to the most important *t-norms* that are the main tool to combine fuzzy information [15].

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In [12], Hájek defined a logic — called basic fuzzy logic $\mathbf{BL}$ — capturing the tautologies of all *continuous t-norms* and their residua. The fact that a t-norm has residuum if and only if the t-norm is left-continuous lead Esteva and Godo to define a logic weaker than $\mathbf{BL}$, namely monoidal t-norm based logic $\mathbf{MTL}$ [9, 10]. The idea behind $\mathbf{MTL}$ has been to obtain the most general residuated logic based on t-norm calculi, i.e., the logic of *left-continuous t-norms* and their residua. This conjecture was proved in [14] for propositional $\mathbf{MTL}$ and in [16] for $\mathbf{MTL}\forall$, i.e. its (safe) first-order extension.

$\mathbf{MTL}\forall$ is built up from two conjunctions: $\odot$ (multiplicative[1] ) and $\wedge$ (additive[1]), an implication $\rightarrow$, the constant $\perp$ and the quantifiers $\exists$ and $\forall$ (see [10]). The conjunction $\odot$ is interpreted by a left continuous $t$-norm $*$ and the implication by the residuum of $*$. $\mathbf{MTL}\forall$ is directly related to first-order Gödel logic $\mathbf{G}\forall$, $\mathbf{BL}$, and intuitionistic logic without contraction $\mathbf{H_{BCK}}$ [19]. Indeed, on the one hand, first-order $\mathbf{BL}$ can be obtained by adding to $\mathbf{MTL}\forall$ axiom $A \wedge B \rightarrow A \odot (A \rightarrow B)$, allowing one to define the additive conjunction $\wedge$ from the multiplicative connectives. On the other hand, adding to $\mathbf{MTL}\forall$ axiom $(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$, corresponding to the sequent rule of contraction, one gets $\mathbf{G}\forall$. Finally, $\mathbf{MTL}\forall$ can be obtained by extending first-order $\mathbf{H_{BCK}}$ (also known as $\mathbf{FL_{ew}}$ – Full Lambek calculus with exchange and weakening [17]) with both the linearity axiom $(A \rightarrow B) \vee (B \rightarrow A)$ and the shifting law of universal quantifiers w.r.t. $\vee$, i.e., $\forall x(P(x) \vee Q) \rightarrow (\forall x P(x) \vee Q)$, where $x$ does not occur free in $Q$.

Most of the work done on $\mathbf{MTL}\forall$ was concerned with problems of axiomatizability and algebraic or model-theoretic issues. With respect to these analysis, the proof-theoretical investigation has not been pursued to the same extent. In particular, no analytic calculi, i.e., in which proof search proceeds by a stepwise decomposition of the formula to be proved, have been provided even for $\mathbf{MTL}$. As is well known, such calculi are the key to all applications that require automated reasoning.

In this paper we introduce two analytic calculi for $\mathbf{MTL}\forall$. These calculi use hypersequents, a simple and natural generalization of Gentzen's sequents to *multisets* of sequents. To define both calculi we proceed in a modular manner as in the case of the hypersequent calculus for $\mathbf{G}\forall$ introduced in [7]. This calculus –which we call here $\mathbf{HG}^{\forall}$– was defined by embedding sequents of Gentzen's calculus $\mathbf{LJ}$ for intuitionistic logic into hypersequents and by adding suitable structural rules. In particular, the crucial rule of $\mathbf{HG}^{\forall}$, added to $\mathbf{LJ}$, is Avron's communication rule $(com)$ [2]. This rule enables to prove both the linearity axiom and the shifting law of universal quantifiers w.r.t. $\vee$. In analogy with Gödel logic, we first identify appropriate analytic sequent calculi for fragments of $\mathbf{MTL}\forall$ without these two axioms to obtain cut-free hypersequent calculi for $\mathbf{MTL}\forall$. We then lift these calculi from sequents to hypersequents and we add $(com)$. The resulting calculi formalize $\mathbf{MTL}\forall$. Remarkably enough, those different calculi become equivalent once one adds $(com)$.

Our calculi for $\mathbf{MTL}\forall$ enable to prove (a suitable version of) Gentzen's mid-sequent theorem that fails in their underlying sequent calculi. As a corollary of this theorem we show how to effectively associate with each prenex formula $P$ of $\mathbf{MTL}\forall$ a propositional (variable free) formula $P^*$ which is valid if and only if $P$ is valid (Herbrand's theorem). Using Herbrand's theorem we prove

1. the decidability[2] of the $\forall\exists$-fragment of $\mathbf{MTL}\forall$

2. reverse Skolemization for $\mathbf{MTL}\forall$, i.e., the possibility to re-introduce quantifiers in Herbrand expansions replacing Skolem terms.

[1]See [20] for the terminology.

[2]The undecidability of the prenex fragment of $\mathbf{MTL}\forall$ follows as a particular case of a more general undecidability result proved in [6].

## 2. Monoidal t-norm based logic

The language of $\mathbf{MTL}$ contains two conjunctions: $\odot$ (multiplicative or monoidal conjunction) and $\wedge$ (additive or lattice conjunction), an implication $\rightarrow$ and the constant $\perp$. $\mathbf{MTL}$ consists of the following axioms, see [9]:

| | | | |
|---|---|---|---|
| $\mathbf{A1}$ | $(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$ | $\mathbf{A6}$ | $(A \odot (A \rightarrow B)) \rightarrow (A \wedge B)$ |
| $\mathbf{A2}$ | $(A \odot B) \rightarrow A$ | $\mathbf{A7}$ | $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \odot B) \rightarrow C)$ |
| $\mathbf{A3}$ | $(A \odot B) \rightarrow (B \odot A)$ | $\mathbf{A8}$ | $((A \odot B) \rightarrow C) \rightarrow (A \rightarrow (B \rightarrow C))$ |
| $\mathbf{A4}$ | $(A \wedge B) \rightarrow A$ | $\mathbf{A9}$ | $((A \rightarrow B) \rightarrow C) \rightarrow (((B \rightarrow A) \rightarrow C) \rightarrow C)$ |
| $\mathbf{A5}$ | $(A \wedge B) \rightarrow (B \wedge A)$ | $\mathbf{A10}$ | $\perp \rightarrow A$ |

together with the rule of modus ponens. Disjunction and negation are defined as follows: $\neg A = A \rightarrow \perp$ and $A \vee B = (A \rightarrow (A \rightarrow B)) \wedge (B \rightarrow (B \rightarrow A))$.

As shown, e.g., in [11], an alternative axiomatization of $\mathbf{MTL}$ is obtained by adding the linearity axiom (Lin) $(A \rightarrow B) \vee (B \rightarrow A)$ to Ono and Komori's system $\mathbf{H_{BCK}}$ [19] characterizing intuitionistic logic without contraction. $\mathbf{H_{BCK}}$ –also known as Höhle's Monoidal Logic [13] or $IPC^* \setminus c$ [1]– is an extension of Full Lambek calculus **FL** with axioms $(A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C))$ and $A \rightarrow (B \rightarrow A)$ corresponding to the sequent rules of exchange and weakening, respectively.

In this work we identify a logic with the set of its derivable formulas.

**Remark 2.1.** As shown in [8], the $\{\wedge, \rightarrow\}$-fragment of $\mathbf{MTL}$ coincides with the (new version of the) logic **C** introduced by Urquhart in the chapter devoted to many-valued logics of the Handbook of Philosophical Logic, see [21].

The algebraic semantics of $\mathbf{MTL}$ is based on particular residuated lattices called MTL-algebras.

**Definition 2.1.** A *(commutative, integral, bounded) residuated lattice* is a system $\mathcal{L} = \langle L, \cdot, \Rightarrow, \sqcup, \sqcap, 0, 1 \rangle$ where $\langle L, \cdot, 1 \rangle$ is a commutative monoid, $\langle L, \sqcup, \sqcap, 0, 1 \rangle$ is a lattice with minimum 0 and maximum 1, and the residuation property holds:

$(res)$ For all $x, y, z \in L,\ x \leq y \Rightarrow z$ iff $x \cdot y \leq z$.

**Definition 2.2.** An *MTL-algebra* is a (commutative, integral, bounded) residuated lattice satisfying

$$(x \Rightarrow y) \sqcup (y \Rightarrow x) = 1$$

Commutative, integral, bounded residuated lattices are the equivalent algebraic semantics of $\mathbf{H_{BCK}}$ while MTL-algebras are the equivalent algebraic semantics of $\mathbf{MTL}$.

First-oder monoidal t-norm based logic $\mathbf{MTL}\forall$ was defined in [10]. Its language extends that of $\mathbf{MTL}$ with the quantifiers $\forall$ and $\exists$. $\mathbf{MTL}\forall$ consists of axioms and rules of $\mathbf{MTL}$ with in addition the generalization rule and the following axioms

| | | | |
|---|---|---|---|
| $(\forall 1)$ | $\forall x P(x) \rightarrow P(t)$ | $(\exists 1)$ | $P(t) \rightarrow \exists x P(x)$ |
| $(\forall 2)$ | $\forall x (P^{(x)} \rightarrow Q) \rightarrow (P^{(x)} \rightarrow \forall x Q)$ | $(\exists 2)$ | $\forall x (P \rightarrow Q^{(x)}) \rightarrow (\exists x P \rightarrow Q^{(x)})$ |
| $(\forall 3)$ | $\forall x (P(x) \vee Q^{(x)}) \rightarrow (\forall x P(x) \vee Q^{(x)})$ | | |

where in ($\forall$1) and ($\exists$1) $t$ is substitutable for $x$ in $P(x)$ and the notation $A^{(x)}$ indicates that $x$ does not occur free in $A$.

Hence, an alternative axiomatization of $\mathbf{MTL}\forall$ is obtained by adding axioms (Lin) and ($\forall$3) to the first-order extension of $\mathbf{H_{BCK}}$ [19]. Note that both axioms are not valid in intuitionistic logic.

It can be shown that every MTL-algebra decomposes into a subdirect product of linearly ordered (w.r.t. the order induced by the lattice structure) MTL-algebras. In particular, the lattice reduct of an MTL-algebra is distributive. Thus when dealing with interpretations of $\mathbf{MTL}\forall$ we can restrict ourselves to linearly ordered MTL-algebras.

**Definition 2.3.** Let $\mathcal{A} = (A, \cdot, \Longrightarrow, \leq, 0, 1)$ be a linearly ordered MTL-algebra. An *evaluation* of $\mathbf{MTL}$ is a mapping $e$ from $\mathbf{MTL}$-formulas into $\mathcal{A}$ such that for every formulas $A$ and $B$ of $\mathbf{MTL}$, the following conditions hold:

$$\begin{array}{llll} (1) & e(\bot) = 0 & (2) & e(A \to B) = e(A) \Longrightarrow e(B) \\ (3) & e(A \odot B) = e(A) \cdot e(B) & (4) & e(A \wedge B) = \min\{e(A), e(B)\} \end{array}$$

Let $D$ be a non-empty set. An *evaluation of* $\mathbf{MTL}\forall$ on $\mathcal{A}$ with domain $D$ is a partial map, which by abuse of language we still call $e$, from the set of closed formulas with parameters in $D$ into $\mathcal{A}$ such that:

$e$ is defined on all closed atomic formulas.

$e$ satisfies clauses (1)-(4) (where it is meant that for every connective $\circ$, $e(A \circ B)$ is defined iff both $e(A)$ and $e(B)$ are defined).

$e(\forall x A)$ is defined iff for all $d \in D$, $e(A(d/x)$ is defined and $\inf\{e(A(d/x)) : d \in D\}$ exists in $\mathcal{A}$. In this case, $e(\forall x A) = \inf\{e(A(d/x)) : d \in D\}$.

$e(\exists x A)$ is defined iff for all $d \in D$, $e(A(d/x)$ is defined and $\sup\{e(A(d/x)) : d \in D\}$ exists in $\mathcal{A}$. In this case, $e(\exists x A) = \sup\{e(A(d/x)) : d \in D\}$.

A first-order evaluation $e$ is said to be *safe* if it is defined on all formulas. A $\mathbf{MTL}\forall$-formula $P$ is said to be *valid* in $\mathcal{A}$ iff for every domain $D$ and safe evaluation $e$ into $\mathcal{A}$, $e(P) = 1$.

It turns out that $\mathbf{MTL}\forall$ is strongly complete with respect to the class of linearly ordered MTL-algebras, that is:

**Theorem 2.1. ([10])**
Let $\Gamma$ be a set of $\mathbf{MTL}\forall$-formulas and $A$ be a $\mathbf{MTL}\forall$-formula. $A$ is derivable from $\Gamma$ in $\mathbf{MTL}\forall$ iff for every linearly ordered MTL-algebra $\mathcal{A}$, for every non-empty domain $D$ of individuals and every safe evaluation $e$ in $\mathcal{A}$ with domain $D$, if $e(B) = 1$ for all $B \in \Gamma$, then $e(A) = 1$.

An alternative semantics for $\mathbf{MTL}\forall$ is given in terms of left continuous t-norms [14, 16]. In fuzzy logic, this semantics is called *standard*.

**Definition 2.4.** A *triangular norm* (*t-norm* for short) is a binary operation $*$ on $[0, 1]$ satisfying the following conditions:

- $*$ is commutative and associative

- $*$ is non-decreasing in both arguments
- $1 * x = x$ and $0 * x = 0$, for all $x \in [0, 1]$.

$*$ is a *continuous t-norm* iff it is a t-norm and is a continuous mapping (in the usual sense) of $[0, 1]^2$ into $[0, 1]$. A t-norm $*$ is *left continuous* iff whenever $(x_n : n \in N)$ and $(y_n : n \in N)$ are increasing sequences of reals in $[0, 1]$ such that $\sup\{x_n : n \in N\} = x$ and $\sup\{y_n : n \in N\} = y$, then $\sup\{x_n * y_n : n \in N\} = x * y$.

It is easy to prove that $*$ is a left continuous $t$-norm iff it has a residuum $\Longrightarrow$ defined by $x \Longrightarrow y = \sup\{z \mid x * z \leq y\}$, see, e.g., [11]. Thus $([0, 1], *, \Longrightarrow, \leq, 0, 1)$ is a linearly ordered MTL-algebra. We will call such algebras *t-norm MTL-algebras*.

**Theorem 2.2. ([16])**
Let $\Gamma$ be a set of $\mathbf{MTL}\forall$-formulas and $A$ be a $\mathbf{MTL}\forall$-formula. $\Gamma \models_{\mathbf{MTL}\forall} A$ iff for every t-norm MTL-algebra $\mathcal{A}$, for every non-empty domain $D$ of individuals and every safe evaluation $e$ in $\mathcal{A}$ with domain $D$, if $e(B) = 1$ for all $B \in \Gamma$, then $e(A) = 1$.

# 3. Hypersequent formulations for $\mathrm{MTL}\forall$

In this section we define two analytic calculi for $\mathbf{MTL}$. Both calculi use hypersequents — a natural generalization of ordinary sequents (see, e.g., [3] for an overview on hypersequent calculi).

**Definition 3.1.** A *hypersequent* is an expression of the form $\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$, where for all $i = 1, \ldots n$, $\Gamma_i \Rightarrow \Delta_i$ is an ordinary sequent. $\Gamma_i \Rightarrow \Delta_i$ is called a *component* of the hypersequent.

Henceforth we consider sequents and hypersequents as *multisets* of formulas and *multisets* of sequents[3], respectively. Moreover, we only deal with hypersequents containing at most one formula on the right hand side of the involved sequents.

The intended interpretation of the symbol "|" is disjunctive as shown by the following definitions:

**Definition 3.2.** Let $S$ be the sequent $A_1, \ldots, A_n \Rightarrow B$. Its *generic interpretation* $\mathcal{I}(S)$ is the formula $(A_1 \to \ldots \to (A_n \to B) \ldots)$. Let $G$ be the hypersequent $S_1 \mid \cdots \mid S_n$. Then its *generic interpretation* $\mathcal{I}(G)$ is defined as $\mathcal{I}(S_1) \vee \ldots \vee \mathcal{I}(S_n)$.

**Definition 3.3.** A (Hyper)sequent rule

$$\frac{S_1 \quad \ldots \quad S_n}{S_0}\ (r) \qquad \text{with } n \geq 1$$

is *sound* for a Hilbert style system $\mathbf{L}$, if whenever $\mathbf{L}$ derives the generic interpretations of its premises, $\mathbf{L}$ derives the generic interpretation of its conclusion too. $(r)$ is *strongly sound* for $\mathbf{L}$ if $\mathbf{L}$ derives the formula $\mathcal{I}(S_1) \to (\ldots(\mathcal{I}(S_n) \to \mathcal{I}(S_0))\ldots)$. A (hyper)sequent calculus $\mathbf{HL}$ is called *sound* (resp. *strongly sound*) for $\mathbf{L}$ if all the axioms and rules of $\mathbf{HL}$ are sound (resp. *strongly sound*) for $\mathbf{L}$. $\mathbf{HL}$ is called *complete* for $\mathbf{L}$ if for all formulas $A$ derivable in $\mathbf{L}$, the (hyper)sequent $\Rightarrow A$ is derivable in $\mathbf{HL}$.

[3]If one prefers sequences over multisets as basic objects of inference then permutation rules have to be considered.

### 3.1. The calculus $\mathbf{HL}^{\forall}_{\mathbf{BCK}}$

As mentioned in Section 2 a Hilbert style axiomatization for $\mathbf{MTL}\forall$ can be obtained extending that of first-order $\mathbf{H_{BCK}}$ with both linearity axiom (Lin) and the shifting law of quantifiers ($\forall 3$). In [2] Avron showed that the former axiom can be derived in the hypersequent version of $\mathbf{LJ}$ with in addition the rule he called communication $(com)$. This construction led to the definition of an analytic calculus for propositional Gödel logic. By introducing suitable quantifier rules in Avron's calculus, Baaz and Zach defined the hypersequent calculus $\mathbf{HG}^{\forall}$ for first-order Gödel logic (see [7]). In particular, $(com)$ enable to prove axiom ($\forall 3$) too. In analogy with the investigation on Gödel logic, to define an analytic calculus for $\mathbf{MTL}\forall$ we add to the hypersequent version of the $\mathbf{L_{BCK}}$ sequent calculus for $\mathbf{H_{BCK}}$ (see [19]) Avron's $(com)$ rule and the quantifier rules of [7]. We call the resulting calculus $\mathbf{HL}^{\forall}_{\mathbf{BCK}}$.

Axioms and rules of $\mathbf{HL}^{\forall}_{\mathbf{BCK}}$ are those of $\mathbf{L_{BCK}}$ augmented with an external context $H$ representing a (possibly empty) hypersequent. Additional rules are external structural rules and quantifier rules. The former, that manipulate whole components of a hypersequent, are the "standard" weakening (EW) and contraction (EC) together with $(com)$. More precisely, $\mathbf{HL}^{\forall}_{\mathbf{BCK}}$ consists of axioms $P \Rightarrow P$ and $\bot \Rightarrow P$ and of the following rules:

$(Internal\ and\ External)\ Structural\ Rules$

$$(w,l)\ \frac{H \mid \Gamma \Rightarrow C}{H \mid \Gamma, B \Rightarrow C} \qquad (w,r)\ \frac{H \mid \Gamma \Rightarrow \bot}{H \mid \Gamma \Rightarrow C}$$

$$(EW)\ \frac{H}{H \mid \Gamma \Rightarrow A} \qquad (EC)\ \frac{H \mid \Gamma \Rightarrow A \mid \Gamma \Rightarrow A}{H \mid \Gamma \Rightarrow A}$$

$$(com)\ \frac{H \mid \Pi_1, \Gamma_1 \Rightarrow A \quad H \mid \Pi_2, \Gamma_2 \Rightarrow B}{H \mid \Pi_1, \Pi_2 \Rightarrow A \mid \Gamma_1, \Gamma_2 \Rightarrow B} \qquad (cut)\ \frac{H \mid \Gamma \Rightarrow A \quad H \mid A, \Gamma' \Rightarrow C}{H \mid \Gamma, \Gamma' \Rightarrow C}$$

$Logical\ rules\ (Multiplicative\ fragment)$

$$\frac{H \mid \Gamma, A, B \Rightarrow C}{H \mid \Gamma, A \odot B \Rightarrow C}\ (\odot, l) \qquad \frac{H \mid \Gamma \Rightarrow A \quad H \mid \Gamma' \Rightarrow B}{H \mid \Gamma, \Gamma' \Rightarrow A \odot B}\ (\odot, r)$$

$$\frac{H \mid \Gamma \Rightarrow A \quad H \mid \Gamma', B \Rightarrow C}{H \mid \Gamma, \Gamma', A \to B \Rightarrow C}\ (\to, l) \qquad \frac{H \mid \Gamma, A \Rightarrow B}{H \mid \Gamma \Rightarrow A \to B}\ (\to, r)$$

$Logical\ rules\ (Additive\ fragment)$

$$\frac{H \mid \Gamma, A_i \Rightarrow C}{H \mid \Gamma, A_1 \wedge A_2 \Rightarrow C}\ (\wedge, l)_{i:\, i=1,2} \qquad \frac{H \mid \Gamma \Rightarrow A \quad H \mid \Gamma \Rightarrow B}{H \mid \Gamma \Rightarrow A \wedge B}\ (\wedge, r)$$

$$\frac{H \mid \Gamma, A \Rightarrow C \quad H \mid \Gamma, B \Rightarrow C}{H \mid \Gamma, A \vee B \Rightarrow C}\ (\vee, l) \qquad \frac{H \mid \Gamma \Rightarrow A_i}{H \mid \Gamma \Rightarrow A_1 \vee A_2}\ (\vee, r)_{i:\, i=1,2}$$

$Quantifier\ rules$

$$\frac{H \mid A(t), \Gamma \Rightarrow B}{H \mid (\forall x)A(x), \Gamma \Rightarrow B}\ (\forall, l) \qquad \frac{H \mid \Gamma \Rightarrow A(a)}{H \mid \Gamma \Rightarrow (\forall x)A(x)}\ (\forall, r)$$

$$\frac{H \mid A(a), \Gamma \Rightarrow B}{H \mid (\exists x)A(x), \Gamma \Rightarrow B}\ (\exists, l) \qquad \frac{H \mid \Gamma \Rightarrow A(t)}{H \mid \Gamma \Rightarrow (\exists x)A(x)}\ (\exists, r)$$

The rules $(\forall, r)$ and $(\exists, l)$ must obey the eigenvariable condition: the variable $a$ must not occur in the lower *hypersequent*.

**Remark 3.1.** By the presence of both $(EW)$ and $(EC)$ one can equivalently formulate all the above two premises rules using multiplicative external contexts ($H$ and $H'$) instead of additive ones.

**Remark 3.2.** As is well known, using the (internal) contraction rule, i.e.

$$\frac{H \mid \Gamma, A, A \Rightarrow C}{H \mid \Gamma, A \Rightarrow C}\ (c, l)$$

together with weakening rules, the above rules for conjunctions are interderivable. Thus, adding $(c, l)$ to $\mathbf{HL}^{\forall}_{\mathbf{BCK}}$ one obtains Baaz and Zach's calculus for $\mathbf{G}\forall$.

**Proposition 3.1.** In $\mathbf{HL}^{\forall}_{\mathbf{BCK}}$ non-atomic axioms are derivable from atomic axioms.

**Lemma 3.1. ([1, 19])**
$\mathbf{L_{BCK}}$ is strongly sound for $\mathbf{H_{BCK}}$.

**Theorem 3.1.** $\mathbf{HL}^{\forall}_{\mathbf{BCK}}$ is sound and complete for $\mathbf{MTL}\forall$.

**Proof:**
(*Soundness*) The soundness of logical and internal structural rules follows by Lemma 3.1 and the derivability in $\mathbf{MTL}$ of $(A \to B) \to (H \vee A \to H \vee B)$ and $(A \to (B \to C)) \to (H \vee A \to (H \vee B \to H \vee C))$. We therefore must only take care of external structural rules and quantifier rules. $(H \vee A \vee A) \to H \vee A$ corresponds to (EC). $H \vee A \to (H \vee A \vee B)$ corresponds to (EW). These formulas are derivable in $\mathbf{MTL}$. The soundness of the $(com)$ rule follows by modus ponens from (Lin) and the formula $(*)$ $((\Gamma_1^{\odot} \to \Pi_2^{\odot}) \vee (\Pi_2^{\odot} \to \Gamma_1^{\odot})) \to \mathcal{I}(com)$, where $\Sigma^{\odot}$ stands for the (multiplicative) conjunction of formulas in $\Sigma$ and $\mathcal{I}(com)$ abbreviates

$$\mathcal{I}(H \mid \Pi_1, \Gamma_1 \Rightarrow A) \to (\mathcal{I}(H \mid \Pi_2, \Gamma_2 \Rightarrow B) \to (\mathcal{I}(H \mid \Pi_1, \Pi_2 \Rightarrow A \mid \Gamma_1, \Gamma_2 \Rightarrow B)).$$

Note that $(*)$ is derivable already in $\mathbf{L_{BCK}}$. The cases $(\forall, l)$ and $(\exists, r)$ are easy. For $(\forall, r)$ we may argue as follows: If $\mathcal{I}(H) \vee \mathcal{I}(\Gamma \Rightarrow A(a))$ is derivable in $\mathbf{MTL}\forall$, so is $\forall x(\mathcal{I}(H) \vee \mathcal{I}(\Gamma \Rightarrow A(x)))$. Since $a$ did not occur in $\mathcal{I}(H)$ or in $\mathcal{I}(\Gamma \Rightarrow A(a))$, we may now assume that $x$ does not either. Using axiom $(\forall 3)$ we obtain $\mathcal{I}(H) \vee \forall x \mathcal{I}(\Gamma \Rightarrow A(x))$. The result follows since $\forall x \mathcal{I}(\Gamma \Rightarrow A(x)) \to \mathcal{I}(\Gamma \Rightarrow \forall x A(x))$ is derivable in $\mathbf{MTL}\forall$. The soundness of $(\exists, l)$ can be proved in a similar way.
(*Completeness*) It is enough to prove that $\mathbf{HL}^{\forall}_{\mathbf{BCK}}$ derives axioms (Lin) and $(\forall 3)$ together with axioms

and rules of first-order $\mathbf{H_{BCK}}$. Observe that modus ponens corresponds to the derivability of $A, A \to B \Rightarrow B$ and of the cut rule while the generalization rule is a particular case of $(\forall, r)$. Axioms of $\mathbf{H_{BCK}}$ are derivable in $\mathbf{L_{BCK}}$ (see [19]) and hence in $\mathbf{MTL}\forall$. The derivations of the remaining axioms are straightforward but for (Lin) and ($\forall$3) whose derivations are:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A \qquad B \Rightarrow B}{A \Rightarrow B \mid B \Rightarrow A}\,(\mathrm{com})}{A \Rightarrow B \mid \ \Rightarrow B \to A}\,(\to,\mathrm{r})}{\Rightarrow A \to B \mid \ \Rightarrow B \to A}\,(\to,\mathrm{r})}{\Rightarrow A \to B \mid \ \Rightarrow (A \to B) \vee (B \to A)}\,(\vee,\mathrm{r})_2}{\Rightarrow (A \to B) \vee (B \to A) \mid \ \Rightarrow (A \to B) \vee (B \to A)}\,(\vee,\mathrm{r})_1}{\Rightarrow (A \to B) \vee (B \to A)}\,(\mathrm{EC})$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A(a) \Rightarrow A(a) \qquad \dfrac{A(a) \Rightarrow A(a) \quad B \Rightarrow B}{B \Rightarrow A(a) \mid A(a) \Rightarrow B}\,(com) \qquad B \Rightarrow B}{A(a) \vee B \Rightarrow A(a) \mid A(a) \vee B \Rightarrow B}\,2x(\vee,l)+(EW)'s}{\forall x(A(x) \vee B) \Rightarrow A(a) \mid \forall x(A(x) \vee B) \Rightarrow B}\,2x(\forall,l)}{\forall x(A(x) \vee B) \Rightarrow \forall x A(x) \mid \forall x(A(x) \vee B) \Rightarrow B}\,(\forall,r)}{\forall x(A(x) \vee B) \Rightarrow \forall x A(x) \vee B \mid \forall x(A(x) \vee B) \Rightarrow \forall x A(x) \vee B}\,2x(\vee,r)}{\dfrac{\forall x(A(x) \vee B) \Rightarrow \forall x A(x) \vee B}{\Rightarrow \forall x(A(x) \vee B) \to (\forall x A(x) \vee B)}\,(\to,r)}\,(EC)$$

$\square$

**Theorem 3.2.** $\mathbf{HL^{\forall}_{BCK}}$ admits cut-elimination.

**Proof:**
See Appendix. $\square$

### 3.2. The calculus $\mathbf{HL^{\forall}_{BCK*}}$

Here we introduce an alternative hypersequent calculus for $\mathbf{MTL}\forall$. We call this calculus $\mathbf{HL^{\forall}_{BCK*}}$. $\mathbf{HL^{\forall}_{BCK*}}$ is based on a sequent calculus $\mathbf{L_{BCK^-}}$ whose axioms and rules are exactly those of $\mathbf{L_{BCK}}$ but for the right rule of the $\wedge$ connective that, in the case of $\mathbf{L_{BCK^-}}$, is splitted into the following rules:

$$\frac{\Gamma \Rightarrow A_1 \quad \Gamma', A_1 \Rightarrow A_2}{\Gamma, \Gamma' \Rightarrow A_1 \wedge A_2}\ (\wedge, r)_1 \qquad \frac{\Gamma \Rightarrow A_2 \quad \Gamma', A_2 \Rightarrow A_1}{\Gamma, \Gamma' \Rightarrow A_1 \wedge A_2}\ (\wedge, r)_2$$

$\mathbf{HL^{\forall}_{BCK*}}$ is defined by adding $(com)$ and the quantifier rules of $\mathbf{HL^{\forall}_{BCK}}$ to the hypersequent version of $\mathbf{L_{BCK^-}}$.

**Proposition 3.2.** Each rule of $\mathbf{HL}^{\forall}_{\mathbf{BCK}*}$ is derivable in $\mathbf{HL}^{\forall}_{\mathbf{BCK}}$ and vice versa.

**Proof:**
$\mathbf{HL}^{\forall}_{\mathbf{BCK}*}$ and $\mathbf{HL}^{\forall}_{\mathbf{BCK}}$ differ only in their right rules for $\wedge$. On the one hand, the $(\wedge, r)_1$ rule of $\mathbf{HL}^{\forall}_{\mathbf{BCK}*}$ can be derived in $\mathbf{HL}^{\forall}_{\mathbf{BCK}}$ as follows:

$$\dfrac{\dfrac{H \mid \Gamma', A_1 \Rightarrow A_2 \quad H \mid \Gamma \Rightarrow A_1}{H \mid \Gamma, \Gamma' \Rightarrow A_2}\,(\text{cut}) \qquad \dfrac{H \mid \Gamma \Rightarrow A_1}{H \mid \Gamma, \Gamma' \Rightarrow A_1}\,(\text{w,l})}{H \mid \Gamma, \Gamma' \Rightarrow A_1 \wedge A_2}\,(\wedge,\text{r})$$

Similarly for the $(\wedge, r)_2$ rule.

On the other hand, a (cut-free) derivation in $\mathbf{HL}^{\forall}_{\mathbf{BCK}*}$ of $(\wedge, r)$ is:

$$\dfrac{\dfrac{\dfrac{H \mid \Gamma \Rightarrow A}{H \mid \Gamma \Rightarrow A \mid \Gamma \Rightarrow A \wedge B}\,(\text{EW}) \qquad \dfrac{\dfrac{\dfrac{A \Rightarrow A}{H \mid A \Rightarrow A}\,(\text{EW}) \quad \dfrac{B \Rightarrow B}{H \mid B \Rightarrow B}\,(\text{EW})}{H \mid A \Rightarrow B \mid B \Rightarrow A}\,(\text{com}) \qquad \dfrac{H \mid \Gamma \Rightarrow B}{H \mid A \Rightarrow B \mid \Gamma \Rightarrow B}\,(\text{EW})}{H \mid A \Rightarrow B \mid \Gamma \Rightarrow A \wedge B}\,(\wedge,\text{r})_2}{H \mid \Gamma \Rightarrow A \wedge B \mid \Gamma \Rightarrow A \wedge B}\,(\wedge,\text{r})_1}{H \mid \Gamma \Rightarrow A \wedge B}\,(\text{EC})$$

□

**Corollary 3.1.** $\mathbf{HL}^{\forall}_{\mathbf{BCK}*}$ admits cut-elimination.

**Proof:**
By the above proposition and Theorem 3.2 follows that any $\mathbf{HL}^{\forall}_{\mathbf{BCK}*}$-derivation can be transformed into a cut-free $\mathbf{HL}^{\forall}_{\mathbf{BCK}}$-derivation. The claim follows from the cut-free derivability of $(\wedge, r)$ in $\mathbf{HL}^{\forall}_{\mathbf{BCK}*}$.
□

**Corollary 3.2.** $\mathbf{HL}^{\forall}_{\mathbf{BCK}*}$ is sound and complete for $\mathbf{MTL}\forall$.

**Proof:**
Follows by Proposition 3.2 and Theorem 3.1. □

Albeit the hypersequent versions of $\mathbf{L_{BCK}}$ and $\mathbf{L_{BCK^-}}$ with in addition $(com)$ are equivalent, $\mathbf{L_{BCK}}$ and $\mathbf{L_{BCK^-}}$ are not. As we shall see, $\mathbf{L_{BCK^-}}$ is weaker than $\mathbf{L_{BCK}}$.

**Proposition 3.3.** $\mathbf{L_{BCK^-}}$ admits cut-elimination.

**Proof:**
It is enough to show that if $\pi$ is a proof in $\mathbf{L_{BCK^-}}$ of a sequent $S$ containing only one cut which occurs as the last inference of $\pi$, then $S$ is derivable in $\mathbf{L_{BCK^-}}$ without the cut rule. Following Gentzen's cut-elimination method the proof will proceed by induction on the complexity of the cut formula (i.e. the number of its connectives), and on the sum of the length of the proofs of the premises of the cut rule.

We shall argue by cases according to which inference rule is being applied immediately before the application of the cut rule. Most of cases are as in the cut-elimination proof of $\mathbf{L_{BCK}}$ (see [19]). Here we outline the "critical" case in which the premises of the cut rule are lower sequents of the rules for the $\wedge$ connective and the cut formula is the introduced formulas in both rules. Suppose, e.g., $\pi$ ends as follows:

$$\dfrac{\dfrac{\Gamma' \Rightarrow A \quad \Sigma, A \Rightarrow B}{\Sigma, \Gamma' \Rightarrow A \wedge B}\,(\wedge,r)_1 \quad \dfrac{(\star)}{\Gamma, A \wedge B \Rightarrow C}\,(\wedge,l)}{\Sigma, \Gamma, \Gamma' \Rightarrow C}\,(cut)$$

Two possibilities arise:

- $(\star)$ is $\Gamma, A \Rightarrow C$. Then the above proof can be replaced by the following

$$\dfrac{\dfrac{\Gamma, A \Rightarrow C \quad \Gamma' \Rightarrow A}{\Gamma, \Gamma' \Rightarrow C}\,(cut)}{\Sigma, \Gamma, \Gamma' \Rightarrow C}\,(w,l)$$

  containing one cut whose complexity of the cut formula is strictly smaller than the one of the cut in $\pi$. Thus this cut can be eliminated by induction hypothesis.

- $(\star)$ is $\Gamma, B \Rightarrow C$. Then the above proof can be replaced by the following

$$\dfrac{\dfrac{\Gamma' \Rightarrow A \quad \Sigma, A \Rightarrow B}{\Sigma, \Gamma' \Rightarrow B}\,(cut) \quad \Gamma, B \Rightarrow C}{\Sigma, \Gamma, \Gamma' \Rightarrow C}\,(cut)$$

  containing two cuts whose complexity of the cut formula is strictly smaller than the one of the cut in $\pi$. Hence all these cuts can be eliminated by induction hypothesis.

□

**Corollary 3.3.** $\mathbf{L_{BCK^-}}$ is weaker than $\mathbf{L_{BCK}}$.

**Proof:**
Trivially follows from the non derivability of $A \wedge B \to B \wedge A$ in $\mathbf{L_{BCK^-}}$. □

# 4. Consequences

As an immediate consequence of the cut-elimination theorem (e.g., Theorem 3.2) one has the *subformula property*: every formula $P$ valid in $\mathbf{MTL}\forall$ has a proof in $\mathbf{HL}^{\forall}_{\mathbf{BCK}}$ or $\mathbf{HL}^{\forall}_{\mathbf{BCK}*}$ which only contains subformulas of $P$. Another important result that follows from cut-elimination (and the presence of $(com)$) is the *mid-(hyper)sequent theorem*. Indeed, in analogy with Baaz and Zach's result for the hypersequent calculus $\mathbf{HG}^{\forall}$ for $\mathbf{G}\forall$ (see [7]), one can prove that in $\mathbf{HL}^{\forall}_{\mathbf{BCK}}$ and $\mathbf{HL}^{\forall}_{\mathbf{BCK}*}$ a certain separation between propositional and quantifier inferences can be achieved in deriving a prenex hypersequent.

**Theorem 4.1. (Mid-hypersequent)**
Any $\mathbf{HL}^{\forall}_{\mathbf{BCK}}$(or $\mathbf{HL}^{\forall}_{\mathbf{BCK}*}$)-derivation $\pi$ of a hypersequent $G$ only containing prenex formulas can be transformed into one in which no propositional rule is applied below any application of a quantifier rule.

**Proof:**
By Proposition 3.2 (and the cut-free derivability of $(\wedge, r)$ in $\mathbf{HL}^{\forall}_{\mathbf{BCK}*}$) it suffices to prove the claim for $\mathbf{HL}^{\forall}_{\mathbf{BCK}}$. Here, Theorem 3.2 and Proposition 3.1 provide us with a cut-free $\mathbf{HL}^{\forall}_{\mathbf{BCK}}$-derivation $\pi'$ of $G$ from atomic axioms. One can check that in $\mathbf{HL}^{\forall}_{\mathbf{BCK}}$ the only problematic cases arise when $\pi'$ contains a $(\exists, r)$ inference above a $(\vee, l)$ inference and when $\pi'$ contains a $(\forall, l)$ inference above a $(\wedge, r)$ inference. Note that in $\mathbf{HL}^{\forall}_{\mathbf{BCK}}$ rules $(\vee, l)$ and $(\wedge, r)$ can be simulated – without using cuts – by the following ones:

$$\frac{H \mid A, \Gamma \Rightarrow C_1 \quad H \mid B, \Gamma' \Rightarrow C_2}{H \mid A \vee B, \Gamma \Rightarrow C_1 \mid A \vee B, \Gamma' \Rightarrow C_2}\ (\vee', l) \qquad \frac{H \mid \Gamma \Rightarrow A \quad H \mid \Gamma' \Rightarrow B}{H \mid \Gamma \Rightarrow A \wedge B \mid \Gamma' \Rightarrow A \wedge B}\ (\wedge', r)$$

The above rules are cut-free derivable in $\mathbf{HL}^{\forall}_{\mathbf{BCK}}$. E.g., in the case of $(\vee', l)$ one has:

$$\dfrac{\dfrac{\dfrac{H \mid B, \Gamma' \Rightarrow C_2 \quad H \mid A, \Gamma \Rightarrow C_1}{H \mid B, \Gamma \Rightarrow C_1 \mid A, \Gamma' \Rightarrow C_2}\,(\mathrm{com}) \quad \dfrac{H \mid A, \Gamma \Rightarrow C_1}{H \mid A, \Gamma' \Rightarrow C_2 \mid A, \Gamma \Rightarrow C_1}\,(\mathrm{EW})}{H \mid A \vee B, \Gamma \Rightarrow C_1 \mid A, \Gamma' \Rightarrow C_2}\,(\vee,\mathrm{l}) \quad \dfrac{H \mid B, \Gamma' \Rightarrow C_2}{H \mid A \vee B, \Gamma \Rightarrow C_1 \mid B, \Gamma' \Rightarrow C_2}\,(\mathrm{EW})}{H \mid A \vee B, \Gamma \Rightarrow C_1 \mid A \vee B, \Gamma' \Rightarrow C_2}\,(\vee,\mathrm{l})$$

Let us define the order of a quantifier inference in $\pi'$ to be the number of propositional inferences under it, and the order of $\pi'$ as the sum of the orders of its quantifier inferences. The proof proceeds by induction on the order of $\pi'$ as follows: In $\pi'$ we replace all the applications of $(\vee, l)$ and $(\wedge, r)$ by applications of $(\vee', l)$ and $(\wedge', r)$, respectively. As shown in [7], $(\vee', l)$ can be shifted upward any quantifier inference. It is easy to check that the same holds for $(\wedge', r)$. ☐

**Remark 4.1.** The midsequent theorem does not hold for the sequent calculi underlying $\mathbf{HL}^{\forall}_{\mathbf{BCK}}$ and $\mathbf{HL}^{\forall}_{\mathbf{BCK}*}$, i.e., first-order $\mathbf{L}_{\mathbf{BCK}}$ and $\mathbf{L}_{\mathbf{BCK}^-}$. Indeed, as in the case of $\mathbf{LJ}$, in these calculi one cannot shift upward, e.g., the following application of $(\vee, l)$

$$\dfrac{\dfrac{A(b) \Rightarrow A(b)}{A(b) \Rightarrow \exists x A(x)}\,(\exists,\mathrm{r}) \quad \dfrac{A(a) \Rightarrow A(a)}{A(a) \Rightarrow \exists x A(x)}\,(\exists,\mathrm{r})}{A(a) \vee A(b) \Rightarrow \exists x A(x)}\,(\vee,\mathrm{l})$$

In addition, due to the lacking of the internal contraction rule in first-order $\mathbf{L}_{\mathbf{BCK}}$ and $\mathbf{L}_{\mathbf{BCK}^-}$, the same problem arises when one tries to shift upward certain $(\wedge, r)$ inferences over $(\forall, l)$.

As a corollary of the above theorem follows that one can effectively associate with each prenex formula $P$ of $\mathbf{MTL}\forall$ a propositional (variable free) formula $P^*$ which is valid if and only if $P$ is valid. (Note that $\mathbf{MTL}\forall$ does not admit an equivalent prenex normal form as in classical logic).

Recall that the *Herbrand universe* $U(P)$ of a formula $P$ is the set of all *ground* terms (those with no variables) which can be constructed from the set of constants and function symbols occurring in $P$. (To prevent $U(P)$ from being empty we add a constant if no such symbols appear in $P$.) A *Herbrand expansion* of $P$ is a disjunction of instances of $P$ where free variables are replaced with terms in $U(P)$.

In the following we write $\vdash_{\mathcal{L}} F$ for "the formula $F$ is derivable in the logic $\mathcal{L}$", i.e., $F \in \mathcal{L}$.

**Corollary 4.1. (Herbrand's theorem)**
Let $P$ be a quantifier-free formula. If $\vdash_{\mathbf{MTL}\forall} \exists\overline{x}P(\overline{x})$ then there exist tuples $\overline{t_1}, \ldots \overline{t_m}$ of terms in $U(P)$, such that $\vdash_{\mathbf{MTL}\forall} \bigvee_{i=1}^{m} P(\overline{t_i})$

Herbrand's theorem enable us to prove the decidability of the $\forall\exists$-fragment of $\mathbf{MTL}\forall$. Note that $\mathbf{MTL}\forall$ is undecidable, even if one only restricts to its prenex fragment. This follows as a corollary of a (more general) undecidability result proved in [6]. Indeed, there sufficient conditions to embed (undecidable fragments of) classical logic **CL** into sub-classical logics[4] were established. As a central notion for the "simulation of two-valued-ness" in non-classical contexts the following notion was introduced in [6]:

**Definition 4.1.** A schema of form $\alpha[p] \vee \overline{\alpha}[p]$ with exactly one formula variable $p$ is called a *duality principle* of a logic $\mathcal{L}$ if for all formulas $F$ : $\vdash_{\mathcal{L}}\alpha[F/p] \vee \overline{\alpha}[F/p]$ but there exists a formula $G$ such that $\not\vdash_{\mathbf{CL}}\alpha[G/p]$ and $\not\vdash_{\mathbf{CL}}\overline{\alpha}[G/p]$, where $\alpha[Q/p]$ (resp. $\overline{\alpha}[Q/p]$) stands for the formula that arises by instantiating the formula variable $p$ of the schema $\alpha[p]$ (resp. $\overline{\alpha}[p]$).

Following [6], we write rules of derivation in the form $\vdash A_1, \ldots, \vdash A_n \Longrightarrow \vdash B$.

**Definition 4.2.** A rule $(r) \vdash A_1, \ldots, \vdash A_n \Longrightarrow \vdash B$ is *sound* in a logic $\mathcal{L}$ if $\vdash_{\mathcal{L}}B'$ whenever $\vdash_{\mathcal{L}}A'_i$ for all $1 \leq i \leq n$, where $B'$ and $A'_i$ are corresponding instances of the schematic formulas of the rule.

$(r)$ is $\vee$-*normal* in $\mathcal{L}$ if its soundness in $\mathcal{L}$ implies the soundness of $\vdash A_1 \vee C, \ldots, \vdash A_n \vee C \Longrightarrow \vdash B \vee C$ (in the case of rules for introducing quantifiers, the eigenvariable condition must be obeyed, i.e. if a variable $x$ is free in $A_i$ while it is not in $B$, then $x$ is not free in $C$).

**Lemma 4.1. ([6])**
The prenex fragment of any sub-classical logic $\mathcal{L}$ (even without function symbols) that has a duality principle and in which the following rules

| | | | |
|---|---|---|---|
| $\vdash A \Longrightarrow \vdash A \vee B$ | (weak) | $\vdash A \vee (B \vee C) \Longrightarrow \vdash (A \vee B) \vee C$ | (assoc-l) |
| $\vdash A \vee B \Longrightarrow \vdash B \vee A$ | (comm) | $\vdash (A \vee B) \vee C \Longrightarrow \vdash A \vee (B \vee C)$ | (assoc-r) |
| $\vdash A \vee A \Longrightarrow \vdash A$ | (idem) | $\vdash A \vee C, \vdash B \vee C \Longrightarrow \vdash (A \wedge B) \vee C$ | (distr) |
| $\vdash A(t) \Longrightarrow \vdash \exists x A(x)$ | ($\exists$-in') | $\vdash A(y) \Longrightarrow \vdash \forall x(A(x))^{(y)}$ | ($\forall$-in') |

are sound and $\vee$-normal, is undecidable.

**Corollary 4.2.** The prenex fragment of $\mathbf{MTL}\forall$ is undecidable.

**Proof:**
It is easy to check that the rules (weak), (assoc-l), (assoc-r),(comm), (idem, (distr), ($\exists$-in') are both: sound and $\vee$-normal, in $\mathbf{MTL}\forall$. The soundness of ($\forall$-in') is straightforward. To prove that ($\forall$-in') is $\vee$-normal, one uses axiom ($\forall 3$). Finally, the schema $(p \to \neg p) \vee (\neg p \to p)$, which is an instance of axiom (Lin), is a duality principle of $\mathbf{MTL}\forall$. The claim follow by Lemma 4.1. □

**Definition 4.3.** Let $\mathsf{Q}_1 y_1 \ldots \mathsf{Q}_n y_n P$, with $\mathsf{Q}_i \in \{\forall, \exists\}$ be a (prenex) formula. Its *Skolem form*, denoted by $\exists\overline{x}P^F(\overline{x})$[5], is obtained as follows: for each universal quantifier $\mathsf{Q}_t$ in $\mathsf{Q}_1 y_1 \ldots \mathsf{Q}_n y_n$, (1) one replaces

[4]By this we mean logics based on (a subset of) the signature of classical logic that can be extended to classical logic.
[5]The notation hides the fact that the Skolem form also depends on the quantifier prefix. However, below, the context will always provide the relevant information.

$y_t$ in $P$ by a *Skolem term*, i.e. a new function symbol $f(y'_1, \ldots, y'_r)$ where $y'_1, \ldots, y'_r$ are the existentially quantified variables preceding $y_t$ in $\mathsf{Q}_1 y_1 \ldots \mathsf{Q}_n y_n$, or a (new) constant $c_t$ if no such $y'_1, \ldots, y'_r$ exist; (2) one deletes $\mathsf{Q}_t y_t$ from $\mathsf{Q}_1 y_1 \ldots \mathsf{Q}_n y_n$.

**Lemma 4.2.** Let $P$ be a quantifier free formula of $\mathbf{MTL}\forall$:

$$\vdash_{\mathbf{MTL}\forall} \mathsf{Q}_1 y_1 \ldots \mathsf{Q}_n y_n P(y_1, \ldots, y_n) \quad \Rightarrow \quad \vdash_{\mathbf{MTL}\forall} \exists \overline{x} P^F(\overline{x}).$$

**Proof:**
Follows from the usual laws of quantification. □

**Corollary 4.3.** The fragment of $\mathbf{MTL}\forall$ only containing formulas of the kind $\forall \overline{x} \exists \overline{y} P$, where $P$ is quantifier-free, is decidable.

**Proof:**
Note that Herbrand universe for such formulas is finite. The claim therefore follows by Herbrand's theorem and the decidability of $\mathbf{MTL}$ (see [18]). □

To show that $\mathbf{MTL}\forall$ allows for *reverse Skolemization* (i.e., the re-introduction of quantifiers in Herbrand expansions replacing Skolem terms) we recall the following lemma establishing sufficient conditions for a logic to allow this property.

**Lemma 4.3. ([5])**
Let $\mathcal{L}$ be any sub-classical logic in which the rules (weak), (assoc-l), (assoc-r), (comm), (idem), (distr), ($\exists$-in') and ($\forall$-in') are sound and $\vee$-normal and let $\exists \overline{x} P^F(\overline{x})$ be the Skolem form of the formula $\mathsf{Q}_1 y_1 \ldots \mathsf{Q}_n y_n P(y_1, \ldots, y_n)$. For all tuples of terms $\overline{t_1}, \ldots, \overline{t_m}$ of the Herbrand universe of $P^F(\overline{x})$

$$\vdash_{\mathcal{L}} \bigvee_{i=1}^{m} P^F(\overline{t_i}) \quad \Rightarrow \quad \vdash_{\mathcal{L}} \mathsf{Q}_1 y_1 \ldots \mathsf{Q}_n y_n P(y_1, \ldots, y_n).$$

**Corollary 4.4.** Let $P$ be a quantifier free formula of $\mathbf{MTL}\forall$:

$$\vdash_{\mathbf{MTL}\forall} \exists \overline{x} P^F(\overline{x}) \quad \Rightarrow \quad \vdash_{\mathbf{MTL}\forall} \mathsf{Q}_1 y_1 \ldots \mathsf{Q}_n y_n P(y_1, \ldots, y_n).$$

**Proof:**
Follows from Herbrand's theorem and Lemma 4.3. □

# Appendix: Proof of Theorem 3.2

The proof of cut-elimination for $\mathbf{HL}^{\forall}_{\mathbf{BCK}}$ is similar to the same proof for the hypersequent calculus for first-order Gödel logic outlined in [4]. It proceeds by selecting for elimination a largest cut (w.r.t. the number of connectives and quantifiers) and replacing it by smaller cuts in the place in which the cut-formula is introduced. This requires to trace the cut-formula through a derivation. To this purpose we introduce the notion of *decorated* formulas in a derivation $d$ of $\mathbf{HL}^{\forall}_{\mathbf{BCK}}$.

In the following we write $d, G' \vdash \mathrm{G}$ if $d$ is a proof in $\mathbf{HL}^{\forall}_{\mathbf{BCK}}$ of $G$ from the assumption $G'$.

**Definition 4.4.** Let $d \vdash$ G and $A$ be a formula in $G$ that is not the cut-formula of any cut in $d$. The *decoration* of $A$ (in $d$) is inductively defined as follows: we denote by $A^*$ a decorated occurrence of $A$. Given a hypersequent $G'$ in $d$ with possibly some $A$'s decorated. Let $(r)$ be the rule introducing $G'$. We distinguish some cases according to $(r)$.

1. $(r)$ is a logical or a quantifier rule, e.g.,

$$\frac{H \mid \Gamma' \Rightarrow C'}{H \mid \Gamma \Rightarrow C}$$

   (a) $A^*$ is the introduced formula in $(r)$. Suppose $A^* \in \Gamma$. $A^* \in \Gamma'$ if and only if $A^* \in$ [$\Gamma$ without the introduced formula]. Moreover, the decoration of $H$ in the premise of $(r)$ is as in the conclusion, i.e., for each component $\{\Sigma \Rightarrow B\} \in H$, $A^* \in \Sigma$ if and only if $A^* \in \Sigma$ of the corresponding component belonging to the conclusion of $(r)$.
   Suppose $C$ is $A^*$. The decoration of $H$ in the premise of $(r)$ is as in the conclusion.

   (b) $A^*$ is not the introduced formula in $(r)$. The decoration of the components in the premise of $(r)$ is as in the conclusion.

   If $(r)$ is a two premises rule, the definition is analogous.

2. $(r)$ is an external structural rule or $(cut)$. The decoration of the formulas in the components of the premise(s) of $(r)$ is as in the conclusion.

3. $(r)$ is $(w, l)$ or $(w, r)$. Analogous to case 1.

**Definition 4.5.** The *complexity* $|A|$ of a formula $A$ is the number of its connectives. The *length* $|d|$ of a derivation $d$ is the maximal number of inference rules + 1 occurring on any branch of $d$. The *cut-rank* $\rho(d)$ of $d$ is the maximal complexity of cut-formulas in $d$ + 1.

Let $d({}^s/_x)$ and $G({}^s/_x)$ denote the result of substituting the term $s$ for all free occurrences of $x$ in the proof $d(x)$ and in the hypersequent $G(x)$, respectively.

**Lemma 4.4. (Substitution)**
If $d(x) \vdash G(x)$, then $d({}^s/_x) \vdash G({}^s/_x)$, with $|d({}^s/_x)| = |d(x)|$ and $\rho(d({}^s/_x)) = \rho(d(x))$, where $s$ only contains variables that do not occur in $d(x)$.

**Lemma 4.5. (Inversion)**

(i) If $d \vdash \mathrm{H} \mid \Gamma, \mathrm{A} \vee \mathrm{B} \Rightarrow \mathrm{C}$ then one can find $d_1 \vdash \mathrm{H} \mid \Gamma, \mathrm{A} \Rightarrow \mathrm{C}$ and $d_2 \vdash \mathrm{H} \mid \Gamma, \mathrm{B} \Rightarrow \mathrm{C}$

(iii) If $d \vdash \mathrm{H} \mid \Gamma \Rightarrow \mathrm{A} \wedge \mathrm{B}$ then one can find $d_1 \vdash \mathrm{H} \mid \Gamma \Rightarrow \mathrm{A}$ and $d_2 \vdash \mathrm{H} \mid \Gamma \Rightarrow \mathrm{B}$

(iv) If $d \vdash \mathrm{H} \mid \Gamma, \mathrm{A} \odot \mathrm{B} \Rightarrow \mathrm{C}$ then one can find $d_1 \vdash \mathrm{H} \mid \Gamma, \mathrm{A}, \mathrm{B} \Rightarrow \mathrm{C}$

(v) If $d \vdash \mathrm{H} \mid \Gamma \Rightarrow \mathrm{A} \rightarrow \mathrm{B}$ then one can find $d_1 \vdash \mathrm{H} \mid \Gamma, \mathrm{A} \Rightarrow \mathrm{B}$

(vi) If $d \vdash \mathrm{H} \mid \Gamma, \exists \mathrm{x} \mathrm{A}(\mathrm{x}) \Rightarrow \mathrm{C}$ then one can find $d_1 \vdash \mathrm{H} \mid \Gamma, \mathrm{A}(\mathrm{a}) \Rightarrow \mathrm{C}$

(vii) If $d \vdash \mathrm{H} \mid \Gamma \Rightarrow \forall \mathrm{x} \mathrm{A}(\mathrm{x})$ then one can find $d_1 \vdash \mathrm{H} \mid \Gamma \Rightarrow \mathrm{A}(\mathrm{a})$
such that $\rho(d_i) \leq \rho(d)$ and $|d_i| \leq |d|$, for $i = 1, 2$.

**Proof:**
The proof is by induction on $|d|$. As an example we illustrate (i) in detail.

We consider the last inference $(r)$ in $d$. $(r)$ is a logical or a quantifier rule

(a) The indicated occurrence of $A \vee B$ is the formula introduced by $(r)$. E.g.,

$$\frac{\begin{matrix} \vdots\, d' & \vdots\, d'' \\ H \mid \Gamma, B \Rightarrow C & H \mid \Gamma, A \Rightarrow C \end{matrix}}{H \mid \Gamma, A \vee B \Rightarrow C}$$

The required derivations $d_1$ and $d_2$ are just $d''$ and $d'$.

(b) The indicated occurrence of $A \vee B$ is not the formula introduced by $(r)$. Suppose, e.g., that $d$ ends in a rule $(\rightarrow, l)$ whose premises are

$$d' \vdash H \mid \Gamma, Y, A \vee B \Rightarrow C \ \text{ and } \ d'' \vdash H \mid \Gamma' \Rightarrow X$$

and the conclusion is $H \mid \Gamma, \Gamma', X \rightarrow Y, A \vee B \Rightarrow C$. Applying the induction hypothesis to $d'$ we obtain $d'_1 \vdash H \mid \Gamma, Y, A \Rightarrow C$ and $d'_2 \vdash H \mid \Gamma, Y, B \Rightarrow C$, with $|d'_i| < |d|$ and $\rho(d'_i) \leq \rho(d)$, for $i = 1, 2$. Then $(\rightarrow, l)$ can be applied to $d'_1$ and $d''$ (resp. $d'_2$ and $d''$) to obtain the proof $d_1$ (resp. $d_2$) with the required properties.

$(r)$ is a structural rule. The case in which $(r)$ is (cut) is analogous to case (b) above. If $(r)$ is an internal or external weakening rule or $(EC)$ and the indicated occurrence of $A \vee B$ is in $H$, the claim trivially follows by induction hypothesis and subsequent application of $(r)$. For otherwise, assume $(r)$ is (EC) and $d$ ends as follows

$$\frac{\begin{matrix} \vdots\, d' \\ H \mid \Gamma, A \vee B \Rightarrow C \mid \Gamma, A \vee B \Rightarrow C \end{matrix}}{H \mid \Gamma, A \vee B \Rightarrow C}$$

By induction hypothesis one obtains the proofs $d'_1 \vdash H \mid \Gamma, A \Rightarrow C \mid \Gamma, A \vee B \Rightarrow C$ and $d'_2 \vdash H \mid \Gamma, B \Rightarrow C \mid \Gamma, A \vee B \Rightarrow C$ with $|d'_i| \leq |d'| < |d|$ and $\rho(d'_i) \leq \rho(d)$, for $i = 1, 2$. The required derivations $d_1$ and $d_2$ are obtained applying again the induction hypothesis to $d'_1$ and $d'_2$ followed by an application of $(EC)$. If $(r)$ is $(com)$ and $d$ ends as follows:

$$\frac{\begin{matrix} \vdots\, d' & \vdots\, d'' \\ H \mid \Gamma', \Pi_2 \Rightarrow D & H \mid \Gamma, \Pi_1, A \vee B \Rightarrow C \end{matrix}}{H \mid \Gamma, \Gamma', A \vee B \Rightarrow C \mid \Pi_1, \Pi_2 \Rightarrow D}$$

By induction hypothesis one obtains the proofs $d''_1 \vdash H \mid \Gamma, \Pi_1, A \Rightarrow C$ and $d''_2 \vdash H \mid \Gamma, \Pi_1, B \Rightarrow C$ with $|d''_i| < |d|$ and $\rho(d'_i) \leq \rho(d)$, for $i = 1, 2$. Then $(com)$ can be applied to $d''_1$ and $d'$ (resp. $d''_2$ and $d'$) to obtain the proof $d_1$ (resp. $d_2$) with the required properties. □

**Lemma 4.6.** Let $d \vdash \mathrm{H} \mid \Gamma, \mathrm{A}^* \Rightarrow \mathrm{B}$, where $A^*$ is an atomic formula decorated in $d$ that is not the cut formula of any cut in $d$. One can find a proof $d', H \mid \Sigma \Rightarrow A \vdash \mathrm{H} \mid \Gamma, \Sigma \Rightarrow \mathrm{B}$ such that $\rho(d') = \rho(d)$.

**Proof:**
We first replace $A^*$ everywhere in $d$ with $\Sigma$ and we add $H$ to all hypersequents in $d$. In the resulting tree, for each leaf $H \mid B \Rightarrow B$ (resp. $H \mid \bot \Rightarrow B$), we add an application of $(EW)$ to recover the original axiom $B \Rightarrow B$ (resp. $\bot \Rightarrow B$) of $d$. Note that $A^*$ can originate

1. in an axiom $A \Rightarrow A$ or $A \Rightarrow \bot$. Then the axiom is transformed into $H \mid \Sigma \Rightarrow A$ or $H \mid \Sigma \Rightarrow \bot$, respectively.
2. by an internal weakening. We then replace the weakening on $A^*$ by stepwise weakenings of formulas in $\Sigma$.
3. by an external weakening. We then replace the weakening on the component $S$ is replaced by a weakening on $S$ in which we uniformly replace $A^*$ by $\Sigma$.

One can check that this procedure results in a derivation $d_1, H \mid \Sigma \Rightarrow A \vdash H \mid H \mid \Gamma, \Sigma \Rightarrow B$ with $\rho(d_1) = \rho(d)$. $d'$ is therefore obtained applying $(EC)$. □

**Lemma 4.7. (Reduction)**
Let $d_0 \vdash \mathrm{H} \mid \Sigma \Rightarrow \mathrm{A}$ and $d_1 \vdash \mathrm{H} \mid \Gamma, \mathrm{A} \Rightarrow \mathrm{C}$ both with cut-rank $\rho(d_i) \leq |A|$. We can find a derivation $d \vdash \mathrm{H} \mid \Gamma, \Sigma \Rightarrow \mathrm{C}$ with $\rho(d) \leq |A|$.

**Proof:**
By Proposition 3.1 we can assume $d_0$ and $d_1$ have atomic axioms. If $A$ is atomic, by Lemma 4.6 one can find a derivation $d', H \mid \Sigma \Rightarrow A \vdash H \mid \Gamma, \Sigma \Rightarrow C$ such that $\rho(d') = \rho(d_1)$. The required derivation $d$ is obtained concatenating $d_0$ and $d'$. Suppose $A$ not atomic. We outline here a proof for some relevant cases.

- $A = B \to D$. Let us consider the decoration of $A$ in $d_1$ starting from $H \mid \Gamma, (B \to D)^* \Rightarrow C$. We first replace in $d_1$ all the components $\{\Psi, (B \to D)^* \Rightarrow C'\}$ by $\{\Psi, \Sigma \Rightarrow C'\}$. Note that the resulting tree is not a proof anymore. We have then to consider the following "correction steps" according to the cases in which the decorated formula originates:

  (i) as introduced formula of a logical inference,
  (ii) by an internal weakening,
  (iii) by an external weakening.

  (i) We replace every ("uncorrect") inference step of the kind

$$\dfrac{\begin{matrix}\vdots \\ H' \mid \Psi \Rightarrow B\end{matrix} \quad \begin{matrix}\vdots \\ H' \mid \Psi', D \Rightarrow C'\end{matrix}}{H' \mid \Psi, \Psi', \Sigma \Rightarrow C'}\,(\to, l) \qquad \text{by}$$

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots \\ H' \mid \Psi \Rightarrow B\end{matrix}}{H \mid H' \mid \Psi \Rightarrow B}\,(EW) \quad \dfrac{\begin{matrix}\vdots\, d_0' \\ H \mid \Sigma, B \Rightarrow D\end{matrix}}{H \mid H' \mid \Sigma, B \Rightarrow D}\,(EW)}{H \mid H' \mid \Psi, \Sigma \Rightarrow D}\,(cut) \quad \dfrac{\begin{matrix}\vdots \\ H' \mid \Psi', D \Rightarrow C'\end{matrix}}{H \mid H' \mid \Psi', D \Rightarrow C'}\,(EW)}{H' \mid H' \mid \Psi, \Psi', \Sigma \Rightarrow C'}\,(cut)$$

where $d'_0 \vdash \mathrm{H} \mid \Sigma, \mathrm{B} \Rightarrow \mathrm{D}$ is obtained from $d_0$ by Inversion Lemma (Lemma 4.5).

(ii)The weakening on $(B \rightarrow D)^*$ is replaced by stepwise weakenings of formulas $X \in \Sigma$.

(iii) The weakening on the component $S$ is replaced by a weakening on $S$ in which we uniformly replace $(B \rightarrow D)^*$ by $\Sigma$.

One can check that this procedure results in a proof $d \vdash \mathrm{H} \mid \Gamma, \Sigma \Rightarrow \mathrm{C}$ with $\rho(d) \leq |B \rightarrow C|$.

- Cases $A = B \wedge D$ and $A = \forall x A(x)$ are treated analogously.
- $A = \exists x A(x)$. Let us consider the decoration of $A$ in $d_0$ starting from $H \mid \Gamma \Rightarrow (\exists x A(x))^*$. We first replace in $d_0$ all the components $\{\Psi \Rightarrow (\exists x A(x))^*\}$ by $\{\Gamma, \Psi \Rightarrow C\}$. As in the previous case, this does not result in a proof anymore. Correction steps (ii) and (iii) proceed similarly as above. If $\exists x A(x)$ originates as the introduced formula of a logical inference, we replace every ("uncorrect") inference step of the kind

$$\dfrac{\begin{array}{c}\vdots\\ H' \mid \Psi \Rightarrow A(t)\end{array}}{H' \mid \Gamma, \Psi \Rightarrow C}\,{\scriptstyle(\exists,l)} \qquad \text{by}$$

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ H' \mid \Psi \Rightarrow A(t)\end{array}}{H \mid H' \mid \Psi \Rightarrow A(t)}\,{\scriptstyle(\mathrm{EW})} \qquad \dfrac{\begin{array}{c}\vdots\, d'_1(t)\\ H \mid \Gamma, A(t) \Rightarrow C\end{array}}{H \mid H' \mid \Gamma, A(t) \Rightarrow C}\,{\scriptstyle(\mathrm{EW})}}{H' \mid H \mid \Psi, \Gamma \Rightarrow C}\,{\scriptstyle(\mathrm{cut})}$$

where $d'_1(t) \vdash \mathrm{H} \mid \Gamma, \mathrm{A(t)} \Rightarrow \mathrm{C}$ is obtained by the Inversion Lemma (and Substitution Lemma).

- Cases $A = B \vee D$ and $A = B \odot D$ are treated analogously.

$\square$

**Proof of Theorem 3.2:**
Let $d \vdash \mathrm{G}$ and $\rho(d) > 0$. The proof proceeds by induction on $(\rho(d), \#\rho(d))$, where $\#\rho(d)$ is the number of cuts in $d$ with complexity $\rho(d)$, using the Reduction Lemma.

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