Proof Search and Co-NP Completeness for Many-Valued Logics

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Abstract

We provide a methodology to introduce proof search oriented calculi for a large class of many-valued logics, and a sufficient condition for their Co-NP completeness. Our results apply to many well known logics including Gödel, Łukasiewicz and Product Logic, as well as Hájek's Basic Fuzzy Logic.

1 Introduction

The invertibility of rules [1] in a proof system is an important feature for guiding proof search; in addition it turns out to be very useful to settle the computational complexity of the formalized logic. For many-valued logics, calculi with invertible rules (*proof search oriented calculi*) have been provided for all finite-valued logics. These calculi are defined by generalizing Gentzen sequents $A_1, \ldots, A_n \Rightarrow B_1, \ldots, B_m$ to many placed (or labelled) sequents, each corresponding to a truth value of the logic, see e.g. the survey [9] ([12], for the non-deterministic case). The construction of these calculi, out of the truth

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[1] The premises are derivable whenever their conclusions are.

tables of the connectives, is even computerized, see [10]. This design does not apply to infinite-valued logics where, excepting Gödel logic [8,5], proof search oriented calculi – when available – are introduced on a logic by logic basis and their construction requires some ingenuity; this is for instance the case of the calculi for Łukasiewicz and Product logic [29,28,27], defined using *hypersequents*, which are finite "disjunctions" of Gentzen sequents [4,3].

An important step towards the automated construction of proof search oriented calculi for many-valued logics was taken in [8] with the introduction of *sequents of relations*, that are disjunctions of semantic predicates over formulas, and of a methodology to construct such calculi for all *projective* logics. Intuitively a logic is projective if for each connective $\Box$, the value of $\Box(x_1, \ldots, x_n)$ is equal to a constant or to one of the $x_1, \ldots, x_n$. The methodology was extended in [19] to handle *semi-projective logics* where the value of each $\Box(x_1, \ldots, x_n)$ can also be a term of the form $p(x_i)$ with p unary function symbol and $i \in \{1, \ldots, n\}$. Projective logics are quite interesting, but perhaps not general enough: among many-valued logics, only the finite-valued logics and Gödel logic are projective. Semi-projective logics constitute a slightly larger class, and they capture, for instance, Nilpotent and Weak Nilpotent Minimum logic [21], the relevance logic *RM* [2] and, by considering conservative extensions, Hájek's Basic Fuzzy Logic BL extended with n-contraction [14]. All semi-projective logics have a locally finite variety as their equivalent algebraic semantics while important many-valued logics such as Łukasiewicz logic, Product Logic and BL do not, despite the fact that they have suitable calculi with invertible rules [18,30,16] and are Co-NP complete. The calculi in [18,30,16] are defined using relational hypersequents (*r-hypersequents* for short) that generalize hypersequents by considering finite disjunctions of two different types of sequents, where Gentzen's sequent arrow $\Rightarrow$ is replaced in one by $<$ and in the other by $\leq$.

In this paper we generalize r-hypersequents to disjunctions of arbitrary semantic predicates (not only $<$ and $\leq$) over *multisets* of formulas, rather than single formulas as in the case of sequents of relations. We introduce a methodology to define r-hypersequent calculi for a large class of many-valued logics (*hyperprojective logics*) and identify sufficient conditions on these calculi that guarantee the Co-NP completeness of the formalized logics. Our methodology applies to projective and semi-projective logics as well as to Łukasiewicz, Product Logic and BL; it subsumes all existing results on sequent of relations and on r-hypersequent calculi (e.g. [8,6,19,18,30,16]), and provides a unified perspective on most of the known complexity results for many-valued logics. Moreover, our method can be applied to new logics (or already known logics not having yet proof search oriented calculi), provided that they are hyperprojective.

In a hyperprojective logic the value of each connective $\Box(x_1, \ldots, x_n)$ is defined by cases this time expressed by relations on *multisets* of constants, of terms $x_1, \dots, x_n$ and of $p(x_i)$, for p unary functions and $i \in \{1, \dots, n\}$.

We illustrate the idea behind hyperprojective logics and the way we define r-hypersequent calculi for them with the example of Product Logic. For this logic, as in (the projective presentation of) Gödel Logic [8,6,7] it is natural to consider the relations $<$ and $\leq$. Product Logic is neither projective nor semi-projective, because if $x, y \notin \{0, 1\}$ then the product $x\&y$ depends on *both* x and y. The idea is to represent the product by a monoidal operation $\oplus$ standing for the union of multisets, i.e. $x\&y = x \oplus y$.

In general, to define invertible rules for a connective $\Box(x_1, \dots, x_n)$ of a hyperprojective logic L we will consider "reductions" (based on the relations in the semantic theory of L) that act on multisets of formulas, i.e., on $\Gamma \oplus \Box(x_1, \dots, x_n)$, where Γ is a multiset, and in which the formula $\Box(x_1, \dots, x_n)$ is decomposed into a *multiset* of smaller terms (constants, x_i or $p(x_i)$).

In the particular case of the connective $x\&y$ of Product Logic we consider the following "reduction cases": for $\Gamma \oplus x\&y \lhd \Delta$ as $\Gamma \oplus x \oplus y \lhd \Delta$ and for $\Gamma \lhd \Delta \oplus x\&y$ as $\Gamma \lhd \Delta \oplus x \oplus y$, where $\lhd$ denotes either $<$ or $\leq$. Our calculus for Product Logic will then contain r-hypersequents consisting of disjunctions of sequents of the form $\Gamma < \Delta$ or $\Gamma \leq \Delta$, where Γ and Δ are multisets of formulas. As in the case of hypersequents [4,3] the disjunction will be denoted by "|" and the union of multisets by ",". With this notation we have that $\phi\&\psi, \Gamma \lhd \Delta$ reduces (and it is indeed equivalent to) to $\phi, \psi, \Gamma \lhd \Delta$ and $\Gamma \lhd \Delta, \phi\&\psi$ reduces to $\Gamma \lhd \Delta, \phi, \psi$, which naturally lead to the following left and right rules for the connective & w.r.t. the relation $\lhd$ (below H stands for an arbitrary r-hypersequent)

$$\frac{H|\phi, \psi, \Gamma \lhd \Delta}{H|\phi\&\psi, \Gamma \lhd \Delta} \qquad \frac{H|\Gamma \lhd \Delta, \phi, \psi}{H|\Gamma \lhd \Delta, \phi\&\psi}$$

Now since

$$x \wedge y = \begin{cases} x & \text{if } x \leq y \\ y & \text{if } y < x \end{cases} \qquad x \vee y = \begin{cases} y & \text{if } x \leq y \\ x & \text{if } y < x \end{cases}$$

we have:

- $\Gamma, \phi \vee \psi \lhd \Delta$ reduces to $\psi < \phi | \Gamma, \psi \lhd \Delta$ and to $\psi \leq \phi | \Gamma, \phi \lhd \Delta$;
- $\Gamma \lhd \Delta, \phi \vee \psi$ reduces to $\psi < \phi | \Gamma \lhd \Delta, \psi$ and to $\phi \leq \psi | \Gamma \lhd \Delta, \phi$;
- $\Gamma, \phi \wedge \psi \lhd \Delta$ reduces to $\psi < \phi | \Gamma, \phi \lhd \Delta$ and to $\phi \leq \psi | \Gamma, \psi \lhd \Delta$;
- $\Gamma \lhd \Delta, \phi \wedge \psi$ reduces to $\psi < \phi | \Gamma \lhd \Delta, \phi$ and to $\phi \leq \psi | \Gamma \lhd \Delta, \psi$.

Finally, recalling that in Product Logic

$$x \to y = \begin{cases} \frac{y}{x} & \text{if } \; y < x \\ 1 & \text{if } \; x \leq y \end{cases}$$

and that "," represents the product, we have the following reductions for $\to$:

- $\Gamma, \phi \to \psi \lhd \Delta$ reduces to $\psi < \phi | \Gamma, 1 \lhd \Delta$ and to $\phi \leq \psi | \Gamma, \psi \lhd \Delta, \phi$, to be read as: either $\phi \leq \psi$ or ($\psi < \phi$ and then) $\Gamma, \frac{\psi}{\phi} \lhd \Delta$, which is equivalent to $\Gamma, \psi \lhd \Delta, \phi$; moreover, either $\psi < \phi$ or ($\phi \leq \psi$ and hence) $\phi \to \psi = 1$, and $\Gamma, 1 \lhd \Delta$.
- $\Gamma \lhd \Delta, \phi \to \psi$ reduces to $\psi < \phi | \Gamma, 1 \lhd \Delta$ and $\phi \leq \psi | \Gamma, \phi \lhd \Delta, \psi$, whose explanation is similar.

Note that the above reductions are nothing but invertible rules introducing a connective $\star \in \{\to, \wedge, \vee\}$ in the left position ($\Gamma, \phi \star \psi \lhd \Delta$) or in the right position ($\Gamma \lhd \Delta, \phi \star \psi$).

In this paper we provide a methodology to introduce relational hypersequent calculi for all hyperprojective logics, and a sufficient condition for their Co-NP completeness.

The paper is organized as follows: Section 2 introduces hyperprojective logics. Similarly to the case of projective and semi-projective logics the definition is based on the shape of their underlying first-order semantic theory. Since proving that a logic is hyperprojective is a non trivial task, Section 2.3 contains a general result on how to build such a semantic theory for a large class of many-valued logics (algebraizable, whose equivalent semantics is generated by a single algebra, and whose first-order theory satisfies suitable properties). Section 3 connects hyperprojective logics to r-hypersequent calculi. Section 3.1 shows how to transform (the semantic theory behind) a hyperprojective logic into invertible r-hypersequent rules. Soundness and completeness of the resulting calculi is contained in Section 3.2. Section 4 proves the co-NP completeness of the validity problem of hyperprojective logics whose r-hypersequent calculi satisfy suitable and easily checkable properties. As case studies we apply our methodology to Gödel logic, the relevance logic *RM*, Product logic, Łukasiewicz logic and Hájek's BL by presenting for them r-hypersequent calculi and alternative proofs (w.r.t., e.g., [11,24,1]) of Co-NP completeness.

2 Hyperprojective logics

We define the class of many-valued logics that we consider in this paper and begin this section recalling some basic properties of multisets. For all concepts of universal algebra we refer to [17] and for many-valued logics to [23].

2.1 Preliminaries on multisets

A *finite multiset* on a set S is a map μ from S into $\mathbb{N}$ such that the set $S(\mu) = \{s \in S \mid \mu(s) > 0\}$, called the *support* of μ, is finite. Moreover for every $s \in S(\mu)$, $\mu(s)$ is called the *multiplicity* of s. Notice that a multiset μ is completely determined by its support $S(\mu)$ and the multiplicity of each $s \in S(\mu)$.

The *union* $\mu_1 \oplus \mu_2$ of two finite multisets μ_1 and μ_2 of S is defined as $(\mu_1 \oplus \mu_2)(s) = \mu_1(s) + \mu_2(s)$. Note that $\oplus$ is commutative and associative and its neutral element is the zero function on S, indicated by ε. In addition, $S(\mu_1 \oplus \mu_2) = S(\mu_1) \cup S(\mu_2)$. Given two sequences of multisets $\overline{\mu} = \mu_1, \ldots, \mu_n$ and $\overline{\nu} = \nu_1, \ldots, \nu_n$, both with length n, then $\overline{\mu} \oplus \overline{\nu}$ denotes the sequence of multisets $\mu_1 \oplus \nu_1, \ldots, \mu_n \oplus \nu_n$.

Definition 1. *$\Theta_i^n(\nu)$ is the sequence $\varepsilon, \ldots, \nu, \ldots, \varepsilon$ of n multisets, such that each multiset is the empty one except at position i, where it is ν.*

Thus, given a sequence of multisets $\overline{\mu} = \mu_1, \ldots, \mu_n$, we have $\overline{\mu} \oplus \Theta_i^n(\nu) = \mu_1, \ldots, \mu_i \oplus \nu, \ldots, \mu_n$.

In the sequel, when there is no danger of confusion we identify the element $x \in S$ with the multiset μ with support $\{x\}$ such that $\mu(x) = 1$. Moreover, given (not necessarily distinct) elements $a_1, \ldots, a_n$, $a_1 \oplus \cdots \oplus a_n$ (written without parentheses) denotes the multiset μ such that $S(\mu) = \{x : x = a_i \text{ for some } i \text{ with } 1 \leq i \leq n\}$, and for $x \in S(\mu)$, $\mu(x)$ is the cardinality of the set $\{i : x = a_i\}$.

2.2 Definition and examples

As in the case of projective and semi-projective logics [8,19], a hyperprojective logic L has an associated semantic first-order theory T_L whose intended range of discourse are sets of truth values. We will make some assumptions on T_L. The first is on its language and we assume that:

(TL0) The language of T_L consists of: (a) an n-ary operation for each [2] n-ary connective of L and a constant for each propositional constant of L; (b) possibly, finitely many constants and finitely many unary function symbols, denoted in the sequel by $c_1, \ldots, c_s$ and $f_1, \ldots, f_h$ respectively, not in the language of L; (c) a constant ε for the empty multiset and a binary operation $\oplus$ for the union of multisets; (d) some relation symbols, not in the language of L; (e) for each n-ary relation symbol P, we assume that either the negation of $P(x_1, \ldots, x_n)$ is equivalent in T_L to a disjunction of atomic formulas without function symbols, or that the language of T_L has a relation symbol P^*, along with the axiom $P^*(x_1, \ldots, x_n) \Leftrightarrow \neg P(x_1, \ldots, x_n)$. In any case, the negation of an atomic formula P is equivalent to a formula which is either atomic or the disjunction of atomic formulas, and which will be denoted by P^*. Moreover, we set $P^{**} = P$.

Before introducing the other conditions, we briefly comment on (TL0). Condition (a) allow us to treat formulas of L as terms of T_L. As in the definition of semi-projective logics [19], the unary function symbols and constant symbols in condition (b) may help to "reduce" the connectives to the right format, (see condition (TL4) below); when present in T_L, we assume that the extension L' of L by the unary connectives is a conservative extension of L. Condition (c) allows us to represent not only formulas, but also multisets of formulas, while condition (d) will permit to represent the proof theory of L inside T_L. Finally, condition (e) allows us to eliminate negations and to keep the language finite.

Example 2. *The semantic theory T_L for product logic is the theory of product chains with large order $\leq$ and strict order $<$; $\oplus$ is interpreted as product, and ε is interpreted as 1. Moreover $x \leq^* y$ is equivalent to $y < x$, and hence we may eliminate negations.*

To introduce the next conditions on T_L, we need some auxiliary definitions.

Definition 3. *A formula of T_L is called* simple *if does not contain any quantifier, negation, implication and function symbol with the exception of $\oplus$. The formula is called* weakly simple *if it has no quantifiers, negation or implication and its* terms *are either variables, or constants, or of the form $f_i(x)$, where f_i is a unary function symbol not in L, or of the form $t_1 \oplus \cdots \oplus t_k$, where $t_1, \ldots, t_k$ are terms of the form shown above.*

The next three assumptions on T_L are:

(TL1) There is a weakly simple formula $Des(x)$ of T_L such that for every formula ϕ, $T_L \models Des(\phi)$ iff $L \models \phi$.

[2] We assume that L has finitely many constants and connectives, and we identify each connective with its corresponding operation and each propositional constant with its corresponding constant symbol.

(TL2) The set of valid formulas of T_L that are universal closures of weakly simple formulas is decidable.

(TL3) If T_L has unary function symbols $f_1, \dots, f_h$ not in L, then for all $i, j = 1, \dots, h$, T_L has an axiom of the form $f_i(f_j(x)) = x$ or an axiom of the form $f_i(f_j(x)) = f_r(x)$ for some r.

The next (and last) condition on T_L describes the connectives of L and will determine the logical rules of the proof systems defined in the next section. We introduce some notation and terminology first: an overlined letter will denote the sequence consisting of the same letter with subscripts. Thus for instance, $\overline{x}$ will denote $x_1, \dots, x_n$ for some $n > 0$ and $\overline{\delta}$ will denote $\delta_1, \dots, \delta_m$, for some $m > 0$. As usual whenever writing $P(\overline{x})$ or $f(\overline{x})$ we implicitly assume that the arities of P and f coincide with the number of x_i.

Definition 4. *The formulas $Q_1, \dots, Q_s$ of T_L form a partition of the unit if $T_L \models Q_i \to \neg Q_j$ for $i \neq j$ and $T_L \models \bigvee_{j=1}^{s} Q_j$.*

We denote by L' the logic L extended by the symbols of constant and of unary function in T_L but not in L, to be interpreted as propositional constants and as unary connectives, respectively.

The connectives of hyperprojective logics have a case-reduction of the following form: for every connective $\Box$ in the language of L, for every unary connective u in the language of T_L but not in the language of L, for every predicate symbol P of T_L with arity k and every $1 \leq i \leq k$

$$P(\overline{\mu} \oplus \Theta_i^k(\Box(\overline{x}))) = \begin{cases} P(\overline{\mu} \oplus \overline{\nu}_1^{(\Box P i)}) \text{ if } Q_1^{(\Box P i)}(\overline{x}) \\ \dots \qquad\qquad \dots \\ P(\overline{\mu} \oplus \overline{\nu}_\ell^{(\Box P i)}) \text{ if } Q_\ell^{(\Box P i)}(\overline{x}) \end{cases}$$

$$P(\overline{\mu} \oplus \Theta_i^k(u(\Box(\overline{x})))) = \begin{cases} P(\overline{\mu} \oplus \overline{\nu}_1^{(u(\Box) P i)}) \text{ if } Q_1^{(u(\Box) P i)}(\overline{x}) \\ \dots \qquad\qquad \dots \\ P(\overline{\mu} \oplus \overline{\nu}_{\ell'}^{(u(\Box) P i)}) \text{ if } Q_{\ell'}^{(u(\Box) P i)}(\overline{x}) \end{cases}$$

where:

- $\overline{\mu}$ is a sequence of multisets of formulas of L';
- $\overline{\nu}_a^{(\Box P i)}$, $a = 1, \dots, \ell$, and $\overline{\nu}_b^{(u(\Box) P i)}$, $b = 1, \dots, \ell'$, are sequences of length k of multisets whose support is contained in the set

$$Z(\overline{x}) = \{x, c_v, f_w(x) \mid x \in \{\overline{x}\} \cup \{c_1, \dots, c_s\}, v = 1, \dots, s, w = 1, \dots, h\};$$

- $Q_a^{(\Box P i)}(\overline{x})$, $a = 1, \dots, \ell$ and $Q_b^{(u(\Box) P i)}(\overline{x})$, $b = 1, \dots, \ell'$, are partitions of the unit consisting of weakly simple formulas.

The dependency of $\overline{\nu}_a^{(\Box Pi)}$, $\overline{\nu}_b^{(u(\Box)Pi)}$, $Q_b^{(u(\Box)Pi)}(\overline{x})$ and $Q_a^{(\Box Pi)}(\overline{x})$ on $\Box, P, i$ will be omitted in what follows. Thus, with the notation just introduced, (TL4) reads:

(TL4) For each k-ary predicate P, for each position i with $1 \leq i \leq k$, for each connective $\Box$ and for each unary function symbol u in T_L and not in L, there are two partitions of the unit $Q_1(\overline{x}), \dots, Q_\ell(\overline{x})$ and $Q_1^u(\overline{x}), \dots, Q_{\ell'}^u(\overline{x})$, consisting of weakly simple formulas, and sequences of multisets $\overline{\nu}_a : a = 1, \dots, \ell$, $\overline{\nu}_b^u : b = 1, \dots, \ell'$, with support $Z(\overline{x})$, such that, for $a = 1, \dots, \ell$, for $b = 1, \dots, \ell'$, for every sequence $\overline{\mu}$ of multisets of formulas of L' and for every substitution σ of variables with formulas of L', the following conditions hold:

(TL4/1) $T_L \models Q_a(\sigma(\overline{x})) \Rightarrow (P(\overline{\mu} \oplus \sigma(\Theta_i^k(\Box(\overline{x})))) \Leftrightarrow P(\overline{\mu} \oplus \sigma(\overline{\nu}_a)))$,

(TL4/2) $T_L \models Q_b^u(\sigma(\overline{x})) \Rightarrow (P(\overline{\mu} \oplus \sigma(\Theta_i^k(u(\Box(\overline{x}))))) \Leftrightarrow P(\overline{\mu} \oplus \sigma(\overline{\nu}_b^u)))$

where $\Rightarrow$ and $\cap$ indicate the implication and the conjunction in T_L, and $Q \Leftrightarrow R$ stands for $(Q \Rightarrow R) \cap (R \Rightarrow Q)$.

Definition 5. *A logic L is hyperprojective if its semantic theory T_L satisfies conditions (TL0), ..., (TL4) above. L is said to be* regular *if all unary function symbols or constants in T_L are already in L.*

Remark 2.1. *Projective and semi-projective logics are particular cases of hyperprojective logics (see Example 7). Indeed, every connective of a projective logic has the following form*

$$\Box(\overline{x}) = \begin{cases} t_1 & \text{if } Q_1(\overline{x}) \\ \dots & \dots \\ t_k & \text{if } Q_k(\overline{x}) \end{cases}$$

where $Q_1(\overline{x}), \dots, Q_k(\overline{x})$ are simple formulas which form a partition of the unit and $t_1, \dots, t_k$ are either variables among $\overline{x}$ or constants. In the case of a semi-projective logic L, we have instead that for every n-ary connective $\Box$ of L and every unary connective u in the language of the semantic theory T_L:

$$\Box(\overline{x}) = \begin{cases} t_1 & \text{if } Q_1(\overline{x}) \\ \dots & \dots \\ t_k & \text{if } Q_k(\overline{x}) \end{cases} \qquad \textit{and} \qquad u(\Box(\overline{x})) = \begin{cases} t'_1 & \text{if } Q_1^u(\overline{x}) \\ \dots & \dots \\ t'_k & \text{if } Q_k^u(\overline{x}) \end{cases}$$

where $Q_1(\overline{x}), \dots, Q_k(\overline{x})$, and $Q_1^u(\overline{x}), \dots, Q_k^u(\overline{x})$ respectively, still form a partition of the unit, but are now weakly simple (i.e., with unary function symbols in T_L), and t_i, t'_i are either variables in $\overline{x}$, constants or terms of the form $f_i(x)$, x a variable in $\overline{x}$ and f_i a unary function symbols in T_L.

After identifying a multiset consisting of just one formula with the formula itself, we see that projective or semi-projective logics are a special case of *hyperprojective logics.*

In regular hyperprojective logics weakly simple formulas are also simple, condition (TL3) becomes irrelevant, and (TL4/2) does not apply.

A non regular hyperprojective logic is a logic that admits a conservative extension which is a regular hyperprojective logic. Regular hyperprojective logics are the three most famous Fuzzy Logics [23]: Gödel, Łukasiewicz and Product logic (see Example 7), while a hyperprojective logic that is not regular is Hájek's Basic Fuzzy Logic BL [23], see the example below.

Example 6. *Recall that BL is complete with respect to the ordinal sum* $\mathbf{A}$ *of* ω *copies of* $[0,1]_{MV}$*. This algebra can be defined as follows: The domain of* $\mathbf{A}$ *is the set* $R^+ \cup \{\infty\}$ *of non-negative reals extended with a new element* ∞*. Let* $<^+$ *and* $\leq^+$ *stand for the usual strict and non-strict total order on* R^+*, and* $\lfloor x \rfloor$ *denote the greatest integer* z *such that* $z \leq x$ *if* $x \in R^+$ *and* ∞ *if* $x = \infty$*. Set:*

- $x \leq y$ *iff either* $x, y \in R^+$ *and* $x \leq^+ y$*, or* $y = \infty$*;*
- $x < y$ *iff either* $x, y \in R^+$ *and* $x <^+ y$*, or* $x \in R^+$ *and* $y = \infty$*;*
- $x \ll y$ *iff either* $x \in R^+$ *and* $y = \infty$ *or* $x, y \in R^+$*, and* $\lfloor x \rfloor <^+ \lfloor y \rfloor$*;*
- $x \ll_= y$ *iff* $y \ll x$ *is false;*
- $x \prec y$ *iff* $x, y \in R^+$*,* $x <^+ y$ *and* $\lfloor x \rfloor = \lfloor y \rfloor$*;*
- $x \preceq y$ *if either* $x \prec y$ *or* $x = y$*;*
- $x \equiv y$ *iff either* $x \preceq y$ *or* $y \prec x$*.*

The relation symbol $\leq$ *induces a total order on* $\mathbf{A}$ *with bottom element* 0 *and top element* ∞*: hence the constant* 0 *of BL is interpreted as the number* 0 *and the constant* 1 *is interpreted as* ∞*. To avoid ambiguity, in the sequel we will denote the real number* 1 *by* 1^+*, in order to distinguish it from the constant* 1 *of BL.*

Join and meet are naturally induced by $\leq$*, while conjunction and implication can be defined on* $\mathbf{A}$ *with the help of the newly introduced relation symbols as follows:*

$$x \& y = \begin{cases} \infty & \text{if } 1 \preceq x \text{ and } 1 \preceq y, \\ x & \text{if } x \ll y, \\ y & \text{if } y \ll x, \\ \lfloor x \rfloor & \text{if } x \equiv y,\ x \ll 1 \text{ and } x + y < 2\lfloor x \rfloor + 1^+, \\ x + y - \lfloor x \rfloor - 1^+ & \text{otherwise.} \end{cases}$$

$$x \to y = \begin{cases} \infty & \text{if } x \leq y, \\ y & \text{if } y \ll x, \\ y + \lfloor x \rfloor + 1^+ - x & \text{otherwise.} \end{cases}$$

Notice that, whenever $x \equiv y$ and $x \ll 1$, the operation $x + y$ is well defined since $x, y \in R^+$, and that $1 \preceq x$ is equivalent to $x = \infty$.

We now introduce relations between multisets whose elements are formulas in the language of BL enriched with the constant 1^+. In the sequel we identify formulas with their truth values. Thus, with reference to an arbitrary valuation v, we write for instance $\phi \ll \psi$ to mean that $v(\phi) \ll v(\psi)$, and we write $\phi + \psi$ for $v(\phi) + v(\psi)$.

Valuations can be extended to multisets of formulas in the following way: if $\sigma = (S_\sigma, \mu_\sigma)$ is a multiset, then

$$v(\sigma) = \sum_{\phi \in S_\sigma} \mu_\sigma(\phi) \cdot (\phi - \lfloor \phi \rfloor) + \mu_\sigma(1^+).$$

This extension is consistent with the union of multisets $\oplus$, in the sense that if σ and ν are two multisets then $v(\sigma \oplus \nu) = v(\sigma) + v(\nu)$.

Relations of the form $\sigma \ll \nu$ are only allowed when σ and ν both consist of a single formula, that is, they are of the form $\phi \ll \psi$, where both ϕ and ψ are formulas in the extended language of BL. For the sake of definiteness, we will consider false by definition any relation of the form $\sigma \ll \nu$ where either σ or ν or both do not consist of a single formula. By contrast, we allow relations $\sigma \preceq \nu$ and $\sigma \prec \nu$ where σ and ν are multisets whose elements are formulas in the language of BL enlarged with 1^+.

If $\sigma = (S_\sigma, \mu_\sigma)$ and $\nu = (S_\nu, \mu_\nu)$ are multisets whose elements are formulas in the language of BL enriched with 1^+, then we set $\sigma \prec \nu$ to be true if the following conditions hold:

(1) for all $\phi, \psi \in S_\sigma \cup S_\nu$ $\phi, \psi \neq \infty$ and $\lfloor \phi \rfloor = \lfloor \psi \rfloor$;
(2) $\sum_{\phi \in S_\sigma} \mu_\sigma(\phi) \cdot (\phi - \lfloor \phi \rfloor) + \mu_\sigma(1^+) < \sum_{\psi \in S_\nu} \mu_\nu(\psi)(\psi - \lfloor \psi \rfloor) + \mu_\nu(1^+)$.

Moreover we set $\sigma \preceq \nu$ if either for all $\phi \in S_\sigma \cup S_\nu$, $\phi = 1$, or condition (1) above holds and

$$\sum_{\phi \in S_\sigma} \mu_\sigma(\phi) \cdot (\phi - \lfloor \phi \rfloor) + \mu_\sigma(1^+) \leq \sum_{\psi \in S_\nu} \mu_\nu(\psi)(\psi - \lfloor \psi \rfloor) + \mu_\nu(1^+).$$

Notice that if σ is the singleton of ϕ and ν is the singleton of ψ, then $\sigma \preceq \nu$ iff $\phi \preceq \psi$, and $\sigma \prec \nu$ iff $\phi \prec \psi$.

We shall indicate with $\mathbf{A}^*$ the structure $\mathbf{A}$ enriched with the new relation and function symbols, and we take T_{BL} to be the first-order theory of $\mathbf{A}^*$.

With the help of the relation symbols introduced above, we shall now see how to reduce any formula $\Box(\phi, \psi)$ occurring in a relation $\sigma \lhd \nu$ to a multiset whose elements are ϕ, ψ, 1^+ or 1, for any binary connective $\Box$ of BL and any relation symbol $\lhd$ of T_{BL}. We have the following reductions:

(1) *If $\phi \leq \psi$, then $\phi \wedge \psi$ reduces to ϕ and $\phi \vee \psi$ reduces to ψ.*

(2) *If $\psi < \phi$, then $\phi \wedge \psi$ reduces to ψ and $\phi \vee \psi$ reduces to ϕ.*
Hence, the partition of the unit used in the reduction of $\phi \wedge \psi$ or of $\phi \vee \psi$ is $\phi \leq \psi$; $\psi < \phi$.

(3) *If $\phi \ll_= \psi$, then the condition $\phi\&\psi \ll \gamma$ is equivalent to $\phi \ll \gamma$, and the condition $\gamma \ll \phi\&\psi$ is equivalent to $\gamma \ll \phi$. Moreover if $\psi \ll \phi$, then $\phi\&\psi \ll \gamma$ is equivalent to $\psi \ll \gamma$ and $\gamma \ll \phi\&\psi$ is equivalent to $\gamma \ll \psi$.*

(4) *If $\phi \ll \psi$, then for $\lhd \in \{\prec, \preceq\}$ the condition $\sigma \oplus (\phi\&\psi) \lhd \nu$ is equivalent to $\sigma \oplus \phi \lhd \nu$, and the condition $\sigma \lhd \nu \oplus (\phi\&\psi)$ is equivalent to $\sigma \lhd \nu \oplus \phi$.*

(5) *Likewise, if $\psi \ll \phi$, then for $\lhd \in \{\prec, \preceq\}$ the condition $\sigma \oplus (\phi\&\psi) \lhd \nu$ is equivalent to $\sigma \oplus \psi \lhd \nu$, and the condition $\sigma \lhd \nu \oplus (\phi\&\psi)$ is equivalent to $\sigma \lhd \nu \oplus \psi$.*

(6) *If $\phi \equiv \psi$, if $\phi \neq 1$ and $\phi + \psi \leq 2\lfloor\phi\rfloor + 1^+$, that is, if $\phi \equiv \psi$, $\phi \ll 1$ and $\phi \oplus \psi \preceq 1^+$, then for $\lhd \in \{\prec, \preceq\}$, both $\sigma \oplus (\phi\&\psi) \lhd \nu$ and $\sigma \lhd \nu \oplus (\phi\&\psi)$ are equivalent to $\sigma \lhd \nu$.*

(7) *If $\phi \equiv \psi$, if $\phi \neq 1$ and $\phi + \psi > 2\lfloor\phi\rfloor + 1^+$, that is, if $\phi \equiv \psi$, $\phi \ll 1$ and $1^+ \prec \phi \oplus \psi$, then for $\lhd \in \{\prec, \preceq\}$, $\sigma \oplus (\phi\&\psi) \lhd \nu$ is equivalent to $\sigma \oplus \phi \oplus \psi \lhd \nu \oplus 1^+$ and $\sigma \lhd \nu \oplus (\phi\&\psi)$ is equivalent to $\sigma \lhd 1^+ \prec \nu \oplus \phi \oplus \psi$.*

(8) *If $\phi = \psi = 1$, that is, if $1 \preceq \phi$ and $1 \preceq \psi$, then for $\lhd \in \{\prec, \preceq, \ll\}$, $\sigma \oplus (\phi\&\psi) \lhd \nu$ (where if $\lhd$ is $\ll$, then ν is the singleton of a formula and σ is empty) is equivalent to $\sigma \oplus 1 \lhd \nu$. Moreover $\sigma \lhd \nu \oplus (\phi\&\psi)$ is equivalent to $\sigma \lhd \nu \oplus 1$ (where if $\lhd$ is $\ll$, then σ is the singleton of a formula and ν is empty).*
Hence, the partition of the unit used in the reduction of $\phi\&\psi$ is: $1 \preceq \phi$ and $1 \preceq \psi$; $\phi \ll \psi$; $\psi \ll \phi$; $\phi \equiv \psi$ and $\phi \ll 1$ and $\phi \oplus \psi \preceq 1^+$; $\phi \equiv \psi$ and $\phi \ll 1$ and $1^+ \prec \phi \oplus \psi$. This is really a partition of the unit: the first three elements of the partition are clearly pairwise incompatible and cover the case where either $\phi = \psi = \infty$, or $\phi \in R^+$ and $\psi = \infty$ (because then $\phi \ll \psi$), or $\phi = \infty$ and $\psi \in R^+$ (because then $\psi \ll \phi$) or $\phi, \psi \in R^+$ and $\lfloor\phi\rfloor \neq \lfloor\psi\rfloor$ (because then either $\phi \ll \psi$ or $\psi \ll \phi$). It remains to consider the case where $\phi, \psi \in R^+$ and $\lfloor\phi\rfloor = \lfloor\psi\rfloor$, that is, the case where $\phi \equiv \psi$ and $\phi \ll 1$. This case splits into two mutually incompatible subcases, namely, $\phi - \lfloor\phi\rfloor + \psi - \lfloor\psi\rfloor \leq 1^+$ and $1^+ < \phi - \lfloor\phi\rfloor + \psi - \lfloor\psi\rfloor$, that is, $\phi \oplus \psi \preceq 1^+$ and $1^+ \prec \phi \oplus \psi$. Hence, the cases in the reduction of $\phi\&\psi$ form a partition of the unit.

(9) *If $\phi \leq \psi$, then for $\lhd \in \{\prec, \preceq, \ll\}$, $\sigma \oplus (\phi \to \psi) \lhd \nu$ is equivalent to $\sigma \oplus 1 \lhd \nu$*

(again, if $\lhd$ is $\ll$, then σ is empty and ν is the singleton of a formula). Moreover $\sigma \lhd \nu \oplus (\phi \to \psi)$ is equivalent to $\sigma \lhd \nu \oplus 1$ (if $\lhd$ is $\ll$, then σ is a singleton and ν is empty).

(10) If $\psi \ll \phi$, then $\sigma \oplus (\phi \to \psi) \lhd \nu$ is equivalent to $\sigma \oplus \psi \lhd \nu$ and $\sigma \lhd \nu \oplus (\phi \to \psi)$ is equivalent to $\sigma \lhd \nu \oplus \psi$ (usual restrictions when $\lhd$ is $\ll$).

(11) If $\psi \prec \phi$, then $\sigma \lhd \nu \oplus (\phi \to \psi)$ reduces to $\sigma \oplus \phi \lhd \nu \oplus 1^+ \oplus \psi$ and $\sigma \oplus (\phi \to \psi) \lhd \nu$ reduces to $\sigma \oplus \psi \oplus 1^+ \lhd \nu \oplus \phi$.

Hence, the partition of the unit used in the reduction of $\phi \to \psi$ is $\phi \leq \psi$; $\psi \ll \phi$; $\psi \prec \phi$.

Each reduction has the form $(C_1 \cap C_2 \cap \cdots \cap C_n) \Rightarrow C$, and we represent it as a disjunction $C_1^* \cup C_2^* \cup \cdots \cup C_n^* \cup C$, where C_i^* is the negation of C_i. If we write $\lhd^*$ for the negation of the relation symbol $\lhd$, we see that for $\lhd \in \{\ll, \ll_=, \equiv, \preceq, \prec\}$ we can rewrite $\lhd^*$ in the following way:

$$
\begin{aligned}
\phi \ll^* \psi &\Leftrightarrow (\psi \ll \phi) \cup (\psi \preceq \psi) \cup (\psi \prec \phi).\\
\phi \ll_=^* \psi &\Leftrightarrow \psi \ll \phi.\\
\phi \equiv^* \psi &\Leftrightarrow (\phi \ll \psi) \cup (\psi \ll \phi).\\
\phi \preceq^* \psi &\Leftrightarrow (\phi \ll \psi) \cup (\psi \ll \phi) \cup (\psi \prec \phi).\\
\phi \oplus \psi \preceq^* 1^+ &\Leftrightarrow (\phi \ll \psi) \cup (\psi \ll \phi) \cup (1^+ \preceq \phi \oplus \psi).\\
1^+ \prec^* \phi \oplus \psi &\Leftrightarrow (\phi \ll \psi) \cup (\psi \ll \phi) \cup (\phi \oplus \psi \preceq 1^+).
\end{aligned}
$$

We thus immediately see that T_{BL} satisfies condition (TL0), and we put emphasis on the fact that with the exception of 1^+ and of the multiset union $\oplus$, all function and constant symbols of T_{BL} are already in the language of BL. Regarding condition (TL1), we may take $Des(x) := 1 \preceq x$. As regards to condition (TL2), the axioms of the theory are the formulas Φ such that:

(1) Φ is a disjunction of formulas of the form $\phi \ll \psi$ or $\sigma \preceq \nu$ or $\sigma \prec \nu$, where ϕ and ψ are atomic formulas, σ and ν are multisets consisting of atomic formulas or 1^+;

(2) Φ is true in $\mathbf{A}^*$.

In [16, Lemma 4.5], it is proved that the set of axioms is in the complexity class P.

Condition (TL3) is empty since we do not add any extra unary connective to T_{BL}.

Notice that the above reductions (1),…,(11) are neither projective nor semi-projective, since there are terms appearing in the decomposition which involve $\oplus$. As an example we show how to obtain the conditions in (TL4) in the case *of the binary predicate $\prec$, the position 1 (thus at the left hand side of $\prec$) and the connective &.*

Consider a substitution of variables with formulas which associates the (generic) formulas ϕ and ψ in L' (whose language coincides with that of BL enriched with the constant 1^+) to x and y, respectively. Taking into account all we have said, for every pair of multisets of formulas Γ and Δ we have the following conditions for (TL4/1) (below we write a comma "," instead of $\oplus$):

- $(\phi \ll \psi) \Rightarrow ((\Gamma, \phi \& \psi \prec \Delta) \Leftrightarrow (\Gamma, \phi \prec \Delta))$
- $(\psi \ll \phi) \Rightarrow ((\Gamma, \phi \& \psi \prec \Delta) \Leftrightarrow (\Gamma, \psi \prec \Delta))$
- $((1 \preceq \phi) \cap (1 \preceq \psi)) \Rightarrow ((\Gamma, \phi \& \psi \prec \Delta) \Leftrightarrow (\Gamma, 1 \prec \Delta))$
- $((\phi \equiv \psi) \cap (\phi \ll 1) \cap (\psi, \phi \preceq 1^+)) \Rightarrow ((\Gamma, \phi \& \psi \prec \Delta) \Leftrightarrow (\Gamma \prec \Delta))$
- $((\phi \equiv \psi) \cap (\phi \ll 1) \cap (1^+ \prec \phi, \psi)) \Rightarrow ((\Gamma, \phi \& \psi \prec \Delta) \Leftrightarrow (\Gamma, \phi, \psi \prec \Delta, 1^+))$

The remaining reductions are obtained in a similar way and can be extracted from the r-hypersequent calculus for BL displayed in Section 4.3.1.

2.3 How to build the semantic theory for a hyperprojective logic

Proving that a logic is hyperprojective is, in general, a non trivial task. We show below how to automate this process for a large class of many-valued logics, which includes Łukasiewic and Product Logic.

To show that a logic L is hyperprojective we need indeed to provide a semantic theory T_L satisfying conditions (TL0), ..., (TL4) (cf. Definition 5). Though this process could be tricky in general, it can be automated for every logic that is algebraizable and its equivalent algebraic semantics is a variety $\mathcal{V}$ generated by a single algebra $\mathbf{A}$, and whose first-order theory satisfies suitable conditions, which will be described below.

If L is algebraizable, then the connectives of L may be regarded as operation symbols and formulas of L as terms in the language of $\mathcal{V}$. Let $\mathbf{A}$ be an algebra which generates $\mathcal{V}$ as a variety, with universe A. Let A^m be the family of all multisets with support included in A (without loss of generality, we may assume that $A \cap A^m = \emptyset$). We equip $A \cup A^m$ with the operations of $\mathbf{A}$, extended by the clause $f(a_1, \ldots, a_n) = \varepsilon$ if some a_i is in A^m, with the operation $\oplus$ for the union of multisets, extended by the condition $a \oplus b = \varepsilon$ if either $a \notin A^m$ or $b \notin A^m$, and, possibly, with finitely many additional unary operations and constants. Moreover we equip $A \cup A^m$ with some relations, thus getting a structure $\mathbf{A}^*$. We assume that:

($\mathcal{A}$1) There is a weakly simple formula $Des(x)$ such that for each formula ϕ of L we have: $L \models \phi$ iff $\mathbf{A}^* \models Des(\phi)$.

($\mathcal{A}2$) The set of weakly simple formulas valid in $\mathbf{A}^*$ is decidable.

($\mathcal{A}3$) If f, g are unary operations not in the language of L, there is a unary operation h not in the language of L such that $\mathbf{A}^* \models \forall x(f(g(x)) = h(x))$.

($\mathcal{A}4$) Let $\overline{t} = (t_1, \ldots, t_n)$ be a sequence of terms. Given a valuation v into $\mathbf{A}$ and a multiset $\mu = y_1 \oplus \ldots \oplus y_k$ with $y_i \in \{t_1, \ldots, t_n\}$ we set $v(\mu)$ to be the multiset $v(y_1) \oplus \ldots \oplus v(y_k)$. Moreover, let $\overline{x} = (x_1, \ldots, x_n)$ be a sequence of variables and let $U_{\overline{x}}$ be the set of all terms which are either of the form x_i or of the form $u(x_i)$, u a unary operation symbol in the language of T_L but not in the language of L. Then, with the notation used in the definition of hyperprojective logic, for each k-ary predicate P, for each position i with $1 \leq i \leq k$, for each connective $\Box$, and for each unary function symbol u in T_L and not in L, there are two partitions of the unit $Q_1(\overline{x}), \ldots, Q_\ell(\overline{x})$ and $Q_1^u(\overline{x}), \ldots, Q_{\ell'}^u(\overline{x})$, consisting of weakly simple formulas, and sequences of multisets $\overline{\nu}_a : a = 1, \ldots, \ell$, $\overline{\nu}_b^u : b = 1, \ldots, \ell'$, whose support is included in $U_{\overline{x}}$, such that, for $a = 1, \ldots, \ell$, for $b = 1, \ldots, \ell'$, for every sequence $\overline{\mu}$ of multisets of elements of A and for every valuation v into $\mathbf{A}$, the following conditions hold:
($\mathcal{A}4/1$) $\mathbf{A}^* \models Q_a(v(\overline{x})) \Rightarrow (P(\overline{\mu} \oplus \Theta_i^k(\Box(v(\overline{x})))) \Leftrightarrow P(\overline{\mu} \oplus v(\overline{\nu}_a)))$, and
($\mathcal{A}4/2$) $\mathbf{A}^* \models Q_b^u(v(\overline{x})) \Rightarrow (P(\mu \oplus \Theta_i^k(u(\Box(v(\overline{x}))))) \Leftrightarrow P(\overline{\mu} \oplus v(\overline{\nu}_b^u)))$.

When conditions ($\mathcal{A}1$), ..., ($\mathcal{A}4$) are satisfied, we take T_L to be the first-order theory of the model $\mathbf{A}^*$.
Example 7.

(1) As seen in Remark 2.1, every projective [8] or semi-projective logic [19] is hyperprojective, provided that we identify any formula ϕ with the multiset with cardinality 1 whose support is $\{\phi\}$.

(2) Though Product Logic is neither projective nor semi-projective, it is hyperprojective. To see this, let $\mathbf{A}$ be the product algebra on $[0, 1]$, (see [23]), and let us extend $\mathbf{A}$ to a structure $\mathbf{A}^*$ on $A \cup A^m$ along the lines indicated at the beginning of this section. We introduce the relations $\preceq$ and $\prec$ on multisets, defined as follows:

$$\sigma \preceq \nu \quad \textit{iff} \quad \prod_{x \in S(\sigma)} x^{\mu(x)} \leq \prod_{y \in S(\nu)} y^{\mu(y)}$$

$$\sigma \prec \nu \quad \textit{iff} \quad \prod_{x \in S(\sigma)} x^{\mu(x)} < \prod_{y \in S(\nu)} y^{\mu(y)}$$

(an empty product is 1 by definition).

Let T_L be the first-order theory of $\mathbf{A}^*$, where we write $<$ instead of $\prec$ and $\leq$ instead of $\preceq$. Then T_L, along with the decomposition rules for the connectives of Product Logic presented in the introduction, witnesses the fact that product logic is hyperprojective. The predicate Des can be defined by $Des(x) := 1 \preceq x$.

(3) Łukasiewicz Logic [20] is hyperprojective. To build its semantic theory, let $\mathbf{A}$ be the standard MV-algebra on $[0, 1]$, and construct $\mathbf{A}^*$ following our

strategy, with the extra binary predicates $\prec$ and $\preceq$ on multisets, defined as follows:

$$\sigma \prec \nu \quad \textit{iff} \quad \sum_{x\in S(\sigma)} \mu(x)\cdot x < \sum_{y\in S(\nu)} \mu(y)\cdot y$$
$$\sigma \preceq \nu \quad \textit{iff} \quad \sum_{x\in S(\sigma)} \mu(x)\cdot x \leq \sum_{y\in S(\nu)} \mu(y)\cdot y$$

(where an empty sum is 0 *by definition).*

The predicate Des can be defined by $Des(x) := 1 \preceq x$. Moreover, recalling that $x * y = \max\{x + y - 1, 0\}$ and $x \to y = \min\{1 - x + y, 1\}$, and writing "," instead of $\oplus$, $<$ instead of $\prec$ and $\leq$ instead of $\preceq$, we can easily build the case reductions for its connectives.

3 Proof search oriented calculi for hyperprojective logics

Proof systems for various many-valued logics have been defined using hypersequents [3], that are finite "disjunctions" of standard sequents; these logics include Gödel, Łukasiewicz and Product logic, as well as Monoidal T-norm based logic MTL [21]; see [27] for an overview. A hypersequent is a multiset of the form

$$\Gamma_1 \Rightarrow \Delta_1 \mid \ \ldots \ \mid \Gamma_n \Rightarrow \Delta_n,$$

where each component $\Gamma_i \Rightarrow \Delta_i$ is an ordinary sequent, i.e. of the form $\phi^i_1, \ldots \phi^i_n \Rightarrow \psi^i_1, \ldots, \psi^i_m$. In contrast with the above mentioned logics, Hájek's basic fuzzy logic BL (cf. Example 6) seems to escape an analytic formalization using hypersequents, that is a hypersequent calculus whose proofs proceed by stepwise decomposition of the formulas to be proved. Moreover, even when analytic, hypersequent calculi are in general not suitable for proof search. The main reason being that their rules are usually not invertible (exceptions are the calculi in [29,28]). For instance, termination is still an open problem for the hypersequent calculus of MTL, that also does not help characterizing the computational complexity of the logic.

Relational hypersequents are a generalization of hypersequents introduced in [18] to define proof search oriented calculi for Gödel, Łukasiewicz and Product logic. A relational hypersequent (r-hypersequent for short) is defined in [18] as a multiset of two different types of sequents, where Gentzen's sequent arrow $\Rightarrow$ is replaced in one by $<$ and in the other by $\leq$. Relational hypersequents were also used in [30] to define analytic calculi for (a conservative extension of) BL. All these calculi are however defined on a logic by logic basis and their discovery has required some ingenuity.

In this section we introduce a *methodology* to define relational hypersequent calculi for *all* hyperprojective logics. To this purpose along the line of sequents of relations [8,19] we generalize r-hypersequents to objects understood as a disjunction of *arbitrary* predicates (not only the binary ones $<$ and $\leq$) belonging to a chosen semantic theory.

3.1 From hyperprojective logics to r-hypersequent calculi

Let L be a hyperprojective logic with semantic theory T_L. A *relational hypersequent* for T_L, has the form

$$P_1(\overline{\mu}_1) \mid \ldots \mid P_\ell(\overline{\mu}_\ell)$$

(each $P_i(\overline{\mu}_i)$ is called a *component* of the r-hypersequent) where $P_1, \ldots P_\ell$ are predicate symbols of T_L and $\overline{\mu}_1, \ldots, \overline{\mu}_\ell$ are sequences of multisets of formulas of the conservative extension L' of L. In what follow, according to a long standing tradition in proof theory we will often write "," for $\oplus$ and multisets of formulas $\phi_1 \oplus \cdots \oplus \phi_n$ will be represented as $\phi_1, \ldots, \phi_n$. In a relational hypersequent, $\mid$ is interpreted as a disjunction, that is, for any sequence of multisets $\overline{\mu} = \mu_1, \ldots, \mu_r$,

$$\mathcal{M}, v \models P_1(\overline{\mu}_1) \mid \ldots \mid P_\ell(\overline{\mu}_\ell) \qquad \text{iff} \qquad \mathcal{M}, v \models P_1(\overline{\mu}_1) \cup \ldots \cup P_\ell(\overline{\mu}_\ell).$$

The T_L formula $P_1(\overline{\mu}_1) \cup \ldots \cup P_\ell(\overline{\mu}_\ell)$ will be called the *formula associated* to the r-hypersequent $P_1(\overline{\mu}_1) \mid \ldots \mid P_\ell(\overline{\mu}_\ell)$.

Conditions (TL4/1) and (TL4/2) can be translated into logical rules as follows: first of all, we can assume that $Q_a(\overline{x})$ and $Q_b^u(\overline{x})$ are conjunctions of multisets of atomic formulas. Since the negation of a conjunction of atomic formulas is equivalent to a disjunction of negations of atomic formulas (and since our semantic theory satisfies condition (TL0), each negation of an atomic formula is equivalent to an atomic formula), the negations of $Q_a(\overline{x})$ and $Q_b^u(\overline{x})$ can be written as r-hypersequents. We denote such r-hypersequents by $(Q_a)^*(\overline{x})$ and $(Q_b^u)^*(\overline{x})$. Hence we have

Definition 8 (Logical rules)**.** *For any n-ary connective $\Box$ of L (and for any unary function symbol u in the language of T_L and not in that of L), any predicate symbol P of T_L with arity r, and for any position i with $1 \leq i \leq r$ we have the rule $(P, \Box, i)$ (resp. $(P, u(\Box), i)$) for introducing $\Box(\overline{x})$ (resp. $u(\Box(\overline{x}))$) at position i into a component of an r-hypersequent containing the symbol P.*

Let $Q_1(\overline{x}), \ldots, Q_\ell(\overline{x})$ and $Q_1^u(\overline{x}), \ldots, Q_{\ell'}^u(\overline{x})$ be two partitions of the unit, consisting of weakly simple formulas, and $\overline{\nu}_a, \overline{\nu}_b^u$ be sequences of multisets with support in $Z(\overline{x})$, such that, for $a = 1, \ldots, \ell$, for $b = 1, \ldots, \ell'$, for any sequence $\overline{\mu}$ of multisets of formulas of L' and for every substitution σ of variables with *formulas of L', conditions (TL4/1) and (TL4/2) hold. Then we have the rules*

$$\frac{H|(Q_1)^*(\sigma(\overline{x}))|P(\overline{\mu}\oplus\sigma(\overline{\nu}_1)) \quad \dots \quad H|(Q_\ell)^*(\sigma(\overline{x}))|P(\overline{\mu}\oplus\sigma(\overline{\nu}_\ell))}{H|P(\overline{\mu}\oplus\Theta_i^r(\Box(\sigma(\overline{x}))))}\ (P,\Box,i)$$

$$\frac{H|(Q_1^u)^*(\sigma(\overline{x}))|P(\overline{\mu}\oplus\sigma(\overline{\nu}_1^u)) \quad \dots \quad H|(Q_{\ell'}^u)^*(\sigma(\overline{x}))|P(\overline{\mu}\oplus\sigma(\overline{\nu}_{\ell'}^u))}{H|P(\overline{\mu}\oplus\Theta_i^r(u(\Box(\sigma(\overline{x})))))}\ (P,u(\Box),i)$$

where H is any side r-hypersequent.

In the above rules, the formula $P(\overline{\mu}\oplus\Theta_i^r(\Box(\sigma(\overline{x}))))$ (resp. $P(\overline{\mu}\oplus\Theta_i^r(u(\Box(\sigma(\overline{x}))))))$ is called *main formula of the rule*, while $(Q_a)^*(\sigma(\overline{x}))$ (resp., $(Q_b^u)^*(\sigma(\overline{x})))$) are called *contexts of the rule*. For each $a = 1,\dots,\ell$ (resp., for $b = 1,\dots,\ell'$), $P(\overline{\mu}\oplus\sigma(\overline{\nu}_a))$ (resp. $P(\overline{\mu}\oplus\sigma(\overline{\nu}_b^u)))$ is called *reduced formula* of the main formula.

Example 9. *The calculus rules for Łukasiewicz logic (see Example 7) are given below (we omit the side hypersequent H for space reasons):*

$$\frac{1\le\phi,\psi|\Gamma\lhd\Delta \quad \phi,\psi<1|\Gamma,\phi,\psi\lhd\Delta,1}{\Gamma,\phi\&\psi\lhd\Delta} \qquad \frac{1\le\phi,\psi|\Gamma\lhd\Delta \qquad \phi,\psi<1|\Gamma,1\lhd\Delta,\phi,\psi}{\Gamma\lhd\Delta,\phi\&\psi}$$

$$\frac{\phi<\psi|\Gamma,\psi\lhd\Delta \qquad \psi\le\phi|\Gamma,\phi\lhd\Delta}{\Gamma,\phi\wedge\psi\lhd\Delta} \qquad \frac{\phi<\psi|\Gamma\lhd\Delta,\psi \qquad \psi\le\phi|\Gamma\lhd\Delta,\phi}{\Gamma\lhd\Delta,\phi\wedge\psi}$$

$$\frac{\phi<\psi|\Gamma,\phi\lhd\Delta \qquad \psi\le\phi|\Gamma,\psi\lhd\Delta}{\Gamma,\phi\vee\psi\lhd\Delta} \qquad \frac{\phi<\psi|\Gamma\lhd\Delta,\phi \qquad \psi\le\phi|\Gamma\lhd\Delta,\psi}{\Gamma\lhd\Delta,\phi\vee\psi}$$

$$\frac{\psi<\phi|\Gamma,1\lhd\Delta \qquad \phi\le\psi|\Gamma,\psi,1\lhd\Delta,\phi}{\Gamma,\phi\to\psi\lhd\Delta} \qquad \frac{\psi<\phi|\Gamma\lhd\Delta,1 \qquad \psi\le\phi|\Gamma,\phi\lhd\Delta,\psi,1}{\Gamma\lhd\Delta,\phi\to\psi}$$

where $\lhd$ stands for $<$ or $\le$ uniformly in each rule.

Given a hyperprojective logic L, we denote by $\mathbb{H}L$ the r-hypersequent calculus whose rules are defined as indicated above and whose axioms are

Definition 10 (Axioms)**.** *Suppose that $P_1(\overline{\mu}_1),\dots,P_\ell(\overline{\mu}_\ell)$ are weakly simple atomic formulas. Then the r-hypersequent $P_1(\overline{\mu}_1)\mid\dots\mid P_\ell(\overline{\mu}_\ell)$ is an axiom of $\mathbb{H}L$ iff the universal closure of the formula associated to it is valid in T_L.*

Remark 3.1. *As it often happens for semantic-based calculi (e.g. in the case of display logic [13]) the calculus $\mathbb{H}L$ is formulated in the language of the conservative extension L' of L.*

From condition (TL4) of T_L immediately follows that each rule constructed as in Definition 8 is *sound and invertible* for L, i.e. for any rule

$$\frac{H_1 \quad \dots \quad H_n}{H}$$

in $\mathbb{H}L$ and for any model $\mathcal{M}$ of T_L, $\mathcal{M}\models H_i$ for any $i=1,\dots,n$ iff $\mathcal{M}\models H$.

Let L be a hyperprojective logic. We show that the calculus $\mathbb{H}L$ is sound and complete for L and use it to show that L is decidable. Towards this section we fix a hyperprojective logic L with semantic theory T_L and r-hypersequent calculus $\mathbb{H}L$. An r-hypersequent H of T_L is said to be *provable in* $\mathbb{H}L$ if there is an upward tree of r-hypersequents rooted in H, such that every leaf is an axiom of $\mathbb{H}L$ and every other r-hypersequent is obtained from the ones standing immediately above it by application of one of the rules of $\mathbb{H}L$. Such a tree is called a *derivation of* H; we define the *length* of a derivation as the number of inferences in a maximal branch of that derivation.

Being Des a simple formula (cf. condition (TL1)), it can be written as a conjunctions of disjunctions of atomic simple formulas, indicated in what follows as $Des_1, \ldots, Des_k$. The two results below establish the soundness and completeness of $\mathbb{H}L$ with respect to L.

Theorem 11 (Soundness)**.** *Let ϕ be any formula in the language of L. If $Des_1(\phi), \ldots, Des_k(\phi)$ are provable in $\mathbb{H}L$ then ϕ is valid in L.*

Proof. Arguing by induction on the length l of the derivation of H, we prove the more general statement: if H is provable in $\mathbb{H}L$ then for every model $\mathcal{M}$ of T_L and for every valuation v of $\mathcal{M}$, $\mathcal{M}, v \models H$.

Base step: if $l = 0$ then H is an axiom of $\mathbb{H}L$, and the associated formula is valid in T_L.

Inductive step: assume that the claim holds for derivations with length n and let $l = n + 1$. If the r-hypersequents above H are $H_1, \ldots, H_n$, then by the inductive hypothesis, $\mathcal{M}, v \models H_i$ for $i = 1, \ldots, n$. The claim follows by condition (TL4).

Thus, if $Des_1(\phi), \ldots, Des_k(\phi)$ are provable in $\mathbb{H}L$ then $T_L \models Des(\phi)$, and (TL1) proves the claim. □

Theorem 12 (Completeness)**.** *Let ϕ be any formula in the language of L. If ϕ is valid in L then $Des_1(\phi), \ldots, Des_k(\phi)$ are provable in $\mathbb{H}L$.*

Proof. If ϕ is valid in L then, by (TL1), $Des_1(\phi), \ldots, Des_k(\phi)$ are provable in T_L, and hence for every model $\mathcal{M}$ of T_L and for every valuation v of $\mathcal{M}$, $\mathcal{M}, v \models Des_i(\phi)$ for $i = 1, \ldots k$. Applying the rules of $\mathbb{H}L$ backwards to every $Des_i(\phi)$ we can build a tree, called *the reduction tree of $Des_i(\phi)$*. The leaves of the reduction tree are r-hypersequents $P_1(\overline{\mu}_1) \mid \ldots \mid P_\ell(\overline{\mu}_\ell)$ such that $P_1(\overline{\mu}_1) \cup \ldots \cup P_\ell(\overline{\mu}_\ell)$ is a weakly simple formula of T_L, and by the invertibility of the rules of $\mathbb{H}L$, their universal closure is valid in T_L. Hence the reduction tree of $Des_i(\phi)$ is actually a derivation of $Des_i(\phi)$, and thus $Des_1(\phi), \ldots, Des_k(\phi)$ are provable in $\mathbb{H}L$. $\square$

A first, easy but important property of hyperprojective logics is given by the following result.

Theorem 13. *Any hyperprojective logic L is decidable.*

Proof. Given a formula ϕ of L, we apply the rules of $\mathbb{H}L$ backwards, starting from $Des(\phi)$. We can assume that no consecutive occurrences of unary functions not in L occur in ϕ (otherwise, we eliminate them using (TL3)). Let $a(\phi)$ denote the number of occurrences of function symbols in ϕ in the language of L, and $b(\phi)$ be the number of occurrences of unary function symbols in ϕ and not in L, and set $c(\phi) = 2a(\phi) + b(\phi)$ (so, function symbols in L count more than unary function symbols not in L in the computation of $c(\phi)$). We call $c(\phi)$ the *complexity* of ϕ. Now let for each r-hypersequent H, $c(H)$ denote the maximum complexity of the formulas in H and $k(H)$ denote the number of occurrences of formulas of maximum complexity. We reduce first the formulas with maximal complexity. It is easily seen that if H' is any premise of a rule acting on a formula of maximal complexity and H is its conclusion, then $(c(H'), k(H')) < (c(H), k(H))$. It follows that every path of the reduction tree terminates with a r-hypersequent containing formulas which are either atomic or of the form $f_j(p)$ with p a propositional variable or a constant and f_j a unary operation not in L. These r-hypersequents are weakly simple formulas and hence they are decidable by our assumptions on T_L. This shows that L is decidable. $\square$

Remark 3.2. *Being invertible, the rules of our calculi decompose the main formula in a set of reduced formulas which have strictly lower complexity (in the sense of the above proof) than the main formula. This justifies our use of the word* reduction *in place of* rule*, to underline the fact that whenever we read a reduction tree starting from the root, each rule is actually* reducing *the complexity of the starting r-hypersequent.*

4 Co-NP completeness

We identify sufficient conditions for a hyperprojective logic L to be in Co-NP. The conditions are on T_L and on the r-hypersequent calculus $\mathbb{H}L$. Since we manly deal with substructural logics, i.e. axiomatic extensions of Full Lambek calculus, the logics treated in this paper are also Co-NP hard by [25], and hence, when they are in Co-NP they are also Co-NP complete.

We start discussing *uniformity of contexts*, a useful property of r-hypersequent rules, and show that all calculi for hyperprojective logics can be modified in order to fulfill it. Intuitively, having a uniform set of rules means that the rules for introducing the same connective have all the same form, i.e. they only depend on the formula and not on the particular predicate symbol or the position inside it.

Definition 14. *Let L be an hyperprojective logic with semantic theory T_L and r-hypersequent calculus $\mathbb{H}L$. We say that $\mathbb{H}L$ has a* uniform set of rules *if for each rule $(P, \Box, i)$ (resp., $(P, u(\Box), i)$), the contexts $(Q_a)^*(\sigma(\overline{x}))$ (resp., $(Q^u_b)^*(\sigma(\overline{x}))$) and the number of premises in the rule $(P, \Box, i)$ (resp., $(P, u(\Box), i)$) only depend on $\Box$ (resp., on u and on $\Box$) but not on P or on i.*

Uniform rules allow us to reduce several occurrences of the same formula simultaneously and make the proof search algorithm more efficient, as the following example shows.

Example 15. *In Product Logic we can simultaneously reduce all occurrences of $A \to B$ in the sequent $H := A \to B \leq C|D < A \to B$, where A, B, C and D are propositional variables, taking advantage of the fact that the contexts are the same in each rule. As already outlined in Section 1, the rules for $\to$ are (where $\lhd$ is either $\leq$ or $<$ and H' is any side-sequent)*

$$\frac{H'|\psi < \phi|\Gamma, 1 \lhd \Delta \qquad H'|\phi \leq \psi|\Gamma, \psi \lhd \Delta, \phi}{H'|\Gamma, \phi \to \psi \lhd \Delta}\ (\lhd, \to, left)$$

$$\frac{H'|\psi < \phi|\Gamma, 1 \lhd \Delta \qquad H'|\phi \leq \psi|\Gamma, \phi \lhd \Delta, \psi}{H'|\Gamma \lhd \Delta, \phi \to \psi}\ (\lhd, \to, right)$$

Thus, by repeatedly applying them to the sequent H we have the following (compact) reduction:

$$\frac{B < A|1 \leq C|D < 1 \qquad A \leq B|B \leq A, C|D, A < B}{A \to B \leq C|D < A \to B}.$$

Our first result is that in *any* hyperprojective logic we can always get uniformity for free. This is based on the following facts:

(1) Given two partitions of the unit $Q_1, \ldots, Q_n$ and $R_1, \ldots, R_k$, the partition consisting of all formulas $W_{i,j} = Q_i \cap R_j$, $i = 1, \ldots, n$ $j = 1, \ldots, k$ is a common refinement of the original partitions.
(2) Hence, for every connective $\Box$ of L and every unary function symbol u in T_L and not in L, we can find a common refinement $W(\overline{x})$ of *all* partitions used for the rules of all sequents of the form $P(\overline{\mu} \oplus \Theta^n_i(\Box(\overline{x})))$ for any $i = 1, \ldots, n$, and, respectively, a common refinement $W^u(\overline{x})$ of *all* partitions of unit used in the rules of all sequents of the form $P(\overline{\mu} \oplus \Theta^n_i(u(\Box(\overline{x}))))$ for

any $i = 1, \ldots, n$. Now we may suppose that each element $W_a(\overline{x})$ of W ($a = 1, \ldots, \ell$) and each element $W_b^u(\overline{x})$ ($b = 1, \ldots, \ell'$) of W^u is a conjunction of atomic formulas, and hence using our assumptions on negations we may write their negations as r-hypersequents, which will be denoted by $(W_a)^*(\overline{x})$ and $(W_b^u)^*(\overline{x})$, respectively. It follows that the original rules can be replaced by the uniform rules

$$\frac{H|(W_1(\sigma(\overline{x})))^*|P(\overline{\mu} \oplus \sigma(\overline{\nu}_1)) \quad \ldots \quad H|(W_\ell(\sigma(\overline{x})))^*|P(\overline{\mu} \oplus \sigma(\overline{\nu}_\ell))}{H|P(\overline{\mu} \oplus \Theta_i^n(\Box(\sigma(\overline{x}))))} \ (P, \Box, i)$$

$$\frac{H|(W_1^u(\sigma(\overline{x})))^*|P(\overline{\mu} \oplus \sigma(\overline{\nu}_1^u)) \quad \ldots \quad H|(W_{\ell'}^u(\sigma(\overline{x})))^*|P(\overline{\mu} \oplus \sigma(\overline{\nu}_{\ell'}^u))}{H|P(\overline{\mu} \oplus \Theta_i^n(u(\Box(\sigma(\overline{x})))))} \ (P, u(\Box), i)$$

Hence it follows
Proposition 16. *Every hyperprojective logic L has a uniform set of rules.*

In the sequel, we tacitly assume to apply rules for (different occurrences of) the same formula simultaneously, whenever possible.

4.2 Resource-boundedness

We define the size of an r-hypersequent as the number of symbols occurring in each formula contained in it. To guarantee that the size of each leaf of the reduction tree is polynomial in the size of the end r-hypersequent we need a further assumption on the calculus' rules.
Example 17. *Assume to have a hyperprojective logic L with a binary connective $\Box$ such that $P(x)$ is unary predicate symbol of T_L and that its r-hypersequent calculus $\mathbb{H}L$ has the rule*

$$\frac{P(\Gamma, \phi, \psi, \psi)}{P(\Gamma, \Box(\phi, \psi))} \ (P, \Box, 1)$$

where Γ is an arbitrary multiset of formulas and ϕ and ψ are metavariables for formulas in L'. Clearly the rule is uniform (it has no context). Now let $\Phi_0 = \phi$ and $\Phi_{n+1} = \Box(\phi, \Phi_n)$. Then the size of Φ_n is linear in n, but, writing $h\phi$ for $\phi, \ldots, \phi$ h times, the nodes in the unique branch of the reduction tree with root $P(\Phi_n)$ are

$$P(\phi, 2\Phi_{n-1}), \ P(3\phi, 4\Phi_{n-2}), \ P(7\phi, 8\Phi_{n-3}), \ldots, P((2^{n+1} - 1)\phi).$$

Hence, the size of the leaf of the tree is exponential in the size of the root.

To avoid such situations, we introduce the following constraint.
Definition 18. *A rule of the form $(P, \Box, i)$ (resp., $(P, u(\Box), i)$), cf. Definition 8, is said to be* resource-bounded *if for all P, i (resp., P, i, u, where u is a unary*

function in the language of T_L but not in L), every element of the union of the multisets in $\overline{\nu}_a$, $a = 1, \dots, \ell$ (resp. $\overline{\nu}^u_b$, $b = 1, \dots, \ell'$) in condition (TL4) has multiplicity at most 1.

A steady example of a family of logics with resource-bounded r-hypersequent calculi is given by semi-projective logics (and thus also projective logics). We have seen that, for the calculi of these logics, multisets of formulas can be replaced simply by formulas, and hence Definition 18 is trivially true for their rules.

An easy corollary of Proposition 16 states that resource-boundedness is stable when replacing a set of rules with a uniform set of rules.
Corollary 19. *If L has a resource bounded proof system, then it has a proof system which is both uniform and resource bounded.*

4.3 Main Theorem

A final condition for the Co-NP containment of a hyperprojective logic is that the set of valid weakly simple formulas of its semantic theory is in Co-NP. The main result of this section reads:
Theorem 20. *Let L be a hyperprojective logic with semantic theory T_L and r-hypersequent calculus $\mathbb{H}L$ with uniform and resource bounded rules. Suppose further that the set of weakly simple formulas which are valid in T_L is in P. Then the set of theorems of L is in Co-NP.*

Proof. We show that the set of formulas of L that are not valid is in NP. Let ϕ be any such formula. Again, write $Des(\phi)$ as a conjunction of disjunctions of atomic formulas $Des_1(\phi), \dots, Des_k(\phi)$; then, for some $1 \leq i \leq k$, the reduction tree starting from $Des_i(\phi)$ has a leaf which is not an axiom. Let J be the maximum cardinality of all multisets occurring in some context of a rule of L (remember that a context is a disjunction of atomic formulas of the form $Q(\mu_1, \dots, \mu_n)$, where each μ_i is a multiset of the form $\phi_1 \oplus \cdots \oplus \phi_h$, with $\phi_1, \dots, \phi_h$ formulas of L'). Let K be the sum of the arities of all predicates occurring in a context of some rule (if a predicate occurs n times, its arity is multiplied by n). Note that, due to the fact that each rule is resource-bounded, the total size of a reduced formula does not exceed the size of the main formula. Hence, denoting, for each expression E, the size of E by $s(E)$, if H is the conclusion of a rule and H' is one of its premises, we have $s(H') \leq J{\cdot}K{\cdot}s(H)$. If we reduce all occurrences of the main formula simultaneously, the situation does not change, because the contexts are the same, and the size of reduced formulas does not exceed the size of the main formula after performing all possible reductions. Hence, if the length of a branch starting from $Des_i(\phi)$ is I, the maximal size of a node in the branch is bounded by $s(Des_i(\phi)) + I{\cdot}J{\cdot}K{\cdot}s(\phi)$.

Proving the following lemmas will yield the desired result.

Lemma 21. *Each branch of the reduction tree starting from $Des_i(\phi)$ has length linear in the size of ϕ.*

Proof. Since we have uniform rules, we reduce all occurrence of a formula together, starting from the formula of highest complexity. So, if a subformula of ϕ is reduced in one node, it does not appear in the nodes above it. Hence, the length of the branch does not exceed the number of subformulas of ϕ, and thus it is linear in the size of ϕ. □

Continuing with the proof of Theorem 20, the total size of a branch starting from $Des_i(\phi)$ is bounded by $s(Des_i(\phi)) + J \cdot K \cdot M \cdot (s(\phi))^2$, where $M \cdot s(\phi)$ is a bound for the length of the branch I, by the claim above. Hence, it is possible to guess a branch of the reduction tree and to reach its leaf in polynomial time (since the length of the branch is linear in there size of ϕ). It follows that a non-deterministic polynomial-time algorithm for non-provability in L is the following:

(1) Guess an i with $1 \leq i \leq k$.

(2) Guess a maximal branch in the reduction tree of $Des_i(\phi)$ and reach its leaf (in polynomial time). Note that this leaf is a weakly simple formula.

(3) Apply a deterministic polynomial algorithm to check if the leaf is not valid in T_L (by assumption, this algorithm exists). □

Lemma 22. *If an r-hypersequent calculus is resource bounded and uniform and the set of its axioms is in Co-NP (and possibly not in P), then the set of its theorems is in Co-NP.*

Proof. Let us encode formulas, r-hypersequents, reduction trees, and in general any procedure involving them, by binary strings via an encoder $\langle \cdot \rangle$, and let $|w|$ denote the length of the binary string w. If the set of axioms is in Co-NP, then there are a P-time relation $R(H, w)$ and a polynomial $Q(x)$ such that H is *not* an axiom iff there is a binary string w with $|w| \leq Q(|\langle H \rangle|)$ such that $R(H, w)$ (which can be meta-interpreted as "w encodes a procedure which provides a counterexample for H").

Now we have seen that the size of any branch b in the reduction tree of any formula ϕ is polynomially bounded in the size of ϕ, and hence, there is a polynomial $P(x)$ such that $|\langle b \rangle| \leq P(|\langle \phi \rangle|)$. Let $l(b)$ denote the only leaf of b. Then ϕ is *not* a theorem iff there is a branch b in the reduction tree such that $|\langle b \rangle| \leq P(|\langle \phi \rangle|)$ and there is w with $|w| \leq Q(|\langle l(b) \rangle|) \leq Q(P(|\langle \phi \rangle|))$ such that $R(l(b), w)$.

Hence, a non-deterministic polynomial time algorithm to check non-provability of theorems is:

(a) guess non-deterministically a branch b with $|\langle b\rangle| \leq P(|\langle\phi\rangle|)$ in the reduction tree of ϕ;
(b) guess non-deterministically a w with $|w| \leq Q(P(|\langle\phi\rangle|))$ such that $R(l(b), w)$.

Of course, the two guesses may be replaced by a single guess: guess a pair (b, w) such that b is a branch of the reduction tree of ϕ and $R(l(b), w)$. It follows that being unprovable is in NP, and hence, being provable is in Co-NP. $\square$

Corollary 23. *Each hyperprojective logic L having a resource bounded proof system and whose axioms are in Co-NP is in Co-NP. If in addition L is a (consistent) substructural logic, then L is Co-NP complete.*

Proof. Proposition 16 and Theorem 20 ensure that L is in Co-NP. The Co-NP completeness follows from [25]. $\square$

Remark 4.1. *Although assuming that the axioms are in Co-NP is a sufficient condition for a hyperprojective resource-bounded logic to be in Co-NP, we believe that in a reasonable proof system the axiom set should be in P. R-hypersequent calculi whose axiom set are in Co-NP (but possibly not in P) are the calculus for Weak Nilpotent Minimum WNM of [19] and that for Hájek's BL in [30].*

4.3.1 Examples

We discuss some known logics that fall into our framework. We provide an r-hypersequent calculus for them and prove that they are Co-NP complete.

Projective and semi-projective logics:

From the projective definition of connectives it follows that projective and hyperprojective logics have a resource bounded and uniform set of rules. Hence when their axioms are in Co-NP, the logics are Co-NP complete. We discuss below three of them: Gödel and classical logic (both projective logics) and the logic *RM* (semi-projective logic).

Uniform and resource-bounded rules for Gödel Logic are ($\lhd$ stands for either $<$ or $\leq$, uniformly in each rule)

$$\frac{H|\phi < \psi|\psi \lhd \chi \qquad H|\psi \leq \phi|\phi \lhd \chi}{H|\phi \wedge \psi \lhd \chi} \qquad \frac{H|\phi < \psi|\chi \lhd \psi \qquad H|\psi \leq \phi|\chi \lhd \phi}{H|\chi \lhd \phi \wedge \psi}$$

$$\frac{H|\phi < \psi|\phi \lhd \chi \qquad H|\psi \leq \phi|\psi \lhd \chi}{H|\phi \vee \psi \lhd \chi} \qquad \frac{H|\phi < \psi|\chi \lhd \phi \qquad H|\psi \leq \phi|\chi \lhd \psi}{H|\chi \lhd \phi \vee \psi}$$

$$\frac{H|\phi \leq \psi|\psi \lhd \chi \qquad H|\psi < \phi|1 \lhd \chi}{H|\phi \to \psi \lhd \chi} \qquad \frac{H|\phi \leq \psi|\chi \lhd \psi \qquad H|\psi < \phi|\chi \lhd 1}{H|\chi \lhd \phi \to \psi}$$

The above rules differ from those of the calculus in [8] and from the r-hypersequent rules in [18] that are not uniform.

Axioms for Gödel Logic are all r-hypersequents that either contain $0 \leq \phi$ or $\phi \leq 1$ or a cycle $\phi_1 \lhd_1 \phi_2|\phi_2 \lhd_2 \phi_3 \ldots |\phi_n \lhd_n \phi_1$ where for all i $\lhd_i$ is either $<$ or $\leq$ and at least one $\lhd_i$ is $\leq$, see [7].
Remark 4.2. *Being a two-valued logic, Classical Logic CL is a regular hyperprojective logic with an r-hypersequent calculus which has a uniform, resource-bounded set of rules. An alternative sequent-style calculus for CL is obtained by adding to the above calculus for Gödel logic axioms of the form $\phi \leq \psi|\psi \leq \chi$. Note that, in contrast with Gentzen sequent calculus LK [22] for CL, the logical rules above are uniform in the sense of Definition 14 (i.e., the left and right rules for the same connective have the same structure).*

R-mingle:

The next example of an hyperprojective logic is the relevant logic R-Mingle [2], indicated in the following by *RM*. This logic has binary connectives $\&, \to, \wedge, \vee$ and unary connective $\neg$, axiomatized Hilbert-style by the following set of formulas:

(B) $(\phi \to \psi) \to ((\psi \to \chi) \to (\phi \to \chi))$	$(\wedge 1)$ $(\phi \wedge \psi) \to \phi$
(C) $(\phi \to (\psi \to \chi) \to (\psi \to (\phi \to \chi))$	$(\wedge 2)$ $(\phi \wedge \psi) \to \psi$
(I) $\phi \to \phi$	$(\wedge 3)$ $((\phi \to \psi) \wedge (\phi \to \chi)) \to (\phi \to (\psi \wedge \chi))$
(& 1) $\phi \to (\psi \to (\phi \& \psi))$	$(\vee 1)$ $\phi \to (\phi \vee \psi)$
(& 2) $\psi \to (\phi \to (\phi \& \psi))$	$(\vee 2)$ $\psi \to (\phi \vee \psi)$
(DIS) $(\phi \wedge (\psi \vee \chi)) \to ((\phi \wedge \psi) \vee (\phi \wedge \chi))$	$(\vee 3)$ $((\phi \to \chi) \wedge (\psi \to \chi)) \to ((\phi \vee \psi) \to \chi)$
$(\neg 1)$ $(\phi \to \neg\psi) \to (\psi \to \neg\phi)$	(C) $\phi \to (\phi \& \phi)$
$(\neg 2)$ $\neg\neg\phi \to \phi$	(M) $(\phi \& \phi) \to \phi$

and rules (modus ponens and adjunction):

$$\frac{\phi \qquad \phi \to \psi}{\psi} \qquad \frac{\phi \qquad \psi}{\phi \wedge \psi}$$

A first analytic calculus for this logic was defined in [4] using hypersequents. The rules of this calculus are however not invertible and do not help proving that the validity problem of *RM* is Co-NP complete (see, e.g., [26] for a semantic-based proof).

It is a well-known fact that the logic *RM* has the variety of Sugihara algebras as equivalent algebraic semantics. This variety is generated by the algebra

$$\mathbb{Z}_{\circ}^{\neg} = \langle \mathbb{Z} \backslash \{0\}, \&, \rightarrow, \vee, \wedge, -, 1 \rangle$$

where $\mathbb{Z}$ is the ordered set of integers, $\wedge$ and $\vee$ are min and max respectively, and the other connectives are defined as follows:

$$x \& y = \begin{cases} x \wedge y & \text{if } |x| = |y| \\ x & \text{if } |x| < |y| \\ y & \text{if } |y| < |x| \end{cases} \qquad x \rightarrow y = \begin{cases} (-x) \vee y & \text{if } x \leq y \\ (-x) \wedge y & \text{if } y < x \end{cases}$$

where $|\cdot|$ is the absolute value function. The following format of the above connectives makes evident the semi-projective nature of *RM*:

$$x \& y = \begin{cases} x & \text{if } x < -y \text{ and } y \leq x \\ y & \text{if } x < -y \text{ and } x < y \\ y & \text{if } -y \leq x \text{ and } y \leq x \\ x & \text{if } -y \leq x \text{ and } x < y \end{cases} \qquad x \rightarrow y = \begin{cases} -x & \text{if } x \leq y \text{ and } y \leq -x \\ y & \text{if } x \leq y \text{ and } -x < y \\ y & \text{if } y < x \text{ and } y \leq -x \\ -x & \text{if } y < x \text{ and } -x < y \end{cases}$$

Hence, *RM* is semi-projective, and multisets are not needed. The rules for $\vee$, $\wedge$, $-\vee$ and $-\wedge$ are as in Nilpotent Minimum logic *NM* (with $-$ in place of $\neg$, see [19]), and the rules for the connectives $\&$, $-(\&)$, $\rightarrow$ and $-(\rightarrow)$ are as follows (having used "," in place of $\oplus$):

The premises of the rule $(\triangleleft, \&, left)$ are

$$\begin{array}{ll} H \mid -\psi \leq \phi \mid \phi < \psi \mid \phi \triangleleft \chi & H \mid -\psi \leq \phi \mid \psi \leq \phi \mid \phi \triangleleft \chi \\ H \mid \phi < -\psi \mid \phi < \psi \mid \phi \triangleleft \chi & H \mid \phi < -\psi \mid \psi \leq \phi \mid \phi \triangleleft \chi \end{array}$$

and the conclusion is $H \mid \phi \& \psi \triangleleft \chi$. The premises of the $(\triangleleft, \&, left)$ rule are:

$$\begin{array}{ll} H \mid -\psi \leq \phi \mid \phi < \psi \mid \chi \triangleleft \phi & H \mid -\psi \leq \phi \mid \psi \leq \phi \mid \chi \triangleleft \phi \\ H \mid \phi < -\psi \mid \phi < \psi \mid \chi \triangleleft \phi & H \mid \phi < -\psi \mid \psi \leq \phi \mid \chi \triangleleft \phi \end{array}$$

and the conclusion is $H \mid \chi \triangleleft \phi \& \psi$. The premises of the $(\triangleleft, -(\&), left)$ rule are:

$$\begin{array}{ll} H \mid -\psi \leq \phi \mid \phi < \psi \mid -\phi \triangleleft \chi & H \mid -\psi \leq \phi \mid \psi \leq \phi \mid -\phi \triangleleft \chi \\ H \mid \phi < -\psi \mid \phi < \psi \mid -\phi \triangleleft \chi & H \mid \phi < -\psi \mid \psi \leq \phi \mid -\phi \triangleleft \chi \end{array}$$

and the conclusion is $H \mid -(\phi \& \psi) \lhd \chi$. The premises of the $(\lhd, -(\&), right)$ rule are:

$$H \mid -\psi \leq \phi \mid \phi < \psi \mid \chi \lhd -\phi \qquad H \mid -\psi \leq \phi \mid \psi \leq \phi \mid \chi \lhd -\phi$$
$$H \mid \phi < -\psi \mid \phi < \psi \mid \chi \lhd -\phi \qquad H \mid \phi < -\psi \mid \psi \leq \phi \mid \chi \lhd -\phi$$

and the conclusion is $H \mid \chi \lhd -(\phi \& \psi)$. The premises of the $(\lhd, \to, left)$ rule are:

$$H \mid \psi < \phi \mid -\phi < \psi \mid -\phi \lhd \chi \qquad H \mid \psi < \phi \mid \psi \leq \phi \mid -\phi \lhd \chi$$
$$H \mid \phi \leq \psi \mid -\phi < \psi \mid -\phi \lhd \chi \qquad H \mid \phi \leq \psi \mid \psi \leq -\phi \mid -\phi \lhd \chi$$

and the conclusion is $H \mid \phi \to \psi \lhd \chi$. The premises of the $(\lhd, \to, right)$ rule are:

$$H \mid \psi < \phi \mid -\phi < \psi \mid \chi \lhd -\phi \qquad H \mid \psi < \phi \mid \psi \leq \phi \mid \chi \lhd -\phi$$
$$H \mid \phi \leq \psi \mid -\phi < \psi \mid \chi \lhd -\phi \qquad H \mid \phi \leq \psi \mid \psi \leq -\phi \mid \chi \lhd -\phi$$

and the conclusion is $H \mid \chi \lhd \phi \to \psi$. The premises of the $(\lhd, -(\to), left)$ rule are:

$$H \mid \psi < \phi \mid -\phi < \psi \mid \phi \lhd \chi \qquad H \mid \psi < \phi \mid \psi \leq \phi \mid \phi \lhd \chi$$
$$H \mid \phi \leq \psi \mid -\phi < \psi \mid \phi \lhd \chi \qquad H \mid \phi \leq \psi \mid \psi \leq -\phi \mid \phi \lhd \chi$$

and the conclusion is $H \mid -(\phi \to \psi) \lhd \chi$. The premises of the $(\lhd, -(\to), right)$ rule are:

$$H \mid \psi < \phi \mid -\phi < \psi \mid \chi \lhd \phi \qquad H \mid \psi < \phi \mid \psi \leq \phi \mid \chi \lhd \phi$$
$$H \mid \phi \leq \psi \mid -\phi < \psi \mid \chi \lhd \phi \qquad H \mid \phi \leq \psi \mid \psi \leq -\phi \mid \chi \lhd \phi$$

and the conclusion is $H \mid \chi \lhd -(\phi \to \psi)$.

The semantic theory, T_{RM}, for RM is just the first-order theory of the structure $\mathbb{Z}_\circ^-$ and the designated truth predicate $Des(x) := 1 \leq x$. Clearly, for every formula ϕ of RM, we have $RM \models \phi$ iff $T_{RM} \models Des(\phi)$.

Theorem 24. *There is a P-time procedure for deciding whether a weakly simple formula of T_{RM} is valid.*

Proof. Any weakly simple formula Q of T_{RM} is equivalent to a formula of the form $Q_1 \cap \ldots \cap Q_n$, such that for every $i = 1, \ldots, n$ there is an $m > 0$ such that

$$Q_i = (u_{i1} \lhd_{i1} v_{i1}) \cup \cdots \cup (u_{im} \lhd_{im} v_{im})$$

where u_{ij}, v_{ij} are either variables or constants or terms of the form $-x$, where x is a variable, and $\lhd_i \in \{\leq, <\}$. Then the negation of Q is the disjunction of conjunctions of atomic formulas of the form $u \lhd v$ (since the negation of an atomic formula is equivalent to another atomic formula in T_{RM}). Denote by Σ_i the set of atomic formulas in Q_i, and let $\Sigma_i^\sharp$ be the set of inequalities obtained from Σ_i by adding, for each inequality $t \lhd s$, the inequality $-s \overline{\lhd} -t$, where $\overline{\lhd}$ is

$\leq$ if $\lhd$ and is $<$ vice versa, and where we identify $-(-t)$ with t. It is readily seen that $\Sigma_i^\sharp$ is satisfiable if and only if Σ_i is. Hence, Q is valid in T_{RM} if and only if all sets $\Sigma_i^\sharp$ are satisfiable.

To check whether $\Sigma_i^\sharp$ is satisfiable, set $SL(t,t')$ iff $t < t'$ is in $\Sigma_i^\sharp$, and $LE(t,t')$ iff $t \leq t'$ is in $\Sigma_i^\sharp$. Set $s \leq^\sharp t$ if there is a finite sequence $s = u_1, \ldots, u_n = t$ such that for $i = 1, \ldots, n-1$, either $LE(u_i, u_{i+1})$ or $SL(u_i, u_{i+1})$ and $s <^\sharp t$ if there is a sequence $s = u_1, \ldots, u_n = t$ as above, such that in addition for at least one i, $SL(u_i, u_{i+1})$. Then $\Sigma_i^\sharp$ is satisfiable in T_{RM} if and only if for every term t in $\Sigma_i^\sharp$, we do not have $t <^\sharp t$ or $t \leq^\sharp -t$ and $-t \leq^\sharp t$ (remember that in $\mathbb{Z}_0^\neg$ the function $-$ has no fixed point). If these conditions are satisfied, then we may map the set of terms in $\Sigma_i^\sharp$ into $\mathbb{Z}_0^\neg$ so that the relations $\leq^\sharp$ and $<^\sharp$, as well as the function $-$, are preserved.

Since the procedure outlined above is polynomial in the size of Q, the theorem is proved. $\square$

Product, Łukasiewicz and Hájek's Basic Logic:

As shown before, Product Logic and Łukasiewicz Logic are examples of regular hyperprojective logics having uniform rules. These rules are easily proved to be resource-bounded. Moreover the set of weakly simple formulas of the corresponding semantic theory is in P (which is the complexity of linear programming [31]), and hence these logics are Co-NP complete.

One of the most important Co-NP complete many-valued logics is Hájek's Basic Logic BL. We are going to introduce a uniform and resource-bounded proof system for this logic whose axiom set is in P. Note that Vetterlein introduced in [30] a simpler system for BL, which is uniform and resource-bounded. However, it is not clear whether his axiom set is in P (it is clearly in Co-NP, because the whole logic BL is in Co-NP).

With reference to the semantic theory for BL described in Example 6, and taking reductions (1), ..., (11) into account, we have the following rules:

For $\lhd \in \{\ll, \prec, \preceq\}$, we have the rules (we use "," in place of $\oplus$; moreover $\lhd \in \{\ll, \preceq, \prec\}$, but when $\lhd$ is $\ll$, in the left rules Γ is empty and Δ is a singleton, and in the right rules Δ is empty and Γ is a singleton):

$$\frac{H|\phi < \psi|\Gamma, \psi \lhd \Delta \quad H|\psi \leq \phi|\Gamma, \phi \lhd \Delta}{H|\Gamma, \phi \wedge \psi \lhd \Delta} \qquad \frac{H|\phi < \psi|\Gamma \lhd \Delta, \psi \quad H|\psi \leq \phi|\Gamma \lhd \Delta, \phi}{H|\Gamma \lhd \Delta, \phi \wedge \psi}$$

$$\frac{H|\phi < \psi|\Gamma, \phi \lhd \Delta \quad H|\psi \leq \phi|\Gamma, \psi \lhd \Delta}{H|\Gamma, \phi \vee \psi \lhd \Delta} \qquad \frac{H|\phi < \psi|\Gamma \lhd \Delta, \phi \quad H|\psi \leq \phi|\Gamma \lhd \Delta, \psi}{H|\Gamma \lhd \Delta, \phi \vee \psi}$$

For the &-rules and for the $\rightarrow$-rules we must distinguish between the case where $\lhd$ is $\ll$ and the case where $\lhd$ is $\preceq$ or $\prec$ (the partition of the unit is the same, but the reductions are different). If $\lhd$ is $\preceq$ or $\prec$, the rules are as follows: The premises of the $(\lhd, \&, left)$ rule are

$$
\begin{array}{ll}
H|\phi \ll 1|\psi \ll 1|\Gamma, 1 \lhd \Delta & H|\psi \ll_= \phi|\Gamma, \phi \lhd \Delta \\
H|\phi \ll_= \psi|\Gamma, \psi \lhd \Delta & H|\phi \equiv^* \psi|1 \preceq \phi|1^+ \prec \phi, \psi|\Gamma \lhd \Delta \\
H|\phi \equiv^* \psi|1 \preceq \phi|\phi, \psi \preceq 1^+|\Gamma, \phi, \psi \lhd \Delta, 1^+ &
\end{array}
$$

and the conclusion is $H|\Gamma, \phi\&\psi \lhd \Delta$.

The premises of the $(\lhd, \&, right)$ rule are

$$
\begin{array}{ll}
H|\phi \ll 1|\psi \ll 1|\Gamma \lhd \Delta, 1 & H|\psi \ll_= \phi|\Gamma \lhd \Delta, \phi \\
H|\phi \ll_= \psi|\Gamma \lhd \Delta, \psi & H|\phi \equiv^* \psi|1 \preceq \phi|1^+ \prec \phi, \psi|\Gamma \lhd \Delta \\
H|\phi \equiv^* \psi|1 \preceq \phi|\phi, \psi \preceq 1^+|\Gamma, 1^+ \lhd \Delta, \phi, \psi &
\end{array}
$$

and the conclusion is $H|\Gamma \lhd \phi\&\psi, \Delta$.

When $\lhd \in \{\preceq, \prec\}$, the rules for $(\lhd, \rightarrow)$ are:

$$
\frac{H|\psi < \phi|\Gamma, 1 \lhd \Delta \quad H|\phi \ll_= \psi|\Gamma, \psi \lhd \Delta \quad H|\phi \leq \psi|\psi \ll \phi|\Gamma, 1^+, \psi \lhd \Delta, \phi}{H|\Gamma, \phi \rightarrow \psi \lhd \Delta}
$$

$$
\frac{H|\psi < \phi|\Gamma, 1 \lhd \Delta \quad H|\phi \ll_= \psi|\Gamma \lhd \Delta, \psi \quad H|\phi \leq \psi|\psi \ll \phi|\Gamma, \phi \lhd \Delta, 1^+, \psi}{H|\Gamma \lhd \Delta, \phi \rightarrow \psi}
$$

The premises of the rule $(\ll, \&, left)$ are

$$
\begin{array}{ll}
H|\phi \ll 1|\psi \ll 1|1 \ll \gamma & H|\psi \ll_= \phi|\phi \ll \gamma \\
H|\phi \ll_= \psi|\psi \ll \gamma & H|\phi \equiv^* \psi|1 \preceq \phi|1^+ \prec \phi, \psi|\phi \ll \gamma \\
H|\phi \equiv^* \psi|1 \preceq \phi|\phi, \psi \preceq 1^+|\phi \ll \gamma &
\end{array}
$$

and the conclusion is $H|\phi\&\psi \ll \gamma$.

The premises of the rule $(\ll, \&, right)$ are

$$
\begin{array}{ll}
H|\phi \ll 1|\psi \ll 1|\gamma \ll 1 & H|\psi \ll_= \phi|\gamma \ll \phi \\
H|\phi \ll_= \psi|\gamma \ll \psi & H|\phi \equiv^* \psi|1 \preceq \phi|1^+ \prec \phi, \psi|\gamma \ll \phi \\
H|\phi \equiv^* \psi|1 \preceq \phi|\phi, \psi \preceq 1^+|\gamma \ll \phi &
\end{array}
$$

and the conclusion is $H|\gamma \ll \phi\&\psi$.

Finally, we have the rules

$$\frac{H|\psi < \phi|1 \ll \gamma \quad H|\phi \ll_{=} \psi|\psi \ll \gamma \quad H|\phi \leq \psi|\psi \ll \phi|\psi \ll \gamma}{H|\phi \to \psi \ll \gamma}$$

$$\frac{H|\psi < \phi|\gamma \ll 1 \quad H|\phi \ll_{=} \psi|\gamma \ll \psi \quad H|\phi \leq \psi|\psi \ll \phi|\gamma \ll \psi}{H|\gamma \ll \phi \to \psi}$$

Remark 4.3.

(1) The rules might be simplified considerably: for instance, in the $\ll$-rules the same reduction corresponds to different contexts, and in the $\prec$ rule for $\to$, the condition $\Gamma, 1 \prec \Delta$ is impossible and may be deleted. However, we have chosen this more complicated formalization in order to make the rules of each connective uniform.

(2) An important advantage of our system w.r.t. that in [30] is that its axiom set is in P (see [16, Lemma 4.5] for a proof).

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