Automated Generation of Analytic Calculi for Logics with Linearity

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Abstract. We show how to automatically generate analytic hypersequent calculi for a large class of logics containing the linearity axiom (lin) $(A \supset B) \vee (B \supset A)$ starting from existing (single-conclusion) cut-free sequent calculi for the corresponding logics without (lin). As a corollary, we define an analytic calculus for Strict Monoidal T-norm based Logic **SMTL**.

1 Introduction

A central task of logic in computer science is to provide *automated generation* of suitable *analytic calculi* for a wide range of non-classical logics. By analytic calculi we mean calculi in which the proof search proceeds by step-wise decomposition of the formula to be proved. The most famous examples of such calculi are the Gentzen sequent calculus **LK** and its single-conclusion version **LJ** for classical and intuitionistic logic respectively. Cut-free "Gentzen-style" calculi serve as a basis for automated deduction, and allow the extraction of important implicit information from proofs such as numerical bounds and programs in proof-style.

The presence of the linearity axiom (lin) $(A \supset B) \vee (B \supset A)$ in the Hilbert-style axiomatization of a logic ensures a total ordering among the elements of its intended models (e.g., Kripke structures, truth-value interpretations). Several logics have been defined adding (lin) to well known systems. E.g., all fuzzy logics based on t-norm[1] connectives [12] – a prominent example being Gödel logic[2] [11, 8, 19] which arises by extending intuitionistic logic **IL** with (lin). Weaker logics such as Monoidal T-norm based Logic **MTL** [9] – the logical counterpart of left continuous t-norms and their residua – or both versions of Urquhart's **C** [21], have also been defined adding (lin) to suitable contraction-free versions of **IL**.

In this paper we show how to automatically generate analytic Gentzen style calculi for a large class of logics containing (lin). To this end we consider a natural generalization of sequent calculi: hypersequent calculi. Hypersequent calculi arise

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[1] T-norms are the main tool in fuzzy logic to combine vague information.

[2] Gödel logic is also known as Dummett's **LC** [8] or Intuitionistic Fuzzy Logic [19].

by extending Gentzen calculi to refer to whole contexts of sequents instead of single sequents. They are particularly suitable for dealing with logics including (*lin*). Indeed, as shown by Avron in [2], this axiom can be enforced in **LJ**, once one embeds sequents into hypersequents and adds suitable rules to manipulate the additional layer of structure. In particular, the crucial rule added to **LJ** is the communication rule (*com*). This design resulted in an analytic calculus for Gödel logic. The same methodology was used e.g. in [6, 5] to introduce analytic hypersequent calculi for some basic fuzzy logics, including **MTL** and Urquhart's **C**, arising by adding (*lin*) to suitable contraction-free versions of **IL**.

Here we generalize these results showing that (*com*) can be viewed, in fact, as a *transfer principle* that translates (single-conclusion) cut-free sequent calculi for a *large class of* logics that do not satisfy (*lin*) into cut-free hypersequent calculi for the corresponding logics with (*lin*). This will give us the means to derive systematically analytic deduction methods for logics whose Hilbert-style axiomatizations contain (*lin*), starting from existing analytic calculi for the corresponding logics without (*lin*). To do this,

– we first introduce a general cut-elimination method for sequent calculi (*cut-elimination by substitutions*) that can be easily transferred to the hypersequent level. Sufficient conditions a calculus has to satisfy in order to admit cut-elimination by substitution are also provided. Among other things, these conditions render our cut-elimination procedure easier to verify than "ad hoc" procedures. (The verification of *unstructured* cut-elimination procedures for hypersequent calculi has been shown to be problematic in the literature.)
– We characterize *which* logics admit this transfer principle, providing some general conditions (on their sequent calculi/Hilbert-style systems) they have to satisfy both at the propositional and at the first-order level.
– As an easy corollary of the transfer principle we define an analytic hypersequent calculus for Strict Monoidal T-norm based Logic **SMTL** [9] – the logic of left-continuous t-norms satisfying the pseudo-complementation property.

2 Sequent and Hypersequent Calculi

The aim of this section is to settle the (hyper)sequent calculi we will deal with. We start by recalling some basic definitions in order to fix the notation and terminology we shall use throughout the paper.

The sequent calculus was introduced by Gentzen [10] in 1934 (see [18] or [20] for a detailed overview). Gentzen sequents are expressions of the form $\Gamma \Rightarrow \Delta$ where Γ and Δ are finite sequences of formulas, respectively called the antecedent and succedent of the sequent. If in a sequent calculus, succedents of all sequents contain at most one formula, the calculus is said to be *single-conclusion*.

In general, in a sequent calculus there are *axioms* (or initial sequents) and inference *rules*. The latter are divided into structural rules, logical rules and cut.

In each logical rule, the introduced formula and the corresponding auxiliary formula(s) are called *principal formula* and *active formula(s)*, respectively. We will refer to the remaining formulas in logical rules as well as to the formulas that remain unchanged in structural rules as (internal) *contexts*.

We call *additive* a multi-premises rule whose contexts in its premises are the same. If those contexts are different and simply merged in the conclusion, the rule is said to be *multiplicative*.

Recall that the structural rules introduced by Gentzen are exchange, weakening and contraction, with single-conclusion versions:

$$\frac{\Gamma, B, A, \Gamma' \Rightarrow C}{\Gamma, A, B, \Gamma' \Rightarrow C}\ (e) \qquad \frac{\Gamma, A, A \Rightarrow C}{\Gamma, A \Rightarrow C}\ (c) \qquad \frac{\Gamma \Rightarrow C}{\Gamma, A \Rightarrow C}\ (w, l) \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow C}\ (w, r)$$

As is well known, their presence or absence determines completely different systems. For instance, a sequent formulation $\mathbf{ScFL}_{ew}$ for Full Lambek calculus with exchange and weakenings[3] $\mathbf{FL}_{ew}$ is obtained by eliminating (c) from the **LJ** sequent calculus for **IL** see [13]. This entails the splitting of the connective "and" of **IL**, into (the additive version) $\wedge$ and (the multiplicative version) $\odot$.

Further structural rules can be defined. Here below are some examples of weaker forms of contraction i.e. *weak contraction* and *n-contraction*:

$$\frac{\Gamma, A, A \Rightarrow}{\Gamma, A \Rightarrow}\ (wc) \qquad \frac{\Gamma, A^n \Rightarrow C}{\Gamma, A^{n-1} \Rightarrow C}\ (nc)$$

where A^k stands for $A, \ldots, A$, k times.

A *derivation* in a sequent calculus is a labelled finite tree with a single root (called *end sequent*), with axioms at the top nodes, and each node-label connected with the label of the (immediate) successor nodes (if any) according to one of the rules. We refer to those connections as (correct) *inferences*.

Definition 1. *We call any propositional single-conclusion sequent calculus* standard *when it satisfies the following conditions:*

1. *antecedents of each sequent are multisets of formulas (or, equivalently, the calculus contains rule (e));*
2. *axioms have the form $A \Rightarrow A$ or $\bot \Rightarrow$;*
3. *each logical rule*
   (a) *has left and right versions, according to the side of the sequent it modifies;*
   (b) *introduces only one connective at a time;*
   (c) *has no side conditions limiting its application (besides, possibly, a condition saying that succedents of some sequents are empty)*
   (d) *has active formulas that are immediate subformulas of the principal formula;*
4. *the cut rule is multiplicative, i.e., it has the form*

$$\frac{\Gamma \Rightarrow A \quad A, \Gamma' \Rightarrow C}{\Gamma, \Gamma' \Rightarrow C}\ (cut)$$

5. *structural rules do not mention any connective.*

[3] $\mathbf{FL}_{ew}$ also coincides with the exponential-free fragment of affine Intuitionistic Linear Logic **ILL**, i.e. **ILL** with weakenings.

Definition 2. *We call a standard sequent calculus containing the rules for quantifiers of Gentzen* **LJ** *calculus for* **IL**, *a* first-order standard sequent calculus.

Henceforth we will only consider (first-order) standard sequent calculi.

Hypersequent calculi were introduced in [1] and [14]. They are a natural generalization of Gentzen sequent calculi.

Definition 3. *A* hypersequent *is a multiset* $\Gamma_1 \Rightarrow \Pi_1 \mid \dots \mid \Gamma_n \Rightarrow \Pi_n$ *where, for all* $i = 1, \dots n$, $\Gamma_i \Rightarrow \Pi_i$ *is a Gentzen sequent.* $\Gamma_i \Rightarrow \Pi_i$ *is called a* component *of the hypersequent. A hypersequent is called* single-conclusion *if so are its components.*

The symbol "|" is intended to denote disjunction at the meta-level.

Like ordinary sequent calculi, hypersequent calculi consist of initial hypersequents (i.e., axioms) as well as logical, structural rules and cut. Axioms, logical rules and cut are essentially the same as in sequent calculi. The only difference is the presence of a *side hypersequent*, denoted by G, representing a (possibly empty) hypersequent. E.g. the hypersequent version of the **LJ** rules $(\supset, r), (\vee, r)_{1,2}$ and $(\vee, l)$ are[4] respectively:

$$\frac{G \mid \Gamma, A \Rightarrow B}{G \mid \Gamma \Rightarrow A \supset B}\ (\supset, r) \qquad \frac{G \mid \Gamma \Rightarrow A_i}{G \mid \Gamma \Rightarrow A_1 \vee A_2}\ (\vee, r)_i \qquad \frac{G \mid \Gamma, A \Rightarrow C \quad G \mid \Gamma, B \Rightarrow C}{G \mid \Gamma, A \vee B \Rightarrow C}\ (\vee, l)$$

Structural rules are divided into *internal* and *external rules*. The internal structural rules deal with formulas within components. They are the same as in ordinary sequent calculi. The external structural rules manipulate whole components of a hypersequent. Examples of this kind of rules are external weakening (ew) and external contraction (ec):

$$\frac{G}{G \mid \Gamma \Rightarrow A}\ (ew) \qquad\qquad \frac{G \mid \Gamma \Rightarrow A \mid \Gamma \Rightarrow A}{G \mid \Gamma \Rightarrow A}\ (ec)$$

Let **Sc** be any sequent calculus. We refer to its *hypersequent version* **HSc** as the calculus containing axioms and rules of **Sc** augmented with side hypersequents and in addition (ew) and (ec). (Note that **HSc** has the same expressive power as **Sc**.) However, in hypersequent calculi it is possible to define *additional external structural rules* which simultaneously act on several components of one or more hypersequents. It is this type of rule which increases the expressive power of hypersequent calculi compared to ordinary sequent calculi. A remarkable example of this kind of rules is Avron's communication rule [2]:

$$\frac{G \mid \Gamma, \Gamma' \Rightarrow A \quad G \mid \Gamma_1, \Gamma_1' \Rightarrow A'}{G \mid \Gamma, \Gamma_1 \Rightarrow A \mid \Gamma', \Gamma_1' \Rightarrow A'}\ (com)$$

Adding this rule to **HLJ** yields an analytic calculus for Gödel logic [11].

[4] We will use the same notation both for sequent and hypersequent rules. However, the context will always provide the relevant information.

The hypersequent version of the quantifier rules we will consider are:

$$\frac{G \mid A(t), \Gamma \Rightarrow B}{G \mid (\forall x)A(x), \Gamma \Rightarrow B}\ (\forall, l) \qquad \frac{G \mid \Gamma \Rightarrow A(a)}{G \mid \Gamma \Rightarrow (\forall x)A(x)}\ (\forall, r)$$

$$\frac{G \mid A(a), \Gamma \Rightarrow B}{G \mid (\exists x)A(x), \Gamma \Rightarrow B}\ (\exists, l) \qquad \frac{G \mid \Gamma \Rightarrow A(t)}{G \mid \Gamma \Rightarrow (\exists x)A(x)}\ (\exists, r)$$

where the eigenvariable condition in $(\exists, l)$ and $(\forall, r)$ has to apply to the whole hypersequent conclusion of the rule, i.e., the free variable a must not occur in the lower *hypersequent*. Indeed, in hypersequent calculi with (com), if one requires the weaker condition that a must not occur (only) in the lower *sequent*, then $\exists x F(x) \Rightarrow \forall x F(x)$ turns out to be derivable.

Definition 4. *We call a single-conclusion hypersequent calculus satisfying the conditions of Definition 1 (and containing the above quantifier rules) a (first-order)* standard hypersequent calculus.

Let **HS** be any sequent or hypersequent calculus. In the following we write $d, S' \vdash_{\mathbf{HS}} S$ if d is a derivation in **HS** of the (hyper)sequent S from the assumption S', i.e. a labelled tree whose nodes are applications of rules of **HS** and whose leaves are either S' or axioms.

Definition 5. *The* length $|d|$ *of a derivation d in* **HS** *is (the maximal number of inference rules) + 1 occurring on any branch of d. The* complexity $|A|$ *of a formula A is defined as the number of occurrences of its connectives and quantifiers. The* cut-rank $\rho(d)$ *of d is (the maximal complexity of cut-formulas in d) + 1. ($\rho(d) = 0$ if d is cut free).*

3 Cut-elimination by Substitutions

Cut-elimination is one of the most important procedures in logic. The removal of cuts corresponds to the elimination of "lemmas" from derivations. This renders a derivation *analytic*, in the sense that all formulas occurring in the derivation are subformulae of the formula to be proved.

Here we prove that if a standard (first-order) sequent calculus **Sc** admits cut-elimination, $\mathbf{HSc} + (com)$ i.e. its hypersequent version with in addition (com), admits cut-elimination too. For this purpose, we introduce a cut-elimination method for sequent calculi (*cut-elimination by substitutions*) that can be easily transferred to the hypersequent level (and in particular to the corresponding hypersequent calculi with (com)).

We start discussing which of, and how, the main cut-elimination methods for sequent calculi can be used in hypersequent context. Recall that Gentzen's cut-elimination method proceeds by eliminating a *uppermost cut* in a derivation by a double induction on the complexity c of the cut formula (+1) and on the sum l of the lengths of its left and right derivations. In his original proof of the cut-elimination theorem for sequent calculus [10], Gentzen met the following problem: If the cut formula is derived by (c), the permutation of cut with (c) does not necessarily move the cut higher up in the derivation. To solve this problem, he introduced the mix rule – a derivable generalization of cut.

In hypersequent calculi a similar problem arises when one tries to permute cut with (ec). (Note that the solution proposed in [6], i.e., to proceed by induction on $(\#(ec), c, l)$ where $\#(ec)$ is the number of applications of (ec) in a derivation, does not work.) In analogy with Gentzen's solution, a way to overcome the problem due to (ec) is to introduce suitable "ad hoc" (derivable) generalizations of the mix rule for each hypersequent calculus. These rules should allow certain cuts to be reduced *in parallel*. E.g. to prove cut-elimination in the hypersequent calculus for propositional Gödel logic, Avron used the following induction hypothesis [2] (generalized mix rule):

If $H \mid \Gamma_1 \Rightarrow A \mid \ldots \mid \Gamma_n \Rightarrow A$ and $H \mid \Sigma_1, A^{n_1} \Rightarrow B_1 \mid \ldots \mid \Sigma_k, A^{n_k} \Rightarrow B_k$ are cut-free provable, so is $H \mid \Gamma, \Sigma_1 \Rightarrow B_1 \mid \ldots \mid \Gamma, \Sigma_k \Rightarrow B_k$, where $\Gamma = \Gamma_1, \ldots, \Gamma_n$ and A^{n_l} stands for $A, \ldots, A$, n_l times.

However, this generalized mix rule does not work for calculi not admitting, e.g., (c) or (w). (Note that to shift upward a cut in which a component $\Gamma_i \Rightarrow A$, with $i \in \{1, \ldots, n\}$, is derived by (ec) or (ew), one needs to use rules (c) and (w), respectively).

A different cut-elimination method for sequent calculus was introduced by Schütte-Tait [15, 17]. This proceeds by eliminating a *largest cut* in a derivation (w.r.t. the number of connectives and quantifiers). The main feature of this method is that a cut with a non-atomic cut formula is not shifted upward but simply reduced (i.e., replaced by smaller cuts) using the inversion(s) of the premises of the original cut (see, e.g. [16]). This renders the presence of (ec) unproblematic once one uses this method in hypersequent calculi. Proofs of cut-elimination à la Schütte-Tait for the hypersequent calculi for (first-order) Gödel logic and **MTL** can be found, e.g., in [3, 5]. There in fact to eliminate a cut with a non-atomic cut formula only *one* premise of this cut is inverted and used to replace the cut by smaller ones exactly in the place(s) in which the cut formula (of the remaining premise of the cut) is introduced.

However, cut-elimination à la Schütte-Tait cannot be straightforwardly transferred from a sequent to the corresponding hypersequent calculus. Moreover, demanding the invertibility (even) of (only) one of the premises of cuts seems to be a rather strong condition. Indeed, there do exist (hyper)sequent calculi in which cuts are eliminable but in which none of the premises of a cut is invertible. An example of such a calculus is obtained by replacing the right rule introducing $\wedge$ in the $\mathbf{ScFL}_{ew}$ calculus for $\mathbf{FL}_{ew}$ by the following rules:

$$\frac{\Gamma \Rightarrow A_1 \quad \Gamma', A_1 \Rightarrow A_2}{\Gamma, \Gamma' \Rightarrow A_1 \wedge A_2}\ (\wedge, r)_1 \qquad \frac{\Gamma \Rightarrow A_2 \quad \Gamma', A_2 \Rightarrow A_1}{\Gamma, \Gamma' \Rightarrow A_1 \wedge A_2}\ (\wedge, r)_2$$

This calculus admits cut-elimination (e.g., using Gentzen's method, see [5]) but neither of the premises of a cut with cut formula $A \wedge B$ can be inverted in the usual way.

In the proof of Theorem 1 below, we introduce *cut-elimination by substitutions*. This proceeds by eliminating a *largest uppermost* cut in a derivation. The idea behind this method is to eliminate a cut via suitable substitutions in the derivations $d_0 \vdash_{\mathbf{Sc}} \Sigma \Rightarrow A$ and $d_1 \vdash_{\mathbf{Sc}} \Gamma, A \Rightarrow C$ of its premises. We substitute all the occurrences of the cut formula. When we do this we have also to replace all the subproofs of d_0 and d_1 ending in an inference whose principal formula is an occurrence of the cut-formula. This requires us to trace up the occurrences of the cut formula through d_0 and d_1. For this purpose we use below the notion of *decoration of a formula A in a (hyper)sequent derivation d*. This essentially amounts to the (marked) derivation obtained by following up and marking in d all occurrences of the considered formula A starting from the end sequent of d: if at some stage any marked occurrence of A –indicated by A^*– is multiplied by a certain (internal or external) structural rule we mark and trace up all these occurrences of the formula from the premise(s). In outline, two cases can occur.

- If the cut formula (A^*) was not introduced by *any* logical (or quantifier) rule in d_0 (respectively d_1), the cut is replaced by the derivation d_0 (respectively d_1) in which one substitutes all A^* by Γ and C (respectively Σ) ($\star$).
- Suppose A^* was introduced by some logical (or quantifier) rules in d_0 and d_1. The required derivation is obtained from d_0 and d_1 by replacing all A^*s via suitable substitutions ($\star$), and replacing the inferences which introduced A^* with suitable cuts on subformulas of A ($\star\star$).

The applicability of cut elimination by substitutions relies on the fact that the considered (standard) sequent calculus satisfies ($\star\star$) and ($\star$), namely, its rules allow the replacement of cuts by smaller ones (i.e. logical and quantifier rules are *reductive*) and they lead to correct inferences once one uniformly replaces any formula in their premises and (some occurrences of this formula in their) conclusions by multisets of formulas (i.e., rules are *substitutive*). The latter condition can be equivalently expressed as: the rules allow any cut to be shifted upward replacing the cut formula in their premises by the contexts of the remaining premise of the cut.

Before introducing the formal definition of reductive and substitutive rules let us consider the following explanatory example:

Example 1. The contraction rule (c) is substitutive. Indeed the sequents obtained by replacing any formula $X \in \Gamma$ (or by replacing A) with a multiset Σ in its conclusion, can be derived by applying (c) to the sequent $\Gamma, A, A \Rightarrow C$ after having replaced $X \in \Gamma$ (or the two occurrences of A) with Σ. Moreover, the sequent $\Gamma, A, \Sigma \Rightarrow D$, obtained by substituting C in the conclusion of (c) with Σ and D, can be derived by applying (c) to $\Gamma, A, A \Rightarrow C$ in which one carries out the same substitution. By contrast, the n-contraction rule (nc) is not substitutive. Indeed e.g. the sequent $\Gamma, A^{n-2}, \Sigma \Rightarrow C$, obtained by substituting one occurrence of A with Σ in its conclusion cannot be derived by applying (nc) to $\Gamma, \Sigma^n \Rightarrow C$.

Definition 6. *Let* **HS** *be any standard (hyper)sequent calculus.*

We call its (logical or quantifier) rules $\{(\star, r)_1, \ldots, (\star, r)_n\}$ *and* $\{(\star, l)_1, \ldots, (\star, l)_m\}$ *for introducing a connective (or a quantifier)* $\star$ reductive, *whenever the sequent obtained via (cut) on the principal formula of the conclusions of* $(\star, l)_i$ *and* $(\star, r)_j$ *(for each* $i = 1, \ldots, m$ *and* $j = 1, \ldots m$*) can be derived from their premises using (cut) and the structural rules of* **HS**. *Any* **HS***-rule (*$n \geq 1$ *and* $C \neq C'$*)*

$$\frac{(G \mid \Gamma_1' \Rightarrow C_1' \mid)\Gamma_1 \Rightarrow C_1 \ldots \quad \ldots (G \mid \Gamma_n' \Rightarrow C_n' \mid)\Gamma_n \Rightarrow C_n}{(G \mid \Gamma' \Rightarrow C' \mid)\Gamma \Rightarrow C} \; (R)$$

is said to be substitutive *whenever the following conditions hold:*

1. *Let* X *be any formula that is not principal in (R) occurring in* Γ *(or* Γ'*) and let* H *be the (hyper)sequent arising by replacing some occurrences of* X *in* Γ *or* Γ' *with any multiset of formulas* Σ. H *can be derived using only (R) and the structural rules of* **HS** *from the premises of (R) with* Σ *uniformly substituted for every occurrence of* X *in each* Γ_i *and* Γ_i' *(*$i = 1, \ldots, n$*).*
2. *If* C *(respectively* C'*) is neither empty nor principal in (R), the (hyper)sequent* $(G \mid \Gamma' \Rightarrow C' \mid)\Sigma, \Gamma \Rightarrow D$ *(respectively* $(G \mid \Gamma', \Sigma \Rightarrow D) \mid \Gamma \Rightarrow C$*), for any* Σ *and* D*, is derivable only using (R) and the structural rules of* **HS** *from the premises of (R) with* $\Gamma_i^{(')}, \Sigma \Rightarrow D$ *uniformly substituted for each* $\Gamma_i^{(')} \Rightarrow C_i^{(')}$ *in which* $C_i^{(')} = C$ *(respectively* $C_i^{(')} = C'$*).*

Let $d(s)$ and $H(s)$ denote the results of substituting the term s for all free occurrences of x in the derivation $d(x)$ and in the (hyper)sequent $H(x)$.

Lemma 1 (Substitution Lemma). *Let* **HS** *be any standard first-order (hyper)sequent calculus. If* $d(x) \vdash_{\mathbf{HS}} H(x)$*, then* $d(s) \vdash_{\mathbf{HS}} H(s)$*, with* $|d(s)| = |d(x)|$ *and* $\rho(d(s)) = \rho(d(x))$*, where* s *only contains variables that do not occur in* $d(x)$*.*

Using the above lemma one can show

Lemma 2. *The (hyper)sequent rules* $(\forall, \lhd)$ *and* $(\exists, \lhd)$*, with* $\lhd \in \{l, r\}$*, are substitutive in any standard first-order (hyper)sequent calculus.*

Theorem 1. *Any standard (first-order) sequent calculus* **Sc** *in which (a) logical rules are reductive and (b) rules are substitutive, admits cut-elimination.*

Proof. Let $d \vdash_{\mathbf{Sc}} S$, with $\rho(d) > 0$. The proof proceeds by induction on the pair $(\rho(d), \#\rho(d))$, where $\#\rho(d)$ is the number of cuts in d with cut-rank $\rho(d)$. Suppose $\rho(d) = |A| + 1$ and let

$$d_0 \vdash_{\mathbf{Sc}} \Sigma \Rightarrow A \quad \text{and} \quad d_1 \vdash_{\mathbf{Sc}} \Gamma, A \Rightarrow C$$

be the premises of the uppermost cut in d with cut-formula A. We can find a derivation $d' \vdash_{\mathbf{Sc}} \Gamma, \Sigma \Rightarrow C$ with $\rho(d') < \rho(d)$. Hence, replacing in d the subderivation ending in this largest uppermost cut by d', results in a derivation $\overline{d}$ such that either $\rho(\overline{d}) < \rho(d)$ or $\#\rho(\overline{d}) = \#\rho(d) - 1$. Two cases can occur:

1. The cut-formula A is not introduced by any logical (or quantifier) inference in d_0 or d_1. Assume first that this is the case in d_1. We consider the decoration of A in d_1 starting from $d_1 \vdash_{\mathbf{Sc}} \Gamma, A^* \Rightarrow C$. We then substitute A^* everywhere in d_1 by Σ. Let us call d_1^* the obtained labelled tree. Since A is not introduced by any logical (or quantifier) inference in d_1 and **Sc** is a (first-order) standard sequent calculus whose rules are substitutive, all the inferences in d_1^* are correct (upon adding some structural inferences, if needed). Note that if A^* originates in an axiom $A^* \Rightarrow A$, this is transformed into $\Sigma \Rightarrow A$. Hence d_1^* is a derivation in **Sc** and either $d_1^*, \Sigma \Rightarrow A \vdash_{\mathbf{Sc}} \Gamma, \Sigma \Rightarrow C$ or $d_1^* \vdash_{\mathbf{Sc}} \Gamma, \Sigma \Rightarrow C$. A derivation $d' \vdash_{\mathbf{Sc}} \Gamma, \Sigma \Rightarrow C$ with $\rho(d') < \rho(d)$ is thus obtained by replacing d_0 and d_1 in d by (the juxtaposition of d_0 and) d_1^*. The case where A is not introduced by any logical (or quantifier) inference in d_0 is symmetric. Here we consider the decoration of A in d_0 starting from $d_0 \vdash_{\mathbf{Sc}} \Sigma \Rightarrow A^*$ and we substitute in d_0 each sequent of the form $\Pi \Rightarrow A^*$ with $\Pi, \Gamma \Rightarrow C$ possibly adding suitable structural inferences, if needed. The rest of the proof proceeds (similarly) as above.

2. The cut-formula A is introduced by logical (or quantifier) inferences both in d_0 and d_1. Let us consider the decoration of A in d_0 and d_1 starting from $d_0 \vdash_{\mathbf{Sc}} \Sigma \Rightarrow A^*$ and $d_1 \vdash_{\mathbf{Sc}} \Gamma, A^* \Rightarrow C$ respectively. Suppose $A = \star(A_1, \dots A_p)$, where $\star$ is any connective, or $A = \forall x B(x)$. Let $\Sigma_1 \Rightarrow A^*, \dots, \Sigma_n \Rightarrow A^*$ and $\Gamma_1, A^* \Rightarrow C_1 \dots \Gamma_m, A^* \Rightarrow C_m$ be the conclusions of the logical (or $\forall$) inferences introducing A^* in d_0 and d_1. We first replace A^* with Σ_1 everywhere in d_1. Note that the resulting tree is not a derivation anymore. However, since the rules of (first-order) **Sc** are substitutive, all the inferences – except those that introduced A^* in d_1 – are correct (upon adding some structural inferences, if needed). These incorrect inferences have the following form (assume w.l.o.g. that $(\star, l)$ is a one-premise rule)

$$\dfrac{\begin{array}{c}\vdots\, d_1' \\ \Gamma_1', A_l, \dots A_t \Rightarrow B_1'\end{array}}{\Gamma_1, \Sigma_1 \Rightarrow B_1}\,{\scriptstyle(\star,1)}$$

We replace them by cut(s) with $d_1' \vdash_{\mathbf{Sc}} \Gamma_1', A_l, \dots A_t \Rightarrow B_1'$ and the premise(s) of the inference rule introducing A^* in d_0, with conclusion $\Sigma_1 \Rightarrow A^*$, (previously applying the Substitution Lemma and), adding some structural inferences, if needed. We call the resulting tree $d_{1_1}^*$. Note that if d_1 also contains axioms $A^* \Rightarrow A$, these are transformed into sequents $\Sigma_1 \Rightarrow A$ in $d_{1_1}^*$. These are simply replaced by the subderivation of d_0 ending in $\Sigma_1 \Rightarrow A$. Since the rules of **Sc** are reductive, $d_{1_1}^*$ is a derivation in **Sc**. Moreover, it is easy to check that $d_{1_1}^* \vdash_{\mathbf{Sc}} \Gamma, \Sigma_1 \Rightarrow C$. Similarly, we can obtain derivations $d_{1_2}^*, \dots d_{1_n}^*$ of $\Gamma, \Sigma_2 \Rightarrow C, \dots \Gamma, \Sigma_n \Rightarrow C$, with $\rho(d_{1_i}^*) < \rho(d)$, for $i = 1, \dots, n$. This is not yet what we were looking for. Let us substitute in (the decorated version of) d_0 each sequent of the form $\Pi \Rightarrow A^*$ with $\Pi, \Gamma \Rightarrow C$, possibly adding suitable structural inferences, if needed. (If d_0 also contains axioms $A \Rightarrow A^*$, these are replaced by the derivation d_1). As before, the resulting tree is not a derivation anymore and the only incorrect inferences are those which introduced A^* that now have the form (assume w.l.o.g. that $(\star, r)$ is a one-premise rule)

$$\dfrac{\begin{array}{c}\vdots \\ \Sigma_i' \Rightarrow A_k\end{array}}{\Sigma_i, \Gamma \Rightarrow C} \ (\star,\mathrm{r})$$

To correct these inferences we replace the whole subtree ending in $\Sigma_i, \Gamma \Rightarrow C$ with the derivation $d^*_{1_i}$ obtained before. Iterating this procedure for all the n inferences introducing A^* in d_0, leads to the required derivation $d' \vdash_{\mathbf{Sc}} \Gamma, \Sigma \Rightarrow B$ with $\rho(d') < \rho(d)$.

If $A = \exists x B(x)$, the proof proceeds as above exchanging, however, the role of d_0 and d_1. This way, one can replace the incorrect $(\exists, r)$ inferences by introducing $(\exists x B(x))^*$ with a cut from their premises and the premises of the $(\exists, l)$ inferences introducing $(\exists x B(x))^*$ in d_1, previously applying the Substitution Lemma to the latter.

Cut-elimination by substitutions can be easily used in hypersequent calculi. First note that (ew) and (ec) are substitutive in any hypersequent calculus.

Theorem 2. *Any (first-order) standard hypersequent calculus* **HL** *in which (a) logical rules are reductive and (b) rules are substitutive, admits cut-elimination.*

Proof. Let $d \vdash_{\mathbf{HL}} H$, with $\rho(d) = |A| + 1$ and let $d_0 \vdash_{\mathbf{HL}} G \mid \Sigma \Rightarrow A$ and $d_1 \vdash_{\mathbf{HL}} G \mid \Gamma, A \Rightarrow C$ be the premises of the uppermost cut in d with cut-formula A. We show that we can find a derivation $d' \vdash_{\mathbf{HL}} G \mid \Gamma, \Sigma \Rightarrow C$ with $\rho(d') < \rho(d)$. The proof proceeds by induction on $(\rho(d), \#\rho(d))$. We sketch below the (few) additional steps – w.r.t. those outlined in the proof of Theorem 1 – needed to cope with side hypersequents.

1. The cut-formula A is not introduced by any logical (or quantifier) inference in d_0 or d_1. Assume w.l.o.g. that this is the case in d_1. We first add G to all the hypersequents in d_1 and for each newly generated hypersequent $G \mid B \Rightarrow B$ or $G \mid \bot \Rightarrow$ (if any), we add an application of (ew) to recover the original axiom $B \Rightarrow B$ or $\bot \Rightarrow$ of d_1. The remaining steps are as in the proof of Theorem 1. The required derivation is finally obtained by applying (ec) to d_1^*.

2. The cut-formula A is introduced by logical (or quantifier) inferences both in d_0 and d_1. Let $G_1 \mid \Sigma_1 \Rightarrow A^*, \ldots, G_n \mid \Sigma_n \Rightarrow A^*$ (and $H_1 \mid \Gamma_1, A^* \Rightarrow C_1 \ldots H_m \mid \Gamma_m, A^* \Rightarrow C_m$) be the conclusions of the logical (or quantifier) inferences introducing A^* in d_0 and d_1, respectively. Assume, w.l.o.g., $A = \star(A_1, \ldots A_p)$ or $A = \forall x B(x)$. We first add G_i to all the hypersequents in d_1 and we add applications of (ew) to recover the original axioms of d_1, if needed. Following the same steps as in the proof of Theorem 1, we obtain the derivations $d^*_{1_i} \vdash_{\mathbf{HL}} G_i \mid G \mid \Gamma, \Sigma_i \Rightarrow C$, for $i = 1, \ldots, n$. We now first add G to all the hypersequents in d_0 and we then proceed as in the proof of Theorem 1. This leads to $d'' \vdash_{\mathbf{HL}} G \mid G \mid \Gamma, \Sigma \Rightarrow B$. The required derivation is finally obtained by applying (ec) to d''.

Corollary 1. *Let* **Sc** *be a standard (first-order) sequent calculus in which (a) logical rules are reductive and (b) rules are substitutive.* **HSc** $+$ *(com) admits cut-elimination.*

Proof. It is easy to verify that **HSc** with in addition (*com*) satisfies conditions (a) and (b) too. The claim follows by Theorem 2.

4 Transfer Principle

Let **Sc** be a (first-order) standard sequent calculus that admits cut-elimination by substitutions. Here we show that if **Sc** (or, equivalently, the formalized logic **L**) is "expressive enough", then **HSc** + (*com*) is an analytic calculus for **L**+ axiom schemata $(A \supset B) \vee (B \supset A)$ (+, in the first-order case, $\forall x(P(x) \vee Q) \supset (\forall x P(x) \vee Q)$, where x does not occur free in Q).

Henceforth we assume logics to be specified by Hilbert-style systems. A logic **L** is identified with the set of its provable formulas. By a first-order logic **L** we mean a Hilbert system whose rules are *modus ponens* and *generalization* and whose axioms for quantifiers are those of first-order intuitionistic logic.

In order to interpret (hyper)sequents into the language of the considered logics, we assume these contain a disjunction connective $\vee$, an implication $\supset$ and the constant $\perp$. Since sequents (respectively hypersequents) are multisets of formulas (respectively sequents), we assume $\vee$ is commutative and $\supset$ satisfies exchange (i.e. $(A \supset (B \supset C)) \supset (B \supset (A \supset C))$). Moreover, $\perp \supset A$ belongs to the provable formulas.

Definition 7. *Let $A_1, \ldots, A_n \Rightarrow B$ be a sequent. Its* generic interpretation $\mathcal{I}$ *is defined as follows:*

$$\mathcal{I}(\Rightarrow B) := B$$
$$\mathcal{I}(A_1, \ldots, A_n \Rightarrow B) := (A_1 \supset \ldots \supset (A_n \supset B) \ldots)$$
$$\mathcal{I}(A_1, \ldots, A_n \Rightarrow) := (A_1 \supset \ldots \supset (A_n \supset \perp) \ldots)$$

Let G be the hypersequent $S_1 \mid \cdots \mid S_n$. Then its generic interpretation $\mathcal{I}(G)$ *is defined as $\mathcal{I}(S_1) \vee \ldots \vee \mathcal{I}(S_n)$.*

Definition 8. *A (Hyper)sequent rule*

$$\frac{S_1 \quad \ldots \quad S_n}{S_0}\ (r) \qquad \text{with } n \geq 1$$

is sound *for a Hilbert style system* **L**, *if whenever* **L** *derives the generic interpretations of its premises,* **L** *derives the generic interpretation of its conclusion too.* (r) *is* strongly sound *for* **L** *if* **L** *derives the formula $\mathcal{I}(S_1) \supset (\ldots(\mathcal{I}(S_n) \supset \mathcal{I}(S_0))\ldots)$. A (hyper)sequent calculus* **HL** *is called* sound *(resp.* strongly sound*) for* **L** *if all the axioms and rules of* **HL** *are sound (resp.* strongly sound*) for* **L**. **HL** *is called* complete *for* **L** *if for all formulas A derivable in* **L**, *the (hyper)sequent $\Rightarrow A$ is derivable in* **HL**.

Lemma 3. *Let* **Sc** *be a standard sequent calculus in which the* **LJ** *rules $(\supset, r)$, $(\vee, r)_{1,2}$, $(\vee, l)$ as well as the rule*

$$\frac{\Gamma \Rightarrow A \quad \Gamma', B \Rightarrow C}{\Gamma, \Gamma', A \supset B \Rightarrow C}\ (\supset, l)$$

are derivable. If **Sc** *is strongly sound and complete for* **L** *then the following properties hold:*

1. $(A \supset B) \supset ((B \supset C) \supset (A \supset C)) \in \mathbf{L}$
2. $A \supset (G \vee A) \in \mathbf{L}$
3. *If* $A \supset B \in \mathbf{L}$ *then* $(H \vee A) \supset (H \vee B) \in \mathbf{L}$,
4. $(A \vee A) \supset A \in \mathbf{L}$
5. *If* $A \in \mathbf{L}$, $B \in \mathbf{L}$ *and* $A \supset X \vee B \supset X \in \mathbf{L}$, *then* $X \in \mathbf{L}$.
6. *If* $A \supset B \in \mathbf{L}$ *and* $C \supset D \in \mathbf{L}$ *then* $(A \vee C) \supset (B \vee D) \in \mathbf{L}$,
7. *If* $(A \supset B) \vee H \in \mathbf{L}$ *and* $A \in \mathbf{L}$, *then* $B \vee H \in \mathbf{L}$,
8. *If* $A \vee B \in \mathbf{L}$ *and* $A \supset X \in \mathbf{L}$, *then* $X \vee B \in \mathbf{L}$,
9. *If* $A \supset (B \supset C) \in \mathbf{L}$, $A \vee H$, $B \vee H \in \mathbf{L}$, *then* $C \vee H \in \mathbf{L}$.
10. *If* $A_1 \supset (A_2 \supset \ldots (A_n \supset B) \ldots)) \in \mathbf{L}$ *and* $A_i \vee H \in \mathbf{L}$, *for each* $i = 1, \ldots n$, *then* $B \vee H \in \mathbf{L}$.

Proof. 3. By Property 2, $B \supset (H \vee B) \in \mathbf{L}$, hence by Property 1 and modus ponens, $A \supset (H \vee B) \in \mathbf{L}$. Since $H \supset (H \vee B) \in \mathbf{L}$, follows that $(H \vee A) \supset (H \vee B) \in \mathbf{L}$.

5. From $A \in \mathbf{L}$ and $B \in \mathbf{L}$ follows $(A \supset X \vee B \supset X) \supset X \in \mathbf{L}$. The claim follows by modus ponens.

7. From $A \in \mathbf{L}$ we get $[(A \supset B) \vee H] \supset B \vee H$. The claim follows by modus ponens.

9. By Property 3, $(B \supset C) \supset [(B \vee H) \supset (C \vee H)] \in \mathbf{L}$. By Property 1 and modus ponens follows $A \supset [(B \vee H) \supset (C \vee H)] \in \mathbf{L}$. By Property 3 and modus ponens we get $(A \vee H) \supset [(B \vee H) \supset (C \vee H) \vee H] \in \mathbf{L}$. By modus ponens we obtain $[(B \vee H) \supset (C \vee H)] \vee H$ and by Property 7 $(C \vee H) \vee H \in \mathbf{L}$. The claim follows since $[(C \vee H) \vee H] \supset (C \vee H) \in \mathbf{L}$.

10. Follows by repetedely applying Properties 3, 7 and 9.

Theorem 3. *Let* **Sc** *be a standard sequent calculus in which the rules* $(\supset, r), (\supset, l), (\vee, r)_{1,2}, (\vee, l)$ *are derivable. If* **Sc** *is strongly sound and complete for* **L**, *then* $\mathbf{HSc} + (com)$ *is sound and complete for*

$$\mathbf{L} + (A \supset B) \vee (B \supset A)$$

Proof. (*Soundness*) The soundness of logical and internal structural rules of **HSc** follows by the strongly soundness of **Sc** w.r.t. **L** together with Property 10. The soundness of (ec) is ensured by Properties 3 and 4, while that of (ew) follows by Property 2. For (*com*) we can argue as follows: Assume $\mathcal{I}(\Gamma, \Gamma' \Rightarrow A) \vee H \in \mathbf{L}$ and $\mathcal{I}(\Gamma_1, \Gamma_1' \Rightarrow A') \vee H \in \mathbf{L}$. We show that

$$(*) \quad \mathcal{I}(\Gamma, \Gamma_1 \Rightarrow A) \vee \mathcal{I}(\Gamma', \Gamma_1' \Rightarrow A') \vee H \in \mathbf{L}$$

Indeed, let the notation $[\Sigma]$, where $\Sigma = \Sigma_1, \ldots \Sigma_n$, stand for $[(\Sigma_1 \supset (\ldots (\Sigma_{n-1} \supset \Sigma_n) \ldots))$. We have

$$([\Gamma_1] \supset [\Gamma']) \supset (\mathcal{I}(\Gamma, \Gamma' \Rightarrow A) \supset \mathcal{I}(\Gamma, \Gamma_1 \Rightarrow A)) \quad \text{and}$$

$$([\Gamma'] \supset [\Gamma_1]) \supset (\mathcal{I}(\Gamma_1, \Gamma_1' \Rightarrow A') \supset \mathcal{I}(\Gamma', \Gamma_1' \Rightarrow A')).$$

By Properties 2, 3, 1 and modus ponens follow

$$(\mathcal{I}(\Gamma, \Gamma' \Rightarrow A) \supset \mathcal{I}(\Gamma, \Gamma_1 \Rightarrow A)) \supset ((\mathcal{I}(\Gamma, \Gamma' \Rightarrow A) \vee H) \supset (*)) \in \mathbf{L} \quad \text{and}$$

$$(\mathcal{I}(\Gamma_1, \Gamma_1' \Rightarrow A') \supset \mathcal{I}(\Gamma', \Gamma_1' \Rightarrow A')) \supset ((\mathcal{I}(\Gamma_1, \Gamma_1' \Rightarrow A') \vee H) \supset (*)) \in \mathbf{L}$$

By Properties 1, 6 and axiom $(A \supset B) \vee (B \supset A)$ we get

$$((\mathcal{I}(\Gamma, \Gamma' \Rightarrow A) \vee H) \supset (*)) \vee ((\mathcal{I}(\Gamma_1, \Gamma_1' \Rightarrow A') \vee H) \supset (*)) \in \mathbf{L}$$

the claim follows by Property 5.

(*Completeness*) Since **Sc** (and hence **HSc**) is complete for **L**, the claim follows by the derivability of the linearity axiom in **HSc** + (*com*):

$$\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A \qquad B \Rightarrow B}{A \Rightarrow B \mid B \Rightarrow A}\,(\text{com})}{\Rightarrow A \supset B \mid \;\Rightarrow B \supset A}\,2\text{x}(\supset,\text{r})}{\Rightarrow (A \supset B) \vee (B \supset A) \mid \;\Rightarrow (A \supset B) \vee (B \supset A)}\,2\text{x}(\vee_\text{i},\text{r})}{\Rightarrow (A \supset B) \vee (B \supset A)}\,(\text{ec})$$

Corollary 2 (Transfer Principle). *Let* **Sc** *be a standard sequent calculus whose logical rules are reductive and all its rules are substitutive and in which the rules* $(\supset, r), (\vee, r)_{1,2}, (\vee, l)$ *and* $(\supset, l)$ *are derivable. If* **Sc** *is strongly sound and complete for* **L** *then* **HSc**+(*com*) *is an analytic calculus sound and complete for* $\mathbf{L} + (A \supset B) + (B \supset A)$.

If **Sc** contains quantifier rules, this result does not hold anymore. E.g. in **LJ** the rules $(\supset, r), (\vee, r)_{i:\, i=1,2}$, $(\vee, l)$ and $(\supset, l)$ are derivable. However the calculus obtained by adding (*com*) to the hypersequent version of **LJ** is *not* sound for first-order **IL** with the linearity axiom. (This logic, introduced by Corsi in [7], is semantically characterized by linearly ordered Kripke frames.) Indeed in this calculus one can derive the shifting law of universal quantifiers w.r.t. $\vee$, i.e., $(\vee\forall)$ $\forall x(P(x) \vee Q) \supset (\forall x P(x) \vee Q)$, where x does not occur free in Q. This law, that forces the domains of the corresponding Kripke models to be constant, is not valid in Corsi's logic. In fact, **HLJ** + (*com*) turns out to be sound and complete for first-order Gödel logic [4] – whose axiomatization is obtained by adding $(\vee\forall)$ to Corsi's logic. As the theorem below shows, this is not by chance, but follows a general principle (note that $(\vee\forall)$ is needed to prove the soundness of the hypersequent rule $(\forall, r)$).

Theorem 4. *Let* **Sc** *be a standard sequent calculus in which the rules* $(\supset, r), (\vee, r)_{1,2}, (\vee, l)$ *and* $(\supset, l)$ *are derivable. If (propositional)* **Sc** *is strongly sound and complete for* **L**, *then first-order* **HSc** + (*com*) *is sound and complete for*

$$\textit{first-order} \quad \mathbf{L} + (A \supset B) \vee (B \supset A) + \;(\vee\forall)$$

Proof. (*Soundness*) By Theorem 3 it is enough to prove the soundness of the hypersequent rules for quantifiers w.r.t. **L**. The cases $(\forall, l)$ and $(\exists, r)$ are easy. For $(\forall, r)$ we may argue as follows: If $\mathcal{I}(G) \vee \mathcal{I}(\Gamma \Rightarrow A(a)) \in \mathbf{L}$, $\forall x(\mathcal{I}(G) \vee \mathcal{I}(\Gamma \Rightarrow A(x))) \in \mathbf{L}$ too. Since a did not occur in $\mathcal{I}(G)$ or in $\mathcal{I}(\Gamma \Rightarrow A(a))$, we may now assume that x does not either. Hence $\mathcal{I}(G) \vee \forall x \mathcal{I}(\Gamma \Rightarrow A(x)) \in \mathbf{L} + (\vee\forall)$. The result follows by Property 8 since $\forall x \mathcal{I}(\Gamma \Rightarrow A(x)) \supset \mathcal{I}(\Gamma \Rightarrow \forall x A(x)) \in \mathbf{L}$. The soundness of $(\exists, l)$ can be proved in a similar way.

(*Completeness*) Since the generalization rule is a particular case of $(\forall, r)$, by Theorem 3 it is enough to prove that $\vdash_{\mathbf{HSc}+(com)} \Rightarrow (\vee\forall)$. Indeed

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A(a) \Rightarrow A(a) \qquad \dfrac{A(a) \Rightarrow A(a) \qquad B \Rightarrow B}{B \Rightarrow A(a) \mid A(a) \Rightarrow B}\,{\scriptstyle (com)} \qquad B \Rightarrow B}{A(a) \vee B \Rightarrow A(a) \mid A(a) \vee B \Rightarrow B}\,{\scriptstyle 2\mathrm{x}(\vee,\mathrm{l})+(\mathrm{ew})\mathrm{s}}}{\forall x(A(x) \vee B) \Rightarrow A(a) \mid \forall x(A(x) \vee B) \Rightarrow B}\,{\scriptstyle 2\mathrm{x}(\forall,\mathrm{l})}}{\forall x(A(x) \vee B) \Rightarrow \forall x A(x) \mid \forall x(A(x) \vee B) \Rightarrow B}\,{\scriptstyle (\forall,r)}}{\forall x(A(x) \vee B) \Rightarrow \forall x A(x) \vee B \mid \forall x(A(x) \vee B) \Rightarrow \forall x A(x) \vee B}\,{\scriptstyle 2\mathrm{x}(\vee,\mathrm{r})}}{\forall x(A(x) \vee B) \Rightarrow \forall x A(x) \vee B}\,{\scriptstyle (ec)}}{\Rightarrow \forall x(A(x) \vee B) \supset (\forall x A(x) \vee B)}\,{\scriptstyle (\supset,r)}
$$

Corollary 3 (Transfer Principle). *Let* **Sc** *be a standard first-order sequent calculus whose logical rules are reductive and rules are substitutive and in which the rules* $(\supset, r), (\vee, r)_{1,2}, (\vee, l)$ *and* $(\supset, l)$ *are derivable. If* **Sc** *is strongly sound and complete for* **L** *then first-order* $\mathbf{HSc} + (com)$ *is an analytic calculus sound and complete for first-order* $\mathbf{L} + (lin) + (\vee\forall)$*.*

5 SMTL: a case study

As an easy corollary of the transfer principle introduced above, we define here an analytic calculus for Strict Monoidal T-norm based Logic **SMTL**. This logic was defined in [9] by adding axioms $((A \supset \bot) \wedge A) \supset \bot$ and (lin) to $\mathbf{FL_{ew}}$. **SMTL** turns out to be the logic based on left-continuous t-norms satisfying the pseudo-complementation property. To the best of our knowledge no analytic calculi have been provided for **SMTL** so far.

Proposition 1. $\mathbf{ScFL_{ew}} + (wc)$ *is strongly sound and complete for* $\mathbf{FL_{ew}}$ *extended with* $((A \supset \bot) \wedge A) \supset \bot$*.*

Proof. (*Soundness*) $\vdash_{\mathbf{ScFL_{ew}}} \mathcal{I}(\Gamma, A, A \Rightarrow), ((A \supset \bot) \wedge A) \supset \bot \Rightarrow \mathcal{I}(\Gamma, A \Rightarrow)$. Hence the claim follows by the strongly soundness of $\mathbf{ScFL_{ew}}$ w.r.t. $\mathbf{FL_{ew}}$ ([13]) and axiom $((A \supset \bot) \wedge A) \supset \bot$.

(*Completeness*) By the completeness of $\mathbf{ScFL_{ew}}$ w.r.t. $\mathbf{FL_{ew}}$ it is enough to check that $\vdash_{\mathbf{ScFL_{ew}}+(wc)} ((A \supset \bot) \wedge A) \supset \bot$. This is straightforward.

Corollary 4. *The hypersequent version of* $\mathbf{ScFL_{ew}}$ + (wc) *with in addition* (com) *is an analytic calculus for* **SMTL**.

Proof. $\mathbf{ScFL_{ew}}$ + (wc) is a standard sequent calculus in which the rules $(\supset, r), (\vee, r)_{1,2}, (\vee, l)$ and $(\supset, l)$ are derivable. Moreover its rules are reductive and substitutive. The claim follows by Proposition 1 and Corollary 2.

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