Proof theory of witnessed Gödel logic: a negative result*

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Abstract

We introduce a first sequent-style calculus for witnessed Gödel logic. Our calculus makes use of the cut rule. We show that this is inescapable by establishing a general result on the non-existence of suitable analytic calculi for a large class of first-order logics. These include witnessed Gödel logic, (fragments of) Łukasiewicz logic, and intuitionistic logic extended with the quantifiers of classical logic.

1 Introduction

(First-order) Gödel logic is a prominent example of both a many-valued and a super-intuitionistic logic. The importance of Gödel logic is emphasized by the fact that it turns up naturally in a number of different contexts; among them relevance logics, fuzzy logic, and logic programming. Witnessed Gödel logic $\mathbf{G}^w$ arises from Gödel logic by interpreting the quantifier $\forall$ (resp. $\exists$) as minimum (resp. maximum) instead of infimum (resp. supremum). As computers have a limited precision, and reasoning w.r.t. general models is typically harder than reasoning w.r.t. witnessed models [19, 20], $\mathbf{G}^w$ is more appealing than Gödel logic for many applications, see, e.g., [12].

The proof theory of Gödel logic has been well-investigated. As a result various "well-behaved" (or *analytic*) *calculi* –in which proofs proceed by stepwise decomposition of the formulas to be proved– have been introduced, and successfully used to prove important properties of this logic; e.g., the Herbrand theorem for prenex formulas or the admissibility of suitable rules. These calculi are defined in formalisms that generalize sequent calculus; among them hypersequent and labelled calculus [2, 21, 22].

In contrast to Gödel logic, the only existing calculus for $\mathbf{G}^w$ is a Hilbert system [20, 3], which is difficult to use even for finding simple proofs manually. The question that we address is: can we define a "well-behaved" calculus for $\mathbf{G}^w$ using the sequent calculus or a suitable generalization?

To answer this question, we first define what we mean by a "well-behaved" calculus. There are indeed many formalisms for constructing calculi and outside of the

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sequent calculus there is no established notion of what is meant by a "well-behaved", or analytic calculus[1]; for example proofs in the Calculus of Structures [17, 13] or display logic [10] might contain logical or structural connectives that do not appear in the formulas to be proved and are not universally considered "well-behaved", e.g. [23].

In this paper we propose an operational and formalism-independent notion of "well-behaved" calculus ($\exists$-*analytic*). This arises as an attempt to answer the fundamental question *what do we expect from a proof theoretic treatment of a first-order logic*?. Our answer is: a relaxed notion of the subformula property and the extraction of minimal information from proofs of simple existential statements, i.e., a weak form of Herbrand theorem. We call $\exists$-analytic any calculus satisfying these properties. The calculi for Gödel logic in [2, 21, 22] are $\exists$-analytic and so are all calculi in the various formalisms that are presented as "well-behaved" (or called analytic).

Having provided a precise definition of the notion of $\exists$-analyticity, we show that a large class of logics, including $\mathbf{G}^w$, cannot have such calculi.

The paper is organized as follows: Section 3 introduces a first sequent-style calculus $\mathbf{R}\mathrm{G}^{fo}_\infty$ for $\mathbf{G}^w$. Our calculus, defined in the formalism of sequents of relations, is obtained by extending the calculus for propositional Gödel logic in [4] by natural quantifier rules. Though finding proofs in $\mathbf{R}\mathrm{G}^{fo}_\infty$ turns out to be easier than in the Hilbert system for $\mathbf{G}^w$, $\mathbf{R}\mathrm{G}^{fo}_\infty$ makes use of the cut rule; as shown in Section 4, this is the case, e.g., when proving prenex formulas that are valid in $\mathbf{G}^w$ but not in Gödel logic; in the same section it is also shown the undecidability of the problem to determine, given a proof in $\mathbf{R}\mathrm{G}^{fo}_\infty$, if there is a cut-free proof of the same formulas. Section 5 shows that the non redundancy of the cut rule in $\mathbf{R}\mathrm{G}^{fo}_\infty$ is unavoidable by introducing a general criterion for a first-order logic expressed Hilbert-style not to admit any $\exists$-analytic calculus. Our criterion applies to a large class of logics including $\mathbf{G}^w$, the fragment[2] of Łukasiewicz logic axiomatized in [18], and intuitionistic logic extended with the quantifiers of classical logic, thus establishing the non-existence of a "well-behaved" calculus for them.

2 Witnessed Gödel logic

The language of Gödel logic is the same as that of intuitionistic first-order logic and it is based on the binary *connectives* $\wedge, \vee$, and $\supset$, the quantifiers $\exists, \forall$, and the *truth constants* 0 and 1; as usual $\neg A$ abbreviates $A \supset 0$.

Semantically Gödel logic can[3] be viewed as an infinite-valued logic with the real interval $[0, 1]$ as set of truth values, see, e.g., [15]. In this setting an *interpretation* $\mathcal{I}$ consists of a non-empty domain D and a valuation $v_\mathcal{I}$ that maps constant symbols and object variables into elements of D and n-ary function symbols to functions from D^n into D; $v_\mathcal{I}$ extends in the usual way to a function mapping all terms of the language to an element of the domain. Moreover, every n-ary predicate symbol p is mapped to a function $v_\mathcal{I}(p)$ of type $D^n \mapsto [0, 1]$. The truth value of an atomic for-

[1]Even in the sequent calculus this notion is controversial (cf. the dispute cut-free vs calculi with analytic cuts [25]).

[2]This fragment is often called Łukasiewicz logic or general Łukasiewicz logic, see [15].

[3]An alternative semantics is provided by the class of linearly ordered Kripke models with constant domains.

mula $p(t_1, \dots, t_n)$ is defined as

$$\|p(t_1, \dots, t_n)\|_{\mathcal{I}} = v_{\mathcal{I}}(p(v_{\mathcal{I}}(t_1), \dots, v_{\mathcal{I}}(t_n))).$$

The semantics of propositional connectives and truth constants is given by

$$\|A \supset B\|_{\mathcal{I}} = \begin{cases} 1 & \text{if } \|A\|_{\mathcal{I}} \leq \|B\|_{\mathcal{I}} \\ \|B\|_{\mathcal{I}} & \text{otherwise.} \end{cases} \qquad \|0\|_{\mathcal{I}} = 0 \quad \text{and} \quad \|1\|_{\mathcal{I}} = 1$$

$$\|A \wedge B\|_{\mathcal{I}} = \min(\|A\|_{\mathcal{I}}, \|B\|_{\mathcal{I}}), \qquad \|A \vee B\|_{\mathcal{I}} = \max(\|A\|_{\mathcal{I}}, \|B\|_{\mathcal{I}}),$$

For quantification we define the *distribution* of a formula A with respect to a free variable x in an interpretation $\mathcal{I}$ as $\mathrm{distr}_{\mathcal{I}}(A(x)) = \{\|A(x)\|_{\mathcal{I}[d/x]} \mid d \in D\}$, where $\mathcal{I}[d/x]$ denotes the interpretation that is exactly as $\mathcal{I}$, except for insisting on $v_{\mathcal{I}[d/x]}(x) = d$. The universal and existential quantifiers correspond to the infimum and supremum, respectively, in the following sense:

$$\|\forall x A(x)\|_{\mathcal{I}} = \inf \mathrm{distr}_{\mathcal{I}}(A(x)) \qquad \|\exists x A(x)\|_{\mathcal{I}} = \sup \mathrm{distr}_{\mathcal{I}}(A(x)).$$

As usual a formula is valid if it is evaluated to 1 under every interpretation.

Remark 1 Taking different subsets V of $[0, 1]$ closed under infima and suprema and containing both 0 and 1 as truth values give rise to different sets of valid formulas, that is they lead to different Gödel logics. As shown in [9], these sets are recursively axiomatizable if and only if V is finite, or it is either order isomorphic to $[0, 1]$ or to $\{0\} \cup [\frac{1}{2}, 1]$. For example, taking $V = \{0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1\}$ as set of truth values leads to the $n + 1$-valued Gödel logic G_n, which is recursively axiomatizable. The Gödel logic whose valid formulas are those valid in all G_n, for $n \in \mathcal{N}$, is denoted by $\mathbf{G}_\uparrow$ and has $V = \{1 - 1/n : n \geq 1\} \cup \{1\}$. $\mathbf{G}_\uparrow$ is not recursively axiomatizable. First-order Gödel logic with $V = [0, 1]$ is usually referred to as standard Gödel logic (or simply Gödel logic) and denoted by G_∞.

A Hilbert axiomatization of (standard) Gödel logic is obtained by extending that of first-order intuitionistic logic with the axiom of linearity $(P \supset Q) \vee (Q \supset P)$ and the "quantifier shift" axiom $\forall x(P(x) \vee Q^{(x)}) \supset (\forall x P(x)) \vee Q^{(x)}$, where the notation $Q^{(x)}$ indicates that there is no free occurrence of x in Q.

Witnessed Gödel logic $\mathbf{G}^w$ is a natural variant of Gödel logic. It arises by considering only witnessed interpretations, that is interpretations where the truth value of each quantified formula coincides with the truth value of some if its instances, see [20]. In our notation this means an interpretation $v_{\mathcal{I}}$ such that:

$$\|\forall x A(x)\|_{\mathcal{I}} = \min \mathrm{distr}_{\mathcal{I}}(A(x)) \qquad \|\exists x A(x)\|_{\mathcal{I}} = \max \mathrm{distr}_{\mathcal{I}}(A(x)).$$

$\mathbf{G}^w$ was axiomatized in [20, 3] by adding to the Hilbert system for Gödel logic both axioms

$$\exists x(P(x) \supset \forall y P(y)) \quad \text{and} \quad \exists x(\exists y P(y) \supset P(x))$$

expressing that each infimum is a minimum, and each supremum is a maximum, respectively.

Like in intuitionistic logic, also in Gödel logic quantifiers cannot be shifted arbitrarily. In other words, in general, arbitrary formulas are not equivalent to prenex formulas (that is in which quantifiers are always in front). In contrast, $\mathbf{G}^w$ admits an equivalent prenex normal form as in classical logic.

Finally, recall that Skolemization holds for the prenex fragment of Gödel logic [9, 7] and it can be easily proved for witnessed Gödel logic.

3 The calculus RG_∞^{fo}

We introduce a sequent-style calculus for witnessed Gödel logic. Our calculus uses sequents of relations, which are disjunctions of semantic predicates over formulas [4].

Various analytic calculi have been defined for Gödel logic at the propositional level. Among them, Avron's hypersequent calculus [1, 2] and the sequent calculus of relations $\mathbf{R}\mathrm{G}_\infty$ [4]. The basic objects in these calculi have the form

$$S_1 \mid \ldots \mid S_n,$$

where the symbol " | " is interpreted as a (commutative) disjunction at the meta-level and

- (in the hypersequent calculus) each S_i is a standard LJ sequent $\Gamma \Rightarrow \Delta$ where Γ is a multiset of formulas and Δ either a formula or the empty set, while
- (in the sequent calculus of relations) each S_i is a relation $A_i \lhd B_i$, where $\lhd$ stands for either $<$ or $\leq$ and A_i, B_i are formulas.

By adding to Avron's hypersequent calculus the natural hypersequent rules for quantifiers we get the analytic calculus $H\mathrm{G}_\infty$ for first-order Gödel logic depicted[4] in Table 1.

Proposition 2 (e.g. [2, 21]) A formula P is valid in Gödel logic if and only if $\Rightarrow P$ is provable in $H\mathrm{G}_\infty$ without using the cut rule.

Clearly if $\Rightarrow P$ is provable in $H\mathrm{G}_\infty$ then P is valid in $\mathbf{G}^w$. The converse however does not hold. In particular the peculiar axioms $\exists x(P(x) \supset \forall y P(y))$ and $\exists x(\exists y P(y) \supset P(x))$ of witnessed Gödel logic cannot be proved in $H\mathrm{G}_\infty$.

To explain the situation for $\mathbf{R}\mathrm{G}_\infty$ let us first recall its rules from [4] (below H and H' stand for a possibly empty sequent of relations):

Logical rules: For disjunction and conjunction we have (here and below $\lhd$ stands for either $<$ or $\leq$, uniformly in each rule):

$$\frac{\gamma \lhd \alpha \mid H \qquad \gamma \lhd \beta \mid H}{\gamma \lhd (\alpha \wedge \beta) \mid H}\ (\wedge\colon \lhd\colon r) \qquad \frac{\alpha \lhd \gamma \mid \beta \lhd \gamma \mid H}{(\alpha \wedge \beta) \lhd \gamma \mid H}\ (\wedge\colon \lhd\colon l)$$

$$\frac{\gamma \lhd \alpha \mid \gamma \lhd \beta \mid H}{\gamma \lhd (\alpha \vee \beta) \mid H}\ (\vee\colon \lhd\colon r) \qquad \frac{\alpha \lhd \gamma \mid H \qquad \beta \lhd \gamma \mid H}{(\alpha \vee \beta) \lhd \gamma \mid H}\ (\vee\colon \lhd\colon l)$$

[4] Axioms and rules there are schemata.

$$\alpha \Rightarrow \alpha \qquad 0 \Rightarrow \alpha \qquad \frac{G \mid \Gamma \Rightarrow \Delta}{G \mid \Gamma, \alpha \Rightarrow \Delta}\ (w,l) \qquad \frac{G \mid \Gamma, \alpha, \alpha \Rightarrow \Delta}{G \mid \Gamma, \alpha \Rightarrow \Delta}\ (c,l)$$

$$\frac{G \mid \Gamma, \Gamma' \Rightarrow \Delta \quad G \mid \Gamma_1, \Gamma_1' \Rightarrow \Delta'}{G \mid \Gamma, \Gamma_1' \Rightarrow \Delta \mid \Gamma', \Gamma_1 \Rightarrow \Delta'}\ (com) \qquad \frac{G}{G \mid \Gamma \Rightarrow \Delta}\ (ew) \qquad \frac{G \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{G \mid \Gamma \Rightarrow \Delta}\ (ec)$$

$$\frac{G \mid \Gamma \Rightarrow \alpha \quad G \mid \beta, \Gamma \Rightarrow \Delta}{G \mid \Gamma, \alpha \supset \beta \Rightarrow \Delta}\ (\supset, l) \qquad \frac{G \mid \Gamma, \alpha \Rightarrow \beta}{G \mid \Gamma \Rightarrow \alpha \supset \beta}\ (\supset, r) \qquad \frac{G \mid \Gamma \Rightarrow \alpha_i}{G \mid \Gamma \Rightarrow \alpha_1 \vee \alpha_2}\ (\vee_i, r)$$

$$\frac{G \mid \Gamma, \alpha \Rightarrow \Delta \quad G \mid \Gamma, \beta \Rightarrow \Delta}{G \mid \Gamma, \alpha \vee \beta \Rightarrow \Delta}\ (\vee, l) \qquad \frac{G \mid \Gamma, \alpha_i \Rightarrow \Delta}{G \mid \Gamma, \alpha_1 \wedge \alpha_2 \Rightarrow \Delta}\ (\wedge_i, l) \qquad \frac{G \mid \alpha(t), \Gamma \Rightarrow \Delta}{G \mid (\forall x)\alpha(x), \Gamma \Rightarrow \Delta}\ (\forall, l)$$

$$\frac{G \mid \Gamma \Rightarrow \alpha \quad G \mid \Gamma \Rightarrow \beta}{G \mid \Gamma \Rightarrow \alpha \wedge \beta}\ (\wedge, r) \qquad \frac{G \mid \Gamma \Rightarrow \alpha(a)}{G \mid \Gamma \Rightarrow (\forall x)\alpha(x)}\ (\forall, r) \qquad \frac{G \mid \alpha(a), \Gamma \Rightarrow \Delta}{G \mid (\exists x)\alpha(x), \Gamma \Rightarrow \Delta}\ (\exists, l)$$

$$\frac{G \mid \Gamma \Rightarrow \alpha \quad G \mid \Sigma, \alpha \Rightarrow \Delta}{G \mid \Gamma, \Sigma \Rightarrow \Delta}\ (CUT) \qquad \frac{G \mid \Gamma \Rightarrow \alpha(t)}{G \mid \Gamma \Rightarrow (\exists x)\alpha(x)}\ (\exists, r) \qquad \frac{G \mid \Gamma' \Rightarrow \Delta' \mid \Gamma \Rightarrow \Delta \mid G'}{G \mid \Gamma \Rightarrow \Delta \mid \Gamma' \Rightarrow \Delta' \mid G'}\ (ee)$$

a does not occur in G, Γ, α $\qquad$ $i = 1, 2$

Table 1: The calculus $H\mathrm{G}_\infty$ for first-order Gödel logic

The rules for implication are:

$$\frac{\alpha \leq \beta \mid \gamma < \beta \mid H \qquad \gamma < 1 \mid H}{\gamma < (\alpha \supset \beta) \mid H}\ (\supset\colon\ <\colon r) \quad \frac{\beta < \alpha \mid H \qquad \beta < \gamma \mid H}{(\alpha \supset \beta) < \gamma \mid H}\ (\supset\colon\ <\colon l)$$

$$\frac{\alpha \leq \beta \mid \gamma \leq \beta \mid H}{\gamma \leq (\alpha \supset \beta) \mid H}\ (\supset\colon\ \leq\colon r) \quad \frac{1 \leq \gamma \mid \beta < \alpha \mid H \qquad \beta \leq \gamma \mid H}{(\alpha \supset \beta) \leq \gamma \mid H}\ (\supset\colon\ \leq\colon l)$$

Structural rules: (external) weakening, exchange, contraction, and (cut):

$$\frac{H}{\alpha \lhd \beta \mid H}\ \text{(EW)} \qquad \frac{H \mid \alpha' \lhd \beta' \mid \alpha \lhd \beta \mid H'}{H \mid \alpha \lhd \beta \mid \alpha' \lhd \beta' \mid H'}\ \text{(EE)}$$

$$\frac{\alpha \lhd \beta \mid \alpha \lhd \beta \mid H}{\alpha \lhd \beta \mid H}\ \text{(EC)} \qquad \frac{H \mid \alpha \leq \beta \quad H \mid \beta < \alpha}{H}\ (cut)$$

A derivation is considered, as usual, as an upward rooted tree of sequents generated from subtrees by applying the inference rules. A sequent of relations is provable in $\mathbf{R}\mathrm{G}_\infty$ if the leafs in its derivation are **axioms**, that is sequents of relations having one of the following forms:

(a) $P_1 \lhd_n P_n \mid P_n \lhd_{n-1} P_{n-1} \mid \ldots \mid P_3 \lhd_2 P_2 \mid P_2 \leq P_1$, where $\lhd_i \in \{<, \leq\}$ and the case $n = 1$ is defined as $P_1 \leq P_1$,

(b) $P_n \leq P_{n-1} \mid P_{n-1} < P_{n-2} \mid \ldots \mid P_1 < 1$, (the case $n = 1$ is $P_1 \leq 1$),

(c) $0 < P_n \mid \ldots \mid P_3 < P_2 \mid P_2 \leq P_1$, (the case $n = 1$ is $0 \leq P_1$),

(d) $0 < P_1 \mid P_1 < P_2 \mid \ldots \mid P_n < 1$, (the case $n = 0$ is $0 < 1$).

Proposition 3 ([4]) A formula P is valid in propositional Gödel logic if and only if $1 \leq P$ is provable in $\mathbf{R}\mathrm{G}_\infty$ without using (cut).

Let us denote by $\mathbf{R}\mathrm{G}_\infty^{fo}$ the calculus $\mathbf{R}\mathrm{G}_\infty$ extended with the natural sequent of relations rules for quantifiers ($\lhd$ stands for either $<$ or $\leq$):

$$\frac{H \mid \alpha(t) \lhd \beta}{H \mid \forall x \alpha(x) \lhd \beta}\ (\forall : \lhd : l) \qquad \frac{H \mid \beta \lhd \alpha(e)}{H \mid \beta \lhd \forall x \alpha(x)}\ (\forall : \lhd : r)$$

$$\frac{H \mid \alpha(e) \lhd \beta}{H \mid \exists x \alpha(x) \lhd \beta}\ (\exists : \lhd : l) \qquad \frac{H \mid \beta \lhd \alpha(t)}{H \mid \beta \lhd \exists x \alpha(x)}\ (\exists : \lhd : r)$$

where e is an eigenvariable, i.e., it does not appear in H, α and β.

Definition 4 Let $S := A_1 \lhd_1 B_1 \mid \cdots \mid A_n \lhd_n B_n$ be a sequent of relations, where $\lhd$ is either $<$ or $\leq$. We write $\models_{\mathbf{G}^w} S$ if for each interpretation $v_\mathcal{I}$ of $\mathbf{G}^w$, there is a $i \in \{1, \ldots, n\}$ such that $\|A_i\|_\mathcal{I} \lhd_i \|B_i\|_\mathcal{I}$. We say that a $\mathbf{R}\mathrm{G}_\infty^{fo}$ rule

$$\frac{S_0}{S} \quad \text{or} \quad \frac{S_1 \quad S_2}{S}$$

is *sound* for $\mathbf{G}^w$ if whenever $\models_{\mathbf{G}^w} S_j$ ($j = 0$ or $j = 1, 2$) then $\models_{\mathbf{G}^w} S$.

The soundness of a sequent of relations rule for Gödel logic is defined analogously.

Proposition 5 (Soundness and Completeness) A formula P is valid in witnessed Gödel logic if and only if $1 \leq P$ is provable in $\mathbf{R}\mathrm{G}_\infty^{fo}$.

Proof ($\Longleftarrow$) Follows from Proposition 3 and the soundness of the quantifier rules for witnessed Gödel logic (easy check).

($\Longrightarrow$) Follows from the derivability in $\mathbf{R}\mathrm{G}_\infty^{fo}$ of all the axioms of the Hilbert calculus for $\mathbf{G}^w$. This is easy for the axioms of first-order Gödel logic. A proof of $1 \leq \exists x(P(x) \supset \forall y P(y))$ is constructed by cutting the (provable) sequents of relations $1 \leq \forall y P(y) \supset \forall y P(y)$ and $1 \leq \exists x(P(x) \supset \forall y P(y)) \mid \forall y P(y) \supset \forall y P(y) < 1$. A derivation of the latter sequent of relations is the following ($\ldots$ abbreviates a component left unchanged by a rule application):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(a) \leq \forall y P(y) \mid \forall y P(y) < P(a)}{1 \leq \forall y P(y) \mid P(a) \leq \forall y P(y) \mid \forall y P(y) < P(a)}\,(EW)}{1 \leq P(a) \supset \forall y P(y) \mid \forall y P(y) < P(a)}\,(\supset{:}\leq{:}r)}{1 \leq \exists x(P(x) \supset \forall y P(y)) \mid \forall y P(y) < P(a)}\,(\exists{:}\leq{:}\mathrm{r})}{\ldots \mid \forall y P(y) < \forall y P(y)}\,(\forall{:}<{:}\mathrm{r}) \quad \dfrac{\dfrac{\dfrac{1 \leq \forall y P(y) \mid \forall y P(y) < 1}{1 \leq \forall y P(y) \mid P(a) \leq \forall y P(y) \mid \ldots}\,(EW)}{1 \leq P(a) \supset \forall y P(y) \mid \forall y P(y) < 1}\,(\supset{:}\leq{:}\mathrm{r})}{\ldots \mid \forall y P(y) < 1}\,(\exists{:}\leq{:}\mathrm{r})}{1 \leq \exists x(P(x) \supset \forall y P(y)) \mid \forall y P(y) \supset \forall y P(y) < 1}\,(\supset{:}<{:}\mathrm{l})$$

A derivation in $\mathbf{R}\mathrm{G}_\infty^{fo}$ of $1 \leq \exists x(\exists y P(y) \supset P(x))$ is obtained in a similar way by cutting $1 \leq \exists x(\exists y P(y) \supset P(x)) \mid \exists y P(y) \supset \exists y P(y) < 1$ and $1 \leq \exists y P(y) \supset \exists y P(y)$. The generalization rule is simulated by $(\forall :\leq: r)$, while modus ponens

$$\frac{B \vdash C \quad B}{C}$$

is simulated by (cut) as follows ($C < B \mid B < 1 \mid 1 \leq C$ is an axiom):

$$\dfrac{\dfrac{C < B \mid B < 1 \mid 1 \leq C \qquad B \leq C}{B < 1 \mid 1 \leq C}\,(\mathrm{cut}) \qquad 1 \leq B}{1 \leq C}\,(\mathrm{cut})$$

Note that among the quantifier rules of $\mathbf{R}\mathrm{G}_\infty^{fo}$, only $(\exists :<: l)$ and $(\forall :<: r)$ are not sound for Gödel logic. Indeed the value of a formula $\forall x P(x)$ under an interpretation $v_\mathcal{I}$ of Gödel logic is the infimum of the values of $P(a)$, for all elements a of the universe and therefore we might have $v_\mathcal{I}(Q) < v_\mathcal{I}(P(a))$ for all a while $v_\mathcal{I}(Q) \leq v_\mathcal{I}(\forall x P(x))$. (Dually for the rule $(\exists :<: l)$ and supremum).

Proposition 6 A formula P is valid in Gödel logic if and only if $1 \leq P$ is provable in $\mathbf{R}\mathrm{G}_\infty^{fo}$ without the rules $(\exists :<: l)$ and $(\forall :<: r)$.

Proof Soundness is easy. Completeness is shown by deriving in the calculus all the axioms of the Hilbert system for Gödel logic (easy check).

It is not difficult to see that in the calculus $\mathbf{R}\mathrm{G}_\infty^{fo}$ without the rules $(\exists :<: l)$ and $(\forall :<: r)$ the cut rule cannot be removed. Indeed a cut-free sequent of relations calculus for Gödel logic is not known, and we conjecture that it cannot be defined as the cut-elimination theorem for $\mathbf{R}\mathrm{G}_\infty$ strongly relies on the invertibility of all logical rules (see [5]).

4 The analytic content of $\mathrm{R}\mathrm{G}_\infty^{fo}$

We have defined a sequent-style calculus for $\mathbf{G}^w$. Its completeness however relies on the use of (cut). Is this rule really needed? This section provides a positive answer to this question. We indeed show that a prenex formula of $\mathbf{G}^w$ is cut-free provable in $\mathbf{R}\mathrm{G}_\infty^{fo}$ if and only if it is already valid in Gödel logic (Theorem 7). The non-admissibility of the cut rule in $\mathbf{R}\mathrm{G}_\infty^{fo}$ easily follows by the existence of prenex formulas that are valid in $\mathbf{G}^w$ and not in Gödel logic.

As a corollary of Theorem 7 and of the undecidability of the prenex fragment of Gödel logic (Lemma 10) we also show the undecidability of the problem of determining, given a proof in $\mathbf{R}\mathrm{G}_\infty^{fo}$, if there exists a cut-free proof of the same end sequent.

To prove the key theorem below we will use the following results from [7]: Let F be a prenex formula valid in Gödel logic and let $\exists \overline{x} F^S(\overline{x})$ be its Skolem form ($\overline{x}$ abbreviates $x_1, \ldots, x_k$)

(a) Any cut-free proof of $\Rightarrow F$ in the hypersequent calculus $H\mathrm{G}_\infty$ can be stepwise transformed into a cut-free proof containing a quantifier-free hypersequent (called mid-hypersequent) such that below this hypersequent only quantifier rules and (ec) are applied.

(b) Any cut-free proof of $\Rightarrow \exists \overline{x} F^S(\overline{x})$ in $H\mathrm{G}_\infty$ can be stepwise transformed into a cut-free proof of $\Rightarrow F$ in $H\mathrm{G}_\infty$.

Theorem 7 Let F be any prenex formula of $\mathbf{G}^w$. $1 \leq F$ is cut-free provable in $\mathbf{R}\mathrm{G}_\infty^{fo}$ if and only if F is valid in first-order Gödel logic.

Proof ($\Longleftarrow$) By Proposition 2 if F is valid in Gödel logic then there is a cut-free proof d' of $\Rightarrow F$ in the hypersequent calculus $H\mathrm{G}_\infty$. The claim follows by translating this proof into a cut-free proof of $1 \leq F$ in $\mathbf{R}\mathrm{G}_\infty^{fo}$. A methodology to translate cut-free $H\mathrm{G}_\infty$-proofs of *propositional* formulas into cut-free proofs in $\mathbf{R}\mathrm{G}_\infty$ is contained in [6]. The key idea of the translation is to replace in hypersequents the symbol $\Rightarrow$ with $\leq$ and use an equivalent version of $H\mathrm{G}_\infty$ in which (i) hypersequents $A_1, \dots A_n \Rightarrow B_1 \mid \dots$ are fully split, that is they are of the form $A_1 \leq B_1 \mid \dots\dots \mid A_n \leq B_1 \mid \dots$, (ii) the $(\rightarrow, l)$ rule is replaced by

$$\frac{D \leq A \mid H \qquad B \leq C \mid H}{(A \supset B) \leq C \mid D \leq C \mid H}\ (\supset\!:\ \leq\!: l)^*$$

and (iii) the communication rule (com) (cf. Table 1) has the following form

$$\frac{A_1 \leq U \mid \dots \mid A_n \leq U \mid H \qquad B_1 \leq V \mid \dots \mid B_m \leq V \mid H}{A_1 \leq V \mid \dots \mid A_n \leq V \mid B_1 \leq U \mid \dots \mid B_m \leq U \mid H}\ \text{(rcom)}$$

The missing step between the propositional translation sketched above and the proof of $\Rightarrow F$ is the mid-hypersequent theorem of [7] (i.e. the result (a) above) which allows us to transform the proof d' of $\Rightarrow F$ separating the quantifier and the propositional part. We can then translate the propositional part of the proof (that is the proof of the mid-hypersequent) into a proof in $\mathbf{R}\mathrm{G}_\infty$ following the method in [6]. The proof of $1 \leq F$ in $\mathbf{R}\mathrm{G}_\infty^{fo}$ then follows by final applications of the quantifier rules for $\leq$ and (EC), if needed.

($\Longrightarrow$) First recall that a prenex formula $F := \mathsf{Q}_1 y_1 \dots \mathsf{Q}_n y_n P$, with $\mathsf{Q}_i \in \{\forall, \exists\}$ is valid in $\mathbf{G}^w$ if and only if so is its Skolem form $\exists\overline{x}F^S(\overline{x})$. By assumption $\exists\overline{x}F^S(\overline{x})$ is cut-free provable in $\mathbf{R}\mathrm{G}_\infty^{fo}$. We show how to construct a proof of $\Rightarrow F$ in $H\mathrm{G}_\infty$, and hence, by Proposition 2 that F is valid in Gödel logic.

We first transform the cut-free proof d of $1 \leq \exists\overline{x}F^S(\overline{x})$ in $\mathbf{R}\mathrm{G}_\infty^{fo}$ into a proof in $\mathbf{R}\mathrm{G}_\infty^{fo}$ of (the propositional formula) $1 \leq \bigvee_{i=1}^n F^S(\overline{t_i})$ for some $t_1, \dots t_n$. Indeed, let $t_1, \dots t_n$ be the terms that appear in d. A derivation of $1 \leq \bigvee_{i=1}^n F^S(\overline{t_i})$ is obtained by replacing everywhere in d $\exists\overline{x}F^S(\overline{x})$ with $\bigvee_{i=1}^n F^S(\overline{t_i})$ and by replacing all applications of $(\exists :\leq: r)$ with suitable applications of $(\vee :\leq: r)$. Note that $\bigvee_{i=1}^n F^S(\overline{t_i})$ is a propositional formula and it is therefore valid in first-order Gödel logic (recall that the propositional fragments of $\mathbf{G}^w$ and of Gödel logic do coincide). By Proposition 2 there is a proof of $\Rightarrow \bigvee_{i=1}^n F^S(\overline{t_i})$ in the hypersequent calculus $H\mathrm{G}_\infty$, from which we can construct a proof of $\Rightarrow \exists\overline{x}F^S(\overline{x})$ in $H\mathrm{G}_\infty$ as follows: by applying (CUT) n times between $\Rightarrow \bigvee_{i=1}^n F^S(\overline{t_i})$ and the provable[5] hypersequent $\bigvee_{i=1}^n F^S(\overline{t_i}) \Rightarrow F^S(\overline{t_1}) \mid \dots \mid \bigvee_{i=1}^n F^S(\overline{t_i}) \Rightarrow F^S(\overline{t_n})$ we get a proof of the hypersequent

$$\Rightarrow F^S(\overline{t_1}) \mid \dots \mid \Rightarrow F^S(\overline{t_n}),$$

that derives $\Rightarrow \exists\overline{x}F^S(\overline{x})$ by applying $(\exists, r)$ to each component, followed by $n-1$ applications of (ec). As $H\mathrm{G}_\infty$ admits cut-elimination, the introduced applications of (CUT) can be removed. A proof of $F := \mathsf{Q}_1 y_1 \dots \mathsf{Q}_n y_n P$ in $H\mathrm{G}_\infty$ can then be obtained by re-introducing the Skolemized quantifiers in the proof of $\Rightarrow \exists\overline{x}F^S(\overline{x})$, as shown in [7] (see the result (b) above).

[5] A proof in $H\mathrm{G}_\infty$ consists of multiple applications of the rules (com), (ew), (ec), and $(\vee, l)$.

Corollary 8 The (cut) rule is not redundant in the calculus $\mathbf{R}\mathrm{G}_{\infty}^{fo}$.

Proof The prenex formula $S := \exists x \forall y (P(x) \supset P(y))$ is valid in $\mathbf{G}^w$ while it is not in Gödel logic. An interpretation $v_{\mathcal{I}}$ of Gödel logic such that $v_{\mathcal{I}}(S) < 1$ is indeed the following: take $D = \{u_i \mid i \in \mathcal{N} \setminus \{0\}\}$ as its universe and assign $v_{\mathcal{I}}(P(u_i)) = \frac{1}{i}$.

Corollary 9 Let F be a prenex formula valid in $\mathbf{G}^w$. The problem of determining if there is a cut-free proof of $1 \leq F$ in $\mathbf{R}\mathrm{G}_{\infty}^{fo}$ is undecidable.

Proof Let F be $\mathsf{Q}_1 x_1 \ldots \mathsf{Q}_n x_n F'$, with $\mathsf{Q}_i \in \{\exists, \forall\}$ and F' quantifier-free. Consider the prenex formula ($P \notin F'$ and $x, y \neq x_i$, for $i = 1, \ldots, n$)

$$F^p = \exists x \forall y \mathsf{Q}_1 x_1 \ldots \mathsf{Q}_n x_n (F' \vee (P(x) \supset P(y)))$$

obtained by combining F with the formula $S := \exists x \forall y (P(x) \supset P(y))$ which is valid in $\mathbf{G}^w$ and not in Gödel logic G_∞ (cf. the proof of Corollary 8). We show that $1 \leq F^p$ is cut-free provable in $\mathbf{R}\mathrm{G}_{\infty}^{fo}$ if and only if F is valid in G_∞. The claim then follows by the undecidability of the validity problem for the prenex fragment of G_∞ (Lemma 10 below).

($\Longrightarrow$) By Theorem 7 if $1 \leq F^p$ is cut-free provable in $\mathbf{R}\mathrm{G}_{\infty}^{fo}$ then F^p is valid in G_∞. Assuming by contradiction that F is not valid in G_∞, i.e. there is an interpretation v_I in G_∞ such that $v_I(F) < 1$. By [9] we can always assume that the domain D of v_I is infinite, say $D = \{u_i \mid i \in \mathcal{N} \setminus \{0\}\}$. By assigning $v_{\mathcal{I}}(P(u_i)) = \frac{1}{i}$ (note that the predicate $P \notin F'$) we get $v_I(F^p) < 1$, that contradics the validity of F^p in G_∞.

($\Longleftarrow$) If F is valid in G_∞, then F^p is valid in G_∞ and by Theorem 7 there is a cut-free proof of $1 \leq F^p$ in $\mathbf{R}\mathrm{G}_{\infty}^{fo}$.

Lemma 10 The validity of prenex formulas in Gödel logic is undecidable.

Proof Let P be any prenex formula in the language of G_∞, which is the same as that of classical logic. Replace each atomic formula $A_i(\overline{x})$ in P with $\neg\neg A_i(\overline{x})$ and denote the resulting formula by $P^{\neg\neg}$. It is not difficult to see that P is valid in Gödel logic if and only if $P^{\neg\neg}$ is valid in classical logic (the proof is by induction on the complexity of P and uses the fact that, under any G_∞ interpretation, the value of a double negated atom is either 0 or 1). The claim follows by the undecidability of the prenex fragment of classical logic (and the fact that $\neg\neg A_i(\overline{x})$ is classically equivalent to $A_i(\overline{x})$).

5 $\exists$-analytic calculi: a negative result

In the sequent of relations calculus $\mathbf{R}\mathrm{G}_{\infty}^{fo}$, introduced for $\mathbf{G}^w$, the use of (cut) cannot be avoided. Here we show that this is not due to a bad design of our rules but there is a principal obstacle preventing $\mathbf{G}^w$ from having a certain cut-free calculus, no matter which rules or formalism is used. We indeed provide a general sufficient criterion for a first-order logic not to admit a certain analytic calculus. As a corollary of this result follows that witnessed Gödel logic does not admit any such calculus.

Henceforth we identify a logic with the set of its provable formulas. We write $\vdash_{\mathcal{L}} F$ if $\mathcal{L}$ is a logic and $F \in \mathcal{L}$. Moreover by a calculus we mean any system consisting of rules and axioms such that there is a method to determine whether a statement is derivable.

We start by formalizing the notion of analytic calculi we deal with ($\exists$-*analytic calculi*). This arises as an attempt to answer the question *what do we expect from a proof theoretic treatment of a first-order logic*?

Our notion is general enough to include analyticity in most of the known formalisms as particular case, but also allows us to rule out pathological situations. For instance one could at first think of identifying well-behaved calculi (analytic calculi) simply with calculi whose proofs are "analytic in Leibniz's sense", i.e. they consist only of concepts already contained in the results; in this case, however, any logic whose (valid) formulas can be enumerated would admit such an "analytic" calculus, as shown by the example below.

Example 11 Let $\mathcal{L}$ be any logic whose formulas (theorems) can be recursively enumerated. A sequent calculus for $\mathcal{L}$ – which would be "analytic" according to the naive notion above – can be defined by taking contraction as the only inference rule and as axioms all sequents $\Rightarrow F, \ldots, F$ (n times) where $\vdash_{\mathcal{L}} F$ and n is the position of F in the enumeration of all formulas of $\mathcal{L}$.

Definition 12 A calculus $\mathcal{C}$ is $\exists$-*analytic* if each provable existential formula $\exists x B(x)$ (B quantifier-free) has a derivation d in $\mathcal{C}$ satisfying the following conditions:

1. d does not contain any universal quantifier and all predicates in d already appear in B;
2. there is a derivation of $\bigvee_{i=1}^{n} B(t_i)$ in $\mathcal{C}$, where $t_1, \ldots, t_n$ are the ground terms appearing in d.

Example 13 Cut-free calculi in display logic or calculus of structures are $\exists$-analytic. Any calculus that needs (non-analytic) cuts is not $\exists$-analytic, due to condition 1 (generalized subformula property). The calculus in Example 11 is also not $\exists$-analytic, due to condition 2.

The hypothesis of the next proposition will be part of the sufficient criterion.

Proposition 14 Let $\mathcal{L}$ be any first-order logic whose language contains the connectives $\supset$, $\vee$, the quantifiers $\forall, \exists$, and in which the following properties hold:

(a) $\vdash_{\mathcal{L}} A \supset A$, for any formula A in the language of $\mathcal{L}$

(b) $\vdash_{\mathcal{L}} P_i$ for the following quantifier shifting laws P_i, $i \in \{1, 2\}$:

- (b.1) $P_1 := (\forall x A(x)) \supset B^{(x)} \supset \exists x (A(x) \supset B^{(x)})$
- (b.2) $P_2 := (B^{(x)} \supset \forall x A(x)) \supset \forall x (B^{(x)} \supset A(x))$

(c) $\vdash_{\mathcal{L}} \forall x A(x) \supset A(t)$, for any term t.

(d) modus ponens and the following rule are in $\mathcal{L}$

$$\frac{\vdash_{\mathcal{L}} A \supset B}{\vdash_{\mathcal{L}} \exists x A \supset \exists x B}$$

Then $\vdash_{\mathcal{L}} \exists x (A(x) \supset A(f(x)))$ for any atomic formula A.

Proof Let A be any atomic formula, by property (a) it follows that $\vdash_{\mathcal{L}} \forall x A(x) \supset \forall x A(x)$ which gives, by modus ponens and (b.1),

$$\vdash_{\mathcal{L}} \exists x(A(x) \supset \forall y A(y)) \tag{1}$$

Applying the rule in property (d) to $\vdash_{\mathcal{L}} (A(x) \supset \forall y A(y)) \supset \forall y(A(x) \supset A(y))$ (coming from (b.2)) we get

$$\vdash_{\mathcal{L}} \exists x(A(x) \supset \forall y A(y)) \supset \exists x \forall y(A(x) \supset A(y)) \tag{2}$$

Modus ponens together with formulas (1) and (2) yield

$$\vdash_{\mathcal{L}} \exists x \forall y(A(x) \supset A(y)) \tag{3}$$

Applying the rule in property (d) to the formula $\vdash_{\mathcal{L}} \forall y(A(x) \supset A(y)) \supset (A(x) \supset A(f(x))$ (coming from (c), here f is a function symbol of the language of $\mathcal{L}$) yields

$$\vdash_{\mathcal{L}} \exists x \forall y(A(x) \supset A(y)) \supset \exists x(A(x) \supset A(f(x))$$

that derives $\vdash_{\mathcal{L}} \exists x(A(x) \supset A(f(x)))$ together with (3) and modus ponens.

Remark 15 The above properties rely on fixed propositional principles: identity axiom, modus ponens and minimal features of quantifiers (note that (b.2) is already valid in intuitionistic logic). The rule in (d) might be replaced by the axiom $\vdash_{\mathcal{L}} A(t) \supset \exists x A(x)$, for any term t and the rule

$$\frac{\vdash_{\mathcal{L}} A \supset B}{\vdash_{\mathcal{L}} \exists x A \supset B}$$

Definition 16 A logic $\mathcal{L}$ satisfies the *infiniteness condition* if there is an atomic formula A in $\mathcal{L}$ such that for no $n > 0$ and for no sequence of ground terms $t_1, \ldots, t_{n+1}$ in the language of $\mathcal{L}$, the formula

$$\bigvee_{i=1}^{n} (A(t_i) \supset A(t_{i+1})) \quad \text{is provable in } \mathcal{L}.$$

Theorem 17 Let $\mathcal{L}$ be any logic which satisfies the conditions of Proposition 14 and in which the infiniteness condition holds. Then $\mathcal{L}$ does not admit any $\exists$-analytic calculus.

Proof By contradiction. Let A be any atomic formula and assume that there is a $\exists$-analytic calculus which is sound and complete for $\mathcal{L}$. By Proposition 14 there is a proof in this calculus of the formula $\exists x(A(x) \supset A(f(x)))$. Being the calculus $\exists$-analytic there is a proof of $\bigvee_{i=1}^{n}(A(t_i) \supset A(f(t_i)))$, for some ground terms $t_1, \ldots, t_n$, which contradicts the infiniteness condition.

Corollary 18 $\mathbf{G}^w$ does not admit any $\exists$-analytic calculus.

Proof Being $\mathbf{G}^w$ an axiomatic extension of intuitionistic logic it clearly satisfies the conditions (a),(b.2),(c),(d) of Proposition 14. It is not hard to see that $1 \leq \forall x A(x) \supset B^{(x)} \supset \exists x(A(x) \supset B^{(x)}) \mid (\forall x A(x)) \supset \forall x A(x) < 1$ is provable in $\mathbf{R}\mathrm{G}_{\infty}^{fo}$. Condition (b.1) then follows by (cut) and Proposition 5.

Finally, for each $n \in \mathcal{N}$ there is an interpretation $v_{\mathcal{I}}$ in $\mathbf{G}^w$ in which $\|\bigvee_{i=1}^{n} A(t_i) \supset A(t_{i+1})\|_{\mathcal{I}} < 1$ (e.g. $\|A(t_i)\|_{\mathcal{I}} = \frac{n-i+1}{n}$). Hence $\mathbf{G}^w$ satisfies the infiniteness condition.

5.1 Applications

We show that many known superintuitionistic and many-valued logics fulfil the sufficient conditions of the previous section. As a corollary of Theorem 17, these logics do not admit any $\exists$-analytic calculus.

5.1.1 Superintuitionistic logics

Superintuitionistic logics are logics including intuitionistic and included in classical logic. They can be described as the set of formulas that are provable in intuitionistic logic extended with suitable axioms or that are valid in certain (classes of) intuitionistic frames, see e.g. [14]. We will now use the latter notion to provide a semantic characterization of the infiniteness condition over intuitionistic frames. Recall that an intuitionistic frame is a pair $\mathfrak{F}_i = \langle W_i, \leqslant \rangle$ where W_i is a non-empty set (of worlds), and $\leqslant$ is a reflexive and transitive (accessibility) relation on W_i. An intuitionistic propositional model (Kripke model) $\mathfrak{M}_i = \langle \mathfrak{F}_i, \Vdash \rangle$ is a frame $\mathfrak{F}_i$ together with a relation $\Vdash$ (called the forcing) between elements of W_i and atomic formulas, satisfying suitable properties (cf. e.g. [14]). Intuitively, $w \Vdash p$ means that the atom p is true at world w. Forcing is assumed to be monotonic with respect to the relation $\leqslant$, namely, if $w \leqslant w'$ and $w \Vdash p$ then also $w' \Vdash p$.

As usual a formula P is valid in a Kripke model $\mathfrak{M}_i = \langle \mathfrak{F}_i, \Vdash \rangle$ if P is forced in all worlds of W_i, and it is falsified if there is a $w \in W_i$ such that w does not force P. Given a set of intuitionistic frames $\mathfrak{F}$ we will refer as the logic of $\mathfrak{F}$ (and denote it by $\mathcal{L}_{\mathfrak{F}}$) to the set of formulas valid in all Kripke models based on all frames $\mathfrak{F}_i$ in $\mathfrak{F}$.

Notice that the infiniteness condition only refers to ground formulas and can therefore be characterized at the propositional level (each predicate $A(t_i)$ in Definition 16 can be seen as an atomic propositional formula A_i). More precisely, consider a language containing infinitely many atoms A_i. $\mathcal{L}_{\mathfrak{F}}$ satisfies the *infiniteness condition* if for no $n > 0$ the formula

$$\bigvee_{i=1}^{n} (A_i \supset A_{i+1}) \quad \text{is valid in all Kripke models based on all frames in } \mathfrak{F}.$$

Lemma 19 The infiniteness condition holds for the logic $\mathcal{L}_{\mathfrak{F}}$ if and only there is a frame $\langle W_i, \leqslant \rangle$ in $\mathfrak{F}$ that satisfies one of the following conditions:

(a) W_i contains infinitely many worlds and the worlds are linearly ordered, or

(b) there are $w, w_1, w_2 \in W_i$ such that $w_1 \not\leq w_2$, $w_2 \not\leq w_1$, $w \leq w_1$ and $w \leq w_2$.

Proof ($\Longleftarrow$) We show that if $\mathfrak{F}_i = \langle W_i, \leqslant \rangle$ satisfies (a) or (b), we can construct a Kripke model $\mathfrak{M}_i = \langle \mathfrak{F}_i, \Vdash \rangle$ such that for no atom $A_1, \ldots, A_n$, $\bigvee_{i=1}^{n} (A_i \supset A_{i+1})$ is valid in $\mathfrak{M}_i$; hence the infiniteness condition holds for $\mathcal{L}_{\mathfrak{F}}$.

Case (a): to falsify $\bigvee_{i=1}^{n} (A_i \supset A_{i+1})$ it is enough to consider a frame in which W_i contains $n+1$ (linearly ordered) worlds and a Kripke model $\mathfrak{M}_i$ in which each world i forces $\{A_1, \ldots A_{i-1}\}$.

Case (b): take any W_i containing two incomparable worlds w_1, w_2 and a Kripke model $\mathfrak{M}_i$ in which $w_1 \Vdash A_i$, for all i odd and $w_2 \Vdash A_j$, for all j even. Hence w falsifies $\bigvee_{i=1}^{n} (A_i \supset A_{i+1})$.

($\Longrightarrow$) By contradiction. If an intuitionistic frames does not satisfy neither (a) nor (b), then it is either a tree consisting of finite number of linearly ordered worlds or a set of trees having the same property. Assume that none of the frames in $\mathfrak{F}$ satisfies (a) or (b). It is easy to find a set of formulas $A_1, \ldots, A_n$ such that $\bigvee_{i=1}^{n}(A_i \supset A_{i+1})$ is valid in all words of all frames in $\mathfrak{F}$ and hence $\mathcal{L}_{\mathfrak{F}}$ does not satisfy the infiniteness condition.

Corollary 20 Let $\mathcal{L}$ be any propositional superintuitionistic logic satisfying the hypothesis of Lemma 19. The first-order logic obtained by extending $\mathcal{L}$ with axioms and rules for quantifiers satisfying conditions (b)-(d) of Proposition 14 does not admit any $\exists$-analytic calculus.

In particular, intuitionistic logic extended with the quantifiers of classical logic (i.e. quantifiers obeying all classical shifting laws) does not admit any $\exists$-analytic calculus.

5.1.2 Many-valued logics

Beside $\mathbf{G}^w$, Theorem 17 applies to (all interesting fragments[6] of) well known many-valued logics [20, 9, 16, 11], as shown by the following corollary:

Corollary 21 Let $\mathcal{L}$ be any fragment of one of the following first-order logics:

1. any witnessed fuzzy logic extending Hajek's basic logic
2. Łukasiewicz logic
3. the Gödel logic $\mathbf{G}_{\uparrow}$ (see Remark 1),
4. nilpotent minimum logic NM with set of truth values $\{1/n : n \geq 1\} \cup \{1-1/n : n \geq 1\}$.

such that (i) $\mathcal{L}$ includes the propositional fragment of the corresponding logic, and (ii) the properties (a)-(d) of Proposition 14 hold for $\mathcal{L}$. Then $\mathcal{L}$ does not admit any $\exists$-analytic calculus.

Proof The proof that the infiniteness condition holds for $\mathcal{L}$ proceeds as that for $\mathbf{G}^w$ in Corollary 18 (note that $\|A \supset B\|_{\mathcal{I}} < 1$ whenever $\|A\|_{\mathcal{I}} > \|B\|_{\mathcal{I}}$, in each interpretation $v_{\mathcal{I}}$ of any of these logics). The claim follows by Theorem 17.

In particular, Corollary 21 holds for the fragment of first-order Łukasiewicz logic[7] investigated in [18]. Various Gentzen-style analytic calculi have indeed been defined for propositional Łukasiewicz logic but in all these calculi the addition of quantifier rules leads either to incomplete calculi for the mentioned fragment or it destroys cut-admissibility, see [8].

[6] The *full* first-order logics, that is all their valid formulas, are not recursively enumerable, see [9, 24, 11] and Remark 1.

[7] This fragment, often called (general) Łukasiewicz logic [15], is obtained by extending propositional Łukasiewicz logic with the quantifiers of classical logic.

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