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Towards a Semantic Characterization of Cut-elimination

Abstract. We introduce necessary and sufficient conditions for a (single-conclusion) sequent calculus to admit (reductive) cut-elimination. Our conditions are formulated both syntactically and semantically.

Keywords: sequent calculus, cut-elimination, nonclassical logics, substructural logics, phase semantics

1. Introduction

Cut-elimination is one of the most important techniques in proof theory. It was first established by Gentzen [7] for the sequent calculi **LK** and **LJ** for classical and intuitionistic first-order logic and later proved, using different methods, for a wide range of calculi in nonclassical logics.

Checking whether a sequent calculus admits cut-elimination is often a rather tedious task. Indeed, the proof is based on case distinctions and has to be checked for all the possible combinations of the rules. This is usually done using heavy syntactic arguments based on case distinctions, written without filling in the details (note that even Gentzen did not formalize all the cases). This renders the cut-elimination process rather opaque. It is then natural to search for general criteria that a sequent calculus has to satisfy in order to admit cut-elimination. Moreover, such criteria, if given on a suitable level of abstraction, would also provide a deeper understanding of the nature of cut-elimination.

Sufficient formal conditions for sequent calculi to admit cut-elimination using *particular* methods were introduced in [13, 19, 6]. Indeed, Miller and Pimentel introduced in [13] such conditions (together with an algorithm to check them) for first-order sequent calculi possibly without the weakening rules and/or the contraction rules. More general criteria, inspired by Belnap's work on Display Logic [4], were defined in [19] for propositional sequent calculi with various logical and structural rules. Some structural rules that do not fit into Restall's pattern were instead considered in [6] for (first-order)

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single-conclusion sequent calculi with exchange and it was shown that these calculi admit cut-elimination whenever their rules are *reductive* and *substitutive*. Intuitively, logical rules are reductive if they allow the replacement of cuts by smaller ones while any rule is substitutive when it leads to correct inferences once one uniformly replaces all the occurrences of any formula in its premise(s) and (some occurrences of this formula in its) conclusion by any multiset of formulae.

Although rather elegant, if not satisfied by a sequent calculus the conditions in [13, 19, 6] give no information as to whether a cut-elimination proof for the considered calculus can be found at all.

A *characterization of cut-elimination*, i.e. the definition of formal criteria that, when satisfied by a sequent calculus guarantee cut-elimination and, when not satisfied they provide a counterexample to the eliminations of cuts, was instead achieved in [2, 21] for particular families of calculi. Avron and Lev indeed characterized propositional sequent calculi which in addition to identity axioms and structural rules (weakening, exchange and contraction) have only pure logical rules. Syntactic and semantic criteria for (additive) structural rules to preserve cut-elimination once added to full Lambek calculus were introduced in [21]. The semantic criterion –called *propagation property*– was inspired by Girard's *naturality test* (see Appendix C.4 in [11]). Intuitively a set of structural rules satisfies the propagation property if it propagates from an arbitrary set of elements to their infinite joins (i.e. $\vee$) and multiplications (i.e. $\otimes$) in all residuated lattices.

In this paper we give a characterization of cut-elimination for *simple calculi*, a rather general class of propositional single-conclusion sequent calculi encompassing, e.g. propositional **LJ**, intuitionistic linear logic extended with the knotted structural rules of [12] or the Full Lambek Calculi in [17]. The aim is to capture the notion of *stepwise process of local transformations* to eliminate cuts in simple calculi. (Note that cut-freeness is undecidable in general[1]). To this purpose, we consider a generalization of Gentzen's Hauptsatz to sequent calculi with non-logical axioms. Our notion (we call it *reductive cut-elimination*) is a naturally strengthened version of so-called *free-cut elimination* [5] (similar concepts are e.g. in [9, 20]). The criteria we propose have two equivalent forms: syntactic (*reductivity* and being *weakly substitutive*) and semantic (*coherence* and *propagation*). The former arise by weakening the sufficient conditions in [6] while the *propagation property* is

[1]This can be shown by extending any calculus for an undecidable logic with suitable rules that force the problem of deciding whether the calculus is cut-free to be reduced to the problem of deciding whether particular formulae are derivable in the original calculus.

a refinement of the homonym condition in [21] stated in terms of a variant of phase semantics [1, 22, 17]. In analogy with [8, 10] we associate to each logical connective two semantic interpretations depending on whether the connective appears on the right or on the left hand side of a sequent. *Coherence* then imposes a suitable restriction on this "asymmetric" interpretation. Using both syntactic and semantic techniques (the latter mainly suggested by [15]) we show that the following are equivalent:

1. A simple calculus $\mathcal{L}$ admits reductive cut-elimination.
2. Logical rules are reductive and structural rules are weakly substitutive.
3. Logical connectives are coherent and structural rules are propagating.

Furthermore, we prove that an identity axiom containing a logical connective $\star$ can be directly derived from atomic axioms only (i.e. the connective $\star$ admits *axiom expansion*) if and only if $\star$ satisfies *rigidity*, a condition dual to coherence.

2. Basic Notions

Let us indicate with $\alpha, \beta, \gamma, \ldots$ propositional variables and with $\star_1, \star_2, \star_3, \ldots$ logical connectives of suitable arity. A *formula* A is either a propositional variable or a *compound formula* of the form $\star(A_1, \ldots, A_m)$ where $A_1, \ldots, A_m$ are formulae. Let $\Gamma, \Delta, \Pi, \Sigma, \ldots$ stand for sequences of formulae. To specify inference rules we will use *meta-variables* $X, Y, \ldots$, standing for arbitrary formulae, and (possibly empty) sequences $\Theta, \Xi, \Phi, \Psi, \Upsilon \ldots$ of meta-variables. A (meta)sequent $\Gamma \Rightarrow \Delta$ ($\Theta \Rightarrow \Xi$) is called *single-conclusion* if Δ (Ξ) contains at most one formula (meta-variable).

Definition 2.1. We call any propositional single-conclusion sequent calculus $\mathcal{L}$ *simple* whenever $\mathcal{L}$ consists of the *identity axiom* of the form $X \Rightarrow X$, together with: the (multiplicative version of the) *cut rule* (CUT), *structural rules* $\{(R_i)\}_{i \in \Lambda_0}$ and for each logical connective $\star$, *left logical rules* $\{(\star, l)_j\}_{j \in \Lambda_1}$ and *right logical rules* $\{(\star, r)_k\}_{k \in \Lambda_2}$ (Λ_0, Λ_1, Λ_2 can be empty):

$$\frac{\Theta \Rightarrow X \quad \Theta_l, X, \Theta_r \Rightarrow \Xi}{\Theta_l, \Theta, \Theta_r \Rightarrow \Xi}\ (CUT) \qquad \frac{\Upsilon_1 \Rightarrow \Psi_1 \quad \cdots \quad \Upsilon_n \Rightarrow \Psi_n}{\Theta_l \Rightarrow \Xi}\ (R_i)$$

$$\frac{\Upsilon_1 \Rightarrow \Psi_1 \quad \cdots \quad \Upsilon_n \Rightarrow \Psi_n}{\Theta_l, \star(\vec{X}), \Theta_r \Rightarrow \Xi}\ (\star, l)_j \qquad \frac{\Upsilon_1 \Rightarrow \Psi_1 \quad \cdots \quad \Upsilon_n \Rightarrow \Psi_n}{\Theta_l \Rightarrow \star(\vec{X})}\ (\star, r)_k$$

where $\Theta, \Theta_l, \Theta_r$ and Ξ in (CUT) are arbitrary (thus (CUT) actually consists of a countable set of inference rules). In rules (R_i), $(\star, l)_j$ and $(\star, r)_k$, $n \geq 0$ and the meta-variables in Θ_l, Θ_r (called *left context meta-variables*), those in Ξ (called *right context meta-variables*), and the meta-variables in $\vec{X} \equiv X_1, \ldots, X_m$, $m \geq 0$ (called *active meta-variables*) are mutually disjoint. The active meta-variables $X_1, \ldots, X_m$ are mutually distinct. In addition, structural rules satisfy the following condition:

(str) Any meta-variable in $\Upsilon_1, \ldots, \Upsilon_n$ is a left context meta-variable, and any meta-variable in $\Psi_1, \ldots, \Psi_n$ is a right context meta-variable.

while logical rules satisfy:

(log0) Any $\Upsilon_1, \ldots, \Upsilon_n$ is either an active or a left context meta-variable, and any meta-variable in $\Psi_1, \ldots, \Psi_n$ is either an active or a right context meta-variable.

(log1) Each meta-variable occurs at most once in Θ_l, Θ_r.

(log2) If $(I)[X]$ is a logical rule of $\mathcal{L}$ with a left context meta-variable X, then $(I)[\Phi]$ also belongs to $\mathcal{L}$ for any sequence Φ of fresh and distinct meta-variables. Here, $(I)[\Phi]$ denotes the rule obtained from (I) by replacing all the occurrences of X with Φ.

(log3) If $(I)[Y]$ is a logical rule of $\mathcal{L}$ with a right context meta-variable Y, then $(I)[\Phi_l; \Phi_r \Rightarrow \Xi]$ also belongs to $\mathcal{L}$ for any sequent $\Phi_l, \Phi_r \Rightarrow \Xi$ that consists of fresh and distinct meta-variables. Here, $(I)[\Phi_l; \Phi_r \Rightarrow \Xi]$ denotes a logical rule obtained from (I) by replacing all sequents of the form $\Theta \Rightarrow Y$ with $\Phi_l, \Theta, \Phi_r \Rightarrow \Xi$.

Henceforth we will only consider simple sequent calculi. Due to conditions (str) and (log0) their rules do not allow context meta-variables to move from antecedent to consequent of sequents and vice versa. Moreover, with the exception of (CUT), rules satisfy the following (subformula) property: any meta-variable occurring in the premises also occurs in the conclusion. Finally, conditions (log1), (log2) and (log3) ensure that logical rules are *substitutive* in the sense of [6].

As usual, an *instance* of a logical or structural rule is obtained by substituting arbitrary formulae for meta-variables. In an instance of a logical or structural rule, the formulae replacing context meta-variables (active meta-variables, respectively) are called *context formulae* (*active formulae*, respectively) and the formula of the form $\star(\vec{A})$ as well as the formulae replacing X in identity axioms are called *principal formulae*. The two occurrences of the formula instantiating X in (CUT) are called *cut formulae*.

EXAMPLE 2.2. Many well known sequent calculi fit into our framework. Among them, propositional **LJ** [7], intuitionistic linear logic extended with knotted structural rules [12] or the Full Lambek Calculi in [17]. For instance, axiom $\Rightarrow \top$ for the logical constant $\top$ (true) is codified by a right logical rule $\overline{\Rightarrow \top}$ with zero premises. The structural rules *exchange* (e), *weakening* (w) and *contraction* (c) are respectively codified by countable sets of rules as follows:

$$\frac{\Theta_i, X, Y, \Theta'_j \Rightarrow \Xi_k}{\Theta_i, Y, X, \Theta'_j \Rightarrow \Xi_k}\ (e)^{ijk} \qquad \frac{\Theta_i, \Theta'_j \Rightarrow \Xi_k}{\Theta_i, X, \Theta'_j \Rightarrow \Xi_k}\ (w)^{ijk} \qquad \frac{\Theta_i, X, X, \Theta'_j \Rightarrow \Xi_k}{\Theta_i, X, \Theta'_j \Rightarrow \Xi_k}\ (c)^{ijk}$$

where Θ_i, Θ'_j and Ξ_k are sequences of distinct meta-variables of length i, j and k, respectively. In the sequel, the rules $\{(e)^{ijk} \mid i, j \in N,\ k \in \{0,1\}\}$ are collectively denoted by (e), and similar conventions are used for other rules. Useful variations of (c) are *sequence contraction* (sc) [21], *weak contraction* (wc) [6] and n*-contraction* (nc) (for $n \geq 2$) [18]:

$$\frac{\Theta_l, \Phi, \Phi, \Theta_r \Rightarrow \Xi}{\Theta_l, \Phi, \Theta_r \Rightarrow \Xi}\ (sc) \qquad \frac{\Theta_l, X, X, \Theta_r \Rightarrow}{\Theta_l, X, \Theta_r \Rightarrow}\ (wc) \qquad \frac{\Theta_l, X^n, \Theta_r \Rightarrow \Xi}{\Theta_l, X^{n-1}, \Theta_r \Rightarrow \Xi}\ (nc)$$

Let us consider the connective $\sqcap$ defined by the following logical rules:

$$\frac{\Theta \Rightarrow X \quad \Theta \Rightarrow Y}{\Theta \Rightarrow X \sqcap Y}\ (\sqcap, r) \qquad \frac{\Theta_l, X, Y, \Theta_r \Rightarrow \Xi}{\Theta_l, X \sqcap Y, \Theta_r \Rightarrow \Xi}\ (\sqcap, l)$$

Note that $\sqcap$ behaves as additive conjunction when appearing on the right hand side of a sequent and as multiplicative conjunction otherwise. Let $\mathcal{L}_\sqcap$ be the simple sequent calculus that consists of the identity axiom, (CUT) and the logical rules $(\sqcap, r)$ and $(\sqcap, l)$. In $\mathcal{L}_\sqcap$, (CUT) between two principal formulae cannot be replaced by (CUT) between their subformulae, due to the lack of any form of contraction:

$$\frac{\dfrac{\Sigma \Rightarrow A \quad \Sigma \Rightarrow B}{\Sigma \Rightarrow A \sqcap B} \quad \dfrac{\Gamma, A, B, \Delta \Rightarrow \Pi}{\Gamma, A \sqcap B, \Delta \Rightarrow \Pi}}{\Gamma, \Sigma, \Delta \Rightarrow \Pi}\ (CUT) \quad \not\longrightarrow \quad \frac{\dfrac{\Sigma \Rightarrow A \quad \dfrac{\Sigma \Rightarrow B \quad \Gamma, A, B, \Delta \Rightarrow \Pi}{\Gamma, A, \Sigma, \Delta \Rightarrow \Pi}}{\Gamma, \Sigma, \Sigma, \Delta \Rightarrow \Pi}}{\Gamma, \Sigma, \Delta \Rightarrow \Pi}\ (???)$$

Nevertheless, the above (CUT) cannot occur in any $\mathcal{L}_\sqcap$-derivation since no sequent containing more than one formula on the left hand side is derivable from identity axioms only using (CUT) and $(\sqcap, r)$. Hence the rule $(\sqcap, l)$ cannot be used in $\mathcal{L}_\sqcap$-derivations. Thus $\mathcal{L}_\sqcap$ admits cut-elimination although not in a *modular* way. Indeed $\mathcal{L}_\sqcap$ no longer admits cut-elimination when extended with other rules (e.g. with the "harmless" rules for implication in intuitionistic linear logic). To avoid such pathological cases we introduce below a specialized notion of cut-elimination.

Let $\mathcal{L}$ be a simple sequent calculus and $\mathcal{S}$ a set of sequents (considered as non-logical axioms). A *derivation* in $\mathcal{L}$ of a sequent S_0 from $\mathcal{S}$ is a labeled tree whose root is labeled by S_0, the leaves are labeled by an instance of an identity axiom, by an instance of a logical $\mathcal{L}$-rule without premises or by a sequent in $\mathcal{S}$, and the inner nodes are labeled in accordance with the instances of the $\mathcal{L}$-rules. When there exists such a derivation, we say that S_0 is *derivable* from $\mathcal{S}$ in $\mathcal{L}$.

Definition 2.3. An occurrence of (CUT) in a derivation is said to be *reducible* if one of the following holds:

(i) Both cut formulae are the principal formulae of logical rules.

(ii) One of the two cut formulae is a context formula of a rule other than (CUT).

(iii) One of the two premises is an identity axiom.

We say that a simple sequent calculus $\mathcal{L}$ admits *reductive cut-elimination* if whenever a sequent S_0 is derivable in $\mathcal{L}$ from a set $\mathcal{S}$ of non-logical axioms, S_0 has a derivation in $\mathcal{L}$ from $\mathcal{S}$ without any reducible cuts.

Notice that in a derivation without non-logical axioms, uppermost cuts are always reducible. Hence reductive cut-elimination implies the usual cut-elimination. On the other hand, the following cuts are not reducible, even if they are uppermost: (i) (CUT) between two non-logical axioms and (ii) (CUT) between a non-logical axiom and the lower sequent of a logical rule whose principal formula coincides with the cut formula.

Reductive cut-elimination is a naturally strengthened version of *free-cut elimination* [5] (or *cut-elimination with non-logical axioms* [9, 20]). The latter roughly says that one can eliminate cut inferences whose cut formulae are not directly derived from non-logical axioms. Reductive cut-elimination in addition aims to shift upward non-eliminable cuts *as much as possible*.

In the sequel, we sometimes treat meta-variables as if they were propositional variables, and consider derivations with meta-variables.

Definition 2.4. A logical connective $\star$ admits *axiom expansion* if the identity axiom $\star(\vec{X}) \Rightarrow \star(\vec{X})$ with $\vec{X} \equiv X_1, \ldots, X_n$ has a cut-free derivation *from atomic axioms*, i.e. in which the identity axioms are restricted to the form $X_i \Rightarrow X_i$ $(1 \leq i \leq n)$.

Example 2.5.

1. The sequent calculus $\mathcal{L}_\sqcap$ admits neither reductive cut-elimination nor axiom expansion. Indeed e.g. $X \Rightarrow Y$ is derivable from $X, X \Rightarrow Y$ only using reducible cuts as follows:

$$\dfrac{\dfrac{X \Rightarrow X \quad X \Rightarrow X}{X \Rightarrow X \wedge X}\,(\sqcap, r) \qquad \dfrac{X, X \Rightarrow Y}{X \wedge X \Rightarrow Y}\,(\sqcap, l)}{X \Rightarrow Y}\,(CUT)$$

   while axiom expansion fails due to the lack of weakening.

2. $\mathcal{L}_\sqcap + (w)$ admits axiom expansion, indeed

$$\dfrac{\dfrac{\dfrac{X \Rightarrow X}{X, Y \Rightarrow X}\,(w) \qquad \dfrac{Y \Rightarrow Y}{X, Y \Rightarrow Y}\,(w)}{X, Y \Rightarrow X \sqcap Y}\,(\sqcap, r)}{X \sqcap Y \Rightarrow X \sqcap Y}\,(\sqcap, l)$$

   However $\mathcal{L}_\sqcap + (w)$ does not admit reductive cut-elimination.

3. By contrast, $\mathcal{L}_\sqcap + (sc)$ (sequence contraction) admits reductive cut-elimination, but not axiom expansion.

3. Syntactic Criteria: Reductivity and Weak Substitutivity

In this section we introduce reductivity and weak substitutivity and show that these syntactic criteria are satisfied by any simple sequent calculus that admits reductive cut-elimination. Our criteria are obtained by suitably modifying the sufficient conditions defined in [6] that ensure cut-elimination (via suitable substitutions) for single-conclusion sequent calculi with exchange.

Definition 3.6. Let $\mathcal{L}$ be a simple sequent calculus. We call its logical rules $\{(\star, r)_p\}_{p\in\Lambda}$ and $\{(\star, l)_q\}_{q\in\Lambda'}$ for introducing a logical connective $\star$ *reductive* in $\mathcal{L}$ if either Λ or Λ' is empty, or for any $p \in \Lambda$ and $q \in \Lambda'$:

$$\dfrac{\Upsilon_1 \Rightarrow \Psi_1 \quad \cdots \quad \Upsilon_n \Rightarrow \Psi_n}{\Theta \Rightarrow \star(\vec{X})}\,(\star, r)_p \qquad \dfrac{\Upsilon'_1 \Rightarrow \Psi'_1 \quad \cdots \quad \Upsilon'_m \Rightarrow \Psi'_m}{\Theta'_l, \star(\vec{X}), \Theta'_r \Rightarrow \Xi'}\,(\star, l)_q$$

the meta-sequent $\Theta'_l, \Theta, \Theta'_r \Rightarrow \Xi'$ is derivable from $\{\Upsilon_i \Rightarrow \Psi_i\}_{1\le i\le n}$ and $\{\Upsilon'_i \Rightarrow \Psi'_i\}_{1\le i\le m}$ using only identity axioms, (CUT) and the structural rules of $\mathcal{L}$.

EXAMPLE 3.7. $(\sqcap, r)$ and $(\sqcap, l)$ are reductive in $\mathcal{L}_\sqcap + (sc)$:

$$\dfrac{\Theta \Rightarrow X \qquad \dfrac{\Theta \Rightarrow Y \quad \Theta_l, X, Y, \Theta_r \Rightarrow \Xi}{\Theta_l, X, \Theta, \Theta_r \Rightarrow \Xi}\,(CUT)}{\dfrac{\Theta_l, \Theta, \Theta, \Theta_r \Rightarrow \Xi}{\Theta_l, \Theta, \Theta_r \Rightarrow \Xi}\,(sc)}\,(CUT)$$

while they are not in $\mathcal{L}_\sqcap + (w)$.

REMARK 3.8. Reductivity was equivalently defined in [6] using rule *instances* instead of rules *schemas*. This condition also corresponds to the *principal formula condition* in [19].

DEFINITION 3.9. Let $\Theta \Rightarrow \Xi$ be a meta-sequent. Given a meta-variable X and a sequence Φ of fresh meta-variables, $[\Theta \Rightarrow \Xi]_{X \mapsto \Phi}$ is the set of meta-sequents obtained from $\Theta \Rightarrow \Xi$ by replacing some (possibly zero) occurrences of X in Θ with Φ. Likewise, given X and a meta-sequent $\Phi_l, \Phi_r \Rightarrow \Psi$ which consists of fresh meta-variables, $[\Theta \Rightarrow \Xi]_{X \mapsto (\Phi_l; \Phi_r \Rightarrow \Psi)}$ is (i) $\{\Theta \Rightarrow \Xi\}$, when $\Xi \not\equiv X$ and (ii) $\{\Theta \Rightarrow \Xi,\ \Phi_l, \Theta, \Phi_r \Rightarrow \Psi\}$, otherwise.

Notice that any $S \in [\Theta \Rightarrow \Xi]_{X \mapsto \Phi}$ ($S \in [\Theta \Rightarrow \Xi]_{X \mapsto (\Phi_l; \Phi_r \Rightarrow \Psi)}$) is obtained by some, possibly zero, applications of (CUT) to $\Theta \Rightarrow \Xi$ and $\Phi \Rightarrow X$ ($\Phi_l, X, \Phi_r \Rightarrow \Psi$).
Henceforth we will indicate with (R) the following structural rule ($S_0, \dots S_n$ stand for meta-sequents):

$$\frac{S_1 \quad \cdots \quad S_n}{S_0}\ (R)$$

DEFINITION 3.10. Let $\mathcal{L}$ be a simple sequent calculus. (R) is said to be a *derived structural rule* in $\mathcal{L}$ if there exists a derivation in $\mathcal{L}$ of S_0 from $\{S_1, \dots, S_n\}$ using only the structural rules of $\mathcal{L}$.

A structural rule (R) is *weakly substitutive* in $\mathcal{L}$ if for any meta-variable X, any $\mathcal{O} \equiv \Phi$ or $\Phi_l; \Phi_r \Rightarrow \Psi$ (see Def. 3.9) and any $S'_0 \in [S_0]_{X \mapsto \mathcal{O}}$, there exists a derived structural rule in $\mathcal{L}$ of the form

$$\frac{S'_1 \quad \cdots \quad S'_m}{S'_0}$$

where each S'_j $(1 \leq j \leq m)$ belongs to $\bigcup_{1 \leq i \leq n} [S_i]_{X \mapsto \mathcal{O}}$.

REMARK 3.11. Intuitively, weakly substitutive structural rules allow any cut to be shifted upward by replacing *some*[2] occurrences of the cut formula in

[2] *All*, in the case of the substitutivity condition in [6].

their premises by the context of the remaining premise of the cut. Due to conditions (log1) – (log3) in Definition 2.1 logical rules naturally satisfy a stronger version of being weakly substitutive.

EXAMPLE 3.12.

1. (c) is not weakly substitutive in $\mathcal{L}_\sqcap + (c)$. Indeed, let S_0 be $\Theta_l, X, \Theta_r \Rightarrow \Xi$ (the conclusion of (c)) and S_1 be $\Theta_l, X, X, \Theta_r \Rightarrow \Xi$ (the premise of (c)). Then $S_0' \equiv \Theta_l, Y, Z, \Theta_r \Rightarrow \Xi$ belongs to $[S_0]_{X \mapsto Y,Z}$ and there is no derived rule in $\mathcal{L}_\sqcap + (c)$ with conclusion S_0' and premises in
$$\begin{aligned}[S_1]_{X \mapsto Y,Z} \quad = \quad & \{(\Theta_l, X, X, \Theta_r \Rightarrow \Xi), (\Theta_l, X, Y, Z, \Theta_r \Rightarrow \Xi), \\ & (\Theta_l, Y, Z, X, \Theta_r \Rightarrow \Xi), (\Theta_l, Y, Z, Y, Z, \Theta_r \Rightarrow \Xi)\}.\end{aligned}$$
In contrast, (c) (and more generally (sc)) is weakly substitutive in $\mathcal{L}_\sqcap + (sc)$. In particular, $\frac{S_1'}{S_0'}$ with $S_1' \equiv \Theta_l, Y, Z, Y, Z, \Theta_r \Rightarrow \Xi$ is just an instance of (sc).
2. (c) and (wc) are weakly substitutive in $\mathcal{L}_\sqcap + (c) + (e)$ and $\mathcal{L}_\sqcap + (wc) + (e)$ respectively.
3. $(3c)$ (and (nc) with $n \geq 3$ in general) is not weakly substitutive in $\mathcal{L}_\sqcap + (3c) + (e)$. Indeed, let U_0 be $\Theta_l, X, X, \Theta_r \Rightarrow \Xi$ (the conclusion of $(3c)$) and U_1 be $\Theta_l, X, X, X, \Theta_r \Rightarrow \Xi$ (the premise of $(3c)$). Then $U_0' \equiv \Theta_l, Y, X, \Theta_r \Rightarrow \Xi$ belongs to $[U_0]_{X \mapsto Y}$ and there is no derived rule in $\mathcal{L}_\sqcap + (3c) + (e)$ with conclusion U_0' and premises in $[U_1]_{X \mapsto Y}$.
4. $(3c)$ is weakly substitutive in $\mathcal{L}_\sqcap + (c) + (e)$ (although *not* substitutive in the sense of [6]). Indeed e.g., by using (c), U_0' above is derivable from $\Theta_l, Y, Y, X, \Theta_r \Rightarrow \Xi$ that belongs to $[U_1]_{X \mapsto Y}$.

The notion of weak substitutivity can be extended to derived rules in a natural way, as shown by the following lemma.

LEMMA 3.13. *Let $\mathcal{L}$ be a simple sequent calculus. If all the structural rules of $\mathcal{L}$ are weakly substitutive then so are all the derived structural rules of $\mathcal{L}$.*

PROOF. By induction on the length of the derivation for each derived structural rule. ■

In the sequel, we prove that reductive cut-elimination implies reductivity (Theorem 3.14) and weak substitutivity (Theorem 3.15).

THEOREM 3.14. *Let $\mathcal{L}$ be a simple sequent calculus. If $\mathcal{L}$ admits reductive cut-elimination, then its logical rules are reductive.*

PROOF. Let $\star$ be a logical connective and $(\star, r)_p$ and $(\star, l)_q$ (as in Def. 3.6) be a pair of right and left introduction rules for $\star$ in $\mathcal{L}$. By applying (CUT), the meta-sequent $\Theta'_l, \Theta, \Theta'_r \Rightarrow \Xi'$ is derivable from $\mathcal{S} = \{\Upsilon_1 \Rightarrow \Psi_1, \dots \Upsilon_n \Rightarrow \Psi_n, \Upsilon'_1 \Rightarrow \Psi'_1, \dots \Upsilon'_m \Rightarrow \Psi'_m\}$. By reductive cut-elimination, $\Theta'_l, \Theta, \Theta'_r \Rightarrow \Xi'$ is derivable from $\mathcal{S}$ without reducible cuts. Since the sequents in $\mathcal{S}$ and $\Theta'_l, \Theta, \Theta'_r \Rightarrow \Xi'$ all consist of meta-variables, no logical rule is used in it. ■

THEOREM 3.15. *Let $\mathcal{L}$ be a simple sequent calculus. If $\mathcal{L}$ admits reductive cut-elimination, then its structural rules are weakly substitutive.*

PROOF. Let (R) be a structural rule of $\mathcal{L}$, X a meta-variable and $\mathcal{O} \equiv \Phi$ a sequence of fresh meta-variables (the case $\mathcal{O} \equiv \Phi_l; \Phi_r \Rightarrow \Psi$ is similar). Let $S'_0 \in [S_0]_{X \mapsto \Phi}$. Thus S'_0 is obtained by (repeatedly) applying (CUT) between S_0 and $\Phi \Rightarrow X$. I.e., there is a derivation of S'_0 of the form

$$\dfrac{\Phi \Rightarrow X \qquad \dfrac{\Phi \Rightarrow X \qquad \dfrac{S_1 \quad \cdots \quad S_n}{S_0}(R)}{\cdots}(CUT)}{S'_0}(CUT)$$

Since $\mathcal{L}$ admits reductive cut-elimination, we can find a derivation $\mathcal{D}$ of S'_0 from $\{S_1, \dots, S_n, \Phi \Rightarrow X\}$ without reducible cuts. In $\mathcal{D}$, we can find meta-sequents $U_1, \dots, U_m$ such that, for each $1 \leq i \leq m$,

above U_i, only (CUT) is used; below U_i, no (CUT) is used

because otherwise there would be a reducible cut in $\mathcal{D}$. Recall that the left context meta-variables of (R) and the right context meta-variables of (R) are mutually distinct, and that Φ consists of fresh meta-variables. It follows that every occurrence of (CUT) above each U_i has cut formula X and premise $\Phi \Rightarrow X$, i.e. each U_i belongs to some $[S_j]_{X \mapsto \Phi}$. This means that we have a derivation of S'_0 from $\bigcup_{1 \leq j \leq n} [S_j]_{X \mapsto \Phi}$ without using (CUT). Since $U_1, \dots, U_m$ and S'_0 all consist of meta-variables, no logical rule is used in it. Hence (R) is weakly substitutive. ■

4. Semantic Criteria: Coherence, Rigidity and Propagation

Here we introduce the semantic counterparts of the criteria investigated in the previous section: coherence and propagation. These are stated in terms of *prephase structures*, which are another presentation of (intuitionistic, non-commutative) phase structures (see [1, 22, 17], or [14, 15] for a presentation closer to ours).

In a prephase structure, closed sets interpreting formulae are built from a class of more "primitive" objects (as closed sets in a topological space are built from a closed basis). *Propagation*, a refinement of the homonym condition in [21], then arises as a natural property. Intuitively, structural rules propagate if when they hold for primitive objects, then they also hold for all closed sets. This property suitably formalizes Girard's *naturality test* [11] for logical principles.

Coherence arises by considering an "asymmetric" interpretation for connectives (cfr. [8]; see also Section 5.2.3 of [10]). Indeed, to each compound formula $\star(\vec{X})$ we will associate two different interpretations $(\star, r)^{\mathbf{P}}(\vec{X}^{\bullet})$ and $(\star, l)^{\mathbf{P}}(\vec{X}^{\bullet})$, depending on whether the formula appears on the right hand side or on the left hand side of a sequent. Under such asymmetry, it is natural to require $(\star, r)^{\mathbf{P}}(\vec{X}^{\bullet}) \subseteq (\star, l)^{\mathbf{P}}(\vec{X}^{\bullet})$, since otherwise the cut rule cannot be semantically sound[3]. That is the coherence property.

We show that reductivity implies coherence and weak substitutivity implies propagation. Furthermore we consider *rigidity* – the converse condition of coherence – and show that axiom expansion implies rigidity.

We start by introducing some notation and terminology. Let $\mathcal{A}$ be any set, $\mathcal{A}^* = (\mathcal{A}^*, \cdot, 1)$ denotes the free monoid generated by $\mathcal{A}$, and $\wp(\mathcal{A}^*)$ the powerset of $\mathcal{A}^*$. For any $P, Q \subseteq \mathcal{A}^*$, define

$$\begin{array}{rclcrcl} P\backslash Q &=& \{y \mid \forall x \in P(x \cdot y \in Q)\}, & \qquad & P \bullet Q &=& \{x \cdot y \mid x \in P, y \in Q\}, \\ Q/P &=& \{y \mid \forall x \in P(y \cdot x \in Q)\}. & & & & \end{array}$$

For any $x \in \mathcal{A}^*$, we write $x\backslash P$ to denote $\{x\}\backslash P$ and P/x to denote $P/\{x\}$. We then have $y \in x\backslash P \iff x \cdot y \in P \iff x \in P/y$. It follows that the set $(x\backslash P)/y$ coincides with $x\backslash(P/y)$ and will be henceforth denoted by $x\backslash P/y$.

Definition 4.16. A *prephase structure* $\mathbf{P}$ is a triple $(\mathcal{A}, \mathcal{B}, \perp)$ such that $\perp \subseteq \mathcal{A}^*$ and $\mathcal{B} \subseteq \wp(\mathcal{A}^*)$. The set $\mathcal{B} \cup \{\perp\}$ is denoted by $\mathcal{B}_\perp$. A *closed* set is a subset of $\mathcal{A}^*$ of the form $\bigcap_{i\in\Lambda} y_i\backslash Q_i/z_i$ where Λ is an arbitrary index set, $y_i, z_i \in \mathcal{A}^*$ and $Q_i \in \mathcal{B}_\perp$ for each $i \in \Lambda$. The set of all closed sets in $\mathbf{P}$ is denoted by $\mathcal{C}_{\mathbf{P}}$. Given a set $P \subseteq \mathcal{A}^*$, $C_{\mathbf{P}}(P)$ denotes the least closed set containing P, i.e.,

$$C_{\mathbf{P}}(P) = \bigcap\{\, y\backslash Q/z \mid P \subseteq y\backslash Q/z,\ y, z \in \mathcal{A}^* \text{ and } Q \in \mathcal{B}_\perp\}.$$

[3] From a semantic point of view, the cut rule is nothing but transitivity of inclusion. Under asymmetry, it gives rise to the following principle: from $\Theta^{\bullet} \subseteq (\star, r)^{\mathbf{P}}(\vec{X}^{\bullet})$ and $(\star, l)^{\mathbf{P}}(\vec{X}^{\bullet}) \subseteq \Xi^{\bullet}$, deduce $\Theta^{\bullet} \subseteq \Xi^{\bullet}$. This holds if and only if $(\star, r)^{\mathbf{P}}(\vec{X}^{\bullet}) \subseteq (\star, l)^{\mathbf{P}}(\vec{X}^{\bullet})$. It is worth noting that the algebraic proof of reductive cut-elimination in the next section vitally rests upon the soundness of the cut rule.

Intuitively, a prephase structure can be considered as a topology-like structure over a free monoid $\mathcal{A}^*$, where the set $\{x\backslash P/y \mid x, y \in \mathcal{A}^*, P \in \mathcal{B}_\perp\}$ forms a basis for closed sets. The distinguished set $\perp$ is used to interpret the empty right hand side $\Xi \equiv \emptyset$ of a sequent.

Lemma 4.17. *In every prephase structure* $\mathbf{P} = (\mathcal{A}, \mathcal{B}, \perp)$*, the operator* $C_{\mathbf{P}}$ *satisfies the following* closure properties*: (i)* $P \subseteq C_{\mathbf{P}}(P)$*, (ii)* $C_{\mathbf{P}}(C_{\mathbf{P}}(P)) \subseteq C_{\mathbf{P}}(P)$*, (iii)* $P \subseteq Q$ *implies* $C_{\mathbf{P}}(P) \subseteq C_{\mathbf{P}}(Q)$*, (iv)* $C_{\mathbf{P}}(P) \bullet C_{\mathbf{P}}(Q) \subseteq C_{\mathbf{P}}(P \bullet Q)$*, and (v) every* $Q \in \mathcal{B}_\perp$ *is closed.*

Proof. Properties (i) – (iii) are easy to check. As for (iv), it is enough to show $C_{\mathbf{P}}(P) \bullet Q \subseteq C_{\mathbf{P}}(P \bullet Q)$ (as well as $P \bullet C_{\mathbf{P}}(Q) \subseteq C_{\mathbf{P}}(P \bullet Q)$). For this, it is enough to prove that for any $z_l, z_r \in \mathcal{A}^*$ and $O \in \mathcal{B}_\perp$ such that $P \bullet Q \subseteq z_l\backslash O/z_r$, we have $C_{\mathbf{P}}(P) \bullet Q \subseteq z_l\backslash O/z_r$. For every $y \in Q$, $P \bullet \{y\} \subseteq z_l\backslash O/z_r$, i.e., $P \bullet \{y \cdot z_r\} \subseteq z_l\backslash O$, hence $P \subseteq z_l\backslash O/(y \cdot z_r)$. Thus $C_{\mathbf{P}}(P) \subseteq z_l\backslash O/(y \cdot z_r)$, so $C_{\mathbf{P}}(P) \bullet \{y \cdot z_r\} \subseteq z_l\backslash O$, i.e., $C_{\mathbf{P}}(P) \bullet \{y\} \subseteq z_l\backslash O/z_r$. Since this holds for any $y \in Q$, we obtain $C_{\mathbf{P}}(P) \bullet Q \subseteq z_l\backslash O/z_r$, as required. As for (v), just observe that $Q = 1\backslash Q/1$. ■

Remark 4.18. The above lemma shows that every prephase structure ($\mathcal{A}$, $\mathcal{B}, \perp$) induces a (non-commutative) phase structure $(\mathcal{A}^*, C_{\mathbf{P}}, \perp)$ [1, 22, 17].

Although closed under intersection, $\mathcal{C}_{\mathbf{P}}$ is not closed under union. We therefore introduce the following operation: Let $\mathcal{X}$ be a set of closed sets,

$$\bigoplus \mathcal{X} = C_{\mathbf{P}}(\bigcup \mathcal{X}), \qquad \text{in particular, } \bigoplus \mathcal{X} = C_{\mathbf{P}}(\emptyset), \text{ when } \mathcal{X} \text{ is empty.}$$

Let us fix a prephase structure $\mathbf{P} = (\mathcal{A}, \mathcal{B}, \perp)$. We now interpret each logical connective $\star$ based on the logical rules for introducing $\star$. Consider the meta-sequent $\Theta \Rightarrow \Xi$, where Θ and Ξ consist of the meta-variables $X_1, \ldots, X_n$. Suppose that a closed set $X_i^\bullet \in \mathcal{C}_{\mathbf{P}}$ is associated to each X_i $(i = 1, \ldots n)$. We can then associate to

- Θ the closed set $\Theta^\bullet$ denoting $C_{\mathbf{P}}((X_{i_1}^\bullet) \bullet \cdots \bullet (X_{i_k}^\bullet))$ if $\Theta \equiv X_{i_1}, \ldots X_{i_k}$ $(i_1, \ldots, i_k \in \{1, \ldots, n\})$, or $\mathbf{1} = C_{\mathbf{P}}(\{1\})$ if Θ is empty;
- Ξ the closed set $\Xi^\bullet$ denoting either $X_i^\bullet$ if $\Xi \equiv X_i$ $(1 \leq i \leq n)$, or $\perp$ if Ξ is empty.

We say that $X_1^\bullet, \ldots, X_n^\bullet$ *satisfy* $\Theta \Rightarrow \Xi$ if $\Theta^\bullet \subseteq \Xi^\bullet$.

Definition 4.19. For each right logical rule

$$\frac{\Upsilon_1 \Rightarrow \Psi_1 \quad \cdots \quad \Upsilon_m \Rightarrow \Psi_m}{\Theta \Rightarrow \star(\vec{X})}\ (\star, r)_i$$

that consists of active meta-variables $\vec{X} \equiv X_1, \dots, X_n$ and context meta-variables $\vec{Y} \equiv Y_1, \dots, Y_k$, and for any closed sets $X_1^\bullet, \dots, X_n^\bullet \in \mathcal{C}_\mathbf{P}$, we define

$$(\star, r)_i^\mathbf{P}(\vec{X}^\bullet) = \bigoplus\{\Theta^\bullet |\ \vec{Y}^\bullet \in \mathcal{C}_\mathbf{P},\ \Upsilon_1^\bullet \subseteq \Psi_1^\bullet, \dots, \Upsilon_m^\bullet \subseteq \Psi_m^\bullet\}$$

i.e., the largest value $\Theta^\bullet \in \mathcal{C}_\mathbf{P}$ such that $\vec{X}^\bullet, \vec{Y}^\bullet$ satisfy all the premises (with $\vec{Y}^\bullet$ ranging over $\mathcal{C}_\mathbf{P}$). Consequently, we always have $\Theta^\bullet \subseteq (\star, r)_i^\mathbf{P}(\vec{X}^\bullet)$ whenever $\vec{X}^\bullet$, $\vec{Y}^\bullet \in \mathcal{C}_\mathbf{P}$ satisfy the premises. Likewise, for each left logical rule

$$\frac{\Upsilon_1 \Rightarrow \Psi_1 \quad \cdots \quad \Upsilon_m \Rightarrow \Psi_m}{\Theta_l, \star(\vec{X}), \Theta_r \Rightarrow \Xi}\ (\star, l)_j$$

that consists of active meta-variables $\vec{X}$ and context meta-variables $\vec{Y}$, and for any closed sets $\vec{X}^\bullet \in \mathcal{C}_\mathbf{P}$, we define

$$(\star, l)_j^\mathbf{P}(\vec{X}^\bullet) = \bigcap\{\Theta_l^\bullet \backslash \Xi^\bullet / \Theta_r^\bullet |\ \vec{Y}^\bullet \in \mathcal{C}_\mathbf{P},\ \Upsilon_1^\bullet \subseteq \Psi_1^\bullet, \dots, \Upsilon_m^\bullet \subseteq \Psi_m^\bullet\}$$

i.e., the least value $\Theta_l^\bullet \backslash \Xi^\bullet / \Theta_r^\bullet \in \mathcal{C}_\mathbf{P}$ such that $\vec{X}^\bullet, \vec{Y}^\bullet$ satisfy all the premises (with $\vec{Y}^\bullet$ ranging over $\mathcal{C}_\mathbf{P}$). Consequently, we always have $\Theta_l^\bullet \bullet (\star, l)_j^\mathbf{P}(\vec{X}^\bullet) \bullet \Theta_r^\bullet \subseteq \Xi^\bullet$, whenever $\vec{X}^\bullet$, $\vec{Y}^\bullet \in \mathcal{C}_\mathbf{P}$ satisfy the premises. When $\{(\star, r)_i\}_{i \in \Lambda}$ and $\{(\star, l)_j\}_{j \in \Lambda'}$ are the right and left logical rules introducing $\star(\vec{X})$, the *right interpretation* $(\star, r)^\mathbf{P}(\vec{X})$ and the *left interpretation* $(\star, l)^\mathbf{P}(\vec{X})$ of $\star$ are defined as follows:

$$(\star, r)^\mathbf{P}(\vec{X}^\bullet) = \bigoplus_{i \in \Lambda} (\star, r)_i^\mathbf{P}(\vec{X}^\bullet), \qquad (\star, l)^\mathbf{P}(\vec{X}^\bullet) = \bigcap_{j \in \Lambda'} (\star, l)_j^\mathbf{P}(\vec{X}^\bullet).$$

Example 4.20. The logical rules

$$\frac{Z \Rightarrow X \quad Z \Rightarrow Y}{Z \Rightarrow X \sqcap Y}\ (\sqcap, r)_i \qquad \frac{Z_1, X, Y, Z_2 \Rightarrow W}{Z_1, X \sqcap Y, Z_2 \Rightarrow W}\ (\sqcap, l)_j$$

are respectively interpreted as follows: for any $X^\bullet, Y^\bullet \in \mathcal{C}_\mathbf{P}$,

$$\begin{array}{rcl} (\sqcap, r)_i^\mathbf{P}(X^\bullet, Y^\bullet) & = & \bigoplus\{Z^\bullet \mid Z^\bullet \in \mathcal{C}_\mathbf{P}, Z^\bullet \subseteq X^\bullet, Z^\bullet \subseteq Y^\bullet\} = X^\bullet \cap Y^\bullet \\ (\sqcap, l)_j^\mathbf{P}(X^\bullet, Y^\bullet) & = & \bigcap\{Z_1^\bullet \backslash W^\bullet / Z_2^\bullet \mid Z_1^\bullet, Z_2^\bullet, W^\bullet \in \mathcal{C}_\mathbf{P}, Z_1^\bullet \bullet X^\bullet \bullet Y^\bullet \bullet Z_2^\bullet \subseteq W^\bullet\} \\ & & = \mathbf{1} \backslash C_\mathbf{P}(X^\bullet \bullet Y^\bullet)/\mathbf{1} = C_\mathbf{P}(X^\bullet \bullet Y^\bullet) \end{array}$$

Definition 4.21. A structural rule (R) with context meta-variables $\vec{X}$ is *valid* in $\mathbf{P}$ if whenever $\vec{X}^\bullet \in \mathcal{C}_\mathbf{P}$ satisfy the premises of (R), $\vec{X}^\bullet$ also satisfy the conclusion. A prephase structure $\mathbf{P}$ is said to be an $\mathcal{L}$-*structure* if all the structural rules of $\mathcal{L}$ are valid in $\mathbf{P}$.

We have associated *two* interpretations to each logical connective. However, to make both identity axiom and (CUT) sound, we must associate *one* closed set to each formula.

DEFINITION 4.22. Given a simple sequent calculus $\mathcal{L}$ and a prephase structure $\mathbf{P}$, a *valuation* on $\mathbf{P}$ (for $\mathcal{L}$) is a map f from the set of formulae to $\mathcal{C}_{\mathbf{P}}$ such that for each compound formula $\star(\vec{A}) \equiv \star(A_1, \ldots, A_n)$,

$$(\star, r)^{\mathbf{P}}(\overrightarrow{f(A)}) \subseteq f(\star(\vec{A})) \subseteq (\star, l)^{\mathbf{P}}(\overrightarrow{f(A)}),$$

where $\overrightarrow{f(A)}$ denotes $f(A_1), \ldots, f(A_n)$. A valuation f can be extended to a sequence $\Gamma \equiv A_1, \ldots, A_n$ of formulae by letting $f(\Gamma) = C_{\mathbf{P}}(f(A_1) \bullet \cdots \bullet f(A_n))$ when $n \geq 1$, and otherwise $f(\Gamma)$ is either $\mathbf{1}$ or $\perp$ depending on whether Γ appears on the left or right hand side of a sequent (the context will always provide the relevant information). A sequent $\Gamma \Rightarrow \Delta$ is *true* under f if $f(\Gamma) \subseteq f(\Delta)$.

THEOREM 4.23 (Soundness). *Let $\mathcal{L}$ be a simple sequent calculus, $\mathbf{P}$ an $\mathcal{L}$-structure, and f a valuation on $\mathbf{P}$. If all non-logical axioms in $\mathcal{S}$ are true under f and S_0 is derivable in $\mathcal{L}$ from $\mathcal{S}$, then S_0 is also true under f.*

PROOF. By induction on the length of the derivation of S_0. It is easy to see that all instances of the identity axiom are true under any valuation, and all instances of the cut rule and the structural rules in $\mathcal{L}$ preserve truth in $\mathbf{P}$. Now consider an instance of a right logical rule $(\star, r)_i$ with conclusion $\Gamma \Rightarrow \star(\vec{A})$, and suppose that the premises are true under f. A straightforward argument shows that $f(\Gamma) \subseteq (\star, r)_i^{\mathbf{P}}(\overrightarrow{f(A)})$ (cfr. Definition 4.19). Since $(\star, r)_i^{\mathbf{P}}(\overrightarrow{f(A)}) \subseteq (\star, r)^{\mathbf{P}}(\overrightarrow{f(A)}) \subseteq f(\star(\vec{A}))$, the conclusion is true under f. Similarly for left logical rules. ∎

Even though the soundness theorem holds for any simple sequent calculus $\mathcal{L}$, this does not mean that $\mathcal{L}$ can be semantically interpreted in a useful way. In particular, it is not always the case that there *exists* a valuation on a given prephase structure. For instance, when $\mathcal{L}$ involves $\sqcap$, any valuation f must satisfy $f(A) \cap f(B) \subseteq f(A \sqcap B) \subseteq C_{\mathbf{P}}(f(A) \bullet f(B))$ (cfr. Example 4.20). However, such a valuation does not always exist because $P \cap Q \subseteq C_{\mathbf{P}}(P \bullet Q)$ does not hold in general. To ensure the existence of a valuation, a further condition is required.

DEFINITION 4.24. A logical connective $\star$ is *coherent* (*rigid*, respectively) in $\mathcal{L}$ if $(\star, r)^{\mathbf{P}}(\vec{X}^{\bullet}) \subseteq (\star, l)^{\mathbf{P}}(\vec{X}^{\bullet})$ ($(\star, l)^{\mathbf{P}}(\vec{X}^{\bullet}) \subseteq (\star, r)^{\mathbf{P}}(\vec{X}^{\bullet})$, respectively) for any closed sets $\vec{X}^{\bullet}$ in any $\mathcal{L}$-structure.

Coherence of logical connectives guarantees that there exists a valuation in any $\mathcal{L}$-structure. When in addition logical connectives are rigid, the value of a compound formula is uniquely determined by the values of its subformulae; namely, $f(\star(\vec{A})) = (\star, l)^{\mathbf{P}}(\overrightarrow{f(A)}) = (\star, r)^{\mathbf{P}}(\overrightarrow{f(A)})$.

EXAMPLE 4.25. In any $\mathcal{L}_{\sqcap} + (sc)$-structure (cfr. Example 2.5), we have $P \subseteq C_{\mathbf{P}}(P \bullet P)$ for any closed set P. This makes $(\sqcap, r)^{\mathbf{P}}(P, Q) = P \cap Q \subseteq C_{\mathbf{P}}((P \cap Q) \bullet (P \cap Q)) \subseteq C_{\mathbf{P}}(P \bullet Q) = (\sqcap, l)^{\mathbf{P}}(P, Q)$ for any closed sets P and Q. Hence $\sqcap$ is coherent in $\mathcal{L}_{\sqcap} + (sc)$.

On the other hand, we have $P \subseteq \mathbf{1}$ in any $\mathcal{L}_{\sqcap} + (w)$-structure. This makes $C_{\mathbf{P}}(P \bullet Q) \subseteq C_{\mathbf{P}}(P \bullet \mathbf{1}) \cap C_{\mathbf{P}}(\mathbf{1} \bullet Q) = P \cap Q$ for any closed sets P and Q. Therefore $\sqcap$ is rigid in $\mathcal{L}_{\sqcap} + (w)$.

To introduce the propagation property, we need a notion of validity in a "primitive" sense. Let $\Theta \Rightarrow \Xi$ be a meta-sequent in which Θ consists of the meta-variables $X_1, \ldots, X_n$ and Ξ is either empty or $\Xi \equiv Y$. Suppose that an element $X_i^\circ \in \mathcal{A}$ is associated to each X_i $(i = 1, \ldots n)$ and a set $Y^\diamond \in \mathcal{B}$ to Y. We can then associate to

- Θ an element Θ° denoting either $X_{i_1}^\circ \cdots X_{i_k}^\circ \in \mathcal{A}^*$ if $\Theta \equiv X_{i_1}, \ldots, X_{i_k}$, or 1 if Θ is empty;
- Ξ a set $\Xi^\diamond$ denoting either $Y^\diamond$ if $\Xi \equiv Y$, or $\perp$ if Ξ is empty.

We say that $X_1^\circ, \ldots, X_n^\circ \in \mathcal{A}$ and $Y^\diamond \in \mathcal{B}$ *pre-satisfy* $\Theta \Rightarrow \Xi$ if $\Theta^\circ \in \Xi^\diamond$.

DEFINITION 4.26. A structural rule (R) with left context meta-variables $\vec{X}$ and (possibly) a right context meta-variable Y is *pre-valid* in $\mathbf{P}$ if, for each $\vec{X}^\circ \in \mathcal{A}$ (and $Y^\diamond \in \mathcal{B}$ when the right hand side is nonempty) whenever $\vec{X}$ (and $Y^\diamond$) pre-satisfy all the premises of (R), they also pre-satisfy the conclusion.

A structural rule (R) is said to be *propagating* in $\mathcal{L}$ if (R) is valid in all prephase structures in which all the structural rules of $\mathcal{L}$ are pre-valid.

REMARK 4.27. In short, structural rules are propagating when their pre-validity implies validity. Instead of pre-validity, validity with respect to the *atomic closed sets*[4] (i.e. those of the form $C(\{a\})$, $a \in \mathcal{A}$) is implicitly used in [21].

[4]Intuitively, an atomic closed set $C(\{a\})$ with $a \in \mathcal{A}$ can be considered as a first-order object, and an arbitrary closed set $C(P)$ with $P \subseteq \mathcal{A}^*$ as a second-order object. Then the basic meaning of propagation is that the structural rules for second-order objects are "conservative" over those for first-order objects. This accounts for why cut-elimination implies the propagation property in [21], as cut-elimination is well-known to be an effective means to show such a conservativity result (as pointed out by J.-Y. Girard).

Example 4.28. Let $\mathbf{P}$ be any prephase structure in which (c) is pre-valid, i.e. $x \cdot a \cdot a \cdot y \in P_0$ implies $x \cdot a \cdot y \in P_0$ for any $a \in \mathcal{A}$, $x, y \in \mathcal{A}^*$ and $P_0 \in \mathcal{B}_\perp$. (c) is not necessarily valid in $\mathbf{P}$, i.e. it is not always the case that for any closed sets P_1, P_2, P_3, Q, $(*)$ $P_1 \bullet Q \bullet Q \bullet P_2 \subseteq P_3$ implies $P_1 \bullet Q \bullet P_2 \subseteq P_3$. Hence (c) is not propagating in $\mathcal{L}_\sqcap + (c)$.

On the other hand, the above $(*)$ holds when (sc) is pre-valid in $\mathbf{P}$, i.e. $x \cdot w \cdot w \cdot y \in P_0$ implies $x \cdot w \cdot y \in P_0$ for any $w, x, y \in \mathcal{A}^*$ and $P_0 \in \mathcal{B}_\perp$. Indeed, assume $P_1 \bullet Q \bullet Q \bullet P_2 \subseteq P_3$. Let $x \in P_1, w \in Q, y \in P_2$ and $P_3 \subseteq z_l \backslash O / z_r$ with $z_l, z_r \in \mathcal{A}^*$ and $O \in \mathcal{B}_\perp$. Then we have $x \cdot w \cdot w \cdot y \in z_l \backslash O / z_r$, i.e. $z_l \cdot x \cdot w \cdot w \cdot y \cdot z_r \in O$. Since (sc) is pre-valid in $\mathbf{P}$, we have $z_l \cdot x \cdot w \cdot y \cdot z_r \in O$, i.e. $x \cdot w \cdot y \in z_l \backslash O / z_r$. This shows $P_1 \bullet Q \bullet P_2 \subseteq P_3$. Therefore, (c) (and more generally (sc)) is propagating in $\mathcal{L}_\sqcap + (sc)$.

In the sequel, we will show that (i) reductivity implies coherence (Theorem 4.29), (ii) axiom expansion implies rigidity (Theorem 4.30), and (iii) weak substitutivity implies propagation (Theorem 4.31).

Theorem 4.29. *Let $\mathcal{L}$ be a simple sequent calculus. Each connective $\star$ defined by reductive rules in $\mathcal{L}$ is coherent in $\mathcal{L}$.*

Proof. Let $\mathbf{P}$ be an $\mathcal{L}$-structure. If there is no right (left) logical rules for $\star$, then $\star$ is trivially coherent because $(\star, r)^{\mathbf{P}}(\vec{X}^\bullet) = C_{\mathbf{P}}(\emptyset)$ ($(\star, l)^{\mathbf{P}}(\vec{X}^\bullet) = \mathcal{A}^*$) for any closed sets $\vec{X}^\bullet$. Otherwise, it is enough to prove that for every pair of right and left logical rules $(\star, r)_i$ and $(\star, l)_j$, and for any closed sets $\vec{X}^\bullet$, $(\star, r)_i^{\mathbf{P}}(\vec{X}^\bullet) \subseteq (\star, l)_j^{\mathbf{P}}(\vec{X}^\bullet)$. Let the conclusions of $(\star, r)_i$ and $(\star, l)_j$ be $\Theta \Rightarrow \star(\vec{X})$ and $\Theta_l, \star(\vec{X}), \Theta_r \Rightarrow \Xi$ respectively. Without loss of generality, we may assume that the context meta-variables $\vec{Y}$ in Θ are distinct from the meta-variables $\vec{Z}$ in Θ_l, Θ_r and Ξ, thus the only common meta-variables in $(\star, r)_i$ and $(\star, l)_j$ are $\vec{X}$.

Now, the reductivity of $(\star, r)_i$ and $(\star, l)_j$ (together with the assumption that all the structural rules of $\mathcal{L}$ are valid in $\mathbf{P}$) implies that for any closed sets $\vec{Y}^\bullet, \vec{Z}^\bullet$ that satisfy the premises of $(\star, r)_i$ and $(\star, l)_j$, $\Theta_l^\bullet \bullet \Theta^\bullet \bullet \Theta_r^\bullet \subseteq \Xi^\bullet$, i.e., $\Theta^\bullet \subseteq \Theta_l^\bullet \backslash \Xi^\bullet / \Theta_r^\bullet$. Since $\vec{Y}^\bullet$ and $\vec{Z}^\bullet$ can be chosen independently, we obtain $(\star, r)_i^{\mathbf{P}}(\vec{X}^\bullet) \subseteq (\star, l)_j^{\mathbf{P}}(\vec{X}^\bullet)$. ■

Theorem 4.30. *Let $\mathcal{L}$ be a simple sequent calculus. Each connective $\star$ that admits axiom expansion is rigid in $\mathcal{L}$.*

Proof. By assumption, $\star(\vec{X}) \Rightarrow \star(\vec{X})$ with $\vec{X} \equiv X_1, \ldots, X_n$ has a cut-free derivation $\mathcal{D}$ from identity axioms of the form $X_i \Rightarrow X_i$ ($1 \leq i \leq n$). Let $\mathbf{P}$ be an $\mathcal{L}$-structure and $\vec{X}^\bullet$ be closed sets. Interpret the formula

$\star(\vec{X})$ by $(\star, l)^{\mathbf{P}}(\vec{X}^{\bullet})$ when it appears on the left hand side of a sequent and by $(\star, r)^{\mathbf{P}}(\vec{X}^{\bullet})$ otherwise. By induction on $\mathcal{D}$, we can prove that this interpretation satisfies each sequent in $\mathcal{D}$. Thus we obtain $(\star, l)^{\mathbf{P}}(\vec{X}^{\bullet}) \subseteq (\star, r)^{\mathbf{P}}(\vec{X}^{\bullet})$. ■

THEOREM 4.31. *If all the structural rules of $\mathcal{L}$ are weakly substitutive, then they are propagating in $\mathcal{L}$.*

To prove this theorem, we need a syntactic way of representing each element of $\mathcal{A}^*$ and $\mathcal{B}_\perp$ in a given prephase structure $\mathbf{P} = (\mathcal{A}, \mathcal{B}, \perp)$. Let $\mathcal{V}_0$ be a set of meta-variables that is in one-to-one correspondence with $\mathcal{A}$ via $^\circ$. Namely, each element in $\mathcal{A}$ can be denoted by V° for some $V \in \mathcal{V}_0$. This way any element in $\mathcal{A}^*$ can be denoted by Θ° where Θ consists of meta-variables in $\mathcal{V}_0$. Similarly, let $\mathcal{V}_1$ be a set of meta-variables that is in one-to-one correspondence with $\mathcal{B}$ via $^\diamond$. Then any element in $\mathcal{B}_\perp$ can be denoted by $\Xi^\diamond$ where Ξ is either empty or consists of some $W \in \mathcal{V}_1$.

PROOF. Let (R) be a structural rule of $\mathcal{L}$ and $\mathbf{P}$ a prephase structure in which all the structural rules of $\mathcal{L}$ are pre-valid. Suppose that (R) consists of left context meta-variables $X_1, \ldots, X_n$ and a right context metavariable Y, and the conclusion is $S_0 \equiv X_{i_1}, \ldots, X_{i_k} \Rightarrow Y$ with $i_1, \ldots, i_k \in \{1, \ldots, n\}$ (the case when the right hand side is empty is easier). Let $X_1^\bullet, \ldots, X_n^\bullet, Y^\bullet$ be closed sets that satisfy the premises $S_1, \ldots, S_m$ of (R). Our goal is to prove that the conclusion is also satisfied, i.e., $X_{i_1}^\bullet \bullet \cdots \bullet X_{i_k}^\bullet \subseteq Y^\bullet$. It is enough to show the following:

> For any $\Theta_1^\circ \in X_{i_1}^\bullet, \ldots, \Theta_k^\circ \in X_{i_k}^\bullet$, any Θ_0°, $\Theta_{k+1}^\circ \in \mathcal{A}^*$ and $\Phi^\diamond \in \mathcal{B}_\perp$ such that $Y^\bullet \subseteq \Theta_0^\circ \backslash \Phi^\diamond / \Theta_{k+1}^\circ$, we have $\Theta_1^\circ \cdots \Theta_k^\circ \in \Theta_0^\circ \backslash \Phi^\diamond / \Theta_{k+1}^\circ$, i.e., $\Theta_0^\circ \cdots \Theta_{k+1}^\circ \in \Phi^\diamond$.

Here we assume that the sequences $\Theta_0, \ldots, \Theta_{k+1}$ (Φ, respectively) consist of meta-variables $\mathcal{V}_0$ ($\mathcal{V}_1$, respectively). Since (R) is weakly substitutive, there exists a derived rule (R_0) with conclusion

$$S_0' \;\equiv\; \Theta_0, X_{i_1}, \ldots, X_{i_k}, \Theta_{k+1} \Rightarrow \Phi \;\in\; [S_0]_{Y \mapsto (\Theta_0; \Theta_{k+1} \Rightarrow \Phi)}$$

and premises $S_1', \ldots, S_{m'}'$ in $\bigcup_{1 \leq p \leq m} [S_p]_{Y \mapsto (\Theta_0; \Theta_{k+1} \Rightarrow \Phi)}$. By Lemma 3.13, (R_0) is weakly substitutive. Hence there exists a derived rule (R_1) with conclusion

$$\Theta_0, \Theta_1, X_{i_2}, \ldots, X_{i_k}, \Theta_{k+1} \Rightarrow \Phi \;\in\; [S_0']_{X_{i_1} \mapsto \Theta_1}$$

and premises in $\bigcup_{1\le p\le m'}[S'_p]_{X_{i_1}\mapsto\Theta_1}$. By repeating this process k times, we obtain a derived rule (R_k) with conclusion $\Theta_0,\ldots,\Theta_{k+1}\Rightarrow\Phi$ and with premises obtained from $S_1,\ldots,S_m$ by applying substitution operations $Y\mapsto(\Theta_0;\Theta_{k+1}\Rightarrow\Phi)$, $X_{i_1}\mapsto\Theta_1,\ldots,X_{i_k}\mapsto\Theta_k$.

Now (R_k) consists only of meta-variables $\mathcal{V}_0,\mathcal{V}_1$. We claim that all premises of (R_k) are pre-satisfied by $\{V^\circ|V\in\mathcal{V}_0\}$ and $\{V^\diamond|V\in\mathcal{V}_1\}$. Indeed let $\Theta_0,\Theta_{j_1},\ldots,\Theta_{j_l},\Theta_{k+1}\Rightarrow\Phi$ be any premise of (R_k) obtained from some $S_j\equiv X_{h_1},\ldots,X_{h_l}\Rightarrow Y$ by applying substitution operations. By assumption, $\Theta^\circ_{j_1}\in X^\bullet_{h_1},\ldots,\Theta^\circ_{j_l}\in X^\bullet_{h_l}$ and $Y^\bullet\subseteq\Theta^\circ_0\backslash\Phi^\diamond/\Theta^\circ_{k+1}$. Hence $\Theta^\circ_0\cdot\Theta^\circ_{j_1}\cdots\Theta^\circ_{j_l}\cdot\Theta^\circ_{k+1}\in\Phi^\diamond$.

Since only structural rules are used to derive (R_k) and all of them are assumed to be pre-valid, we conclude $\Theta^\circ_0\cdots\Theta^\circ_{k+1}\in\Phi^\diamond$. ■

5. Semantic Reductive Cut-Elimination

In this section we show that coherence and propagation are sufficient conditions for a simple sequent calculus $\mathcal{L}$ to admit reductive cut-elimination. Moreover, if a logical connective $\star$ satisfies rigidity, then $\star$ admits axiom expansion. These results are obtained using a powerful semantic technique introduced by Okada [14, 15, 16] (and also used in [3, 21]) who proved cut-elimination for linear logic by constructing a specific phase structure in which the validity of a formula directly implies its cut-free provability[5].

Let us fix a simple sequent calculus $\mathcal{L}$ and a set $\mathcal{S}$ of non-logical axioms. Let $\mathcal{F}$ be the set of formulae in $\mathcal{L}$. We can identify a finite sequence Γ of formulae with an element of the free monoid $\mathcal{F}^*$. Define

$$\begin{array}{rcl}\llbracket A\rrbracket &=& \{\Gamma\mid\Gamma\Rightarrow A \text{ is derivable in } \mathcal{L} \text{ from } \mathcal{S} \text{ without reducible cuts}\},\\ \llbracket\ \rrbracket &=& \{\Gamma\mid\Gamma\Rightarrow\quad \text{is derivable in } \mathcal{L} \text{ from } \mathcal{S} \text{ without reducible cuts}\}.\end{array}$$

[5] In more detail, the argument goes as follows: we codify formulae by $\mathcal{A}$ and cut-free provability by $\mathcal{B}_\perp$. Since $\mathcal{A}$ and $\mathcal{B}_\perp$ themselves do not constitute an algebraic structure that models $\mathcal{L}$, we build more abstract objects (i.e. closed sets) on them. By soundness, any sequent provable using the cut rule is true on the closed sets. Then *Okada's Lemma* —a key result to be proved for each logical connective— provides a link between truth on the closed sets and cut-free provability codified by $\mathcal{B}_\perp$. For this argument to work, two requirements must be satisfied. First, the structural rules codified in the primitive objects ($\mathcal{A}$ and $\mathcal{B}_\perp$) must extend smoothly to the more abstract objects (closed sets); this holds when rules are propagating. Second, our asymmetric interpretation causes a split-up of Okada's Lemma into two parts. Hence they must be composed together to regain the effect; this is guaranteed when logical connectives are coherent. Therefore propagation and coherence are the keys to establish (reductive) cut-elimination. Rigidity provides us with another way to compose two parts of Okada's Lemma, thus yielding axiom expansion.

Then $\mathbf{L} = (\mathcal{F}, \mathcal{B}, [\![\ \ ]\!])$ with $\mathcal{B} = \{[\![A]\!] | A \in \mathcal{F}\}$ is a prephase structure. In the sequel, the induced closure operator $C_{\mathbf{L}}$ will be simply written as C. Although these devices are enough to prove cut-elimination, for reductive cut-elimination we also need:

- $[\![A]\!]'_p = \{\Gamma \mid \Gamma \Rightarrow A$ is derivable in $\mathcal{L}$ from $\mathcal{S}$ without reducible cuts and without A being a context formula of any logical/structural rule$\}$.
- $C'_p(A) = \bigcap\{\Delta\backslash[\![D]\!]/\Sigma \mid \Delta, A, \Sigma \Rightarrow D$ is derivable in $\mathcal{L}$ from $\mathcal{S}$ without reducible cuts and without A being a context formula of any logical/structural rule$\}$.
- $[\![A]\!]_p = C([\![A]\!]'_p), \quad C_p(A) = C'_p(A) \cap [\![A]\!]_p$.

It is clear that $C_p(A)$, $[\![A]\!]_p$ and $[\![A]\!]$ are closed, and $A \in C_p(A) \subseteq [\![A]\!]_p \subseteq [\![A]\!]$.

We now adapt to our setting *Okada's Lemma* [14, 15, 16], which plays a prominent role in proving algebraic cut-elimination. Since we have two interpretations for each logical connectives, Okada's Lemma is accordingly split into two parts:

LEMMA 5.32. *Let $\star(\vec{A}) \equiv \star(A_1, \dots, A_n)$ and suppose that for every $1 \leq i \leq n$, $X_i^\bullet$ is a closed set such that $A_i \in X_i^\bullet \subseteq [\![A_i]\!]$. We then have:*

(1) $(\star, r)^{\mathbf{L}}(\vec{X}^\bullet) \subseteq [\![\star(\vec{A})]\!]_p$, *(2)* $C_p(\star(\vec{A})) \subseteq (\star, l)^{\mathbf{L}}(\vec{X}^\bullet)$.

PROOF. (1) It suffices to show that for each right logical rule $(\star, r)_i$

$$\frac{\Upsilon_1 \Rightarrow \Psi_1 \quad \cdots \quad \Upsilon_m \Rightarrow \Psi_m}{\Theta \Rightarrow \star(\vec{X})}\ (\star, r)_i$$

$(\star, r)_i^{\mathbf{L}}(\vec{X}^\bullet) \subseteq [\![\star(\vec{A})]\!]_p$. Suppose that $\Theta \equiv Y_1, \dots, Y_h$, and let $Y_1^\bullet, \dots, Y_h^\bullet$ be any closed sets such that $\Upsilon_1^\bullet \subseteq \Psi_1^\bullet, \dots, \Upsilon_m^\bullet \subseteq \Psi_m^\bullet$. Our goal is to show $\Theta^\bullet \subseteq [\![\star(\vec{A})]\!]_p$, from which we can to conclude $(\star, r)_i^{\mathbf{L}}(\vec{X}^\bullet) \subseteq [\![\star(\vec{A})]\!]_p$.

Choose arbitrary $\Gamma_1 \in Y_1^\bullet, \dots, \Gamma_h \in Y_h^\bullet$ and denote each Γ_i by $\check{Y}_i$ (condition (log1) ensures that $\check{Y}_i$ is uniquely determined by Y_i). For each active meta-variable X_j, let $\check{X}_j$ stand for A_j. With this notation, we have $\check{Z} \in Z^\bullet$ for any meta-variable Z occurring in $(\star, r)_i$, and moreover $Z^\bullet \subseteq [\![\check{Z}]\!]$ when Z is an active meta-variable. Let us extend this notation to any sequence $\Upsilon \equiv Z_1, \dots, Z_k$ by letting $\check{\Upsilon}$ be $\check{Z}_1, \dots, \check{Z}_k$ (and empty when Υ is). We then have $\check{\Upsilon}_p \in \Upsilon_p^\bullet \subseteq \Psi_p^\bullet \subseteq [\![\check{\Psi}_p]\!]$ for every $1 \leq p \leq m$, that means that $\check{\Upsilon}_p \Rightarrow \check{\Psi}_p$ is derivable from $\mathcal{S}$ without reducible cuts. Moreover, due to condition (log2) the inference

$$\frac{\check{\Upsilon}_1 \Rightarrow \check{\Psi}_1 \quad \cdots \quad \check{\Upsilon}_m \Rightarrow \check{\Psi}_m}{\check{\Theta} \Rightarrow \star(\vec{A})}$$

is an instance of a right logical rule introducing $\star$. This shows that $\Gamma_1, \dots, \Gamma_h \equiv \check{\Theta} \in [\![\star(\vec{A})]\!]'_p$. Since this holds for any $\Gamma_1 \in Y_1^\bullet$, $\dots$, $\Gamma_h \in Y_h^\bullet$, we conclude that $\Theta^\bullet \subseteq C([\![\star(\vec{A})]\!]'_p) = [\![\star(\vec{A})]\!]_p$.

(2) It suffices to show that for each left logical rule $(\star, l)_j$

$$\frac{\Upsilon_1 \Rightarrow \Psi_1 \quad \cdots \quad \Upsilon_m \Rightarrow \Psi_m}{\Theta_l, \star(\vec{X}), \Theta_r \Rightarrow \Xi}\ (\star, l)_j$$

$C_p(\star(\vec{A})) \subseteq (\star, l)_j^{\mathbf{L}}(\vec{X}^\bullet)$. Suppose that $\Xi \equiv W$ (the case $\Xi \equiv \emptyset$ is easier), $\Theta_l \equiv Y_1, \dots, Y_h$, and $\Theta_r \equiv Y_{h+1}, \dots, Y_k$. As before, let $Y_1^\bullet, \dots, Y_k^\bullet, W^\bullet$ be closed sets such that $\Upsilon_1^\bullet \subseteq \Psi_1^\bullet$, $\dots$, $\Upsilon_m^\bullet \subseteq \Psi_m^\bullet$. Our goal is to show $\Theta_l^\bullet \bullet C_p(\star(\vec{A})) \bullet \Theta_r^\bullet \subseteq W^\bullet$.

Choose $\Gamma_1 \in Y_1^\bullet, \dots, \Gamma_k \in Y_k^\bullet$ and Δ_l, Φ, Δ_r (with $|\Phi| \leq 1$) such that $W^\bullet \subseteq \Delta_l \backslash [\![\Phi]\!] / \Delta_r$. For each meta-variable Z occurring in $(\star, l)_j$, define $\check{Z}$ as before. We again have $\check{Z} \in Z^\bullet$, and moreover $Z^\bullet \subseteq [\![\check{Z}]\!]$ when Z is an active meta-variable. In addition, we define:

$$\begin{array}{rcll} \overline{\Upsilon \Rightarrow \Psi} & \equiv & \Delta_l, \check{\Upsilon}, \Delta_r \Rightarrow \Phi & \text{if } \Psi \equiv W; \\ & \equiv & \check{\Upsilon} \Rightarrow \check{\Psi} & \text{otherwise.} \end{array}$$

Then for each premise $\Upsilon_p \Rightarrow \Psi_p$ $(1 \leq p \leq m)$, $\overline{\Upsilon_p \Rightarrow \Psi_p}$ is derivable from $\mathcal{S}$ without reducible cuts; indeed, when $\Psi_p \equiv W$, we have $\check{\Upsilon}_p \in \Upsilon_p^\bullet \subseteq W^\bullet \subseteq \Delta_l \backslash [\![\Phi]\!] / \Delta_r$, hence $\Delta_l, \check{\Upsilon}_p, \Delta_r \Rightarrow \Phi$ is derivable from $\mathcal{S}$ without reducible cuts. Now the inference

$$\frac{\overline{\Upsilon_1 \Rightarrow \Psi_1} \quad \cdots \quad \overline{\Upsilon_m \Rightarrow \Psi_m}}{\Delta_l, \check{\Theta}_l, \star(\vec{A}), \check{\Theta}_r, \Delta_r \Rightarrow \Phi}\ (\star, l)_j$$

is an instance of a left logical rule introducing $\star$ due to conditions (log2) and (log3). Therefore, $C'_p(\star(\vec{A})) \subseteq (\Delta_l, \check{\Theta}_l) \backslash [\![\Phi]\!] / (\check{\Theta}_r, \Delta_r)$. By noting that $C_p(\star(\vec{A})) \subseteq C'_p(\star(\vec{A}))$, we obtain $\{\check{\Theta}_l\} \bullet C_p(\star(\vec{A})) \bullet \{\check{\Theta}_r\} \subseteq \Delta_l \backslash [\![\Phi]\!] / \Delta_r$. Since this holds for arbitrary $\Gamma_i \in Y_i^\bullet$ $(1 \leq i \leq k)$ and arbitrary Δ_l, Φ, Δ_r such that $W^\bullet \subseteq \Delta_l \backslash [\![\Phi]\!] / \Delta_r$, we conclude $\Theta_l^\bullet \bullet C_p(\star(\vec{A})) \bullet \Theta_r^\bullet \subseteq W^\bullet$. ∎

When logical connectives are coherent we can compose the two parts of the above lemma in a harmonious way. For any formula A, define $f_0(A)$ by induction on the complexity of A as follows:

$$\begin{array}{rcl} f_0(\alpha) & = & [\![\alpha]\!]_p, \\ f_0(\star(\vec{A})) & = & (\star, l)^{\mathbf{L}}(\overrightarrow{f_0(A)}) \cap [\![\star(\vec{A})]\!]_p. \end{array}$$

LEMMA 5.33. *If all the logical connectives in $\mathcal{L}$ are coherent, then f_0 is a valuation such that for any compound formula $\star(\vec{A})$ we have:*

$$\begin{array}{ccccc} & & (\star, r)^{\mathbf{L}}(\overrightarrow{f_0(A)}) & & \\ & & \cap\!\!\mid & & \\ C_p(\star(\vec{A})) & \subseteq & f_0(\star(\vec{A})) & \subseteq & [\![\star(\vec{A})]\!]_p \\ & & \cap\!\!\mid & & \\ & & (\star, l)^{\mathbf{L}}(\overrightarrow{f_0(A)}) & & \end{array}$$

PROOF. By induction on the complexity of A, using coherence, Lemma 5.32 and the fact that $A \in C_p(A) \subseteq [\![A]\!]_p \subseteq [\![A]\!]$. ■

From the horizontal line above, we recover the original form of Okada's Lemma: $A \in f_0(A) \subseteq [\![A]\!]$ for any formula A. As a consequence follows:

THEOREM 5.34 (Completeness). *If $\Gamma \Rightarrow \Delta$ is true under f_0 in $\mathbf{L}$, then $\Gamma \Rightarrow \Delta$ is derivable from $\mathcal{S}$ without reducible cuts.*

PROOF. Let $\Gamma \equiv A_1, \dots, A_n$ and $\Delta \equiv B$ (the case $\Delta \equiv \emptyset$ is similar). Since $f_0(A_1) \bullet \cdots \bullet f_0(A_n) \subseteq f_0(B)$, Okada's Lemma implies $A_1, \dots, A_n \in [\![B]\!]$, i.e., $\Gamma \Rightarrow \Delta$ is derivable from $\mathcal{S}$ without reducible cuts. ■

Note that the above theorem holds independently of the structural rules of $\mathcal{L}$. Their properties, instead, are used to prove the following lemma:

LEMMA 5.35. *If all the structural rules of $\mathcal{L}$ are propagating then $\mathbf{L} = (\mathcal{F}, \mathcal{B}, [\![\ \ ]\!])$ is an $\mathcal{L}$-structure.*

PROOF. We first prove that all the structural rules of $\mathcal{L}$ are pre-valid in $\mathbf{L}$. Let (R) be a structural rule of $\mathcal{L}$ with premises $\Upsilon_1 \Rightarrow \Psi_1, \dots\ \Upsilon_n \Rightarrow \Psi_n$ and conclusion $\Theta \Rightarrow \Xi$. Suppose that Θ consists of meta-variables $X_1, \dots, X_m$ and $\Xi \equiv Y$ (the case $\Xi \equiv \emptyset$ is similar). Let $X_1^\circ, \dots, X_m^\circ$ be elements of $\mathcal{F}$ and $Y^\diamond = [\![Z]\!]$ an element of $\mathcal{B}$ such that $\Upsilon_i^\circ \in \Psi_i^\diamond$ for each $1 \leq i \leq n$. When $\Psi_i \equiv Y$ ($\Psi_i \equiv \emptyset$, respectively), this means that $\Upsilon_i^\circ \Rightarrow Z$ ($\Upsilon_i^\circ \Rightarrow\ $, respectively) is derivable from $\mathcal{S}$ without reducible cuts. Since these are instances of the premises of (R), the conclusion $\Theta^\circ \Rightarrow Z$ is also derivable from $\mathcal{S}$ without reducible cuts. From this, we conclude that $\Theta^\circ \in [\![Z]\!] = Y^\diamond$.

Since the structural rules of $\mathcal{L}$ are propagating, they are valid in $\mathbf{L}$. Therefore $\mathbf{L}$ is an $\mathcal{L}$-structure. ■

LEMMA 5.36. *All non-logical axioms in $\mathcal{S}$ are true under the valuation f_0.*

PROOF. Let $A_1, \dots, A_n \Rightarrow B$ be a non-logical axiom in $\mathcal{S}$. To show that it is true under f_0, by Lemma 5.33 it is enough to prove that $[\![A_1]\!]_p \bullet \cdots \bullet [\![A_n]\!]_p \subseteq C_p(B)$. For this, we have to prove (i) $[\![A_1]\!]'_p \bullet \cdots \bullet [\![A_n]\!]'_p \subseteq C'_p(B)$ and (ii) $[\![A_1]\!]'_p \bullet \cdots \bullet [\![A_n]\!]'_p \subseteq [\![B]\!]'_p$. To show (i), let $\Gamma_i \in [\![A_i]\!]'_p$ for $1 \leq i \leq n$. By definition, $\Gamma_i \Rightarrow A_i$ is derivable from $\mathcal{S}$ without reducible cuts and without A_i being a context formula of any logical/structural rule. Suppose that (*) $\Delta_l, B, \Delta_r \Rightarrow \Sigma$ is derivable from $\mathcal{S}$ without reducible cuts and without B being a context formula of any logical/structural rule. By repeatedly applying (CUT) to these sequents and $A_1, \dots, A_n \Rightarrow B$, we obtain a derivation of $\Delta_l, \Gamma_1, \dots, \Gamma_n, \Delta_r \Rightarrow \Sigma$ without reducible cuts. Thus $\Gamma_1, \dots, \Gamma_n \in \Delta_l \backslash [\![\Sigma]\!] / \Delta_r$. Since this holds for arbitrary Δ_l, Δ_r and Σ satisfying (*), we obtain $\Gamma_1, \dots, \Gamma_n \in C'_p(B)$. (ii) can be shown similarly. ■

THEOREM 5.37. *Let $\mathcal{L}$ be a simple sequent calculus in which structural rules are propagating and logical connectives are coherent. Then $\mathcal{L}$ admits reductive cut-elimination.*

PROOF. Suppose that a sequent S_0 is derivable from $\mathcal{S}$ in $\mathcal{L}$. Lemmas 5.35 and 5.36 ensure that $\mathbf{L} = (\mathcal{F}, \mathcal{B}, [\![\ \ ]\!])$ is an $\mathcal{L}$-structure and that axioms in $\mathcal{S}$ are true under f_0. By Theorem 4.23, S_0 is true under f_0. The claim follows by Theorem 5.34. ■

When a logical connective $\star$ is rigid and the structural rules of $\mathcal{L}$ are propagating, we can modify the above construction to show that $\star$ admits axiom expansion. Define $\mathbf{L}_e = (\mathcal{F}, \mathcal{B}_e, [\![\ \ ]\!]_e)$ by

$$\begin{aligned} [\![A]\!]_e &= \{\Gamma \mid \Gamma \Rightarrow A \text{ has a cut-free derivation from atomic axioms}\} \\ [\![\ \ ]\!]_e &= \{\Gamma \mid \Gamma \Rightarrow \quad \text{has a cut-free derivation from atomic axioms}\} \end{aligned}$$

and $\mathcal{B}_e = \{[\![A]\!]_e | A \in \mathcal{F}\}$. Similarly to Lemma 5.32, we can prove that: whenever $A_i \in X_i^\bullet \subseteq [\![A_i]\!]_e$ for every $1 \leq i \leq n$, we have $(\star, r)^{\mathbf{L}_e}(\vec{X}^\bullet) \subseteq [\![\star(\vec{A})]\!]_e$ and $\star(\vec{A}) \in (\star, l)^{\mathbf{L}_e}(\vec{X}^\bullet)$. Moreover, in analogy with Lemma 5.35 we can also prove that $\mathbf{L}_e$ is an $\mathcal{L}$-structure. Therefore the rigidity of $\star$ allows the composition of the two inclusions above as:

$$\star(\vec{X}) \in (\star, l)^{\mathbf{L}_e}(\overrightarrow{[\![X]\!]_e}) \subseteq (\star, r)^{\mathbf{L}_e}(\overrightarrow{[\![X]\!]_e}) \subseteq [\![\star(\vec{X})]\!]_e.$$

This means that $\star(\vec{X}) \Rightarrow \star(\vec{X})$ has a cut-free derivation from atomic axioms $X_i \Rightarrow X_i$ $(1 \leq i \leq n)$. We therefore have:

THEOREM 5.38. *Let $\mathcal{L}$ be a simple sequent calculus in which structural rules are propagating. If a logical connective $\star$ is rigid in $\mathcal{L}$, then $\star$ admits axiom expansion.*

To sum up, we have obtained:

COROLLARY 5.39. *Let $\mathcal{L}$ be a simple sequent calculus.*
1. The following are equivalent:

(i) $\mathcal{L}$ admits reductive cut-elimination.

(ii) All the logical rules of $\mathcal{L}$ are reductive and all the structural rules of $\mathcal{L}$ are weakly substitutive.

(iii) All the logical connectives of $\mathcal{L}$ are coherent and all the structural rules of $\mathcal{L}$ are propagating.

2. When the structural rules of $\mathcal{L}$ are weakly substitutive, a logical connective $\star$ admits axiom expansion in $\mathcal{L}$ iff $\star$ is rigid.

On the one hand, the equivalence between (i) and (ii) provides us with a syntactic and *modular* (rule-by-rule) way of checking whether a simple sequent calculus admits reductive cut-elimination. On the other hand, the equivalence between (i) and (iii) shows an algebraic perspective of reductive cut-elimination. Finally, the duality between coherence and rigidity together with our main results 1 and 2 suggests a duality between reductive cut-elimination and axiom expansion, for simple sequent calculi whose structural rules are weakly substitutive.

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