Hypersequent Calculi for Gödel Logics — a Survey

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Abstract

Hypersequent calculi arise by generalizing standard sequent calculi to refer to whole contexts of sequents instead of single sequents. We present a number of results using hypersequents to obtain a Gentzen-style characterization for the family of Gödel logics. We first describe analytic calculi for propositional finite and infinite-valued Gödel logics. We then show that the framework of hypersequents allows one to move straightforwardly from the propositional level to first-order as well as propositional quantification. A certain type of modalities, enhancing the expressive power of Gödel logic, is also considered.

1 Introduction

In this paper we survey a number of results in proof theory for Gödel logics, that have been scattered over several works of the authors. We also include some new material. Our aim is to show that a particular type of calculus — based on so-called *hypersequents* — is a simple but versatile tool for handling several important logics in Gentzen's spirit. Indeed, in his seminal paper on the concept of logical inference, Gentzen [44, 45] achieved — among other things — a satisfactory characterization of the relation between classical and intuitionistic proofs in terms of *sequents* as basic objects of derivations. In particular, the sequent calculus **LJ** for *intuitionistic logic* is defined by simply restricting the right hand side of all sequents to contain at most one formula in the sequent calculus **LK** for *classical logic*.

In the presence of the cut rule it is almost trivial to define sound and complete sequent calculi for all kinds of other logics for which a Hilbert style axiomatization is known. One can, e.g., simply extend (a suitable version of) Gentzen's calculi with additional axioms. However, it should be clear that only *cut-free* and therefore *analytic* Gentzen-style systems share the important proof theoretical properties of **LJ** and **LK**. In particular, (some form of) analyticity is a pre-condition for efficient — human or mechanized — proof search. Consequently, a main challenge in proof theory

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is to extend Gentzen's celebrated results to other logics by defining appropriate calculi which enjoy *cut-elimination*. I.e., one should be able to show how any given derivation can be transformed into an equivalent one that does not contain applications of the cut rule.

A large range of variants and extensions of Gentzen's original sequent calculi have been introduced in the last decades to provide analytic proof systems for many types of non-classical logics. As an example, we just mention *labeled* systems [42] to accommodate modalities and the generalization of the (binary) sequent arrow to an n-ary relation ("many placed sequents") in order to describe analytic deduction in finite-valued logics in a uniform manner see, e.g., [60, 56, 25, 48, 19] as well as the more recent survey article [18].

Here we deal with another natural extension of Gentzen's calculi, called hypersequent calculi. A hypersequent calculus is defined by incorporating Gentzen's original calculus (**LJ**, **LK** or a substructural version of it) as a sub-calculus and adding an additional layer of information by considering a single sequent to live in the context of finite multisets of sequents (called hypersequents). This opens the possibility to define new rules that allow to "exchange information" between different sequents. It is this type of rules which increases the expressive power of hypersequent calculi compared to ordinary sequent calculi.

To illustrate the method of hypersequents we investigate the family of (propositional and quantified) *Gödel logics* that is of particular interest in its own.

Propositional finite-valued Gödel logics were introduced (implicitly) by Gödel [47] to show that intuitionistic logic does not have a characteristic finite matrix. Dummett [34] later generalized these to an infinite set of truth-values, and showed that the set of its tautologies is axiomatized by intuitionistic logic extended by the linearity axiom $(A \supset B) \vee (B \supset A)$. Hence infinite-valued Gödel logic **G** is also called Gödel-Dummett logic or Dummett's **LC**.

Gödel logics naturally turn up in a number of different areas of logic and computer science. For instance, Dunn and Meyer [35] pointed out their relation to relevance logics; Visser [68] employed **G** in investigations of the provability logic of Heyting arithmetic; three-valued Gödel logic $\mathbf{G}_3$ has been used to model strong equivalence between logic programs [53]; and more importantly, **G** was recognized as one of the most important formalizations of fuzzy logic [49].

A hypersequent calculus for **G** was introduced in [4]. This calculus — which we call **HG** — is defined by embedding Gentzen's **LJ**-sequents into hypersequents and by adding suitable structural rules to manipulate the additional layer of structure to the basic objects of inferences. **HG** will be described in Section 3.

In Section 3.2 we will present hypersequent calculi $\mathbf{HG}_k$ for the finite-valued Gödel logics $\mathbf{G}_k$ with k truth-values ($k \geq 2$). These calculi, introduced in [30], are simply obtained by adding one more structural rule to **HG**. This is done in a uniform way for all k. A new proof of the cut-elimination theorem for $\mathbf{HG}_k$ is provided.

Section 3.3 contains new results on $\mathbf{G}_\Delta$, i.e., Gödel logic extended by the "projection modality" Δ of [9] (see also [65, 64]). Indeed, by adding suitable rules to **HG** one obtains a hypersequent calculus for $\mathbf{G}_\Delta$.

Finally, in Section 4 we will show that hypersequents allow to extend analytic calculi for *propositional* Gödel logic to include quantifiers. In particular, we will consider two different forms of quantification: *first-order quantifiers* (universal and existential quantification over object variables) and *propositional* or *"fuzzy" quantifiers* (universal and existential quantification over propositions). Refining and restructuring the results contained in [20] and [24], we shall discuss analytic hypersequent calculi for both first-order and quantified propositional Gödel logic. As we shall see, the first-order calculus allows one to prove (a suitable version of) Gentzen's mid-sequent theorem.

2 Hypersequent Calculi

Hypersequent calculi have been introduced in [2] and [54]. They are a natural generalization of Gentzen's sequent calculi.

We take sequents to be expressions of the form $\Gamma \Rightarrow \Pi$ where Γ and Π are finite multisets of formulas [44, 59]. Hypersequent calculi do not alter the definition of a sequent at all, but just add an additional level of *context* to ordinary sequents.

Definition 1 A *hypersequent* is a multiset[1], written as

$$\Gamma_1 \Rightarrow \Pi_1 \mid \ \dots \ \mid \Gamma_n \Rightarrow \Pi_n$$

where, for all $i = 1, \dots n$, $\Gamma_i \Rightarrow \Pi_i$ is an ordinary sequent. $\Gamma_i \Rightarrow \Pi_i$ is called a *component* of the hypersequent. A hypersequent is called *single-conclusioned* if, for every $i = 1, \dots, n$, Π_i consists of at most one formula.

The symbol "|" is intended to denote disjunction at the meta-level. (This will be made precise in Definition 2, below.)

Just as ordinary sequent calculi, hypersequent calculi consist in initial hypersequents (i.e., axioms) as well as *logical* and *structural* rules. The axioms and logical rules are essentially the same as in sequent calculi. The only difference is the presence of *side hypersequents*, denoted by G and G', representing (possibly empty) hypersequents.

The structural rules are divided into *internal* and *external rules*. The internal structural rules deal with formulas within components. When present, they are the same as in ordinary sequent calculi (weakening and contraction). The external structural rules manipulate whole components of a hypersequent. These are external weakening (ew) and external contraction (ec) (see Table 1).

As an example, in Table 1 one can find a hypersequent calculus for intuitionistic logic **IL** which we call **HIL**. The "hyperlevel" of this calculus is in fact redundant, in the sense that a hypersequent $\Gamma_1 \Rightarrow \Pi_1 \mid \ \dots \ \mid \Gamma_k \Rightarrow \Pi_k$ is derivable if and only if for some $i \in \{1, \dots, k\}$, already $\Gamma_i \Rightarrow \Pi_i$ is derivable.

In hypersequent calculi it is possible to define *additional external structural rules* which simultaneously act on several components of one or more hypersequents. Below are some examples of rules of this kind (see [2, 3, 5, 27, 30] for further examples).

- As shown in [31], by adding to **HIL** the following rule

$$\frac{G \mid \Gamma, \Gamma' \Rightarrow}{G \mid \Gamma \Rightarrow \ \mid \Gamma' \Rightarrow} \ (lq)$$

[1] If one prefers sequences over multisets as basic objects of inference then a permutation rule has to be added to the calculus.

Axioms

$$A \Rightarrow A \qquad \bot \Rightarrow A$$

Cut Rule

$$\frac{G \mid \Gamma' \Rightarrow A \quad G' \mid A, \Gamma \Rightarrow C}{G \mid G' \mid \Gamma, \Gamma' \Rightarrow C}\ (cut)$$

External Structural Rules

$$\frac{G}{G \mid \Gamma \Rightarrow A}\ (ew) \qquad \frac{G \mid \Gamma \Rightarrow A \mid \Gamma \Rightarrow A}{G \mid \Gamma \Rightarrow A}\ (ec)$$

Internal Structural Rules

$$\frac{G \mid \Gamma \Rightarrow C}{G \mid \Gamma, A \Rightarrow C}\ (w, l) \qquad \frac{G \mid \Gamma \Rightarrow}{G \mid \Gamma \Rightarrow C}\ (w, r) \qquad \frac{G \mid \Gamma, A, A \Rightarrow C}{G \mid \Gamma, A \Rightarrow C}\ (c, l)$$

Logical Rules

$$\frac{G \mid \Gamma, A \Rightarrow B}{G \mid \Gamma \Rightarrow A \supset B}\ (\supset, r) \qquad \frac{G \mid \Gamma \Rightarrow A \quad G' \mid B, \Gamma \Rightarrow C}{G \mid G' \mid \Gamma, A \supset B \Rightarrow C}\ (\supset, l)$$

$$\frac{G \mid \Gamma \Rightarrow A \quad G' \mid \Gamma \Rightarrow B}{G \mid G' \mid \Gamma \Rightarrow A \wedge B}\ (\wedge, r) \qquad \frac{G \mid \Gamma, A_i \Rightarrow C}{G \mid \Gamma, A_1 \wedge A_2 \Rightarrow C}\ (\wedge_i, l)_{i=1,2}$$

$$\frac{G \mid \Gamma \Rightarrow A_i}{G \mid \Gamma \Rightarrow A_1 \vee A_2}\ (\vee_i, r)_{i=1,2} \qquad \frac{G \mid \Gamma, A \Rightarrow C \quad G' \mid \Gamma, B \Rightarrow C}{G \mid G' \mid \Gamma, A \vee B \Rightarrow C}\ (\vee, l)$$

Table 1: Hypersequent Calculus **HIL** for Intuitionistic Logic

one obtains a cut-free calculus for the intermediate logic **LQ** [51] whose axiomatization is given by adding the weak law of excluded middle, i.e., $(A \supset \bot) \vee ((A \supset \bot) \supset \bot)$ to a Hilbert style calculus for intuitionistic propositional logic (see, e.g., Table 2 below).

- Let us consider the hypersequent calculus for **aMALL**[2], i.e. the calculus whose rules are those of the sequent calculus for linear logic without exponential connectives, augmented by side hypersequents and with in addition internal weakening rules, (ew) and (ec). By adding to it either

$$\frac{G \mid \Gamma_1, \Gamma_2, \Gamma_3 \Rightarrow \Pi_1, \Pi_2, \Pi_3 \quad G' \mid \Gamma_1', \Gamma_2', \Gamma_3' \Rightarrow \Pi_1', \Pi_2', \Pi_3'}{G \mid G' \mid \Gamma_1, \Gamma_1' \Rightarrow \Pi_1, \Pi_1' \mid \Gamma_2, \Gamma_2' \Rightarrow \Pi_2, \Pi_2' \mid \Gamma_3, \Gamma_3' \Rightarrow \Pi_3, \Pi_3'}\ (M) \quad \text{or}$$

$$\frac{G \mid \Sigma, \Gamma_1 \Rightarrow \Pi_1, \Pi \quad G' \mid \Sigma, \Gamma_2 \Rightarrow \Pi_2, \Pi}{G \mid G' \mid \Gamma_1, \Gamma_2 \Rightarrow \Pi_1, \Pi_2 \mid \Sigma \Rightarrow \Pi}\ (3\text{-weak})$$

one obtains a cut-free calculus for 3-valued Łukasiewicz logic [4, 31].

[2]Multiplicative Additive fragment of Linear Logic [46] extended by weakening rules

Note: Henceforth we will consider single-conclusioned hypersequent calculi, i.e. containing only single-conclusioned hypersequents.

To assist a better understanding of hypersequents consider the following definitions:

Definition 2 The *generic interpretation* of a sequent $\Gamma \Rightarrow B$, denoted by $Int(\Gamma \Rightarrow B)$, is defined by $(\bigwedge \Gamma \supset B^*)$, where $\bigwedge \Gamma$ stands for the conjunction of the formulas in Γ or $\top$ if Γ is empty, and B^* is B or $\perp$ if B is empty. The *generic interpretation* of a hypersequent $\Gamma_1 \Rightarrow A_1 \mid \ldots \mid \Gamma_n \Rightarrow A_n$ is defined by $Int(\Gamma_1 \Rightarrow A_1) \vee \ldots \vee Int(\Gamma_n \Rightarrow A_n)$.

Definition 3 A (Hyper)sequent rule is *sound* for a Hilbert style system **sL**, if whenever **sL** derives the generic interpretations of its premises, **sL** derives the generic interpretation of its conclusion too. A (hyper)sequent calculus **HL** is called *sound* for **sL** if all the axioms and rules of **HL** are sound for **sL**. **HL** is called *complete* for **sL** if for all formulas A derivable in **sL**, the (hyper)sequent $\Rightarrow A$ is derivable in **HL**.

3 Hypersequent Calculi for Gödel logics

Gödel logics can be seen both as intermediate logics, i.e., logics including intuitionistic and included in classical logic, and as many-valued logics. On the one hand, they are characterized by the class of all rooted linearly ordered Kripke models (with at most k worlds, $k \geq 1$, in the case of $(k+1)$-valued Gödel logic $\mathbf{G}_{k+1}$), see, e.g., [41, 26]. On the other hand, their connectives can be interpreted as functions over either the real interval $[0, 1]$ (for **G**)[3] or $\{0, \frac{1}{k}, \ldots, \frac{k-1}{k}, 1\}$ (for $\mathbf{G}_{k+1}$)[4]. More precisely, let $v_{\mathcal{I}}$ be a mapping of propositional variables into the set of truth-values. $v_{\mathcal{I}}$ can be extended to formulas of Gödel logics as follows:

$$v_{\mathcal{I}}(A \wedge B) = \min\{v_{\mathcal{I}}(A), v_{\mathcal{I}}(B)\} \qquad v_{\mathcal{I}}(A \vee B) = \max\{v_{\mathcal{I}}(A), v_{\mathcal{I}}(B)\}$$

$$v_{\mathcal{I}}(A \supset B) = \begin{cases} 1 & \text{if } v_{\mathcal{I}}(A) \leq v_{\mathcal{I}}(B) \\ v_{\mathcal{I}}(B) & \text{otherwise} \end{cases} \qquad v_{\mathcal{I}}(\perp) = 0$$

As usual, $\neg A$ can be defined as $A \supset \perp$. A formula A is a *tautology* iff for all $v_{\mathcal{I}}$, $v_{\mathcal{I}}(A) = 1$. Moreover A is a *logical consequence* of a set of formulas Γ iff, for all $v_{\mathcal{I}}$, $\min\{v_{\mathcal{I}}(\gamma) \mid \gamma \in \Gamma\} \leq v_{\mathcal{I}}(A)$.

As mentioned above, a Hilbert style calculus **sG** for **G** is obtained by adding the linearity axiom (Lin) $(A \supset B) \vee (B \supset A)$ to any Hilbert style calculus for **IL**.

For sake of concrete argumentation, we take **IL** to be axiomatized by the Hilbert style system **sIL** presented in Table 2.

Avron's calculus **HG** for **G** ([3], called **HLC** there) is defined by extending the hypersequent calculus **HIL** for intuitionistic logic with the following *communication rule*:

$$\frac{G \mid \Gamma, \Gamma' \Rightarrow A \quad G' \mid \Gamma_1, \Gamma_1' \Rightarrow A'}{G \mid G' \mid \Gamma, \Gamma_1' \Rightarrow A \mid \Gamma', \Gamma_1 \Rightarrow A'}\ (com)$$

[3]Note that Dummett's **LC** was originally defined in [34] using the set of truth-values $\{1\} \cup \{1 - \frac{1}{n} : n \geq 1\}$. However, at the propositional level any infinite set of truth-values gives rise to the same set of tautologies in Gödel logic.

[4]In fact, one can take any set of $k+1$ real numbers from $[0, 1]$, that includes 0 and 1.

Axioms

$I1$	$A \supset (B \supset A)$	$I8$	$(A \supset B) \supset [(C \supset A) \supset (C \supset B)]$
$I2$	$(A \wedge B) \supset A$	$I9$	$[A \supset (C \supset B)] \supset [C \supset (A \supset B)]$
$I3$	$(A \wedge B) \supset B$	$I10$	$(A \supset C) \wedge (B \supset C) \supset ((A \vee B) \supset C)$
$I4$	$A \supset (B \supset (A \wedge B))$	$I11$	$\bot \supset A$
$I5$	$(C \supset A) \wedge (C \supset B) \supset (C \supset (A \wedge B))$	$I12$	$(A \supset (B \supset C)) \supset (A \wedge B \supset C)$
$I6$	$A \supset (A \vee B)$	$I13$	$[A \supset (A \supset B)] \supset (A \supset B)$
$I7$	$B \supset (A \vee B)$		

Rule

(Modus Ponens) $\dfrac{A \quad A \supset B}{B}$

Table 2: Hilbert style calculus **sIL** for Intuitionistic Logic

Remark 4 Avron suggested that a hypersequent can be thought of as a multiprocess ([3]). Under this interpretation, (com) is intended to model the exchange of information within multiprocesses.

Example 5 We display a proof of the linearity axiom $(A \supset B) \vee (B \supset A)$ in **HG**. Recall that this axiom is not valid in intuitionistic logic.

$$\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A \qquad B \Rightarrow B}{A \Rightarrow B \mid B \Rightarrow A}\,{\scriptstyle (com)}}{\Rightarrow A \supset B \mid \ \Rightarrow B \supset A}\,{\scriptstyle 2x(\supset,r)}}{\Rightarrow (A \supset B) \vee (B \supset A) \mid \ \Rightarrow (A \supset B) \vee (B \supset A)}\,{\scriptstyle 2x(\vee_i,r)}}{\Rightarrow (A \supset B) \vee (B \supset A)}\,{\scriptstyle (ec)}$$

To prove the soundness of (propositional) hypersequent calculi we introduce the following notion:

Definition 6 The *generic interpretation* $Int(r)$ of a (hyper)sequent rule

$$\frac{S_1 \quad \ldots \quad S_n}{S_0}\ (r) \qquad \text{with } n \geq 1$$

is defined as $Int(S_1) \supset (\ldots(Int(S_n) \supset Int(S_0))\ldots)$, where $Int(S_i)$, with $i = 0, \ldots, n$ stands for the generic interpretation of the (hyper)sequent S_i (see Definition 2).

Theorem 7 **HIL** is sound and complete for **IL**.

Proof The proof is relative to the Hilbert style calculus **sIL** for **IL**.

(*Soundness*) We prove the stronger claim that the generic interpretation of each rule of **HIL** is derivable in **sIL**. The generic interpretation of the axioms of **HIL** has the form $A \supset A$ and $\bot \supset A$, while that of the rules (ec) and (ew) is $(C \vee A \vee A) \supset$

$(C \vee A)$ and $(C \vee A) \supset (C \vee A \vee B)$, respectively. The corresponding derivations in **sIL** are straightforward. Observe that the derivability in **sIL** of $A \supset B$ implies the derivability of $(A \vee C) \supset (B \vee C)$. Therefore we can disregard side hypersequents in proving the soundness of the rules. The soundness of the remaining rules reduces to the derivation in **sIL** of single formulas obtained according to Definition 2.

(*Completeness*) Observe that Modus Ponens — the only rule of **sIL** — corresponds to the derivability of $A, A \supset B \Rightarrow B$ and the cut rule. It thus suffices to show that all the axioms of **sIL** are derivable in **HIL**. This is straightforward. ☐

Theorem 8 **HG** is sound and complete for **G**.

Proof The proof is relative to the Hilbert style calculus **sG** for **G**.

(*Soundness*) In addition to Theorem 7 one has to prove that the (*com*) rule is sound for **sG**. This amounts to showing, e.g., that **sG** derives the generic interpretation $Int^{-}(com)$ of (*com*) without side hypersequents, i.e. the formula

$$(\bigwedge \Gamma \wedge \bigwedge \Gamma' \supset A) \supset ((\bigwedge \Gamma_1 \wedge \bigwedge \Gamma_1' \supset A') \supset (\bigwedge \Gamma \wedge \bigwedge \Gamma_1' \supset A) \vee (\bigwedge \Gamma' \wedge \bigwedge \Gamma_1 \supset A'))$$

where $\bigwedge \Sigma$ stands for the conjunction of the formulas in Σ.

Using $I8$ and $I9$, together with $I6$ and $I7$ we obtain

$$(\bigwedge \Gamma_1' \supset \bigwedge \Gamma') \supset ((\bigwedge \Gamma \wedge \bigwedge \Gamma' \supset A) \supset (\bigwedge \Gamma \wedge \bigwedge \Gamma_1' \supset A) \vee (\bigwedge \Gamma' \wedge \bigwedge \Gamma_1 \supset A')) \quad \text{and}$$

$$(\bigwedge \Gamma' \supset \bigwedge \Gamma_1') \supset ((\bigwedge \Gamma_1 \wedge \bigwedge \Gamma_1' \supset A') \supset (\bigwedge \Gamma \wedge \bigwedge \Gamma_1' \supset A) \vee (\bigwedge \Gamma' \wedge \bigwedge \Gamma_1 \supset A')).$$

Using $I1, I8$ and $I9$, we obtain $(\bigwedge \Gamma_1' \supset \bigwedge \Gamma') \supset Int^{-}(com)$ and $(\bigwedge \Gamma' \supset \bigwedge \Gamma_1') \supset Int^{-}(com)$. Finally, $I10$ allows us to derive $Int^{-}(com)$ from these formulas by "cut" with axiom (Lin).

(*Completeness*) Directly follows from Theorem 7 and Example 5. ☐

Remark 9 The rule (*lq*) characterizing the intermediate logic **LQ** (see Section 2) can be easily derived in **HG** using (c, l) and (*com*).

Remark 10 As has been shown in [14, 29], the communication rule can be viewed as a *transfer principle* mapping different versions of contraction-free intuitionistic logic into their corresponding extensions containing axiom (Lin). This allowed us to define hypersequent calculi[5] for some basic (fuzzy) logics that can be considered as fragments of contraction-free **G**. Two particular examples of such logics are Urquhart's **C** — introduced in §3 of his Handbook article on many-valued logics [66, 67] — and monoidal t-norm based logic **MTL** [37], the logical counterpart of left-continuous t-norms[6] and their residua [52]. Hilbert style axiomatizations for **C** and **MTL** are obtained by extending the Hilbert style system consisting of axioms $\{I1, \ldots I3, I5, \ldots, I11\}$ of Table 2 and a Hilbert style system for **aMAILL**[7], respectively, with axiom (Lin). In analogy to Avron's work on Gödel logic, analytic hypersequent calculi for **C** and **MTL** were defined in [28] and [15] by adding the communication rule to suitable contraction-free versions of **HIL**.

[5]For contraction-free logics, the *generic interpretation* of a sequent $\Gamma \Rightarrow B$, with $\Gamma = A_1, \ldots, A_k$ should be changed into $A_1 \supset (\ldots (A_{k-1} \supset (A_k \supset B)) \ldots)$.

[6]T-norms are the main tool in fuzzy logic to combine vague information.

[7]Multiplicative Additive fragment of Intuitionistic Linear Logic [46] extended by weakening rules.

In [5], alternative cut-free hypersequent calculi for **G** were introduced. These calculi are obtained by adding to **HIL** either the rule

$$\frac{G \mid \Gamma_1, \Gamma_2 \Rightarrow A \quad G' \mid \Gamma_1, \Gamma_2 \Rightarrow B}{G \mid G' \mid \Gamma_1 \Rightarrow A \mid \Gamma_2 \Rightarrow B}\ (com')$$

or the combination of

$$\frac{G \mid \Pi, \Gamma \Rightarrow A}{G \mid \Pi \Rightarrow A \mid \Gamma \Rightarrow A}\ (S_I) \quad \text{and} \quad \frac{G \mid \Gamma_1 \Rightarrow A \quad G' \mid \Gamma_2 \Rightarrow B}{G \mid G' \mid \Gamma_2 \Rightarrow A \mid \Gamma_1 \Rightarrow B}\ (com'')$$

We call the above calculi $\mathbf{HG}'$ and $\mathbf{HG}''$, respectively. It is not hard to see that the (com) rule is interderivable with the above rules. However, in contrast with (com), (S_I) and (com') cannot be used to define analytic calculi for contraction-free logics (see Remark 10). Indeed, as shown in [14], (internal) contraction is definable from either rule (S_I) or rule (com').

On the other hand, $\mathbf{HG}'$ and $\mathbf{HG}''$ lend themselves to a more natural "computational interpretation" than **HG**. A first step in that direction was achieved in [11], where $\mathbf{HG}''$-proofs have been translated into a special natural deduction format. A different approach has been used in [38]. There, the authors show that any application of (com') corresponds to a merging of suitable parallel Lorenzen type dialogue games.

3.1 Cut-elimination

Recall that the cut-elimination method of Gentzen ([44]) proceeds by eliminating the uppermost cut by a double induction on the complexity of the cut formula and on the sum of its left and right ranks; where the right (left) rank of a cut is the number of consecutive (hyper)sequents containing the cut formula, counting upward from the right (left) upper sequent of the cut.

In fact, in **LJ**, by the presence of the internal contraction rule one has to consider a derivable generalization of the cut rule, namely, the multi-cut rule (see, e.g., [62])

$$\frac{\Gamma \Rightarrow A \quad \Gamma', A^n \Rightarrow B}{\Gamma, \Gamma' \Rightarrow B}\ (mcut)$$

where A^n stands for $A, \dots, A$ (n times). A is called *multi-cut formula*.

Due to the presence of (ec), in hypersequent calculi (and, in particular, in **HG**) one cannot directly apply Gentzen's argument to show that

(∗) If $G' \mid \Gamma \Rightarrow A$ and $H' \mid \Gamma', A^n \Rightarrow B$ are cut-free provable in **HG**, so is $G' \mid H' \mid \Gamma, \Gamma' \Rightarrow B$.

A simple way to overcome this problem, is to modify Gentzen's original *Hauptsatz* allowing to reduce certain cuts *in parallel*. In [3] Avron has used the following induction hypothesis ("extended multi-cut rule"):

(∗∗) If $G' \mid \Gamma_1 \Rightarrow A \mid \dots \mid \Gamma_n \Rightarrow A$ and $H' \mid \Sigma_1, A^{n_1} \Rightarrow B_1 \mid \dots \mid \Sigma_k, A^{n_k} \Rightarrow B_k$ are cut-free provable in **HG**, so is $H' \mid G' \mid \Gamma, \Sigma_1 \Rightarrow B_1 \mid \dots \mid \Gamma, \Sigma_k \Rightarrow B_k$, where $\Gamma = \Gamma_1, \dots, \Gamma_n$.

This formulation is easily seen to be equivalent to $(*)$.

We provide a version of Avron's proof for further reference.

Theorem 11 (Cut-elimination) If a hypersequent S is derivable in **HG** then S is derivable in **HG** without using the cut rule.

Proof Let γ and δ be the proofs of $G := G' \mid \Gamma_1 \Rightarrow A \mid \ldots \mid \Gamma_n \Rightarrow A$ and $H := H' \mid \Sigma_1, A^{n_1} \Rightarrow B_1 \mid \ldots \mid \Sigma_k, A^{n_k} \Rightarrow B_k$, respectively. We show $(**)$ by induction on the pair $[c, r]$, where c is the complexity of the multi-cut formula (A), and r is the sum of the ranks of γ and δ. It suffices to consider the following cases according to which inference rule is being applied just before the application of the multi-cut rule:

1. either G or H is an axiom;
2. either γ or δ ends in an application of a structural rule;
3. both γ and δ end in an application of a logical rule such that the principal formula of both rules is just the multi-cut formula;
4. either γ or δ ends in an application of a logical rule whose principal formula is not the multi-cut formula.

We will give here a proof for some relevant cases.

2. Suppose that γ ends in an application of (ec), e.g.,

$$\dfrac{\begin{array}{c}\vdots\,\gamma_1\\ G' \mid \Gamma_1 \Rightarrow A \mid \ \ldots\ \mid \Gamma_n \Rightarrow A \mid \Gamma_n \Rightarrow A\end{array}}{G' \mid \Gamma_1 \Rightarrow A \mid \ \ldots\ \mid \Gamma_n \Rightarrow A}\ {\scriptstyle (ec)}$$

Applying the induction hypothesis to both δ and γ_1 one obtains a proof of $G' \mid H' \mid \Gamma', \Sigma_1 \Rightarrow B_1 \mid \ \ldots\ \mid \Gamma', \Sigma_k \Rightarrow B_k$, where $\Gamma' = \Gamma_1, \ldots, \Gamma_n, \Gamma_n$. The desired result is obtained by several applications of (c, l).

Suppose that δ ends in an application of (ec), e.g.,

$$\dfrac{\begin{array}{c}\vdots\,\delta_1\\ H' \mid \Sigma_1, A^{n_1} \Rightarrow B_1 \mid \ \ldots\ \mid \Sigma_k, A^{n_k} \Rightarrow B_k \mid \Sigma_k, A^{n_k} \Rightarrow B_k\end{array}}{H' \mid \Sigma_1, A^{n_1} \Rightarrow B_1 \mid \ \ldots\ \mid \Sigma_k, A^{n_k} \Rightarrow B_k}\ {\scriptstyle (ec)}$$

Applying the induction hypothesis to both γ and δ_1 one obtains a proof of $H' \mid G' \mid \Gamma, \Sigma_1 \Rightarrow B_1 \mid \ \ldots\ \mid \Gamma, \Sigma_k \Rightarrow B_k \mid \Gamma, \Sigma_k \Rightarrow B_k$, where $\Gamma = \Gamma_1, \ldots, \Gamma_n$. Hence the claim follows by applying the (ec) rule.

γ or δ ends in another structural inference: These cases are unproblematic applications of the induction hypothesis to the premises followed by applications of structural inferences. E.g., suppose that δ ends in an application of (com), e.g.,

$$\dfrac{\begin{array}{c}\vdots\,\delta_1\\ H_1 \mid \Sigma_1', \Sigma_2', A^{n_2} \Rightarrow B_2\end{array} \qquad \begin{array}{c}\vdots\,\delta_2\\ H_2 \mid \Sigma_1, \Sigma_2, A^{n_1} \Rightarrow B_1\end{array}}{H' \mid \Sigma_1, \Sigma_1', A^{n_1} \Rightarrow B_1 \mid \Sigma_2, \Sigma_2', A^{n_2} \Rightarrow B_2 \mid \ \ldots\ \mid \Sigma_k, A^{n_k} \Rightarrow B_k}\ {\scriptstyle (com)}$$

where H_1 and H_2 stand for $H' \mid \Sigma_{l+1}, A^{n_{l+1}} \Rightarrow B_{l+1} \mid \ldots \mid \Sigma_k, A^{n_k} \Rightarrow B_k$ and $\Sigma_3, A^{n_3} \Rightarrow B_3 \mid \ldots \mid \Sigma_l, A^{n_l} \Rightarrow B_l$, respectively. Applying the induction hypothesis to γ and δ_1 as well as to γ and δ_2 one obtains $G' \mid H' \mid \Sigma_1', \Sigma_2', \Gamma \Rightarrow B_2 \mid \Sigma_{l+1}, \Gamma \Rightarrow B_{l+1} \mid \ldots \mid \Sigma_k, \Gamma \Rightarrow B_k$ and $G' \mid \Sigma_1, \Sigma_2, \Gamma \Rightarrow B_1 \mid \Sigma_3, \Gamma \Rightarrow B_3 \mid \ldots \mid \Sigma_l, \Gamma \Rightarrow B_l$, respectively, where Γ is $\Gamma_1, \ldots, \Gamma_n$. The desired result, namely, $H' \mid G' \mid \Sigma_1, \Sigma_1', \Gamma \Rightarrow B_1 \mid \Sigma_2, \Sigma_2', \Gamma \Rightarrow B_2 \mid \ldots \mid \Sigma_k, \Gamma \Rightarrow B_k$ is obtained by applying (com) and (ec).

3. We first apply the induction hypothesis to the premises (based on the reduced r first, and on the reduced c then). The claim follows by applications of appropriate logical and structural inferences. (See [3]).

4. This case is easily handled by appeal to the induction hypothesis and applications of appropriate logical and structural inferences. We outline the only non-trivial case, i.e., when γ ends in an application of $(\vee, l)$, e.g.,

$$\dfrac{\overset{\vdots\, \gamma_1}{G_1' \mid \Gamma_1, B \Rightarrow A \mid \Gamma_2 \Rightarrow A \mid \ldots \mid \Gamma_l \Rightarrow A} \quad \overset{\vdots\, \gamma_2}{G_2' \mid \Gamma_1, C \Rightarrow A \mid \Gamma_{l+1} \Rightarrow A \mid \ldots \mid \Gamma_n \Rightarrow A}}{G' \mid \Gamma_1, B \vee C \Rightarrow A \mid \ldots \mid \Gamma_n \Rightarrow A}\ (\vee, l)$$

Applying the induction hypothesis to both γ_1 and δ as well as to γ_2 and δ one obtains the proofs of $H' \mid G_1' \mid \Gamma', B, \Sigma_1 \Rightarrow B_1 \mid \ldots \mid \Gamma', B, \Sigma_k \Rightarrow B_k$ and $H' \mid G_1' \mid \Gamma'', C, \Sigma_1 \Rightarrow B_1 \mid \ldots \mid \Gamma'', C, \Sigma_k \Rightarrow B_k$, where $\Gamma' = \Gamma_1, \ldots, \Gamma_l$ and $\Gamma'' = \Gamma_1, \Gamma_{l+1}, \ldots, \Gamma_n$. Hence the desired hypersequent $H' \mid G' \mid \Gamma, B \vee C, \Sigma_1 \Rightarrow B_1 \mid \ldots \mid \Gamma, B \vee C, \Sigma_k \Rightarrow B_k$ follows by several applications of (w, l) and the following lemma. □

Lemma 12 The following generalized rule

$$\dfrac{G \mid A, \Gamma_1 \Rightarrow C_1 \mid \ldots \mid A, \Gamma_n \Rightarrow C_n \quad G' \mid B, \Gamma_1 \Rightarrow C_1 \mid \ldots \mid B, \Gamma_n \Rightarrow C_n}{G \mid G' \mid A \vee B, \Gamma_1 \Rightarrow C_1 \mid \ldots \mid A \vee B, \Gamma_n \Rightarrow C_n}\ (\vee, l)^*$$

is cut-free derivable in **HG**.

Proof For $n = 1$, the claim follows by applying $(\vee, l)$. Otherwise, using only (ec) and (com) one can derive $G \mid G' \mid A, \Gamma_1 \Rightarrow C_1 \mid B, \Gamma_2 \Rightarrow C_2 \mid \ldots \mid B, \Gamma_n \Rightarrow C_n$ from the premises of $(\vee, l)$. Hence by applying $(\vee, l)$ (together with (ec) as necessary) one obtains (a) $G \mid G' \mid A \vee B, \Gamma_1 \Rightarrow C_1 \mid B, \Gamma_2 \Rightarrow C_2 \mid \ldots \mid B, \Gamma_n \Rightarrow C_n$. Similarly one can derive (b) $G \mid G' \mid A \vee B, \Gamma_1 \Rightarrow C_1 \mid A, \Gamma_2 \Rightarrow C_2 \mid \ldots \mid A, \Gamma_n \Rightarrow C_n$. The desired result follows by iteratively applying the above argument to (a) and (b).

Remark 13 Maehara's lemma (see [62]) cannot be established for cut-free derivations in hypersequent calculi with (ec) and (com). It is however possible to construct interpolants for **G** directly by the elimination of propositional quantifiers in quantified propositional Gödel logic over $[0, 1]$ (see Section 4.2).

3.2 Finite-valued Gödel logics

In this section we present cut-free hypersequent calculi for finite-valued Gödel logics. These calculi are obtained by simply adding one more structural rule to the hypersequent calculus for intuitionistic logic.

Recall that a Hilbert style axiomatization $\mathbf{sG}_{k+1}$ for $\mathbf{G}_{k+1}$ is obtained by extending the one of **G** with the axiom (Lin_{k+1}) $A_1 \vee (A_1 \supset A_2) \vee \ldots \vee (A_1 \wedge \ldots \wedge A_k \supset A_{k+1})$.

Let us consider the following rule

$$\frac{\langle G_{i,j} \mid \Gamma_i, \Gamma_j \Rightarrow A_i \rangle_{1 \leq i \leq k,\, i+1 \leq j \leq k+1}}{G_{1,2} \mid \ldots \mid G_{k,k+1} \mid \Gamma_1 \Rightarrow A_1 \mid \ldots \mid \Gamma_k \Rightarrow A_k \mid \Gamma_{k+1} \Rightarrow}\ (Bc_{k+1})$$

In [30] it was shown that extending **HIL** with the above rule results in a cut-free calculus for the *intermediate logics* $\mathbf{Bc}_{k+1}$ (with $k \geq 1$) which are semantically characterized by Kripke models with at most k worlds. As mentioned above, $\mathbf{G}_{k+1}$ can be characterized also by linearly ordered Kripke models with at most k worlds. Therefore, one way to define a hypersequent calculus for $\mathbf{G}_{k+1}$ is simply to add the rule (Bc_{k+1}) to the **HG** calculus for **G**.

An alternative cut-free calculus for $\mathbf{G}_{k+1}$ (called $\mathbf{HG}_{k+1}$) was defined in [30] by adding to **HIL** the following rule

$$\frac{G_1 \mid \Gamma_1, \Gamma_2 \Rightarrow A_1 \quad G_2 \mid \Gamma_2, \Gamma_3 \Rightarrow A_2 \quad \ldots \quad G_k \mid \Gamma_k, \Gamma_{k+1} \Rightarrow A_k}{G_1 \mid \ldots \mid G_k \mid \Gamma_1 \Rightarrow A_1 \mid \ldots \mid \Gamma_k \Rightarrow A_k \mid \Gamma_{k+1} \Rightarrow}\ (G_{k+1})$$

It is not hard to see that both rules (*com*) and (Bc_{k+1}) are derivable in $\mathbf{HG}_{k+1}$.

Theorem 14 ([30]) $\mathbf{HG}_{k+1}$ is sound and complete for $\mathbf{G}_{k+1}$.

Proof The proof is relative to the Hilbert style calculus $\mathbf{sG}_{k+1}$. It is not difficult to show that the rule (G_{k+1}) is sound for $\mathbf{sG}_{k+1}$ and that $\mathbf{HG}_{k+1}$ derives axiom (Lin_{k+1}). Therefore the claim follows from Theorem 8. □

Remark 15 $\mathbf{HG}_2$ is a single-conclusioned hypersequent calculus for classical logic (see [31, 6]).

Theorem 16 (Cut-elimination) If a hypersequent S is derivable in $\mathbf{HG}_{k+1}$ then S is derivable in $\mathbf{HG}_{k+1}$ without using the cut rule.

Proof We show that if both the hypersequents $G := G' \mid \Gamma_1 \Rightarrow A \mid \ldots \mid \Gamma_n \Rightarrow A$ and $H := H' \mid \Sigma_1, A^{n_1} \Rightarrow B_1 \mid \ldots \mid \Sigma_s, A^{n_s} \Rightarrow B_s$ are cut-free provable in $\mathbf{HG}_{k+1}$, then so is $H' \mid G' \mid \Gamma, \Sigma_1 \Rightarrow B_1 \mid \ldots \mid \Gamma, \Sigma_s \Rightarrow B_s$ where $\Gamma = \Gamma_1, \ldots, \Gamma_n$. Let γ and δ be the proofs of G and H, respectively.

In addition to the proof of Theorem 11 we have to consider cases involving the rule (G_{k+1}). We will give here a proof for some relevant cases. Suppose δ ends in an application of such a rule, e.g.,

$$\frac{\begin{array}{c}\vdots\,\delta_1\\ H_1 \mid \Sigma_1, \Sigma_2, A^{n_1}, A^{n_2} \Rightarrow B_1\end{array} \quad \ldots \quad \begin{array}{c}\vdots\,\delta_k\\ H_k \mid \Sigma_k, \Sigma_{k+1}, A^{n_k}, A^{n_{k+1}} \Rightarrow B_k\end{array}}{H_1 \mid \ldots \mid H_k \mid \Sigma_1, A^{n_1} \Rightarrow B_1 \mid \Sigma_2, A^{n_2} \Rightarrow B_2 \mid \ldots \mid \Sigma_k, A^{n_k} \Rightarrow B_k \mid \Sigma_{k+1}, A^{n_{k+1}} \Rightarrow}$$

Applying the induction hypothesis to γ and δ_i ($i = 1, \ldots, k$) one obtains a proof δ'_i of $G' \mid H_i \mid \Sigma_i, \Sigma_{i+1}, \Gamma \Rightarrow B_i$, where Γ is $\Gamma_1, \ldots, \Gamma_n$. Hence the desired result $H_1 \mid \ldots \mid H_k \mid G' \mid \Sigma_1, \Gamma \Rightarrow B_1 \mid \ldots \mid \Sigma_{k+1}, \Gamma \Rightarrow$ can be obtained as follows

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\;\delta_1'\\ G' \mid H_1 \mid \Sigma_1,\Sigma_2,\Gamma \Rightarrow B_1\end{array}}{G' \mid H_1 \mid \Sigma_1,\Sigma_2,\Gamma,\Gamma \Rightarrow B_1}\,{\scriptstyle (w,l)'s} \qquad \dots \qquad \dfrac{\begin{array}{c}\vdots\;\delta_k'\\ G' \mid H_k \mid \Sigma_k,\Sigma_{k+1},\Gamma \Rightarrow B_k\end{array}}{G' \mid H_k \mid \Sigma_k,\Sigma_{k+1},\Gamma,\Gamma \Rightarrow B_k}\,{\scriptstyle (w,l)'s}}{H_1 \mid \dots \mid H_k \mid G' \mid \dots \mid G' \mid \Sigma_1,\Gamma \Rightarrow B_1 \mid \dots \mid \Sigma_{k+1},\Gamma \Rightarrow}\,{\scriptstyle (G_{k+1})}}{H_1 \mid \dots \mid H_k \mid G' \mid \Sigma_1,\Gamma \Rightarrow B_1 \mid \dots \mid \Sigma_{k+1},\Gamma \Rightarrow}\,{\scriptstyle (ec)'s}$$

Suppose γ ends in an application of (G_{k+1}). We outline below the two cases: in (G_{k+1}), more than one component of G is not a side hypersequent and only one component of G is not a side hypersequent. In the former case, assume, e.g., γ ends as follows

$$\dfrac{\begin{array}{c}\vdots\;\gamma_1\\ G_1 \mid \Gamma_1,\Gamma_2 \Rightarrow A\end{array} \qquad \dots \qquad \begin{array}{c}\vdots\;\gamma_k\\ G_k \mid \Pi_k,\Pi_{k+1} \Rightarrow C_k\end{array}}{G_1 \mid \dots \mid G_k \mid \Gamma_1 \Rightarrow A \mid \dots \mid \Gamma_l \Rightarrow A \mid \Pi_{l+1} \Rightarrow C_{l+1} \mid \dots \mid \Pi_{k+1} \Rightarrow}$$

Applying the induction hypothesis to δ and, e.g., γ_1 one obtains a proof of $H' \mid G_1 \mid \Sigma_1, \Gamma_1,\Gamma_2 \Rightarrow B_1 \mid \dots \mid \Sigma_s,\Gamma_1,\Gamma_2 \Rightarrow B_s$. Hence the desired result $H' \mid G_1 \mid \dots \mid G_k \mid \Sigma_1, \Gamma \Rightarrow B_1 \mid \dots \mid \Sigma_s,\Gamma \Rightarrow B_s$, with $\Gamma = \Gamma_1,\dots,\Gamma_l$ follows by several applications of (ew) and (w,l). In the latter case, suppose, e.g., γ ends as follows

$$\dfrac{\begin{array}{c}\vdots\;\gamma_1\\ G_1 \mid \Gamma_1,\Pi_1 \Rightarrow A\end{array} \quad \begin{array}{c}\vdots\;\gamma_2\\ G_2 \mid \Pi_1,\Pi_2 \Rightarrow C_1\end{array} \qquad \dots \qquad \begin{array}{c}\vdots\;\gamma_k\\ G_k \mid \Pi_{k-1},\Pi_k \Rightarrow C_{k-1}\end{array}}{G_1 \mid \dots \mid G_k \mid \Gamma_1 \Rightarrow A \mid \Pi_1 \Rightarrow C_1 \mid \dots \mid \Pi_{k-1} \Rightarrow C_{k-1} \mid \Pi_k \Rightarrow}$$

Applying the induction hypothesis to δ and γ_1 one obtains a proof of $G_1 \mid H' \mid \Sigma_1,\Gamma_1,\ \Pi_1 \Rightarrow B_1 \mid \dots \mid \Sigma_s,\Gamma_1,\Pi_1 \Rightarrow B_s$. Hence the desired result $H' \mid G_1 \mid \dots \mid G_k \mid \Sigma_1,\Gamma_1 \Rightarrow B_1 \mid \dots \mid \Sigma_s,\Gamma_1 \Rightarrow B_s \mid \Pi_1 \Rightarrow C_1 \mid \dots \mid \Pi_k \Rightarrow$ follows by s applications of (G_{k+1}) together with (ec) as necessary. $\square$

Remark 17 In [30] the above proof was formulated without using the "extended multi-cut rule". However, as pointed out by Avron, in hypersequent calculi, Gentzen's argument works only if one can suitably trace the cut formula over the proof. (See, e.g., the "history technique" used in [2] or the notion of "decoration" of formulas introduced in [10]).

3.3 Gödel logic with 0-1 projections

In [9], Gödel logic extended by the "projection modalities" $\bigtriangledown$ and Δ has been investigated:

$$v(\bigtriangledown A) = \begin{cases} 1 & \text{if } v(A) = 0 \\ 0 & \text{if } v(A) \neq 0 \end{cases} \qquad v(\Delta A) = \begin{cases} 1 & \text{if } v(A) = 1 \\ 0 & \text{if } v(A) \neq 1 \end{cases}$$

Whereas $\bigtriangledown A$ can be already defined in $\mathbf{G}$ as $A \supset \bot$, the extension including Δ — called $\mathbf{G}_\Delta$ — is strictly more expressive.

Remark 18 The Δ operator is called globalization (and denoted by $\Box$) in [65, 64].

A Hilbert calculus $\mathbf{sG}_\Delta$ for $\mathbf{G}_\Delta$ was defined in [9], extending the calculus for $\mathbf{G}$ by

$$(\Delta\,1):\ \Delta A \supset A \qquad (\Delta\,2):\ \Delta A \supset \Delta\Delta A \qquad (\Delta\,3):\ \Delta(A \supset B) \supset (\Delta A \supset \Delta B)$$

as well as the following axioms

$$(\Delta\,4):\ \Delta A \vee (\Delta A \supset \bot) \quad \text{and} \quad (\Delta\,5):\ \Delta(A \vee B) \supset \Delta A \vee \Delta B$$

together with the rule

$$\frac{A}{\Delta A}\ (\Delta\ rule)$$

Note that axioms $(\Delta\,1) - (\Delta\,3)$ and $(\Delta\ rule)$ are the modal axioms of the logic **S4** and its necessitation rule, respectively.

To obtain a hypersequent calculus for $\mathbf{G}_\Delta$ we first extend **HG** with the following rules for introducing Δ

$$\frac{G \mid \Gamma, A \Rightarrow C}{G \mid \Gamma, \Delta A \Rightarrow C}\ (\Delta, l) \qquad\qquad \frac{G \mid \Delta\Gamma \Rightarrow A}{G \mid \Delta\Gamma \Rightarrow \Delta A}\ (\Delta, r)$$

where $\Delta\Gamma$ denotes any set of Δ-formulas, i.e. formulas of $\mathbf{G}_\Delta$ prefixed by Δ. The above two rules correspond to the **S4**-rules for Δ (see, e.g., [40]). However, they do not suffice to establish that Δ-formulas behave like boolean formulas. In particular, axiom $(\Delta\,4)$ is not derivable. To this aim, we consider the additional rule

$$\frac{G \mid \Delta\Gamma, \Gamma' \Rightarrow A}{G \mid \Delta\Gamma \Rightarrow\ \mid \Gamma' \Rightarrow A}\ (cl_\Delta, l)$$

We call the resulting calculus $\mathbf{HG}_\Delta$.

Remark 19 By replacing $\Delta\Gamma$ with Γ in (cl_Δ, l), one obtains the rule (G_2) defining a single-conclusioned calculus for classical logic (see Remark 15).

Lemma 20 Let Γ be $A_1, \ldots, A_n$. $(\Delta \bigwedge \Gamma \vee \Delta(\bigwedge \Gamma \supset \bot)) \supset (\bigwedge \Delta\Gamma \vee (\bigwedge \Delta\Gamma \supset \bot))$, where $\bigwedge \Delta\Gamma$ abbreviates $\Delta A_1 \wedge \ldots \wedge \Delta A_n$, is derivable in $\mathbf{sG}_\Delta$.

Theorem 21 $\mathbf{HG}_\Delta$ is sound and complete for $\mathbf{G}_\Delta$.

Proof The proof is relative to the Hilbert style calculus $\mathbf{sG}_\Delta$.

(*Soundness*) Proceeds as in Theorem 8. Proving the soundness of the rules (Δ, l) and (Δ, r) is straightforward. To show that (cl_Δ, l) is sound for $\mathbf{sG}_\Delta$ one can prove that $\mathbf{sG}_\Delta$ derives the generic interpretation $Int^-(\Delta, r)$ of (Δ, r) without side hypersequents, i.e. the formula

$$(\bigwedge \Delta\Gamma \wedge \bigwedge \Gamma' \supset A) \supset ((\bigwedge \Delta\Gamma \supset \bot) \vee (\bigwedge \Gamma' \supset A))$$

where $\bigwedge \Delta\Gamma$ stands for $\Delta A_1 \wedge \ldots \wedge \Delta A_n$. This follows by the derivability in $\mathbf{sG}_\Delta$ of

$$(\bigwedge \Delta\Gamma \vee (\bigwedge \Delta\Gamma \supset \bot)) \supset Int^-(\Delta, r)$$

together with Lemma 20 and axiom $(\Delta\,4)$.

(*Completeness*) $(\Delta\ rule)$ is a particular case of the rule (Δ, r). By Theorem 8 it suffices to prove that $\mathbf{HG}_\Delta$ derives $(\Delta\ i)$ with $i = 1, \ldots, 5$. We display the proofs of axioms $(\Delta\,4)$ and $(\Delta\,5)$ in $\mathbf{HG}_\Delta$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Delta A \Rightarrow \Delta A}{\Rightarrow \Delta A \mid \Delta A \Rightarrow}\ (\mathrm{cl}_\Delta,\mathrm{l})}{\Rightarrow \Delta A \mid \Delta A \Rightarrow \bot}\ (\mathrm{w},\mathrm{r})}{\Rightarrow \Delta A \mid \ \Rightarrow \Delta A \supset \bot}\ (\supset,\mathrm{r})}{\Rightarrow \Delta A \vee (\Delta A \supset \bot) \mid \ \Rightarrow \Delta A \vee (\Delta A \supset \bot)}\ 2\mathrm{x}(\vee,\mathrm{r})}{\Rightarrow \Delta A \vee (\Delta A \supset \bot)}\ (\mathrm{ec})$$

and

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A \qquad \dfrac{A \Rightarrow A \qquad B \Rightarrow B}{B \Rightarrow A \mid A \Rightarrow B}\ (\mathrm{com}) \qquad B \Rightarrow B}{A \vee B \Rightarrow A \mid A \vee B \Rightarrow B}\ 2\mathrm{x}(\vee,\mathrm{l})}{\Delta(A \vee B) \Rightarrow A \mid \Delta(A \vee B) \Rightarrow B}\ (\Delta,\mathrm{l})}{\Delta(A \vee B) \Rightarrow \Delta A \mid \Delta(A \vee B) \Rightarrow \Delta B}\ (\Delta,\mathrm{r})}{\Delta(A \vee B) \Rightarrow \Delta A \vee \Delta B \mid \Delta(A \vee B) \Rightarrow \Delta A \vee \Delta B}\ 2\mathrm{x}(\vee_{\mathrm{i}},\mathrm{r})}{\Delta(A \vee B) \Rightarrow \Delta A \vee \Delta B}\ (\mathrm{ec})}{\Rightarrow \Delta(A \vee B) \supset \Delta A \vee \Delta B}\ (\supset,\mathrm{r})$$

$\square$

Before proving the cut-elimination theorem for $\mathbf{HG}_\Delta$, observe that one cannot directly shift the (*cut*) rule upward over (cl_Δ, l), e.g., in the case below:

$$\dfrac{\dfrac{H' \mid \Delta A, \Delta\Sigma, \Sigma' \Rightarrow B}{H' \mid \Delta A, \Delta\Sigma \Rightarrow \ \mid \Sigma' \Rightarrow B}\ (\mathrm{cl}_\Delta,\mathrm{l}) \qquad G' \mid \Gamma \Rightarrow \Delta A}{G' \mid H' \mid \Delta\Sigma, \Gamma \Rightarrow \ \mid \Sigma' \Rightarrow B}\ (\mathrm{cut})$$

A way to solve this problem is to consider the following cut rule over Δ-formulas as cut formulas

$$\dfrac{G \mid \Gamma \Rightarrow \Delta A \quad G' \mid \Gamma', (\Delta A)^n \Rightarrow B}{G \mid G' \mid \Gamma \Rightarrow \ \mid \Gamma' \Rightarrow B}\ (cut_\Delta)$$

Lemma 22 The rules (cut_Δ) and (*cut*) with a Δ-formula as cut formula are interderivable in $\mathbf{HG}_\Delta$.

Proof On the one hand, (cut_Δ) allows one to derive (*cut*). The derivation proceeds as follows

$$\dfrac{\dfrac{\dfrac{G \mid \Gamma \Rightarrow \Delta A \qquad G' \mid \Gamma', \Delta A \Rightarrow B}{G \mid G' \mid \Gamma \Rightarrow \ \mid \Gamma' \Rightarrow B}\ (\mathrm{cut}_\Delta)}{G \mid G' \mid \Gamma, \Gamma' \Rightarrow B \mid \Gamma, \Gamma' \Rightarrow B}\ (\mathrm{w},\mathrm{l})'\mathrm{s}\,(\mathrm{w},\mathrm{r})}{G \mid G' \mid \Gamma, \Gamma' \Rightarrow B}\ (\mathrm{ec})$$

On the other hand, (cut_Δ) is derivable in $\mathbf{HG}_\Delta$ using (cut):

$$\dfrac{\dfrac{H' \mid \Gamma', \Delta A \Rightarrow B}{H' \mid \Delta A \Rightarrow \ \mid \Gamma' \Rightarrow B}\ (\mathrm{cl}_\Delta, \mathrm{l}) \qquad G' \mid \Gamma \Rightarrow \Delta A}{G' \mid H' \mid \Gamma \Rightarrow \ \mid \Gamma' \Rightarrow B}\ (\mathrm{cut})$$

□

Let $\mathbf{HG}'_\Delta$ be the calculus obtained from $\mathbf{HG}_\Delta$ by replacing the (cut) rule with both (cut_Δ) and

$$\dfrac{G \mid \Gamma' \Rightarrow X \quad G' \mid X, \Gamma \Rightarrow C}{G \mid G' \mid \Gamma, \Gamma' \Rightarrow C}\ (cut')$$

where X is *not* a Δ-formula.

Corollary 23 A hypersequent H is derivable in $\mathbf{HG}'_\Delta$ if and only if H is derivable in $\mathbf{HG}_\Delta$.

Lemma 24 In $\mathbf{HG}'_\Delta$ Non-atomic axioms can be derived from atomic axioms.

Theorem 25 (Cut-elimination) If a hypersequent S is derivable in $\mathbf{HG}_\Delta$ then S is derivable in $\mathbf{HG}_\Delta$ without using the cut rule.

Proof Let π be a proof of a hypersequent H in $\mathbf{HG}_\Delta$. By Corollary 23 one can get a proof π' of H in $\mathbf{HG}'_\Delta$. The proof of the elimination of both types of cuts in $\mathbf{HG}'_\Delta$ ((cut') and (cut_Δ)) is similar to the proof of Theorem 11. Indeed, we will show that if both the hypersequents $G := G' \mid \Gamma_1 \Rightarrow A \mid \ldots \mid \Gamma_n \Rightarrow A$ and $H := H' \mid \Sigma_1, A^{n_1} \Rightarrow B_1 \mid \ldots \mid \Sigma_k, A^{n_k} \Rightarrow B_k$ are cut-free provable in $\mathbf{HG}'_\Delta$, so is

($\star$) $H' \mid G' \mid \Sigma_1, \Gamma \Rightarrow B_1 \mid \ldots \mid \Sigma_k, \Gamma \Rightarrow B_k$, where Γ is $\Gamma_1, \ldots, \Gamma_n$, if A is not a Δ-formula

($\star\star$) $H' \mid G' \mid \Sigma_1 \Rightarrow B_1 \mid \ldots \mid \Sigma_k \Rightarrow B_k \mid \Gamma_1 \Rightarrow \ \mid \ \ldots \ \mid \Gamma_n \Rightarrow$, otherwise.

It is easy to see that ($\star$) and ($\star\star$) are derivable generalizations of the rules (cut') and (cut_Δ), respectively.

The proof proceeds by induction on the pair $[c, r]$ according to the 4 cases indicated in the proof of Theorem 11. We will give here a proof for some relevant cases. Let γ and δ be the proofs of G and H, respectively.

1. If H (resp. G) is an axiom, by Lemma 24 the desired hypersequent is just G (resp. H).

We will present here some examples for cases 2 and 3 involving Δ-formulas (or formulas with Δ-subformulas) as multi-cut formulas.

2. Suppose that γ ends in an application of (cl_Δ, l), e.g.,

$$\dfrac{\begin{array}{c}\vdots\ \gamma_1 \\ H' \mid \Delta A, \Delta\Sigma, \Sigma' \Rightarrow B\end{array}}{H' \mid \Delta A, \Delta\Sigma \Rightarrow \ \mid \Sigma' \Rightarrow B}\ (\mathrm{cl}_\Delta, \mathrm{l})$$

and G is $G' \mid \Gamma \Rightarrow \Delta A$. By applying the induction hypothesis to δ and γ_1 one gets $G' \mid H' \mid \Gamma \Rightarrow \ \mid \Delta\Sigma, \Sigma' \Rightarrow B$, hence the desired result $H' \mid G' \mid \Gamma \Rightarrow \ \mid \Delta\Sigma \Rightarrow \ \mid \Sigma' \Rightarrow B$ is obtained by applying (cl_Δ, l).

3. Suppose that δ and γ end as follows

$$\dfrac{\begin{array}{cc} \vdots\,\delta_1 & \vdots\,\delta_2 \\ H_1 \mid \Sigma \Rightarrow \Delta A & H_2 \mid \Sigma, B \Rightarrow C \end{array}}{H_1 \mid H_2 \mid \Sigma, \Delta A \supset B \Rightarrow C}\ {\scriptstyle (\supset,\mathrm{l})} \qquad \dfrac{\begin{array}{c} \vdots\,\gamma_1 \\ G' \mid \Gamma, \Delta A \Rightarrow B \end{array}}{G' \mid \Gamma \Rightarrow \Delta A \supset B}\ {\scriptstyle (\supset,\mathrm{r})}$$

Applying the induction hypothesis to γ_1 and δ_2 one obtains a proof γ_1' of $G' \mid H_2 \mid \Gamma, \Sigma, \Delta A \Rightarrow C$. Applying the induction hypothesis again, based on the reduced complexity of the multi-cut formula, to γ_1' and δ_1, one obtains a proof of $G' \mid H_1 \mid H_2 \mid \Gamma, \Sigma \Rightarrow C \mid \Sigma \Rightarrow$. The desired result $G' \mid H_1 \mid H_2 \mid \Gamma, \Sigma \Rightarrow C$ follows by several applications of (w, l) and (ec).

For Δ-formulas, case 3 can only occur when both γ and δ end in an application of a introduction rule for Δ. Suppose, e.g., that γ and δ end as follows

$$\dfrac{\begin{array}{c} \vdots\,\gamma_1 \\ G' \mid \Delta\Gamma \Rightarrow A \end{array}}{G' \mid \Delta\Gamma \Rightarrow \Delta A}\ {\scriptstyle (\Delta,\mathrm{r})} \qquad \dfrac{\begin{array}{c} \vdots\,\delta_1 \\ H' \mid \Sigma, A \Rightarrow B \end{array}}{H' \mid \Sigma, \Delta A \Rightarrow B}\ {\scriptstyle (\Delta,\mathrm{l})}$$

By applying the inductive hypothesis to δ_1 and γ_1 one obtains a proof of the hypersequent $H' \mid G' \mid \Sigma, \Delta\Gamma \Rightarrow B$. Hence the desired result $H' \mid G' \mid \Sigma \Rightarrow B \mid \Delta\Gamma \Rightarrow$ follows by applying the rule (cl_Δ, l). □

4. It is easy to check that the claim is true when either γ or δ ends in an application of a logical rule whose principal formula is not the multi-cut formula.

Remark 26 In order to eliminate cuts in proofs containing non atomic axioms (e.g., in the case in which H is the axiom $\Delta A \Rightarrow \Delta A$ and G is $G' \mid \Gamma \Rightarrow \Delta A$) one would need in $\mathbf{HG}_\Delta$ the following additional rule, which is sound for $\mathbf{G}_\Delta$:

$$\frac{G \mid \Gamma, \Gamma' \Rightarrow \Delta A}{G \mid \Gamma' \Rightarrow \Delta A \mid \Gamma \Rightarrow}\ (cl_\Delta, r)$$

4 Quantifiers in Gödel Logic

In [20, 24] extensions of analytic calculi for *propositional* Gödel logic with quantifiers were presented. In particular, two different forms of quantification have been investigated: *first-order quantifiers* (universal and existential quantification over object variables) and *propositional* or *"fuzzy" quantifiers* (universal and existential quantification over propositions).

As already mentioned before, if we consider the set of tautologies, there is only *one*[8] infinite-valued propositional Gödel logic. In contrast, *different* first-order and quantified propositional Gödel logics are induced by different infinite subsets of truth-values over $[0, 1]$ (closed under infima and suprema) [22]. As an example, consider the first-order Gödel logics based on the truth-value sets: $V_\infty = [0, 1], V_\downarrow = \{0\} \cup \{\frac{1}{n} : n \geq 1\}$ and $V_\uparrow = \{1\} \cup \{1 - \frac{1}{n} : n \geq 1\}$, respectively. The one based on V_∞ (i.e. the

[8] This is no longer true for the entailment relation, as shown in [23].

"standard" Gödel logic when viewed as a fuzzy logic [49]) is axiomatizable, while those based on $V_\uparrow$ and $V_\downarrow$ are not [55]. Moreover, the first-order Gödel logic based on $V_\uparrow$ turns out to be the intersection of all finite-valued first-order Gödel logics [22]. (See [32, 33] for alternative axiomatizable first-order extensions of **G** which are defined not via their many-valued semantics but as the class of formulas valid in all linearly ordered Kripke models.)

A similar situation holds for propositionally quantified Gödel logics as shown in [22, 16].

Another feature that makes quantified Gödel logics conceptually interesting is the role of the so-called Takeuti-Titani "density rule",

$$\frac{\Gamma \Rightarrow C \vee (A \supset p) \vee (p \supset B)}{\Gamma \Rightarrow C \vee (A \supset B)}$$

where p is a propositional eigenvariable (i.e., it does not occur in the conclusion). This rule, expressing the density of the ordered set of truth-values, was used in [63] to axiomatize first-order Gödel logic based on the set of truth-values $[0, 1]$ (called "intuitionistic fuzzy logic" **IF** there). Takano [61] has later shown that this rule is in fact redundant in the calculus by referring to semantical arguments already present in Horn [50]. The situation for Gödel logic with propositional quantifiers is different. Here, in contrast to **IF**, an instance of the Takeuti-Titani rule is *essential* to obtain a complete (Hilbert style) axiomatization for quantified propositional Gödel logic based on $[0, 1]$, as was shown in [22]. (The reason is that whereas first-order Gödel logic based on $[0, 1]$ turns out to be the intersection of *all* first-order Gödel logics, the intersection of *all* quantified propositional Gödel logics is not even a Gödel logic at all, see [22, 55].)

Henceforth we only deal with Gödel logic in which the full real interval $[0, 1]$ serves as set of truth-values.

Based on results from [24], in Section 4.1 below we show that Avron's **HG** calculus can be suitably extended to a cut-free hypersequent calculus for first-order Gödel logic for which applications of the Takeuti-Titani rule are (syntactically) eliminable from all derivations. Moreover, for this calculus the mid-hypersequent theorem holds. Along the lines of [20] we present, in Section 4.2, a cut-free hypersequent calculus for quantified propositional Gödel logic and characterize a non-trivial fragment for which the Takeuti-Titani rule is eliminable.

4.1 First-order Gödel logic

The language of first-order Gödel logic is identical to that of classical logic (or intuitionistic logic, for that matter). Free and bound (object) variables are distinguished syntactically using a as meta-variable for the former and x for the latter. Propositional variables are identified with predicate symbols of arity 0 and are denoted with p. Generalizing the many-valued semantics of **G** to the first-order level is straightforward: An *interpretation* $\mathcal{I}$ consists of a non-empty *domain* D and a *valuation function* $v_\mathcal{I}$ that maps constants and object variables to elements of D and n-ary function symbols to functions from D^n into D. $v_\mathcal{I}$ thus extends in the usual way to a function mapping all terms of the language to an element of the domain. Moreover, $v_\mathcal{I}$ maps every n-ary predicate symbol P to a function from D^n into $[0, 1]$. The truth-value of an atomic formula $A := P(t_1, \ldots, t_n)$ is defined as

$$v_{\mathcal{I}}(A) = v_{\mathcal{I}}(P)(v_{\mathcal{I}}(t_1), \ldots, v_{\mathcal{I}}(t_n)).$$

The semantics of propositional connectives remains unchanged. We call *distribution* of $A(x)$, the set $\mathrm{Distr}_{\mathcal{I}}(A(x)) = \{v_{\mathcal{I}}(A(p)) \mid p \in D\}$. The quantifiers are, as usual, defined as infimum and supremum of their distributions, i.e.

$$v_{\mathcal{I}}((\forall x)A(x)) = \inf \mathrm{Distr}_{\mathcal{I}}(A(x)) \quad v_{\mathcal{I}}((\exists x)A(x)) = \sup \mathrm{Distr}_{\mathcal{I}}(A(x))$$

$\mathcal{I}$ *satisfies* a formula A iff $v_{\mathcal{I}}(A) = 1$. A is *valid* iff it is satisfied by every interpretation.

First-order Gödel logic is also characterized by the class of all rooted *linearly ordered* Kripke models with *constant domains*.

A Hilbert style calculus for first-order Gödel logic – which we call **sIF** – is obtained by extending the one for first-order **IL** by axioms (Lin) and the "law of quantifiers shifting" $(\vee\forall)$ $\forall x(A(x) \vee B) \supset (\forall x A(x) \vee B)$, where x does not occur in B (see [50, 61]).

In [24] a hypersequent calculus **HIF**[9] for first-order Gödel logic was defined. It amounts to **HG** extended by the following quantifier rules:

$$\frac{G \mid A(t), \Gamma \Rightarrow B}{G \mid (\forall x)A(x), \Gamma \Rightarrow B}\ (\forall, l) \qquad \frac{G \mid \Gamma \Rightarrow A(a)}{G \mid \Gamma \Rightarrow (\forall x)A(x)}\ (\forall, r)$$

$$\frac{G \mid A(a), \Gamma \Rightarrow B}{G \mid (\exists x)A(x), \Gamma \Rightarrow B}\ (\exists, l) \qquad \frac{G \mid \Gamma \Rightarrow A(t)}{G \mid \Gamma \Rightarrow (\exists x)A(x)}\ (\exists, r)$$

The rules $(\forall, r)$, $(\exists, l)$ must obey the eigenvariable condition: the free variable a must not occur in the lower *hypersequent*.

Theorem 27 ([24]) **HIF** is sound and complete for first-order Gödel logic.

Proof The proof is relative to the Hilbert style calculus **sIF**.

(*Soundness*) By Theorem 8 we only have to show the soundness of quantifier rules. This is easy in the case of $(\forall, l)$ and $(\exists, r)$. For $(\forall, r)$ we may argue as follows: If $Int(G) \vee (\bigwedge \Gamma \supset A(a))$ is derivable in **sIF**, so is $\forall x(Int(G) \vee (\bigwedge \Gamma \supset A(x)))$. Since a did not occur in G or in $\bigwedge \Gamma \supset A(a)$, we may now assume that x does not either. Using axiom $(\vee\forall)$ we obtain $Int(G) \vee \forall x(\bigwedge \Gamma \supset A(x))$. The result follows since $\forall x(\bigwedge \Gamma \supset A(x)) \supset (\bigwedge \Gamma \supset \forall x A(x))$ is derivable in **sIF**. The soundness of $(\exists, l)$ can be proved in a similar way.

(*Completeness*) Proceeds as in Theorem 8. We just show a derivation in **HIF** of

[9] **HIF** stands for Hypersequent calculus for Intuitionistic Fuzzy logic.

axiom $(\vee\forall)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A(a) \Rightarrow A(a) \quad \dfrac{A(a) \Rightarrow A(a) \quad B \Rightarrow B}{B \Rightarrow A(a) \mid A(a) \Rightarrow B}\,{\scriptstyle (com)} \quad B \Rightarrow B}{A(a) \vee B \Rightarrow A(a) \mid A(a) \vee B \Rightarrow B}\,{\scriptstyle 2x(\vee,l)}}{\forall x(A(x) \vee B) \Rightarrow A(a) \mid \forall x(A(x) \vee B) \Rightarrow B}\,{\scriptstyle 2x(\forall,l)}}{\forall x(A(x) \vee B) \Rightarrow \forall x A(x) \mid \forall x(A(x) \vee B) \Rightarrow B}\,{\scriptstyle (\forall,r)}}{\forall x(A(x) \vee B) \Rightarrow \forall x A(x) \vee B \mid \forall x(A(x) \vee B) \Rightarrow \forall x A(x) \vee B}\,{\scriptstyle 2x(\vee,r)}}{\forall x(A(x) \vee B) \Rightarrow \forall x A(x) \vee B}\,{\scriptstyle (ec)}}{\Rightarrow \forall x(A(x) \vee B) \supset (\forall x A(x) \vee B)}\,{\scriptstyle (\supset,r)}$$

□

As already mentioned before, the Takeuti-Titani rule is redundant in an appropriate calculus for first-order Gödel logic. In [61], Takano posed the question whether a syntactical elimination of this rule is also possible. **HIF** allows one to give a positive answer to this question. Indeed, let us consider the following version of Takeuti and Titani's density rule

$$\frac{G \mid \Pi \Rightarrow p \mid p, \Gamma \Rightarrow C}{G \mid \Pi, \Gamma \Rightarrow C}\ (tt)$$

where the propositional variable p must not occur in the lower hypersequent.

Theorem 28 ([24]) Any derivation of a hypersequent S in **HIF** augmented by (tt) can be transformed into a derivation of S in **HIF**.

This follows by induction on the number of applications of (tt) using the following lemma.

Lemma 29 If π is an **HIF**-derivation of

$$G \mid \Phi_1 \Rightarrow p \mid \ \dots \ \mid \Phi_n \Rightarrow p \mid \Pi_1, \Psi_1 \Rightarrow A_1 \mid \ \dots \ \mid \Pi_m, \Psi_m \Rightarrow A_m$$

where p does not occur in G, Φ_i and Ψ_j and if $\bigcup \Pi_j \subseteq \{p\}$ (for $i = 1, \dots, n$ and $j = 1, \dots, m$), then there is an **HIF**-derivation of $G \mid \Phi_1, \dots, \Phi_n, \Psi_1 \Rightarrow A_1 \mid \ \dots \ \mid \Phi_1, \dots, \Phi_n, \Psi_m \Rightarrow A_m$.

Proof By induction on the length of π. We distinguish cases according to the last inference I in π. As an example, consider the case in which I is $(\supset, l)$ and its premises are, say,

$$G' \mid \Phi_1 \Rightarrow p \mid \ \dots \ \mid \Phi_n \Rightarrow p \mid p, \Psi_1 \Rightarrow A_1 \mid \ \dots \ \mid p, \Psi_m \Rightarrow A_m \mid \Gamma \Rightarrow A \quad \text{and}$$

$$G'' \mid \Phi_1 \Rightarrow p \mid \ \dots \ \mid \Phi_n \Rightarrow p \mid p, \Psi_1 \Rightarrow A_1 \mid \ \dots \ \mid p, \Psi_m \Rightarrow A_m \mid B, \Gamma \Rightarrow p.$$

Let $\Phi = \Phi_1, \dots, \Phi_n$. The induction hypothesis provides us with

$$G' \mid \Phi, \Psi_1 \Rightarrow A_1 \mid \ \dots \ \mid \Phi, \Psi_m \Rightarrow A_m \mid \Gamma \Rightarrow A \quad \text{and}$$

$$G'' \mid B, \Gamma, \Phi, \Psi_1 \Rightarrow A_1 \mid \ \dots \ \mid B, \Gamma, \Phi, \Psi_m \Rightarrow A_m.$$

We obtain the desired hypersequent by applying $(\supset, l)$ successively m times, together with some applications of (w, l) and (ec). □

Lemma 30 The following generalized rule

$$\frac{G \mid A(a), \Gamma_1 \Rightarrow C_1 \mid \ \dots \ \mid A(a), \Gamma_n \Rightarrow C_n}{G \mid \exists x A(x), \Gamma_1 \Rightarrow C_1 \mid \ \dots \ \mid \exists x A(x), \Gamma_n \Rightarrow C_n} \ (\exists, l)^*$$

is cut-free derivable in **HIF**.

Proof For $n = 1$, the claim follows by applying the $(\exists, l)$ rule. Otherwise, using only $(\exists, l)$, (com) and (ec), we can derive $(*)$ $G \mid \exists x A(x), \Gamma_1 \Rightarrow C_1 \mid H$, where H stands for $A(a), \Gamma_2 \Rightarrow C_2 \mid \ \dots \ \mid A(a), \Gamma_n \Rightarrow C_n$. Indeed, let (b) be $G \mid A(b), \Gamma_1 \Rightarrow C_1 \mid \ \dots \ \mid A(b), \Gamma_n \Rightarrow C_n$, where b is a new variable. The derivation of the hypersequent $(*)$ is then as follows (we omit contexts that are not involved in the derivation)

$$\dfrac{\dfrac{\dfrac{(a)\ A(a), \Gamma_1 \Rightarrow C_1 \mid \ \dots \ \mid A(a), \Gamma_n \Rightarrow C_n \qquad\qquad (b)}{A(b), \Gamma_1 \Rightarrow C_1 \mid A(b), \Gamma_1 \Rightarrow C_1 \mid A(a), \Gamma_2 \Rightarrow C_2 \mid A(b), \Gamma_i \Rightarrow C_i \mid H} \ {\scriptstyle (com)}}{A(b), \Gamma_1 \Rightarrow C_1 \mid A(b), \Gamma_3 \Rightarrow C_3 \mid \ \dots \ \mid A(b), \Gamma_n \Rightarrow C_n \mid H} \ {\scriptstyle 2x(ec)} \qquad (a)}{\vdots \qquad\qquad \vdots}$$

$$\dfrac{\dfrac{\dfrac{\vdots \qquad\qquad \vdots}{A(b), \Gamma_1 \Rightarrow C_1 \mid A(b), \Gamma_1 \Rightarrow C_1 \mid A(a), \Gamma_n \Rightarrow C_n \mid H} \ {\scriptstyle (com)}}{A(b), \Gamma_1 \Rightarrow C_1 \mid H} \ {\scriptstyle 2x(ec)}}{\exists x A(x), \Gamma_1 \Rightarrow C_1 \mid H} \ {\scriptstyle (\exists, l)}$$

with $i = 3, \dots, k$. The desired result follows by iteratively applying the above argument to $(*)$. □

As shown in [10], using the above lemma one can prove

Theorem 31 (Cut-elimination) If a hypersequent S is derivable in **HIF** then S is derivable in **HIF** without using the cut rule.

Proof We show that if both the hypersequents $G := G' \mid \Gamma_1 \Rightarrow A \mid \dots \mid \Gamma_n \Rightarrow A$ and $H := H' \mid \Sigma_1, A^{n_1} \Rightarrow B_1 \mid \dots \mid \Sigma_k, A^{n_k} \Rightarrow B_k$ are cut-free provable in **HIF**, then so is $H' \mid G' \mid \Gamma, \Sigma_1 \Rightarrow B_1 \mid \dots \mid \Gamma, \Sigma_k \Rightarrow B_k$ where $\Gamma = \Gamma_1, \dots, \Gamma_n$.

In addition to the proof of Theorem 11 we have to consider the cases involving quantifiers. More precisely, let γ and δ be the proofs of G and H, respectively. We consider the following cases:

1. both γ and δ end in applications of rules for quantifiers such that the principal formula of both rules is just the multi-cut formula;

2. either γ or δ ends in an application of a rule for quantifiers whose principal formula is not the multi-cut formula.

1. Suppose that both γ and δ end in an application of a rule for $\forall$ and the principal formulas of both rules are the cut formulas. For instance, δ is

$$\dfrac{\begin{matrix} \vdots\, \delta_1 \\ H' \mid \Sigma_1, A(a), (\forall x A(x))^{n_1 - 1} \Rightarrow B_1 \mid \ \dots \ \mid \Sigma_k, (\forall x A(x))^{n_k} \Rightarrow B_k \end{matrix}}{H' \mid \Sigma_1, (\forall x A(x))^{n_1} \Rightarrow B_1 \mid \ \dots \ \mid \Sigma_k, (\forall x A(x))^{n_k} \Rightarrow B_k} \ {\scriptstyle (\forall, l)}$$

and γ is

$$\dfrac{\begin{array}{c}\vdots\,\gamma_1\\ G' \mid \Gamma_1 \Rightarrow A(a) \mid \ \ldots\ \mid \Gamma_n \Rightarrow \forall x A(x)\end{array}}{G' \mid \Gamma_1 \Rightarrow \forall x A(x) \mid \ \ldots\ \mid \Gamma_n \Rightarrow \forall x A(x)}\ (\forall,\mathrm{r})$$

Applying the induction hypothesis to both γ and δ_1 one obtains a proof δ' of $H' \mid G' \mid \Sigma_1, \Gamma, A(a) \Rightarrow B_1 \mid \ \ldots\ \mid \Sigma_k, \Gamma \Rightarrow B_k$, where $\Gamma = \Gamma_1, \ldots, \Gamma_n$, while applying the induction hypothesis to γ_1 and δ one obtains a proof γ' of $H' \mid G' \mid \Gamma_1 \Rightarrow A(a) \mid \Sigma_1, \Gamma_2, \ldots, \Gamma_n \Rightarrow B_1 \mid \ \ldots\ \mid \Sigma_k, \Gamma_2, \ldots, \Gamma_n \Rightarrow B_k$. We now apply again the induction hypothesis, based on the reduced complexity of the multi-cut formula, to γ' and δ'. The desired result is obtained by several applications of $(c,l), (w,l)$ and (ec).

2. This case is easily handled by appeal to the induction hypothesis and applications of appropriate logical and structural inferences. We outline the only non-trivial case, i.e., when δ ends as follows

$$\dfrac{\begin{array}{c}\vdots\,\delta_1\\ G' \mid \Gamma_1, B(a) \Rightarrow A \mid \ \ldots\ \mid \Gamma_n \Rightarrow A\end{array}}{G' \mid \Gamma_1, \exists x B(x) \Rightarrow A \mid \ \ldots\ \mid \Gamma_n \Rightarrow A}\ (\exists,\mathrm{l})$$

Applying the induction hypothesis to γ and δ_1 one obtains a proof of $H' \mid G' \mid \Sigma_1, \Gamma, B(a) \Rightarrow B_1 \mid \ \ldots\ \mid \Sigma_k, \Gamma, B(a) \Rightarrow B_k$, where $\Gamma = \Gamma_1, \ldots, \Gamma_n$. Hence the desired result follows from Lemma 30. $\square$

Remark 32 The above proof has originally been formulated in [24] without using the "extended multi-cut rule" (see Remark 17).

Another proof of the cut-elimination theorem for **HIF** can be found in [10]. This proof follows the Schütte-Tait method ([57, 59]) that differs from Gentzen's method by its *cut selection rule*: Gentzen selects a highest cut, while in Schütte-Tait style procedure a largest cut (w.r.t. the number of connectives and quantifiers of the cut-formula) is selected. Moreover, the cut-elimination proof in [10] allows one to establish non-elementary primitive recursive bounds for the lengths of cut-free proofs in **HIF** in terms of the length and the maximal complexity of cut-formulas in the original proof. Indeed, let $4^n_0 = n, 4^n_{k+1} = 4^{4^n_k}$. The length of the resulting cut-free derivations in **HIF** is bounded by $4^{|d|}_{\rho(d)}$, where $|d|$ is the depth of the original derivation and $\rho(d)$ the maximal complexity of cut-formulas in it.

Remark 33 The elimination procedure of the rule (tt) (Theorem 28) is directly applicable to cut-free proofs. Note that case (5) in the original proof of Lemma 29, i.e., when the last inference I applied is either $(\vee, l)$ or $(\exists, l)$ (see [24]), can be handled without introducing cuts, since the needed generalized rules $(\vee, l)^*$ and $(\exists, l)^*$ are cut-free derivable in **HIF** (Lemmas 12 and 30).

As is well known, already Gentzen showed that in **LK** — as a consequence of cut-elimination — a separation between propositional and quantificational inferences can be achieved in deriving a prenex sequent (see, e.g., [62]). This result, that does not hold for **LJ**, was extended in [24] to Gödel logic, as follows:

Theorem 34 (Mid-hypersequent) Any **HIF**-derivation π of a prenex hypersequent H can be transformed into one in which no propositional rule is applied below any application of a quantifier rule.

Proof The proof proceeds as in the classical case (see, e.g., [62]). First observe that all non-atomic axioms are cut-free derivable from atomic axioms. Recall that the only case that does not work for **LJ** arises when π contains a quantifier inference above a $(\vee, l)$ inference. In **HIF** the $(\vee, l)$ rule can be simulated without using cuts by the following one

$$\frac{G \mid A, \Gamma \Rightarrow C_1 \quad G \mid B, \Gamma \Rightarrow C_2}{G \mid A \vee B, \Gamma \Rightarrow C_1 \mid A \vee B, \Gamma \Rightarrow C_2}\ (\vee', l)$$

We replace all the applications of $(\vee, l)$ by applications of $(\vee', l)$ in π. We define the order of a quantifier inference in π to be the number of propositional inferences under it, and the order of π as the sum of the orders of its quantifier inferences. The proof then proceeds by induction on the order of π. □

The above theorem can be used to prove Herbrand's theorem for the prenex fragment of Gödel logic. (See [12] for a semantical proof of the latter theorem).

4.2 Quantified propositional Gödel logic

An interesting generalization of propositional Gödel logic is obtained by adding quantifiers over propositional variables. In contrast to classical logic, propositional quantification may increase the expressive power of Gödel logic. More precisely, statements about the topological structure of the set of truth-values (taken as infinite subsets of the real interval $[0, 1]$) can be only expressed using propositional quantifiers [22]. There is yet another reason that renders the investigation of quantified propositional Gödel logic interesting, namely its relation with the interpolation property (see [21]). Indeed, Gödel logic admits elimination of propositional quantifiers which yields an immediate proof of the uniform interpolation property (Corollary 38).

Henceforth let us use the notation $A[X]$ to exhibit the occurrences of the formula X in the formula A.

In *classical* propositional logic one may understand $(\exists q)A[q]$ to abbreviate $A[\bot] \vee A[\top]$ and $(\forall q)A[q]$ to abbreviate $A[\bot] \wedge A[\top]$. In other words, propositional quantification is semantically defined by the supremum and infimum, respectively, of truth functions (with respect to the usual ordering "$0 < 1$" over the classical truth-values $\{0, 1\}$). This correspondence can be extended to Gödel logic by using *propositional quantifiers*. Syntactically, this means that we allow formulas $(\forall q)A$ and $(\exists q)A$ for propositional variables in the language. Again we distinguish free and bound variables syntactically by using a to denote free variables and q to denote bound variables.

The semantics of propositional quantifiers is defined analogously to that of first-order quantifiers as the infimum and supremum of the corresponding *distribution*. In this context the distribution of $A([q])$ is the set $\{v_{\mathcal{I}}(A[p]) \mid p \in D\}$.

To obtain a Hilbert style calculus for quantified propositional logic, we first add to the axiom system for **IL** of Table 2 the following two axioms and rules:

$$\mathit{Implies}{-}\exists : A[X] \supset (\exists q)A[q] \qquad \forall{-}\mathit{Implies} : ((\forall q)A[q]) \supset A[X]$$

$$\frac{Z[a] \supset Y}{((\exists q)Z[q]) \supset Y}\ (\mathrm{R}\exists) \qquad \frac{Y \supset Z[a]}{Y \supset (\forall q)Z[q]}\ (\mathrm{R}\forall)$$

where a does not occur in Y.

Remark 35 *Implies*-$\exists$, $\forall$-*Implies* as well as (R$\exists$) and (R$\forall$) are already sound for intuitionistic logic with propositional quantifiers.

The system **sQG** for quantified propositional Gödel logic is obtained by taking all abovementioned axioms and rules plus the following two axioms:

$$\begin{array}{ll} \vee{-}Shift: & ((\forall q)(A \vee B)) \supset (A \vee (\forall q)B) \\ Density: & [(\forall q')((A \supset q') \vee (q' \supset B))] \supset \ (A \supset B) \end{array}$$

where q does not occur in A and q' occurs neither in A nor in B.

Theorem 36 ([22]) **sQG** admits quantifiers elimination: For every formula A there exists a quantifier-free formula B, all whose variables are in A, such that **sQG** derives $(A \supset B) \wedge (B \supset A)$.

Corollary 37 **sQG** is sound and complete for quantified propositional Gödel logic.

Corollary 38 **G** admits uniform interpolation: For every tautology $P \supset Q$ of **G** there exists a formula C, depending only on P and on the propositional variables of P not occurring in Q, such that $P \supset C$ and $C \supset Q$ are tautologies of **G**.

Proof Let us use $\overline{u}$ to denote a sequence of subformulas in a formula. Let $A[\overline{y}, \overline{x}] \supset B[\overline{z}, \overline{x}]$ be a tautology of **G**. **sQG** derives $A[\overline{y}, \overline{x}] \supset \exists \overline{q} A[\overline{y}, \overline{q}]$ and $\exists \overline{q} A[\overline{y}, \overline{q}] \supset B[\overline{z}, \overline{x}]$. The claim follows from Theorem 36. $\Box$

Remark 39 In fact, it was proved in [22] that instances of axioms $Implies{-}\exists$ and $\forall{-}Implies$, where the formulas denoted by X are quantifier free, suffice for the completeness of the calculus.

The hypersequent calculus **HQG** for quantified propositional Gödel logic is obtained by augmenting **HG** with both the density rule (tt) and the following rules for introducing propositional quantifiers:

$$\frac{G \mid A[X], \Gamma \Rightarrow B}{G \mid (\forall q)A[q], \Gamma \Rightarrow B}\ (\forall, l)^0 \qquad \frac{G \mid \Gamma \Rightarrow A[a]}{G \mid \Gamma \Rightarrow (\forall q)A[q]}\ (\forall, r)^0$$

$$\frac{G \mid A[a], \Gamma \Rightarrow B}{G \mid (\exists q)A[q], \Gamma \Rightarrow B}\ (\exists, l)^0 \qquad \frac{G \mid \Gamma \Rightarrow A[X]}{G \mid \Gamma \Rightarrow (\exists q)A[q]}\ (\exists, r)^0$$

In the rules $(\forall, l)^0$ and $(\exists, r)^0$ the formula X is *quantifier free*. Moreover, the rules $(\forall, r)^0$ and $(\exists, l)^0$ must obey the eigenvariable condition.

Theorem 40 ([20]) **HQG** is sound and complete for quantified propositional Gödel logic.

Proof The proof is relative to the Hilbert system **sQG**. It proceeds as in Theorem 27. For the completeness part, we just show how to prove the density axiom in **HQG**. First notice that for all formulas A, B and C,

$$\textbf{HQG} \quad \text{derives} \quad (A \supset C) \vee (C \supset B), A \Rightarrow C \mid (A \supset C) \vee (C \supset B), C \Rightarrow B$$

Therefore, the derivation of the density axiom proceeds as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{(A \supset q) \vee (q \supset B), A \Rightarrow q \mid (A \supset q) \vee (q \supset B), q \Rightarrow B}{(\forall q)((A \supset q) \vee (q \supset B)), A \Rightarrow q \mid (\forall q)((A \supset q) \vee (q \supset B)), q \Rightarrow B}\ {\scriptstyle 2\mathrm{x}(\forall,\mathrm{l})^0}}{(\forall q)((A \supset q) \vee (q \supset B)), A \Rightarrow B}\ {\scriptstyle (\mathrm{tt})}}{(\forall q)((A \supset q) \vee (q \supset B)) \Rightarrow A \supset B}\ {\scriptstyle (\supset,\mathrm{r})}}{\Rightarrow [(\forall q)((A \supset q) \vee (q \supset B))] \supset \ (A \supset B)}\ {\scriptstyle (\supset,\mathrm{r})}$$

$\square$

Theorem 41 (Cut-elimination) If a hypersequent S is derivable in **HQG** then S is derivable in **HQG** without using the cut rule.

Proof Proceeds as in Theorem 31. We show that if both the hypersequents $G := G' \mid \Gamma_1 \Rightarrow A \mid \ldots \mid \Gamma_n \Rightarrow A$ and $H := H' \mid \Sigma_1, A^{n_1} \Rightarrow B_1 \mid \ldots \mid \Sigma_k, A^{n_k} \Rightarrow B_k$ are cut-free provable in **HQG**, then so is $H' \mid G' \mid \Gamma, \Sigma_1 \Rightarrow B_1 \mid \ldots \mid \Gamma, \Sigma_k \Rightarrow B_k$ where $\Gamma = \Gamma_1, \ldots, \Gamma_n$. Let γ and δ be the proofs of G and H, respectively.

In order to see that reducing cuts on *quantified* formulas does not spoil the termination of the cut-elimination procedure we have to introduce the number nq of quantifier occurrences in the multi-cut formula as an additional parameter to the pair $[c, r]$, where c is the complexity of the multi-cut formula, and r is the sum of the ranks of γ and δ. The lexicographical ordering over the resulting triple $[nq, c, r]$ is an appropriate reduction ordering. As an example, suppose that γ and δ end in

$$\dfrac{\begin{matrix}\vdots\ \gamma_1 \\ G' \mid \Gamma \Rightarrow A[a]\end{matrix}}{G' \mid \Gamma \Rightarrow (\forall q)A[q]}\ {\scriptstyle (\forall,\mathrm{r})^0} \qquad \text{and} \qquad \dfrac{\begin{matrix}\vdots\ \delta_1 \\ H' \mid \Sigma, A[X] \Rightarrow B\end{matrix}}{H' \mid (\forall q)A[q], \Sigma \Rightarrow B}\ {\scriptstyle (\forall,\mathrm{l})^0}$$

Let us denote with $\gamma_1[X/a]$, the proof γ_1 after substituting all the occurrences of a by X. Note that in $A[X]$ the number of occurrences of quantifiers is decreased w.r.t. $(\forall q)A[q]$, since X is quantifier free. Therefore we can apply the induction hypothesis to $\gamma_1[X/a]$ and δ_1 to obtain the desired result, namely, $G' \mid H' \mid \Gamma, \Sigma \Rightarrow B$.

Suppose that γ ends in an application of (tt), e.g.,

$$\dfrac{\begin{matrix}\vdots\ \gamma_1 \\ G' \mid \Gamma_1 \Rightarrow p \mid \Gamma_2, p \Rightarrow A \mid \ \ldots \ \mid \Gamma_n \Rightarrow A\end{matrix}}{G' \mid \Gamma_1, \Gamma_2 \Rightarrow A \mid \ \ldots \ \mid \Gamma_n \Rightarrow A}\ {\scriptstyle (\mathrm{tt})}$$

Applying the induction hypothesis to γ_1 and δ one obtains a proof of $G' \mid H' \mid \Gamma_1 \Rightarrow p \mid \Gamma', \Sigma_1 \Rightarrow B_1 \mid \ \ldots \ \mid \Gamma', \Sigma_k \Rightarrow B_k$, where $\Gamma' = p, \Gamma_2, \ldots, \Gamma_n$. In analogy with case 2 in the proof of Theorem 31, the desired result $G' \mid H' \mid \Gamma, \Sigma_1 \Rightarrow B_1 \mid \ \ldots \ \mid \Gamma, \Sigma_k \Rightarrow B_k$ with $\Gamma = \Gamma_1, \ldots, \Gamma_n$, follows by several applications of (tt), (com) and (ec). $\square$

As shown before, the density rule is needed to derive instances of the density axiom in **HQG**. On the other hand, this rule renders proof search rather problematic. Moreover we conjecture that the fragment of quantified propositional Gödel logic in which the rule (tt) (or a variant thereof) is not actually needed to find a proof is the intersection of *all* quantified propositional Gödel logics. Therefore it is useful to characterize such a fragment. Let $\mathbf{HQG}^-$ be the calculus obtained from **HQG** by dropping the rules $(\forall, l)^0$ and $(\exists, r)^0$.

Theorem 42 ([20]) Every $\mathbf{HQG}^-$-proof π of a hypersequent H can be transformed into a proof π' of H in which no application of the density rule occurs.

Proof The proof is similar to that of Theorem 28.

Remark 43 Due to the eigenvariable condition in (tt), one cannot permute this rule with $(\forall, l)^0$ (or $(\exists, r)^0$) as, e.g., in the derivation of the density axiom (see the proof of Theorem 40).

5 Conclusion

The literature contains various analytic calculi for **G**, see, e.g., [58, 1, 36, 17, 43, 7, 12, 39]. Among them, several calculi are better suited for proof search than hypersequent calculi. This holds in particular for *sequent of relations calculi* [17, 13], *goal-oriented proof procedures* [43], the systems recently defined in [7, 8, 39] or the resolution-style *chaining calculi* used in [12]. However, the mentioned calculi cannot be modified in a simple way to include quantifiers, modalities or to formalize related logics.

The most significant feature of the calculus **HG** is its close relation to Gentzen's sequent calculus **LJ** for intuitionistic logic. **HG** contains **LJ** as a sub-calculus and simply adds an additional layer of information by allowing **LJ**-sequents to live in the context of finite multisets of sequents; suitable structural rules allow to manipulate sequents with respect to their contexts. This design provides a rather flexible framework that allows one to formulate analytic Gentzen-style calculi for a range of logics that bear a similar relation to contraction-free versions of intuitionistic logic as **G** bears to **IL** (e.g., Urquart's **C** [66, 67] or Esteva and Godo's **MTL** [37]). In addition, suitable rules for dealing with the Δ modality, as well as for bounding the number of possible truth-values in Gödel logics, can be naturally defined. Moreover, one can easily go beyond the propositional level by adding the usual quantifier rules (both for first-order and propositional quantifiers). Remarkably enough, in the resulting calculus for first-order Gödel logic one can prove (a version of) Gentzen's classical mid-sequent theorem. This should be contrasted with the fact that no comparable version of this theorem holds for intuitionistic logic. In this sense, the external level of **HG** captures some "classical" features of **G**.

Finally, we remark that in (cut-free) hypersequent calculi the subformula property is retained in its original form. This makes them a nice tool for analyzing and reasoning about proofs in the logics concerned.

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