Analytic proof theory for Åqvist's system F

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Abstract

The key strength of preference-based logics for conditional obligation is their ability to handle contrary-to-duty paradoxes and account for exceptions. Here we investigate Åqvist's system **F**, a well-known logic in this family. **F** has the notable feature that every satisfiable formula has a "best" element. Thus far, the only proof system for **F** was a Hilbert calculus, impeding applications and deeper investigations. We fill this gap, constructing the first analytic calculus for **F**. The calculus possesses good proof-theoretical properties—in particular, cut-elimination, which greatly facilitates proof search. Our calculus is used to provide explanations of logical consequences, as a decision-making tool, and to obtain a preliminary complexity upper bound for **F** (giving a theoretical limit on its automated behavior).

Keywords: Dyadic deontic logic; analytic sequent calculi; hypersequents; system **F**

1 Introduction

This paper deals with so-called preference based dyadic deontic logic, initially put forth by [7,14,28,18]. The syntax contains a conditional obligation operator $\bigcirc(B/A)$, read as "B is obligatory given A". A binary relation ranks the possible worlds in terms of betterness. In that framework, the truth-conditions for $\bigcirc(B/A)$ are phrased in terms of best-antecedents worlds. It has emerged as one of the *de facto* standards for normative reasoning; its key strengths are the ability to handle contrary-to-duty paradoxes [5] and to account for exceptions.

Past research on preference-based dyadic deontic logic has focused on the search for an Hilbert style axiomatization, and on the question of clarifying the correspondence between semantic properties and modal axioms. An overview of the existing findings may be found in, e.g., [11,22]. It is only recently that analytic calculi for these logics have been proposed [24,6]. In an analytic calculus,

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proof search proceeds by step-wise decomposition of the formulas to be proven and this yields practical and theoretical advantages over Hilbert systems. In particular, they can be used to establish important meta-logical properties for the formalized logics (e.g., decidability, complexity and interpolation), and they facilitate the development of automated reasoning methods. The original tool to construct analytic calculi was the sequent calculus; following Gentzen (1933), the key idea was to use the cut rule to establish completeness, and then show elimination (or redundancy) of this rule from derivations to establish analyticity. However, the sequent calculus is not expressive enough to support cut elimination for most logics of interest. Hence various extensions and generalizations have been introduced in the pursuit of analytic proof calculi.

Analytic sequent-style calculi were obtained for two well-known systems proposed by Åqvist [1]: **E** and **G**. For **E** the calculus (called **HE**) was defined in [6], whereas the calculus for **G** appears in [10] (it was in fact introduced for Lewis's VTA, to which **G** is equivalent). In this paper we consider **F** which lies between **E** and **G**. It is obtained by supplementing **E** with axiom (D*) that rules out models without a best element (a 'limit'). Obligations in **E** collapse to triviality when there is no best world: if A does not have a best element element then $\bigcirc(B/A)$ holds for any B. One obtains **G**, by extending **F** with the so-called principle of rational monotony [17].

The main contribution of the paper is an analytic calculus **HF** for **F**, leading to a decision procedure and a CoNEXP upper bound, the first complexity bound for this logic. Of Åqvist's three systems, **F** is the most complex in terms of proof theory. **HF** is obtained in a modular way, by adding to (an equivalent version of) **HE** a new rule corresponding to the (D*) axiom. Surprisingly, this rule shares common structural features with the peculiar rule for the calculus for provability logic GL [23]. As in **HE**, the calculus **HF** employs hypersequents to accommodate the extra S5-type modality used to express settledness. The hypersequent framework [2] consists of multiple sequents in parallel, and it can be seen as the minimal extension of Gentzen's sequent framework permitting a cut-free calculus for the logic $S5$ [19,3,16] (itself a sub-logic of **F**). The analyticity of **HF** is established as a consequence of the algorithmic eliminability of the cut rule from derivations (cut-elimination). The proof is intricate and of technical interest. In particular, the presence in the peculiar rule of **HF** of "diagonal formulas" [23] (i.e., formulas that change polarity from conclusion to premises) makes the proof very challenging; even more than in Valentini's cut-elimination proof [25] for GL (see [12,13] for a survey on cut-elimination proofs for GL).

A potential misunderstanding must be cleared up from the start. As in previous work on modal interpretation of conditionals, e.g., [9,20,26,6], we encode maximality by a unary modal operator $\mathcal{B}et$. It is important to realize that by doing so we are not carrying out a reduction of dyadic deontic logic to some bi-modal logic. Indeed the calculus rules for $\mathcal{B}et$ cannot be understood in isolation, and they do not correspond to any known normal or non-normal modality. The $\mathcal{B}et$ operator is not part of the language of **F**, and it is used just in the hypersequent calculus to define suitable rules for the conditional obligation operator.

2 The system F in a nutshell

We present the logic **F**. Its language is defined by the following BNF:

$$A ::= p \in \text{PropVar} \mid \neg A \mid A \to A \mid \Box A \mid \bigcirc(A/A)$$

$\Box A$ is read as "A is settled as true," and $\bigcirc(B/A)$ as "B is obligatory, given A." The Boolean connectives other than $\neg$ and $\to$ are defined as usual. $\Diamond$ is a derived connective, defined as usual (viz. as the dual of $\Box$).

Definition 2.1 **F** consists of any Hilbert system for S5 supplemented with:

$$\bigcirc(B \to C/A) \to (\bigcirc(B/A) \to \bigcirc(C/A)) \tag{COK}$$
$$\bigcirc(A/A) \tag{Id}$$
$$\bigcirc(C/A \wedge B) \to \bigcirc(B \to C/A) \tag{Sh}$$
$$\Box(A \leftrightarrow B) \to (\bigcirc(C/A) \leftrightarrow \bigcirc(C/B)) \tag{Ext}$$
$$\bigcirc(B/A) \to \Box\bigcirc(B/A) \tag{Abs}$$
$$\Box A \to \bigcirc(A/B) \tag{O-Nec}$$
$$\Diamond A \to \neg\bigcirc(\bot/A) \tag{D*}$$

F extends **E** with one axiom: (D*). This axiom, which is equivalent to the original axiom, $\Diamond A \to \neg(\bigcirc(B/A) \wedge \bigcirc(\neg B/A))$, rules out the possibility of conflicts between obligations (for consistent, or possible, antecedents).

The notions of derivation and theoremhood are defined in the usual way.

The semantics of **F** can be defined in terms of *preference models*. They are possible-world models equipped with a comparative goodness relation $\succ$ on worlds so that $x \succ y$ can be read as "world x is *better* than world y." Conditional obligation is defined by considering "best" worlds: intuitively, $\bigcirc(B/A)$ holds in a model, if all the best worlds in which A is true also make B true.

Definition 2.2 A preference model is a structure $M = (W, \succ, V)$ $(W \neq \emptyset)$ whose members are called possible worlds, $\succ \subseteq W \times W$, $V : W \to \mathcal{P}(PropVar)$. The evaluation rules for the Boolean connectives are as usual. The evaluation rules for $\Box$ and $\bigcirc$ are defined as follows:

- $M, x \vDash \Box A$ iff $\forall y \in W$ $M, y \vDash A$
- $M, x \vDash \bigcirc(B/A)$ iff $\forall y \in \text{best}(A)$ $M, y \vDash B$

Here $\text{best}(A) = \{y \in W \mid M, y \vDash A \text{ and there is no } z \succ y \text{ such that } M, z \vDash A\}$.

When no confusion arises, we write $x \vDash A$ for $M, x \vDash A$.

The distinctive feature of the semantics for **F** (w.r.t **E**) is that $\succ$ is required to be limited, that is if $\exists x$ s.t. $x \models A$, then $\text{best}(A) \neq \emptyset$. Intuitively, if the set of A-worlds is non-empty, then it has a best element. This assumption validates (D*). Observe that the relation $\succ$ is not assumed to be transitive.

Validity in a model and validity over all models are defined as usual.

For the purpose of the calculi developed subsequently, we introduce the modality $\mathcal{B}et$ that will allow us to represent the "Best" worlds: $M, x \vDash \mathcal{B}et\, A$ iff $\forall y \succ x\ M, y \vDash A$. By this definition, we get $x \in \text{best}(A)$ iff $M, x \vDash A$ and $M, x \vDash \mathcal{B}et\, \neg A$. However, the modality $\mathcal{B}et$ is not part of $\mathcal{L}$.

The following applies:

Theorem 2.3 (Soundness and completeness, [21]) **F** *is sound and complete w.r.t. the class of preference models whose relation* $\succ$ *is limited.*

A few words on the *rationale* behind the limitedness condition. As mentioned, it provides a remedy to the fact that obligations collapse to triviality when there is no best world in a given model. This collapse may arise in two typical situations: when there is an infinite sequence of better and better worlds (see Ex. 2.4 below), and when there is a cycle of betterness (see Ex. 2.5).

Example 2.4 [Starvation, [8]] Let $W = \{x_i : i < \omega\}$. Assume that all the worlds share an infinite number of inhabitants, $\{a_i : i < \omega\}$. In each world x_i, all the individuals whose index is less than or equal to i are relieved and saved from starvation, all the other are left dying. Thus, in x_1, only a_1 is relieved or saved, all the other individuals are starving. In world x_2, only a_1 and a_2 are relieved, all the others are starving, and so on. Suppose the worlds are ranked according to the number of individuals saved from starvation. Then, for all $i < \omega$, $x_{i+1} \succ x_i$. There is no best world. In this model, for all $i < \omega$, (sv_a_i stand for "a_i is saved") $\bigcirc(sv\_a_i/\top)$ and $\bigcirc(\neg sv\_a_i/\top)$, contradicting (D*). Note that $\succ$ has been chosen so as to be transitive. But nothing hinges on it. Indeed, what makes the limitedness condition fail in this model, is that $\succ$ is serial, viz. for all x_i, there is a y such that $y \succ x_i$.

Cycles are usually considered irrational, because they lead to a violation of the principles of transitivity and consistency in decision-making. Nevertheless, empirical studies have revealed that cycles can arise in certain contexts, for instance when the ranking is based on multiple criteria. It is customary to rank the possible worlds based on the number of obligations they violate: the less obligations are violated by a world, the better the world is. This mono-criterion becomes a bi-criterion, if one distinguishes between the obligations issued by an authority P from those issued by an authority Q, and use them separately to rank the possible worlds.

Example 2.5 [Multi-criteria ranking, [27]] Suppose the authorities P and Q issue the commands p_1 and p_2, and q_1 and q_2, respectively. Consider two words x_1 and x_2 such that $x_1 \models p_1 \wedge p_2 \wedge q_1 \wedge \neg q_2$, and $x_2 \models \neg p_1 \wedge p_2 \wedge q_1 \wedge q_2$. We have $x_1 \succ x_2$, since x_1 violates less obligations issued by P than x_2. But $x_2 \succ x_1$, because x_2 violates less obligations issued by Q than x_1. This is a cycle of length 1. In this model, e.g. $\Diamond(p_1 \vee p_2)$, $\bigcirc(p_2/p_1 \vee p_2)$ and $\bigcirc(\neg p_2/p_1 \vee p_2)$, contradicting (D*). As in the previous example, even though $\succ$ has been chosen so as to be transitive, nothing hinges on it. To see why, we use the following variant of $\succ$, putting $x \succ y$ whenever x violates *strictly* less obligations issued by one authority than y does. The outcome is the same. In particular we still have $x_2 \succ x_1$ and $x_1 \succ x_2$. But $\succ$ is no longer transitive (since e.g. $x_2 \not\succ x_2$).

Observe that $\succ$ has been chosen to be total or complete, viz. for all x and y, $x \succ y$ or $y \succ x$. To show that nothing hinges on this property, consider the following variant definition, setting $x \succ y$ whenever the set of obligations issued by one authority that are violated by x is a subset of the set of those violated by y. Suppose the model contains two extra words x_3 and x_4 such that $x_3 \models \neg p_1 \wedge p_2 \wedge \neg q_1 \wedge q_2$, and $x_4 \models p_1 \wedge \neg p_2 \wedge q_1 \wedge \neg q_2$. We have in addition $x_1 \succ x_3$, $x_2 \succ x_4$, $x_1 \succ x_4$, $x_2 \succ x_3$, $x_3 \not\succ x_4$, and $x_4 \not\succ x_3$. Observe that the the outcome is the same even though $\succ$ is not total in this setting.

3 A cut-free hypersequent calculus for F

We introduce the hypersequent calculus **HF** for the logic **F**. **HF** is defined by adding to (a slightly modified[4] version of the) calculus for **E** a new rule $(\mathcal{B}et_F)$ corresponding to the $(\mathrm{D}^\star)$ axiom. The resulting calculus extends the hypersequent calculus for $S5$ [3,16] with left and right rules for the dyadic obligation, and two rules for $\mathcal{B}et$ (the **HE** calculus for **E** had only one).

Introduced in [19] to define a cut-free calculus for $S5$, hypersequents consist of sequents working in parallel.

Definition 3.1 [2] A *hypersequent* is a multiset $\Gamma_1 \Rightarrow \Pi_1 \mid \ldots \mid \Gamma_n \Rightarrow \Pi_n$ where, for all $i = 1, \ldots, n$, $\Gamma_i \Rightarrow \Pi_i$ is a multisets-based sequent, called a *component* of the hypersequent.

The hypersequent calculus **HF** is presented in Definition 3.2. **HF** consists of initial hypersequents (i.e., axioms), logical/modal/deontic and structural rules. The latter are divided into *internal* and *external rules.* **HF** incorporates the sequent calculus for the modal logic S4 as a sub-calculus and adds an additional layer of information by considering a single sequent to live in the context of hypersequents. Hence all the axioms and rules of **HF** (but the external structural rules) are obtained by adding to each sequent a context G or H, representing a (possibly empty) hypersequent. For instance, the (hypersequent version of the) axioms are $\Gamma, p \Rightarrow \Delta, p \mid G$. The external structural rules include ext. weakening (ew) and ext. contraction (ec) (see Fig. 1). These behave like weakening and contraction over whole hypersequent components. The hypersequent structure opens the possibility to define new such rules that allow the "exchange of information" between different sequents. These type of rules increases the expressive power of hypersequent calculi compared to sequent calculi, enabling the definition of cut-free calculi for logics that seem to escape a cut-free sequent formulation (e.g., $S5$). An example of external structural rule is the $(s5)$ rule in [16] (reformulated as $(s5')$ in Fig. 1 to account for the presence of $\bigcirc$), that allows the peculiar axiom of $S5$ to be derived.

The rules in Fig. 1 and 2 make use of the following notation:

$$\Sigma^\Box = \{\Box B :\ \Box B \in \Sigma\} \quad \Sigma^O = \{\bigcirc(C/D) : \bigcirc(C/D) \in \Sigma\} \quad \Sigma^{\Box,O} = \Sigma^\Box, \Sigma^O$$

[4] We employ a version of the rule $(\mathcal{B}et)$ that contains exactly one formula on its LHS, see Remark 3.10.

$$\frac{G}{G \mid \Gamma \Rightarrow \Pi}\ (ew) \qquad \frac{G \mid \Gamma \Rightarrow \Pi \mid \Gamma \Rightarrow \Pi}{G \mid \Gamma \Rightarrow \Pi}\ (ec) \qquad \frac{G \mid \Gamma^{\Box}, \Gamma^{O}, \Gamma' \Rightarrow \Pi'}{G \mid \Gamma \Rightarrow \ \mid \Gamma' \Rightarrow \Pi'}\ (s5')$$

Fig. 1. External structural rules

Also, for any set $\mathbb{D}$ of formulae, define $\mathcal{B}et\,\mathbb{D}$ as the set $\{\mathcal{B}et\,D \mid D \in \mathbb{D}\}$.

Definition 3.2 The hypersequent calculus **HF** consists of the hypersequent version of Gentzen LK sequent calculus for propositional classical logic, the external structural rules in Fig. 1, and the modal and deontic rules in Fig. 2.

Lemma 3.3 *The rules* $(\bigcirc R), (\mathcal{B}et), (\Box R)$ *and* (Bet_F) *are equivalent in* **HF** *to their version* $(\bigcirc R)^*, (\mathcal{B}et)^*, (\Box R)^*$*, and* $(Bet_F)^*$ *without the internal contexts* Γ *and* $\Gamma^{\Box,O}$.

Proof. One direction is trivial. For the other direction, consider the case of $(\bigcirc R)$ (the other cases are similar), and the following proof

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma^{\Box,O}, A, \mathcal{B}et\,\neg A \Rightarrow B \mid G}{A, \mathcal{B}et\,\neg A \Rightarrow B \mid \Gamma^{\Box,O} \Rightarrow \ \mid G}\ (s5')}{\Rightarrow \bigcirc(B/A) \mid \Gamma^{\Box,O} \Rightarrow \ \mid G}\ (\bigcirc R)^*}{\Gamma \Rightarrow \bigcirc(B/A) \mid \Gamma \Rightarrow \bigcirc(B/A) \mid G}\ (w)}{\Gamma \Rightarrow \bigcirc(B/A) \mid G}\ (ec)$$

□

Remark 3.4 The $(\mathcal{B}et_F)$ rule corresponds to the condition of limitedness of the betterness relation. A natural way to express this condition as a hypersequent rule is

$$\frac{G \mid \Gamma^{\Box,O}, \mathcal{B}et\,A \Rightarrow A}{G \mid \Gamma \Rightarrow A}$$

The upper sequent encodes the fact that in an arbitrary model $best(\neg A) = \emptyset$ (i.e. for any world x, if $y \vDash A$ for all $y \succ x$, then $x \vDash A$ also). The limitedness condition states that this can only happen if there is no world where $\neg A$ holds. The lower sequent encodes this fact. However, the addition of this rule to the calculus **HE** is not enough to obtain a complete *cut-free* calculus. The same holds for the one premise version of $(\mathcal{B}et_F)$, viz

$$\frac{G \mid \Gamma^{\Box,O}, \mathcal{B}et\,A \Rightarrow A}{G \mid \Gamma \Rightarrow \mathcal{B}et\,A}$$

In a calculus with this sole rule, the following formula (where a, b, and c are propositional variables) cannot be derived without using the cut rule:

$$\bigcirc(b \wedge \neg c/a) \wedge \bigcirc(a \wedge c/b) \rightarrow \bigcirc(\bot/a)$$

Ex. 3.9 shows how $(\mathcal{B}et_F)$ enables to get a cut free derivation of this formula.

$$\frac{\Gamma^{\Box,O}, A, \mathcal{B}et\,\neg A \Rightarrow B \mid G}{\Gamma \Rightarrow \Delta, \bigcirc(B/A) \mid G}\ (\bigcirc R) \qquad \frac{\Gamma^{\Box,O}, B \Rightarrow A \mid G}{\Gamma, \mathcal{B}et\,B \Rightarrow \Delta, \mathcal{B}et\,A \mid G}\ (\mathcal{B}et)$$

$$\frac{\Gamma^{\Box,O} \Rightarrow A \mid G}{\Gamma \Rightarrow \Delta, \Box A \mid G}\ (\Box R) \qquad \frac{\Gamma, A \Rightarrow \Delta \mid G}{\Gamma, \Box A \Rightarrow \Delta \mid G}\ (\Box L)$$

$$\frac{\{\Gamma^{\Box,O}, \mathcal{B}et\,\mathbb{D}, \mathcal{B}et\,B \Rightarrow D_i \mid G\}_{D_i \in \mathbb{D}} \qquad \Gamma^{\Box,O}, \mathcal{B}et\,\mathbb{D}, \mathcal{B}et\,B \Rightarrow B \mid G}{\Gamma, \mathcal{B}et\,\mathbb{D} \Rightarrow \Delta, \mathcal{B}et\,B \mid G}\ (\mathcal{B}et_F)$$

$$\frac{\Gamma, \bigcirc(B/A) \Rightarrow \Delta, A \mid G \quad \Gamma, \bigcirc(B/A) \Rightarrow \Delta, \mathcal{B}et\,\neg A \mid G \quad \Gamma, \bigcirc(B/A), B \Rightarrow \Delta \mid G}{\Gamma, \bigcirc(B/A) \Rightarrow \Delta \mid G}\ (\bigcirc L)$$

Fig. 2. Deontic and modal rules

A *derivation* in **HF** is a (possibly infinite) tree obtained by applying the rules bottom up. A *proof* $\mathcal{D}$ is a finite derivation whose leaves are axioms.

The soundness of **HE** is proved with respect to preference models. Although we can interpret a hypersequent H directly into the semantics, it is easier (and more readable) to interpret it as a formula $I(H)$ of the extended language $\mathcal{L} + \mathcal{B}et$. Then validity of $I(H)$ is defined as usual. We now show the validity of this formula whenever H is provable.

Theorem 3.5 *If there is a proof in* **HF** *of* $H := \Gamma_1 \Rightarrow \Pi_1 \mid \ldots \mid \Gamma_n \Rightarrow \Pi_n$, *then* $I(H) := \Box(\bigwedge \Gamma_1 \to \bigvee \Pi_1) \vee \ldots \vee \Box(\bigwedge \Gamma_n \to \bigvee \Pi_n)$ *is valid.*

Proof. We only need to show the soundness of the new rule $(\mathcal{B}et_F)$. Soundness of the other rules w.r.t. **F** follows from their soundness w.r.t. the weaker logic **E** proved in [6]. This includes the $(\mathcal{B}et)$ rule of **HF**, which is a weakened version of the homonymous rule of **HE**. It is enough to establish soundness for the simplified version of $(\mathcal{B}et_F)$ without internal contexts since the original version can be obtained via its combination with sound structural rules (Lemma 3.3). Suppose all the premises of $(\mathcal{B}et_F)$ are valid but not the conclusion. Thus, for some model M whose relation $\succ$ is limited and some world w in it, $w \not\models \Box(\bigwedge \mathcal{B}et\,\mathbb{D} \to \mathcal{B}et\,B) \vee I(G)$. Thus $w \not\models I(G)$ and therefore (1) $I(G)$ does not hold in any world–$I(G)$ is a disjunction of formulas prefixed with $\Box$, and gets the same truth-value in all worlds. Also, for some world x, $x \not\models \mathcal{B}et\,B$. Therefore, there exists a world $y \succ x$ such that $y \not\models B$, viz. $y \models \neg B$ and so $y \models \neg(B \wedge \bigwedge_{D_i \in \mathbb{D}} D_i)$. By the limitedness condition, there exists a world z in M that belongs to best$(\neg(B \wedge \bigwedge_{D_i \in \mathbb{D}} D_i))$, i.e. (2) $z \not\models B \wedge \bigwedge_{D_i \in \mathbb{D}} D_i$ and (3) for all $u \succ z$, $u \models B \wedge \bigwedge_{D_i \in \mathbb{D}} D_i$. By (2), (4) either $z \not\models B$ or $z \not\models D_j$ for some $D_j \in \mathbb{D}$. Consider the second case. From the opening assumption (using the left-most premise, with D_j on the right) and (1), one gets $z \models \bigwedge \mathcal{B}et\,\mathbb{D} \wedge \mathcal{B}et\,B \to D_j$. By contraposition, one gets that either $z \not\models \mathcal{B}et\,B$ or $z \not\models \mathcal{B}et\,D_k$ for some $D_k \in \mathbb{D}$. This contradicts (3). The case when $z \not\models B$ in (4) is handled analogously (now using the right-most premise, with B on the right hand side of $\Rightarrow$). $\Box$

Lemma 3.6 *For every formula A: $A \Rightarrow A$ is derivable in* **HF**.

Proof. Standard induction on the complexity of A. $\square$

Theorem 3.7 (Completeness with cut) *Each theorem of* **F** *has a proof in* **HF** *with the addition of the cut rule:*

$$\dfrac{G \,|\, \Gamma, A \Rightarrow \Delta \qquad H \,|\, \Sigma \Rightarrow \Pi, A}{G \,|\, H \,|\, \Gamma, \Sigma \Rightarrow \Delta, \Pi}\ (cut)$$

Proof. ($\mathrm{D}^{\star}$) axiom (with $\Diamond$ rewritten as $\neg\Box\neg$) can be derived as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{\Rightarrow \neg A, A}{\scriptstyle(\neg\mathrm{R})} \qquad \dfrac{\dfrac{\dfrac{A \Rightarrow A}{\Rightarrow \neg A, A}{\scriptstyle(\neg\mathrm{R})} \quad \mathcal{B}et\,\neg A \Rightarrow \mathcal{B}et\,\neg A \quad \dfrac{}{\bot \Rightarrow}{\scriptstyle(\bot\mathrm{L})}}{\bigcirc(\bot/A), \mathcal{B}et\,\neg A \Rightarrow \neg A}{\scriptstyle(\bigcirc\mathrm{L})+(\mathrm{w})}}{\bigcirc(\bot/A) \Rightarrow \mathcal{B}et\,\neg A}{\scriptstyle(\mathcal{B}et_{\mathrm{F}})} \qquad \dfrac{}{\bot \Rightarrow}{\scriptstyle(\bot\mathrm{L})}}{\bigcirc(\bot/A) \Rightarrow \neg A}{\scriptstyle(\bigcirc\mathrm{L})+(\mathrm{w})}}{\bigcirc(\bot/A) \Rightarrow \Box\neg A}{\scriptstyle(\Box\mathrm{R})}}{\neg\Box\neg A \Rightarrow \neg\bigcirc(\bot/A)}{\scriptstyle(\neg\mathrm{L})+(\neg\mathrm{R})}}{\Rightarrow \neg\Box\neg A \;\rightarrow\; \neg\bigcirc(\bot/A)}{\scriptstyle(\rightarrow\mathrm{R})}$$

(Nec) and all the remaining axioms are provable in **HF** (without using ($\mathcal{B}et_F$) or (cut)), while modus ponens requires (cut). $\square$

Example 3.8 The Kantian principle "ought implies can" $\bigcirc(B/A) \rightarrow (\Diamond A \rightarrow \Diamond(A \wedge B))$ holds in **F**, as shown by the following **HF** proof (we omit straightforward subderivations of propositional tautologies in the leaves)

$$\dfrac{\dfrac{\dfrac{\Rightarrow \neg A, A \qquad \dfrac{\dfrac{\Rightarrow \neg A, A \quad \mathcal{B}et\,\neg A \Rightarrow \mathcal{B}et\,\neg A \quad \dfrac{\neg(A \wedge B), B \Rightarrow \neg A}{\Box\neg(A \wedge B), B \Rightarrow \neg A}{\scriptstyle(\Box\mathrm{L})}}{\bigcirc(B/A), \Box\neg(A \wedge B), \mathcal{B}et\,\neg A \Rightarrow \neg A}{\scriptstyle(\bigcirc\mathrm{L})+(\mathrm{w})}}{\bigcirc(B/A), \Box\neg(A \wedge B) \Rightarrow \mathcal{B}et\,\neg A}{\scriptstyle(\mathcal{B}et_{\mathrm{F}})} \qquad \dfrac{\neg(A \wedge B), B \Rightarrow \neg A}{\Box\neg(A \wedge B), B \Rightarrow \neg A}{\scriptstyle(\Box\mathrm{L})}}{\bigcirc(B/A), \Box\neg(A \wedge B) \Rightarrow \neg A}{\scriptstyle(\bigcirc\mathrm{L})+(\mathrm{w})}}{\bigcirc(B/A), \Box\neg(A \wedge B) \Rightarrow \Box\neg A}{\scriptstyle(\Box\mathrm{R})}}{\Rightarrow \bigcirc(B/A) \;\rightarrow\; (\neg\Box\neg A \;\rightarrow\; \neg\Box\neg(A \wedge B))}{\scriptstyle(\rightarrow\mathrm{R})\mathrm{x}2+(\neg\mathrm{L})+(\neg\mathrm{R})}$$

Example 3.9 A derivation in **HF** of the formula in Remark 3.4 is as follows. First, we eliminate connectives and modalities in a natural fashion (we omit the premises of the ($\bigcirc L$) rule applications that are propositional tautologies):

$$
\dfrac{
\dots \quad \dots \quad
\dfrac{
\dots \quad
\dfrac{(1) \qquad (2)}{\bigcirc(b \wedge \neg c/a), \bigcirc(a \wedge c/b), \mathcal{B}et\, \neg a \Rightarrow \mathcal{B}et\, \neg b}\,(\mathcal{B}et_{\mathrm{F}})
\quad \dots
}{\bigcirc(b \wedge \neg c/a), \bigcirc(a \wedge c/b), a, \mathcal{B}et\, \neg a, b \wedge \neg c \Rightarrow \bot}\,(\bigcirc\mathrm{L})+(\mathrm{w})
}{
\dfrac{
\dfrac{\bigcirc(b \wedge \neg c/a), \bigcirc(a \wedge c/b), a, \mathcal{B}et\, \neg a \Rightarrow \bot}{\bigcirc(b \wedge \neg c/a), \bigcirc(a \wedge c/b) \Rightarrow \bigcirc(\bot/a)}\,(\bigcirc\mathrm{R})
}{\Rightarrow \bigcirc(b \wedge \neg c/a) \wedge \bigcirc(a \wedge c/b) \ \to \ \bigcirc(\bot/a)}\,(\to\mathrm{R})+(\wedge\mathrm{L})
}\,(\bigcirc\mathrm{L})+(\mathrm{w})
$$

The sequent $\bigcirc(b \wedge \neg c/a), \bigcirc(a \wedge c/b), \mathcal{B}et\, \neg a \Rightarrow \mathcal{B}et\, \neg b$ can be derived by applying the $(\mathcal{B}et_F)$ rule on both Bet-formulas leading to the two premises (1) $O(b \wedge \neg c/a), O(a \wedge c/b), \mathcal{B}et\, \neg a, \mathcal{B}et\, \neg b \Rightarrow \neg a$ and (2) $O(b \wedge \neg c/a), O(a \wedge c/b), \mathcal{B}et\, \neg a, \mathcal{B}et\, \neg b \Rightarrow \neg b$, both of which can be proved by applying $(\bigcirc L)$ once again.

Remark 3.10 The calculus **HF** is obtained by extending the hypersequent calculus for S5 with suitable rules for $\bigcirc$, making use of the auxiliary modality $\mathcal{B}et$. It is easy to see that $\bigcirc$ can be actually defined in the language with $\Box$ and $\mathcal{B}et$ in the sense that we have the following semantic equivalence

$$\bigcirc(B/A) \equiv \Box((A \wedge \mathcal{B}et\, \neg A) \to B)$$

It is possible to obtain a proof system for F by treating $\bigcirc(B/A)$ as an abbreviation. However, we have not done so for two reasons. First of all, this makes it possible to study $\bigcirc$ even in the $\Box$-free fragment of **F**. Moreover, we have a complete calculus for the $\Box$-free fragment of **F** (and this entails that the addition of $\Box$ is conservative). The second reason is that the way the $\mathcal{B}et$-modality is treated in the calculus **HF** does not correspond to any known normal or non-normal modal logic. For instance, it is easily seen that the $(\mathcal{B}et)$-rule does not allow us to derive standard axioms of the modal logic K, like $(\mathcal{B}et\, A \wedge \mathcal{B}et\, B) \to \mathcal{B}et(A \wedge B)$. Therefore the rules for $\mathcal{B}et$ *are not complete* with respect to its semantics for proving arbitrary sequents in the language with $\mathcal{B}et$ (as opposed to sequents containing formulas in the language of **F**). Also observe that the following rule (cf. Remark 3.4) is valid:

$$\frac{\mathcal{B}et\, A \to A}{\mathcal{B}et\, A}$$

This rule is not valid in K, but it is in GL (see, e.g. [25]). In conclusion, **HF** is not the combination of two existing calculi, one for $S5$ and one for $\mathcal{B}et$.

3.1 Cut-elimination

The completeness proof of **HF** makes use of the cut rule. Here we give a constructive proof that cut can be *eliminated* from **HF**+*cut* proofs. This result (cut-elimination) is typically proved by stepwise applications of permutation and principal reductions. The former shifts a cut one step upwards in either the left premise or the right premise. Following repeated applications, the situation is reached of a cut in which the cut-formula is principal (i.e. created by the rule immediately above it) in both premises. The principal reduction is now used to replace that cut with cuts on proper subformulas. An appeal to (transfinite) induction ultimately yields a cut-free proof.

The cut-elimination proof for **HF** + *cut* is not an easy adaptation of the corresponding result for **HE**. Indeed, the presence in (Bet_F) of a formula $\mathcal{B}et\, B$ that changes polarity from conclusion to premises, makes the principal reduction step even more involved than in the modal logic of provability GL.

The immediate corollary of cut-elimination is (a relaxed form of) the *subformula property*: every formula in a **HF** proof is a subformula (possibly negated and under the scope of $\mathcal{B}et$) of the end-formula.

Roadmap of the proof: To reduce the complexity of the cut on a $\rightarrow$ or $\neg$-formula we exploit the invertibility of its introduction rules (Lemma 3.11) and the usual principal reduction steps. Invertibility does not work for formulas of the form $\Box A$, $\mathcal{B}et\, A$, and $\bigcirc(B/A)$ so cuts have to be shifted upward till the cut-formula is introduced. The first challenge, already witnessed in **HE** is that the $(\Box R)$, $(\bigcirc R)$ and $(\mathcal{B}et)$ (as well as $(\mathcal{B}et_F)$) rules cannot be shifted below *every* cut: only those involving hypersequents of a certain "good" shape. Therefore a specific reduction strategy for lifting uppermost cuts is required: first over the premise in which the cut formula appears on the right (Lemma 3.15) and then, when a rule introducing the cut formula is reached (and in this case the sequent has a "good" shape), shifting the cut upwards over the other premise (Lemma 3.14) and then applying the principal reduction. This last reduction step is "standard" for $\Box$, $\bigcirc$-formulas and $\mathcal{B}et$ formulas introduced on both sides by $(\mathcal{B}et)$ (Lemma 3.12), while when $\mathcal{B}et$ formulas are introduced by $(\mathcal{B}et_F)$ a sophisticated argument inspired by the cut-elimination proof for the logic GL [25] is used. This is the second, and main, challenge in proving cut-elimination for **HF**. Note that the hypersequent structure itself does not necessitate major changes: the $(s5')$ rule permits permutation with cuts of a "good" shape, and to handle (ec) we consider the hypersequent version of the multicut: cut one component against (possibly) many components.

Notation and Terminology. The *length* $|\mathcal{D}|$ of an **HF** proof $\mathcal{D}$ is (the maximal number of applications of inference rules) +1 occurring on any branch of d. The *complexity* $\lceil A \rceil$ of a formula A is defined as: $\lceil A \rceil = 1$ if A is atomic, $\lceil \neg A \rceil = \lceil A \rceil + 1$, $\lceil A \rightarrow B \rceil = \lceil A \rceil + \lceil B \rceil + 1$, $\lceil \mathcal{B}et\, A \rceil = \lceil A \rceil + 1$, $\lceil \Box A \rceil = \lceil A \rceil + 1$, and $\lceil \bigcirc(A/B) \rceil = \lceil A \rceil + \lceil B \rceil + 3$. The *cut rank* $\rho(\mathcal{D})$ of $\mathcal{D}$ is the maximal complexity of cut formulas in $\mathcal{D}$, so $\rho(\mathcal{D}) = 0$ if $\mathcal{D}$ is cut-free. We use A^n (resp. Γ^n) to indicate n occurrences of A (resp. of Γ).

The rules of the classical propositional connectives remain invertible.

Lemma 3.11 (invertible connectives) *Every* **HF** *proof $\mathcal{D}$ of a hypersequent containing a formula $\neg A$ (resp. $A \rightarrow B$), can be transformed into a proof $\mathcal{D}'$ of the same hypersequent ending in an introduction rule for $\neg A$ (resp. $A \rightarrow B$) such that $\rho(\mathcal{D}') \leq \rho(\mathcal{D})$.*

As shown below, any cut whose cut formula is immediately introduced in left and right premise can be replaced by smaller cuts. While for compound formulas not introduced by the rule ($\mathcal{B}et_F$) the transformation is easy, this last case requires Lemma 3.13.

Lemma 3.12 (reduce principal cuts) *Let A be a compound formula and $\mathcal{D}_l$ and $\mathcal{D}_r$ be* **HF** *proofs such that $\rho(\mathcal{D}_l) < \lceil A \rceil$ and $\rho(\mathcal{D}_r) < \lceil A \rceil$, and*

(i) *$\mathcal{D}_l$ is a proof of $G \mid \Gamma, A \Rightarrow \Delta$ ending in a rule introducing A*

(ii) *$\mathcal{D}_r$ is a proof of $H \mid \Sigma \Rightarrow A, \Pi$ ending in a rule introducing A*

There is a transformation of these proofs into a **HF** *proof of $G \mid H \mid \Gamma, \Sigma \Rightarrow \Delta, \Pi$ with $\rho(\mathcal{D}) < \lceil A \rceil$.*

Proof. We discuss the only non-standard case: $A = \mathcal{B}et\, B$, and use a simplified version of the rules without internal contexts (cf. Lemma 3.3).

Assume that $\mathcal{B}et\, B$ is introduced by two ($\mathcal{B}et$) rules as in

$$\dfrac{\dfrac{G \mid B \Rightarrow C}{G \mid \Sigma, \mathcal{B}et\, B \Rightarrow \mathcal{B}et\, C, \Pi}\,(\mathcal{B}et) \qquad \dfrac{H \mid D \Rightarrow B}{H \mid \Gamma, \mathcal{B}et\, D \Rightarrow \mathcal{B}et\, B, \Delta}\,(\mathcal{B}et)}{G \mid H \mid \Gamma, \Sigma, \mathcal{B}et\, D \Rightarrow \mathcal{B}et\, C, \Delta, \Pi}\,(\text{cut})$$

the above cut is replaced by

$$\dfrac{\dfrac{G \mid B \Rightarrow C \qquad H \mid D \Rightarrow B}{G \mid H \mid D \Rightarrow C}\,(\text{cut})}{G \mid H \mid \Gamma, \Sigma, \mathcal{B}et\, D \Rightarrow \mathcal{B}et\, C, \Delta, \Pi}\,(\mathcal{B}et)$$

Assume that $\mathcal{B}et\, B$ is introduced on the right hand side by ($\mathcal{B}et_F$) as in

$$\dfrac{\dfrac{G \mid B \Rightarrow C}{G \mid \Sigma, \mathcal{B}et\, B \Rightarrow \mathcal{B}et\, C, \Pi}\,(\mathcal{B}et) \qquad \dfrac{\{H \mid \mathcal{B}et\, \mathbb{D}, \mathcal{B}et\, B \Rightarrow D_i\}_{1 \le i \le n} \qquad H \mid \mathcal{B}et\, \mathbb{D}, \mathcal{B}et\, B \Rightarrow B}{H \mid \Gamma, \mathcal{B}et\, \mathbb{D} \Rightarrow \mathcal{B}et\, B, \Delta}\,(\mathcal{B}et_{\mathrm{F}})}{G \mid H \mid \Gamma, \Sigma, \mathcal{B}et\, \mathbb{D} \Rightarrow \mathcal{B}et\, C, \Delta, \Pi}\,(\text{cut})$$

This case cannot be simply handled by cutting the premises of ($\mathcal{B}et$) and ($\mathcal{B}et_F$), because of the additional formulas $\mathcal{B}et\, B$ *on the left* appearing in the premises of ($\mathcal{B}et_F$). The strategy is to apply Lemma 3.13 to all premises of (Bet_F) to get proofs, with cut-rank $< \lceil \mathcal{B}et\, B_N \rceil$, of the same hypersequents but with $\mathcal{B}et\, B$ on the left removed. Hence we get

$$\dfrac{\dfrac{\{H \mid \mathcal{B}et\, \mathbb{D} \Rightarrow D_i\}_{1 \le i \le n}}{\{G \mid H \mid \mathcal{B}et\, \mathbb{D}, \mathcal{B}et\, C \Rightarrow D_i\}_{1 \le i \le n}}\,(\text{ew})+(\text{w}) \qquad \dfrac{\dfrac{G \mid B \Rightarrow C \qquad H \mid \mathcal{B}et\, \mathbb{D} \Rightarrow B}{G \mid H \mid \mathcal{B}et\, \mathbb{D} \Rightarrow C}\,(\text{cut})}{G \mid H \mid \mathcal{B}et\, \mathbb{D}, \mathcal{B}et\, C \Rightarrow C}\,(\text{w})}{G \mid H \mid \Gamma, \Sigma, \mathcal{B}et\, \mathbb{D} \Rightarrow \mathcal{B}et\, C, \Delta, \Pi}\,(\mathcal{B}et_{\mathrm{F}})$$

□

The following lemma allows us to remove any application of $\mathcal{B}et\, B$ formulas that appear on the left hand side of the ($\mathcal{B}et_F$) rule, via suitable cuts on B. Its proof is inspired by Valentini's cut-elimination argument for provability logic GL [25] where the corresponding lemma provides a constructive proof of Löb's theorem in GL. It requires indeed to perform global transformations: tracing bottom up from all the premises of (Bet_F) all the *occurrences* (ancestors) of the $\mathcal{B}et\, B$ formulas and substituting them with suitable formulas, taking care that the resulting proof is still correct. The tracing works as follows: we denote by $\mathcal{B}et\, B^*$ a *decorated occurrence* of $\mathcal{B}et\, B$. Starting with a hypersequent with one decorated occurrence of $\mathcal{B}et\, B$, we propagate the decoration through the proof to all formulas $\mathcal{B}et\, B$ which are in a predecessor relation [5] with $\mathcal{B}et\, B^*$. The tracing terminates at an *upper sequent* that is either (a) an axiom $\Gamma, \mathcal{B}et\, B^*, p \Rightarrow p, \Delta$, or the conclusion (b) of an internal/external weakening or of a rule with weakening built in (i.e., $(\Box R)$, and $(\bigcirc R)$), or (c) of $(\mathcal{B}et)$. In the following, for $\mathbb{B} = B_1, \ldots, B_N$, we write $\mathbb{B}_j$ to denote $\mathbb{B} \setminus B_j$.

Lemma 3.13 *Let $\mathcal{D}_1, \ldots, \mathcal{D}_n$ be the following* $\mathbf{HF}+$*cut proofs of the premises of a $(\mathcal{B}et_F)$ rule instance.*

$$\begin{array}{ccc} \mathcal{D}_1 & & \mathcal{D}_n \\ G \mid \mathcal{B}et\, \mathbb{B} \Rightarrow B_1 & \ldots & G \mid \mathcal{B}et\, \mathbb{B} \Rightarrow B_n \end{array}$$

Suppose that N satisfies $1 \le N \le n$, and $\rho(\mathcal{D}_i) < \lceil \mathcal{B}et\, B_N \rceil$ for each i ($1 \le i \le n$). There is a transformation of these proofs into a $\mathbf{HF}+$ *cut proof with cut-rank $< \lceil \mathcal{B}et\, B_N \rceil$ of $G \mid \mathcal{B}et\, \mathbb{B}_N \Rightarrow B_i$ for each i.*

Proof. Observe that the lemma is easy to prove using cut if we remove the requirement that the resulting proof has cut-rank $< \lceil \mathcal{B}et\, B_N \rceil$ (apply $(\mathcal{B}et_F)$ to $\mathcal{D}_1, \ldots, \mathcal{D}_n$ to get $\mathcal{B}et\, \mathbb{B}_N \Rightarrow \mathcal{B}et\, B_N$, then apply cut with the latter to each $\mathcal{D}_i$). To reduce clutter we omit the external contexts and the modal internal contexts as they do not play a role in the argument (cf. Lem. 3.3 for the latter).

Trace $\mathcal{B}et\, B_N$ upwards in each $\mathcal{D}_i$ (we indicate with $\mathcal{B}et\, B_N^*$ its decorated version) until the upper sequents ((a)-(c) above) introducing $\mathcal{B}et\, B_N^*$ are encountered. Define the *depth of* $\mathcal{B}et\, B_N^*$ for a proof ending in a $(\mathcal{B}et_F)$ rule as the total number (over all of its premises) of $(\mathcal{B}et_F)$ rules that contain the decorated formula $\mathcal{B}et\, B_N^*$. Note that $\mathcal{B}et\, B_N^*$ can only appear on the LHS of sequents. We prove the claim by induction on the depth K of $\mathcal{B}et\, B_N^*$ in the premises $\mathcal{D}_1, \ldots, \mathcal{D}_n$.

Inductive case. Suppose that the depth $K > 0$. In that case there must be a nearest $(\mathcal{B}et_F)$ rule above the root of some $\mathcal{D}_i$ of the form

$$\frac{\{\mathcal{B}et\, B_N^*, \mathcal{B}et\, \mathbb{D} \Rightarrow D_i\}_{1 \le i \le I} \qquad \mathcal{B}et\, B_N^*, \mathcal{B}et\, \mathbb{D} \Rightarrow B_N}{\mathcal{B}et\, B_N^*, \mathcal{B}et\, \mathbb{D}_I \Rightarrow \mathcal{B}et\, D_I}\,(\mathcal{B}et_F) \qquad (1)$$

Each premise of the above is one $(\mathcal{B}et_F)$ rule away from the root of $\mathcal{D}_i$ and so the depth of $\mathcal{B}et\, B_N^*$ in (1) must be $< K$. Hence we can apply IH to obtain proofs with cut-rank $< \lceil \mathcal{B}et\, B_N \rceil$ of $\mathcal{B}et\, \mathbb{D} \Rightarrow D_i$ for every i ($1 \le i \le I$).

Let $\mathcal{D}_i'$ be obtained from $\mathcal{D}_i$ by replacing the subproof concluding (1) with

[5] This is the familiar parametric ancestor relation of [4] in the setting of hypersequents.

$$\cfrac{\cfrac{\{\mathcal{B}et\,\mathbb{D} \Rightarrow D_i\}_{1\leq i \leq I}}{\mathcal{B}et\,\mathbb{D}_I \Rightarrow \mathcal{B}et\,D_I}\,(\mathcal{B}et_F)}{\mathcal{B}et\,B_N^*, \mathcal{B}et\,\mathbb{D}_I \Rightarrow \mathcal{B}et\,D_I}\,(\mathrm{w})$$

Since we replaced a $(\mathcal{B}et_F)$ rule between the root and the upper sequent with a weakening on $\mathcal{B}et\,B_N^*$, it follows that the depth of $\mathcal{B}et\,B_N^*$ in $\mathcal{D}_1, \ldots, \mathcal{D}'_i, \ldots, \mathcal{D}_n$ (n elements) is $< K$. From the IH we obtain proofs of $\mathcal{B}et\,\mathbb{B}_N \Rightarrow B_i$ with cut-rank $< \lceil \mathcal{B}et\,B_N \rceil$ for every i so the claim is proved.

Base case $K = 0$: there are no $(\mathcal{B}et_F)$ rule instances involving $\mathcal{B}et\,B_N^*$. In this case, when replacing the decorated formula $\mathcal{B}et\,B_N^*$ with suitable formulas, only the upper sequents arising from applications of $(\mathcal{B}et)$ (i.e. case (c)) need some care. We illustrate the proof strategy with a concrete example. See the Appendix for full details.

Suppose that the following upper sequents occur in $\mathcal{D}_1,, \ldots \mathcal{D}_n$.

$$\frac{B_N \Rightarrow C}{\mathcal{B}et\,B_N^* \Rightarrow \mathcal{B}et\,C} \qquad \frac{B_N \Rightarrow D}{\mathcal{B}et\,B_N^* \Rightarrow \mathcal{B}et\,D} \qquad \frac{B_N \Rightarrow B_N}{\mathcal{B}et\,B_N^* \Rightarrow \mathcal{B}et\,B_N}$$

Replace $\mathcal{B}et\,B_N^*$ with $\mathcal{B}et\,\mathbb{B}_N, \mathcal{B}et\,C, \mathcal{B}et\,D$ throughout $\mathcal{D}_1,, \ldots \mathcal{D}_n$. The first two upper sequents above become quasi-axioms (cf. Lem. 3.6) $\mathcal{B}et\,\mathbb{B}_N, \mathcal{B}et\,C, \mathcal{B}et\,D \Rightarrow \mathcal{B}et\,C$ and $\mathcal{B}et\,\mathbb{B}_N, \mathcal{B}et\,C, \mathcal{B}et\,D \Rightarrow \mathcal{B}et\,D$, respectively. The third upper sequent now looks like $\mathcal{B}et\,\mathbb{B}_N, \mathcal{B}et\,C, \mathcal{B}et\,D \Rightarrow \mathcal{B}et\,B_N$; the latter sequent is provable by applying $(\mathcal{B}et_F)$ to the conclusions of $\mathcal{D}_1, \ldots, \mathcal{D}_n$ (followed by some weakening). In this way we obtain proofs of $(*)$ $\mathcal{B}et\,\mathbb{B}_N, \mathcal{B}et\,C, \mathcal{B}et\,D \Rightarrow B_i$ for each i. Now, by two applications of cut on B_N (with the premises $B_N \Rightarrow C$ and $B_N \Rightarrow D$ that appeared in the upper sequents indicated above), we also get $(**)$ $\mathcal{B}et\,\mathbb{B}_N, \mathcal{B}et\,C, \mathcal{B}et\,D \Rightarrow C$ and $(***)$ $\mathcal{B}et\,\mathbb{B}_N, \mathcal{B}et\,C, \mathcal{B}et\,D \Rightarrow D$. An application of $(\mathcal{B}et_F)$ with premises $(*) - (***)$ leads to a proof of $\mathcal{B}et\,\mathbb{B}_N, \mathcal{B}et\,D \Rightarrow \mathcal{B}et\,C$.

Next, replace $\mathcal{B}et\,B_N^*$ with $\mathcal{B}et\,\mathbb{B}_N, \mathcal{B}et\,D$ throughout the original $\mathcal{D}_1, \ldots \mathcal{D}_n$ (once again, as in the previous paragraph, the replacements are made in the *original proofs*; this is a feature of the transformation that is seen also in the next paragraph). Then $\mathcal{B}et\,B_N^* \Rightarrow \mathcal{B}et\,D$ becomes a quasi-axiom once more (i.e., $\mathcal{B}et\,\mathbb{B}_N, \mathcal{B}et\,D \Rightarrow \mathcal{B}et\,D$). Also $\mathcal{B}et\,B_N^* \Rightarrow \mathcal{B}et\,C$ becomes $\mathcal{B}et\,\mathbb{B}_N, \mathcal{B}et\,D \Rightarrow \mathcal{B}et\,C$ whose proof we obtained in the paragraph above. The third upper sequent now looks like $\mathcal{B}et\,\mathbb{B}_N, \mathcal{B}et\,D \Rightarrow \mathcal{B}et\,B_N$ and it is proved as before. Proceeding downwards similarly as before we ultimately obtain a proof of $\mathcal{B}et\,\mathbb{B}_N \Rightarrow \mathcal{B}et\,D$. In analogous fashion we prove $\mathcal{B}et\,\mathbb{B}_N \Rightarrow \mathcal{B}et\,C$.

Finally, replace $\mathcal{B}et\,B_N^*$ with $\mathcal{B}et\,\mathbb{B}_N$ throughout the original $\mathcal{D}_1, \ldots \mathcal{D}_n$. The point is that the first and second upper sequents become $\mathcal{B}et\,\mathbb{B}_N \Rightarrow \mathcal{B}et\,C$ and $\mathcal{B}et\,\mathbb{B}_N \Rightarrow \mathcal{B}et\,D$ and we have already obtained proofs of these (the third upper sequent is handled similarly to before). Proceed downwards to obtain a proof of $\mathcal{B}et\,\mathbb{B}_N \Rightarrow B_i$ for every i. Every introduced cut was on B_N and hence the cut-rank of the final proof is $< \lceil \mathcal{B}et\,B_N \rceil$. □

The following lemma shifts the cut upward on the left premise of a cut when the right premise is principal, and uses Lemma 3.15 to reduce it.

Lemma 3.14 (permutation left) *Let $\mathcal{D}_l$ and $\mathcal{D}_r$ be* **HF** *proofs such that:*

(i) *$\mathcal{D}_l$ is a proof of $G \mid \Gamma_1, A^{\lambda_1} \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n, A^{\lambda_n} \Rightarrow \Delta_n$ and $\rho(\mathcal{D}_l) < \lceil A \rceil$;*

(ii) *A is a compound formula and $\mathcal{D}_r := H \mid \Sigma \Rightarrow A, \Pi$ ends with a right logical rule introducing the indicated occurrence of A, and $\rho(\mathcal{D}_r) < \lceil A \rceil$;*

Here each $\lambda_i > 0$. There is a transformation of these proofs into a **HF** *proof $\mathcal{D}$ of $G \mid H \mid \Gamma_1, \Sigma^{\lambda_1} \Rightarrow \Delta_1, \Pi^{\lambda_1} \mid \ldots \mid \Gamma_n, \Sigma^{\lambda_n} \Rightarrow \Delta_n, \Pi^{\lambda_n}$ with $\rho(\mathcal{D}) < \lceil A \rceil$.*

Proof. We distinguish cases according to the shape of A. If A is $\neg B$ or $B \to C$, the claim follows by Lemmas 3.11 and 3.12. If A is $\Box B$, $\bigcirc(B/C)$ or $\mathcal{B}et\, B$ the proof proceeds by induction on $|\mathcal{D}_l|$. If $\mathcal{D}_l$ ends in an initial sequent, then we are done. If $\mathcal{D}_l$ ends in a left rule introducing one of the indicated cut formulas, the claim follows by (i.h. and) Lemma 3.12. Otherwise, let (r) be the last inference rule applied in $\mathcal{D}_l$. The claim follows by the i.h., an application of (r) and/or weakening. Some care is needed to handle the cases in which r is $(s5')$, $(\Box R)$, $(\bigcirc R)$ or $(\mathcal{B}et)$ and A is not in the hypersequent context G. Notice that when $A = \Box B$ (resp. $A = \bigcirc(B/C)$) the conclusion of $\mathcal{D}_r$ is $\Sigma \Rightarrow \Box B, \Pi$ (resp. $\Sigma \Rightarrow \bigcirc(B/C), \Delta$), but we can safely use the "good"-shaped sequent $\Sigma^{\Box}, \Sigma^{O} \Rightarrow \Box B$ (resp. $\Sigma^{\Box}, \Sigma^{O} \Rightarrow \bigcirc(B/C)$), that allows cuts to be shifted upwards over all **HF** rules, and we apply weakening afterwards.

Let $A = \mathcal{B}et\, B$ and $\mathcal{D}_r$ ends in a $(\mathcal{B}et)$ rule with conclusion $\mathcal{B}et\, C, \Sigma \Rightarrow \mathcal{B}et\, B, \Pi$. If (r) is a $(\mathcal{B}et)$ rule introducing $\mathcal{B}et\, B$, the claim follows by Lemma 3.12. If (r) is (Bet_F), as in the proof below (to simplify the matter we omit both the internal and external contexts)

$$\dfrac{\{\mathcal{B}et\, \mathbb{D}, \mathcal{B}et\, B \Rightarrow D_j\}_{j=1,\ldots N} \qquad \begin{array}{c}\vdots\, d'_l \\ \mathcal{B}et\, \mathbb{D}, \mathcal{B}et\, B \Rightarrow B\end{array}}{\mathcal{B}et\, \mathbb{D}_i, \mathcal{B}et\, B \Rightarrow \mathcal{B}et\, D_i}\ (Bet_F)$$

we apply Lemma 3.13 to its premises (to get rid of the formula $\mathcal{B}et\, B$) and get

$$\{\mathcal{B}et\, \mathbb{D} \Rightarrow D_j\}_{j=1,\ldots N}.$$

The desired hypersequent $\mathcal{B}et\, \mathbb{D}_i, \mathcal{B}et\, C, \Sigma \Rightarrow \mathcal{B}et\, D_i, \Pi$ is simply obtained by applying the rule $(\mathcal{B}et_F)$ followed by (w). The case in which $\mathcal{D}_r$ ends in a $(\mathcal{B}et_F)$ rule is analogous. □

Lemma 3.15 (permutation right) *Let $\mathcal{D}_l$ and $\mathcal{D}_r$ be* **HF** *proofs where*

(i) *$\mathcal{D}_l$ concludes $G \mid \Gamma, A \Rightarrow \Delta$ and $\rho(\mathcal{D}_l) < \lceil A \rceil$*

(ii) *$\mathcal{D}_r$ concludes $H \mid \Sigma_1 \Rightarrow A^{\lambda_1}, \Pi'_1 \mid \ldots \mid \Sigma_n \Rightarrow A^{\lambda_n}, \Pi'_n$ with $\rho(\mathcal{D}_r) < \lceil A \rceil$.*

Here each $\lambda_i > 0$. There is a transformation of these proofs into a **HF** *proof $\mathcal{D}$ of $G \mid H \mid \Sigma_1, \Gamma^{\lambda_1} \Rightarrow \Pi'_1, \Delta^{\lambda_1} \mid \ldots \mid \Sigma_n, \Gamma^{\lambda_n} \Rightarrow \Pi'_n, \Delta^{\lambda_n}$ with $\rho(\mathcal{D}) < \lceil A \rceil$.*

Proof. Let (r) be the last inference rule applied in $\mathcal{D}_r$. If (r) is an axiom, then the claim holds trivially. If (one of) the indicated occurrence(s) of A is principal by (r) then the claim follows from Lemma 3.14. So suppose that no A is principal by (r). Proceed by induction on $|\mathcal{D}_r|$.

Consider the following analysis of (r): it acts only on H or is a rule other than $(s5')$, $(\Box R)$, $(\bigcirc R)$, $(\mathcal{B}et_F)$ and $(\mathcal{B}et)$; if it is $(\Box R)$, $(\bigcirc R)$, $(\mathcal{B}et_F)$ or

($\mathcal{B}et$) then the indicated A cannot be in the active premise component since that would make it principal; if (r) is $(s5')$ and A is in an active component of the conclusion it must be the component without any context restriction (it cannot be other since that should be empty). In all these cases the claim follows by applying the IH to the premise(s) followed by (r). $\square$

Theorem 3.16 (Cut Elimination) *Cut elimination holds for* $\mathbf{HF} + cut$.

Proof. Define the *cut-multiset* $M_{\mathcal{D}}$ of $\mathcal{D}$ to be the multiset over the natural numbers $\mathbb{N}$ such that the multiplicity $M(n)$ of $n \in \mathbb{N}$ is the number of cut-rules in $\mathcal{D}$ with cut-rank n. We establish cut-elimination via induction on the Dershowitz-Manna[6] well-founded ordering over these multisets.

Let $\mathcal{D}$ be a $\mathbf{HF} + cut$ proof. Base case: $M_{\mathcal{D}} = \emptyset$ and hence $\mathcal{D}$ is cut-free. Inductive case: apply Lemma 3.15 to a subproof δ concluding a topmost cut in $\mathcal{D}$ (let the cut-formula be A). We thus obtain a new proof δ' whose cut-rank is $< \lceil A \rceil$. Let $\mathcal{D}'$ be the proof obtained from $\mathcal{D}$ by replacing δ with δ'. By inspection, $M_{\mathcal{D}'} <_m M_{\mathcal{D}}$ and hence the result follows by induction. $\square$

Corollary 3.17 (Completeness) *Each theorem of* $\mathbf{F}$ *has a proof in* $\mathbf{HF}$.

4 A proof search oriented calculus for F

By modifying the calculus presented in Section 3, we obtain a decision procedure for the logic $\mathbf{F}$, and a complexity bound. The modified calculus $\mathbf{HF}^{+}$is based on the following ideas:

(i) Hypersequent component are considered as "set-based": no duplication of formula is allowed within a component $\Gamma \Rightarrow \Delta$ of an hypersequent G.
(ii) In every rule the "principal" component(s) are kept in all premises, but not duplicated; thus hypersequents themselves are considered to be sets of components.
(iii) There are no redundant application of rules, in the sense that a rule is not applied (to a formula/component) if one of the premises of the rules is already contained in the conclusion.
(iv) There are no structural rules, except for the rule $(s5')$.

Restriction (i) is justified by the admissibility of internal contraction. As an example, by this restriction the backward application of $(\wedge L)$ will produce:

$$\frac{\Gamma, A, B \Rightarrow \Delta \,|\, G}{\Gamma, A, A \wedge B \Rightarrow \Delta \,|\, G} \qquad \text{rather than} \qquad \frac{\Gamma, A, A, B \Rightarrow \Delta \,|\, G}{\Gamma, A, A \wedge B \Rightarrow \Delta \,|\, G}$$

We display below the modified rules, we omit propositional rules; notice that the (O-L) rule does not need to be modified:

$$\frac{G \,|\, \Gamma^{\Box}, \Gamma^{O}, \Gamma' \Rightarrow \Pi' \,|\, G \,|\, \Gamma \Rightarrow \Delta}{G \,|\, \Gamma \Rightarrow \Delta \,|\, \Gamma' \Rightarrow \Pi'} \ (s5'_{new})$$

[6] $M <_m N$ iff $M \neq N$ and $M(k) > N(k)$ implies there is $k' > k$ such that $M(k') < N(k')$

$$\frac{\Gamma^{\Box,O}, A, \mathcal{B}et\,\neg A \Rightarrow B \mid G \mid \Gamma \Rightarrow \bigcirc(B/A), \Delta}{\Gamma \Rightarrow \bigcirc(B/A), \Delta \mid G}\ (\bigcirc R) \quad \frac{\Gamma^{\Box,O}, B \Rightarrow A \mid G \mid \Gamma, \mathcal{B}et\,B \Rightarrow \Delta, \mathcal{B}et\,A}{\Gamma, \mathcal{B}et\,B \Rightarrow \Delta, \mathcal{B}et\,A \mid G}\ (\mathcal{B}et)$$

$$\frac{\Gamma^{\Box,O} \Rightarrow A \mid G \mid \Gamma \Rightarrow \Delta, \Box A}{\Gamma \Rightarrow \Delta, \Box A \mid G}\ (\Box R) \qquad \frac{\Gamma, A \Rightarrow \Delta \mid G \mid \Gamma, \Box A \Rightarrow \Delta}{\Gamma, \Box A \Rightarrow \Delta \mid G}\ (\Box L)$$

$$\frac{\{\Gamma^{\Box,O}, \mathcal{B}et\,\mathbb{D}, \mathcal{B}et\,B \Rightarrow D_i \mid G \mid S\}_{D_i \in \mathbb{D}} \qquad \Gamma^{\Box,O}, \mathcal{B}et\,\mathbb{D}, \mathcal{B}et\,B \Rightarrow B \mid G \mid S}{\Gamma, \mathcal{B}et\,\mathbb{D} \Rightarrow \mathcal{B}et\,B, \Delta \mid G}\ (\mathcal{B}et_F)$$

where $S = \Gamma, \mathcal{B}et\,\mathbb{D} \Rightarrow \mathcal{B}et\,B, \Delta$

It is tacitly assumed that contraction is applied in the premises (in particular for $(s5')$ rule), so that $\Box$ and O-formulas are not duplicated).

It is easy to see that the the calculus $\mathbf{HF}^+$is sound and also complete, as a cut-free proof of **HF** can be simulated by $\mathbf{HF}^+$ and *vice versa*.

Proposition 4.1 *Given an hypersequent G: $\vdash_{\mathbf{HF}} G$ iff $\vdash_{\mathbf{HF}^+} G$.*

Furthermore, observe that all rules are invertible, thus the order of application of rules within a derivation does not matter.

In order to obtain a decision procedure based on the calculus $\mathbf{HF}^+$, we must avoid redundant application of rules in a backward proof search. First, let us define for two hypersequents G_1 and G_2 that $G_1 \sqsubseteq G_2$ if for every $\Gamma \Rightarrow \Delta \in G_1$ there is $\Gamma' \Rightarrow \Delta' \in G_2$ such that $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$. We denote by $G_1 \sqsubset G_2$ the strict relation. Observe that for any rule R of $\mathbf{HF}^+$:

$$\frac{G_1 \ldots G_n}{G}\ (R)$$

we have $G \sqsubseteq G_i$ for $i = 1, \ldots, n$. We say that an application of a rule R is *redundant* if for some $i \in \{1, \ldots, n\}$, it holds $G_i \sqsubseteq G$. We say that a hypersequent G is *saturated* if it is not an axiom and all rule applications to it are redundant.

We adopt the following proof-search strategy: (i) no rule can be applied to an axiomatic sequent (ii) no redundant application of rule is allowed. The strategy preserves completeness.

Proposition 4.2 *Given an hypersequent G: if $\vdash_{\mathbf{HF}^+} G$ then G has a proof in* $\mathbf{HF}^+$*according to the proof-strategy.*

From now on we restrict attention to derivations built according to the strategy. We show that any derivation with root sequent $\Rightarrow A$, for a formula A, is finite. To this purpose given a formula $A \in \mathcal{L}$, let $Sub(A)$ be the set of subformulas of A and $Sub^+(A) = Sub(A) \cup \{\mathcal{B}et\,\neg B : \bigcirc(C/B)$ occurs in $A\}$.

We now prove that the calculus $\mathbf{HF}^+$provides a decision procedure for **F**.

Theorem 4.3 *Let $\mathcal{D}$ be a derivation in $\mathbf{HF}^+$ with root $\Rightarrow A$ for a **F**-formula A, then $\mathcal{D}$ is finite.*

Proof. Since the rules are analytic, given any hypersequent G occurring in $\mathcal{D}$, we have that for any $\Gamma \Rightarrow \Delta \in G$ we have $\Gamma \subseteq Sub^+(A)$ and $\Delta \subseteq Sub^+(A)$. But hypersequents are *sets* of components, thus it must be that for any $\Gamma \Rightarrow \Delta \in G$ and $\Gamma' \Rightarrow \Delta' \in G$ either $\Gamma \neq \Gamma'$ or $\Delta \neq \Delta'$. Thus G may have at most $2^{Sub^+(A)} \times 2^{Sub^+(A)}$ components, and each component has a size bounded by $Sub^+(A)$. Thus we can conclude that only finitely-many different hypersequents may occur in a derivation $\mathcal{D}$. By preventing repetitions of the same hypersequent on any branch (loop-checking), we get that every branch of $\mathcal{D}$ is finite. Since $\mathcal{D}$ is a finitely-branching tree, we can conclude that $\mathcal{D}$ is finite. □

Although the previous theorem ensures that any derivation is finite, it does not provide directly a decision algorithm for **F**.

Let n be the length of A as a string of symbols. Here is the decision procedure: we consider a non-deterministic algorithm which takes as input $\Rightarrow A$ and guesses a saturated hypersequent H: if it finds it, the algorithm answers "non-provable", otherwise, it answers "provable". By inspection, the size of the candidate saturated hypersequent H is $O(2^{2n})$. More concretely, the algorithm tries to build the candidate hypersequent H as follows: initialise a derivation with root $H_0 = \Rightarrow A$. Apply the rules backwards in an arbitrary but fixed order, choose non-deterministically a premise if there are more than one. In this way we generate a branch $\mathcal{B} = H_0, H_1, H_2 \ldots$. Observe that by the strategy, an application of a rule R to H_i is allowed only if H_i is not an axiom and that application of R is non-redundant, in this case it must be $H_i \sqsubset H_{i+1}$. The latter together with the observation that every hypersequent has size $O(2^{2n})$ implies that the length of every branch $\mathcal{B}$ is $O(2^{2n})$ and the last hypersequent H_k of $\mathcal{B}$ is either saturated or an axiom. Since every rule of $\mathbf{HF}^+$is invertible, unprovability of a hypersequent coincides with the existence of a branch rooted at that hypersequent whose leaf is saturated. Observe that all checks (whether H_i is an axiom, or is saturated, or whether an application of R to it is non-redundant) take at most quadratic time in the size of H_i.

The previous argument shows that non-provability in **F** can be decided in NEXP time. Whence we get:

Theorem 4.4 *Deciding if a formula is a theorem of* **F** *is in CoNEXP.*

Future work

The proposed calculus provides a preliminary complexity bound (CoNEXP) for theoremhood in **F**. Notice that CoNEXP is a worst-case bound, in practice there are several heuristics and techniques that could be adopted to reduce the complexity and get a more efficient proof system. Moreover, although the complexity of the decision problem was previously unknown, we expect that a better bound can be obtained by refining the rules of the calculus, in particular the $(\mathcal{B}et_F)$ rule which is the source of the exponential blow-up as in principle it has to be applied to any subset of $\mathcal{B}et$-formulas.

Furthermore we would like to investigate how to extract countermodels of non-valid formulas from failed derivations. This is a non-trivial task because of the limitedness condition that countermodels must satisfy.

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5 Appendix

Proof. [Lemma 3.13 Base case $K = 0$]

Base case $K = 0$. There there are no $(\mathcal{B}et_F)$ rule instances containing $\mathcal{B}et\, B_N^*$. Define the *width of* $\mathcal{B}et\, B_N^*$ (terminology due to Valentini [25]) of a proof ending in a $(\mathcal{B}et_F)$ rule as the total number of upper sequents where $\mathcal{B}et\, B_N^*$ is introduced by a $(\mathcal{B}et)$ rule (this is the rule introducing $\mathcal{B}et$ in the antecedent, it should not to be confused with the $(\mathcal{B}et_F)$ rule!) with conclusion $\mathcal{B}et\, B_N^* \Rightarrow \mathcal{B}et\, C_w$ with $C_w \neq B_N$.

We establish the result by induction on the width W of the given proof which ends in a $(\mathcal{B}et_F)$ rule with premises $\mathcal{D}_1, \ldots, \mathcal{D}_n$. We proceed by case analysis on W.

Case $W = 0$. The upper sequents introduce $\mathcal{B}et\, B_N^*$ by weakening, or by a $(\mathcal{B}et)$ rule whose conclusion is $\mathcal{B}et\, B_N^* \Rightarrow \mathcal{B}et\, B_N$. The desired proof is obtained by replacing the occurrences of $\mathcal{B}et\, B_N^*$ in these upper sequents with $\mathcal{B}et\, \mathbb{B}_N$ as follows: the weakening on $\mathcal{B}et\, B_N^*$ is replaced with $\mathcal{B}et\, \mathbb{B}_N$, and the subproof ending in $\mathcal{B}et\, B_N^* \Rightarrow \mathcal{B}et\, B_N$ is replaced by a proof of $\mathcal{B}et\, \mathbb{B}_N \Rightarrow \mathcal{B}et\, B_N$ (itself obtained by applying $(\mathcal{B}et_F)$ to $\mathcal{D}_1, \ldots, \mathcal{D}_n$).

Case $W > 0$. Let $\mathbb{C} = \{C_1, \ldots, C_W\}$ be the set of upper sequents introducing $\mathcal{B}et\, B_N^*$ by a $(\mathcal{B}et)$ rule that conclude as $\mathcal{B}et\, B_N^* \Rightarrow \mathcal{B}et\, C_i$ with $C_i \neq B_N$.

Claim: If $\mathcal{B}et\, \mathbb{B}_N, \mathcal{B}et(\mathbb{C} \setminus S) \Rightarrow \mathcal{B}et\, C$ is provable with cut-rank $< \lceil \mathcal{B}et\, B_N \rceil$ for $S \subseteq \mathbb{C}$ and every $C \in S$, then the following premises of a $(\mathcal{B}et_F)$ rule are provable with cut-rank $< \lceil \mathcal{B}et\, B_N \rceil$:

$$\mathcal{B}et\, \mathbb{B}_N, \mathcal{B}et(\mathbb{C} \setminus S) \Rightarrow D \qquad (D \in \mathbb{B}_N \cup (\mathbb{C} \setminus S))$$

Proof of claim: let $S \subseteq \mathbb{C}$ be given. There are W occurrences of subproofs (spread across $\mathcal{D}_1, \ldots, \mathcal{D}_n$) that end in an upper sequent of the following form.

$$\frac{B_N \Rightarrow C}{\mathcal{B}et\, B_N^* \Rightarrow \mathcal{B}et\, C}\ (\mathcal{B}et) \text{ where } C \in \mathbb{C} \text{ and } C \neq B_N$$

For $C \in S$, replace the above with the following (the premise is the proof provided from the hypothesis).

$$\frac{\mathcal{B}et\,\mathbb{B}_N, \mathcal{B}et(\mathbb{C}\setminus S) \Rightarrow \mathcal{B}et\,C}{\mathcal{B}et\,\mathbb{B}_N, \mathcal{B}et\,B_N, \mathcal{B}et(\mathbb{C}\setminus S) \Rightarrow \mathcal{B}et\,C}\ (\mathrm{w})$$

For $C \in \mathbb{C}\setminus S$, replace instead with the 'obvious' proof (NB. $\mathcal{B}et\,C \in \mathcal{B}et(\mathbb{C}\setminus S)$)

$$\frac{\mathcal{B}et\,C \Rightarrow \mathcal{B}et\,C}{\mathcal{B}et\,\mathbb{B}_N, \mathcal{B}et\,B_N, \mathcal{B}et(\mathbb{C}\setminus S) \Rightarrow \mathcal{B}et\,C}\ (\mathrm{w})$$

In each of the W subproofs, $\mathcal{B}et\,B_N$ has been introduced by weakening. For this reason, proceeding downwards, we obtain the following premises of a $(\mathcal{B}et_F)$ rule with width 0 (the second row is obtained by a cut on $\mathcal{B}et\,\mathbb{B}_N, \mathcal{B}et\,B_N, \mathcal{B}et(\mathbb{C}\setminus S) \Rightarrow B_N$ and $B_N \Rightarrow C$).

$$\begin{array}{ll} \mathcal{B}et\,\mathbb{B}_N, \mathcal{B}et\,B_N, \mathcal{B}et(\mathbb{C}\setminus S) \Rightarrow B_i & \text{every } i \\ \mathcal{B}et\,\mathbb{B}_N, \mathcal{B}et\,B_N, \mathcal{B}et(\mathbb{C}\setminus S) \Rightarrow C & C \in \mathbb{C}\setminus S \end{array}$$

Since the width is 0, we can remove the $\mathcal{B}et\,B_N$ from every sequent above (see Case $W = 0$) and hence the claim is proved.

Returning to the main proof (case $K = 0$), setting $S = \emptyset$, the hypothesis of the above claim is vacuously true and hence we obtain a proof of $\mathcal{B}et\,\mathbb{B}_N, \mathcal{B}et\,\mathbb{C} \Rightarrow D$ for each $D \in \mathbb{B}_N \cup \mathbb{C}$ i.e. starting with the given proof which ends in a $(\mathcal{B}et_F)$ rule with premises $\mathcal{D}_1, \ldots, \mathcal{D}_n$, apply the transformation to every $C \in \mathbb{C}\setminus S (= \mathbb{C})$ that is described in the argument witnessing the claim.

Now apply $(\mathcal{B}et_F)$ to get $\mathcal{B}et\,\mathbb{B}_N, \mathcal{B}et(\mathbb{C}\setminus\{C_1\}) \Rightarrow \mathcal{B}et\,C_1$. Applying the claim we get $\mathcal{B}et\,\mathbb{B}_N, \mathcal{B}et(\mathbb{C}\setminus\{C_1\}) \Rightarrow D$ for each $D \in \mathbb{B}_N \cup (\mathbb{C}\setminus\{C_1\})$ and then from $(\mathcal{B}et_F)$ we get $\mathcal{B}et\,\mathbb{B}_N, \mathcal{B}et(\mathbb{C}\setminus\{C_1, C_2\}) \Rightarrow \mathcal{B}et\,C_2$. We cannot apply the claim yet; we first need $\mathcal{B}et\,\mathbb{B}_N, \mathcal{B}et(\mathbb{C}\setminus\{C_1, C_2\}) \Rightarrow \mathcal{B}et\,C_1$ and this is obtained in a similar manner. Apply the claim to get $\mathcal{B}et\,\mathbb{B}_N, \mathcal{B}et(\mathbb{C}\setminus\{C_1, C_2\}) \Rightarrow D$ for each $D \in \mathbb{B}_N \cup (\mathbb{C}\setminus\{C_1, C_2\})$.

Now apply $(\mathcal{B}et_F)$ to get $\mathcal{B}et\,\mathbb{B}_N, \mathcal{B}et(\mathbb{C}\setminus\{C_1, C_2, C_3\}) \Rightarrow \mathcal{B}et\,C_3$. Similarly obtain $\mathcal{B}et\,\mathbb{B}_N, \mathcal{B}et(\mathbb{C}\setminus\{C_1, C_2, C_3\}) \Rightarrow \mathcal{B}et\,C_1$ and $\mathcal{B}et\,\mathbb{B}_N, \mathcal{B}et(\mathbb{C}\setminus\{C_1, C_2, C_3\}) \Rightarrow \mathcal{B}et\,C_2$, and then apply the claim. Proceeding in this manner we ultimately obtain the statement for $S := \mathbb{C}$ (i.e. $\mathcal{B}et\,\mathbb{B}_N \Rightarrow B_i$ for each i) so the lemma is proved.

□