Uniform Rules and Dialogue Games for Fuzzy Logics*

A. Ciabattoni, C.G. Fermüller, and G. Metcalfe

Technische Universität Wien, A-1040 Vienna, Austria
{agata,chrisf,metcalfe}@logic.at

Abstract. We provide uniform and invertible logical rules in a framework of relational hypersequents for the three fundamental t-norm based fuzzy logics i.e., Łukasiewicz logic, Gödel logic, and Product logic. Relational hypersequents generalize both hypersequents and sequents-of-relations. Such a framework can be interpreted via a particular class of dialogue games combined with bets, where the rules reflect possible moves in the game. The problem of determining the validity of atomic relational hypersequents is shown to be polynomial for each logic, allowing us to develop Co-NP calculi. We also present calculi with very simple initial relational hypersequents that vary only in the structural rules for the logics.

1 Introduction

Fuzzy logics based on t-norms and their residua are formal systems providing a foundation for reasoning under vagueness. Following e.g., [10], conjunction and implication are interpreted on the real unit interval $[0, 1]$ by a continuous t-norm and its residuum, respectively. The most important of these logics are Łukasiewicz logic **Ł**, Gödel logic $\mathbf{G}$, and Product logic $\mathbf{\Pi}$. These three are viewed as fundamental since *all* continuous t-norms can be constructed from their respective t-norms.

A variety of proof methods have been proposed for **Ł**, $\mathbf{G}$, and $\mathbf{\Pi}$. In particular, calculi for many fuzzy logics have been presented in a framework of *hypersequents*, a generalization of Gentzen sequents to multisets of sequents (see e.g., [2]). A very attractive calculus has been defined for $\mathbf{G}$ in [2] by embedding Gentzen's **LJ** for intuitionistic logic into a hypersequent calculus without modifying the rules for connectives. Elegant hypersequent calculi have also been defined for **Ł** [16] and $\mathbf{\Pi}$ [14], but using different rules for connectives. A further calculus for $\mathbf{G}$, which unlike the respective hypersequent calculus has *invertible* rules, has been introduced in a framework of *sequents-of-relations* [5]. More proof search oriented calculi include a tableaux calculus for **Ł** [9], decomposition proof systems for $\mathbf{G}$ [3], and goal-directed systems for **Ł** [15] and $\mathbf{G}$ [13]. Finally, a general approach is presented in [1] where a calculus for any logic based on a continuous t-norm is obtained via reductions to suitable finite-valued logics.

In this paper we introduce a generalization of both hypersequents and sequents-of-relations, that we call *relational hypersequents*. A relational hypersequent, or, for short, r-hypersequent, is a multiset of two different types of sequents, where Gentzen's sequent arrow is replaced in one by $<$ and in the other by $\leq$. Intuitively we may think

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of an r-hypersequent as a meta-level (classical) disjunction of negated and non-negated sequents. Within this framework, we are able to give logical rules for **Ł**, $\mathbf{G}$, and $\mathbf{\Pi}$, that are *uniform* i.e., identical for all three logics. Since these rules are also *invertible*, we thus obtain uniform proof search procedures where the validity problem for r-hypersequents in **Ł**, $\mathbf{G}$, or $\mathbf{\Pi}$ can be reduced to the validity problem in the respective logic for r-hypersequents containing only atomic formulas.[1] Moreover, we show that this latter problem is *polynomial* for each logic. Simple modifications then allow us to use these rules to present Co-NP decision procedures for **Ł**, $\mathbf{G}$, and $\mathbf{\Pi}$, matching the complexity class of the logics (see e.g., [10]). Furthermore, purely syntactic calculi with very simple initial relational hypersequents are obtained by introducing structural rules reflecting the characteristic properties of the particular logic.

We also present an interpretation of the uniform logical rules in terms of *dialogue games combined with bets*, that stems from Giles's game-theoretic characterization of **Ł** in the seventies [7, 8]. Giles defined a Lorenzen-style game for which the existence of winning strategies for a formula corresponds to the validity of that formula in **Ł**. Here we reveal a deep connection between the search for winning strategies in Giles's game and the r-hypersequent rules for **Ł**, and extend this connection to $\mathbf{G}$ and $\mathbf{\Pi}$.

2 t-Norm Based Fuzzy Logics

Continuous t-norms and their residua are defined as follows:

Definition 1. *A* continuous t-norm *is a continuous, commutative, associative, monotonically increasing function* $*: [0,1]^2 \to [0,1]$ *where* $1 * x = x$ *for all* $x \in [0,1]$*. The residuum of* $*$ *is a function* $\Rightarrow_*$: $[0,1]^2 \to [0,1]$ *where* $x \Rightarrow_* y = max\{z \mid x * z \leq y\}$.

The most important examples of continuous t-norms and their residua are:

	t-Norm	Residuum
Łukasiewicz	$x *_{\mathbf{Ł}} y = max(0, x + y - 1)$	$x \Rightarrow_{\mathbf{Ł}} y = min(1, 1 - x + y)$
Gödel	$x *_{\mathbf{G}} y = min(x, y)$	$x \Rightarrow_{\mathbf{G}} y = \begin{cases} 1 \text{ if } x \leq y \\ y \text{ otherwise} \end{cases}$
Product	$x *_{\mathbf{\Pi}} y = x \cdot y$	$x \Rightarrow_{\mathbf{\Pi}} y = \begin{cases} 1 & \text{if } x \leq y \\ y/x & \text{otherwise} \end{cases}$

Any continuous t-norm is an ordinal sum construction of these three, see e.g., [10] for details. Observe also that the functions min and max can be expressed in terms of $*$ and $\Rightarrow_*$, i.e., $min(x, y) = x * (x \Rightarrow_* y)$ and $max(x, y) = min((x \Rightarrow_* y) \Rightarrow_* y, (y \Rightarrow_* x) \Rightarrow_* x)$. Each continuous t-norm determines a *propositional logic* as follows:

Definition 2. *For a continuous* t*-norm* $*$ *with residuum* $\Rightarrow_*$*, we define a logic* $\mathbf{L}_*$ *based on a language with binary connectives* $\to$, $\odot$*, constant* $\bot$*, and defined connectives* $\neg A =_{def} A \to \bot$, $A \wedge B =_{def} A \odot (A \to B)$, $A \vee B =_{def} ((A \to B) \to B) \wedge ((B \to$

[1] These may also be viewed as providing a uniform *normal form* for **Ł**, $\mathbf{G}$, and $\mathbf{\Pi}$.

$A) \to A)$. *A* valuation *for* $\mathbf{L}_*$ *is a function* v *assigning to each propositional variable a truth value from the real unit interval* $[0, 1]$*, uniquely extended to formulas by:*

$$v(A \odot B) = v(A) * v(B) \qquad v(A \to B) = v(A) \Rightarrow_* v(B) \qquad v(\bot) = 0$$

A formula A *is valid in* $\mathbf{L}_*$*, written* $\models_{\mathbf{L}_*} A$*, iff* $v(A) = 1$ *for all valuations* v *for* $\mathbf{L}_*$.

We call the logics $\mathbf{L}_{*_{Ł}}$, $\mathbf{L}_{*_{\mathbf{G}}}$, and $\mathbf{L}_{*_{\mathbf{\Pi}}}$, Łukasiewicz logic **Ł**, Gödel logic $\mathbf{G}$, and Product logic $\mathbf{\Pi}$, respectively.

3 Uniform Rules

We give uniform and invertible logical rules for **Ł**, $\mathbf{G}$, and $\mathbf{\Pi}$ in a framework of *relational hypersequents*, which are defined as follows:

Definition 3. *A* relational hypersequent (r-hypersequent) *is a finite multiset of the form:*

$$G = \Gamma_1 \lhd_1 \Delta_1 \mid \ldots \mid \Gamma_n \lhd_n \Delta_n$$

where $\lhd_i \in \{<, \leq\}$ *and* Γ_i *and* Δ_i *are finite multisets of formulas for* $i = 1, \ldots, n$. G *is* atomic *if all formulas occurring in* G *are atomic. The* size *of* G *is the total number of symbols occurring in formulas of* G.

The use of *multisets* in this definition means that the multiplicity but not the order of elements is important. Hence all set notation will refer to multisets, denoted by the symbols Γ and Δ. Also, we take advantage of standard conventions such as allowing Γ, A and Γ, Δ to stand for $\Gamma \cup \{A\}$ and $\Gamma \cup \Delta$ respectively, $\lambda\Gamma$ for $\Gamma, \ldots, \Gamma$ (λ times), and the empty space for the empty multiset $\emptyset$. Note moreover, that the use of inequality symbols $<$ and $\leq$ in the definition is purely syntactic (although of course also suggestive of the intended meaning). Finally, we remark that a *hypersequent* (see e.g., [2]) may be viewed as an r-hypersequent with just one relation symbol, while a *sequent-of-relations* (see e.g., [5]) may be viewed as an r-hypersequent where all multisets contain exactly one formula.

Below, we define validity for r-hypersequents in each of the three logics, informally understanding $\mid$ as a meta-level "or" and $<$ and $\leq$ as denoting inequalities between combinations (different for each logic) of truth values of formulas. Note that here (and throughout this paper) the symbols $<$ and $\leq$ have *two* uses: a syntactic one as part of an r-hypersequent, and a semantic one as inequalities holding between two mathematical expressions. We rely on context to make clear which use is intended.

Definition 4. *An r-hypersequent* $G = \Gamma_1 \lhd_1 \Delta_1 \mid \ldots \mid \Gamma_n \lhd_n \Delta_n$ *is* valid *for* $L \in \{\text{Ł}, \mathbf{G}, \mathbf{\Pi}\}$*, written* $\models_L G$*, iff for all valuations* v *for* L*,*

$$\#_L^v \Gamma_i \lhd_i \#_L^v \Delta_i \ \text{ for some } i,\ 1 \leq i \leq n,$$

where $\#_L^v \emptyset = 1$ *for* $L \in \{\text{Ł}, \mathbf{G}, \mathbf{\Pi}\}$ *and*

$$\#_{\text{Ł}}^v(\Gamma) = 1 + \sum_{A \in \Gamma} \{v(A) - 1\} \qquad \#_{\mathbf{G}}^v(\Gamma) = min_{A \in \Gamma}\{v(A)\} \qquad \#_{\mathbf{\Pi}}^v(\Gamma) = \prod_{A \in \Gamma} \{v(A)\}$$

Observe that for all formulas A, $\models_L \leq A$ iff $\models_L A$ for $L \in \{\mathbf{Ł}, \mathbf{G}, \mathbf{\Pi}\}$. Below we present uniform logical rules in this framework, using G and H as metavariables to denote (possibly empty) r-hypersequents called *side r-hypersequents*.

Definition 5. *We define the following* uniform logical rules *for* $\triangleleft \in \{<, \leq\}$*:*

$$(\to, \triangleleft, l) \ \frac{G \mid \Gamma \triangleleft \Delta \mid \Gamma, B \triangleleft A, \Delta \quad G \mid \Gamma \triangleleft \Delta \mid B < A}{G \mid \Gamma, A \to B \triangleleft \Delta}$$

$$(\to, \triangleleft, r) \quad \frac{G \mid \Gamma \triangleleft \Delta \quad G \mid \Gamma, A \triangleleft B, \Delta \mid A \leq B}{G \mid \Gamma \triangleleft A \to B, \Delta}$$

$$(\odot, \triangleleft, l) \ \frac{G \mid \Gamma, A, B \triangleleft \Delta \quad G \mid \Gamma, \bot \triangleleft \Delta}{G \mid \Gamma, A \odot B \triangleleft \Delta} \qquad (\odot, \triangleleft, r) \ \frac{G \mid \Gamma \triangleleft \bot, \Delta \mid \Gamma \triangleleft A, B, \Delta}{G \mid \Gamma \triangleleft A \odot B, \Delta}$$

Note that uniform rules for $\wedge$ and $\vee$ are derivable using Definition 2. However we can also give more streamlined versions, i.e., for $\triangleleft \in \{<, \leq\}$:

$$(\wedge, \triangleleft, l) \quad \frac{G \mid \Gamma, A \triangleleft \Delta \mid \Gamma, B \triangleleft \Delta}{G \mid \Gamma, A \wedge B \triangleleft \Delta} \qquad (\wedge, \triangleleft, r) \ \frac{G \mid \Gamma \triangleleft A, \Delta \quad G \mid \Gamma \triangleleft B, \Delta}{G \mid \Gamma \triangleleft A \wedge B, \Delta}$$

$$(\vee, \triangleleft, l) \ \frac{G \mid \Gamma, A \triangleleft \Delta \quad G \mid \Gamma, B \triangleleft \Delta}{G \mid \Gamma, A \vee B \triangleleft \Delta} \qquad (\vee, \triangleleft, r) \quad \frac{G \mid \Gamma \triangleleft A, \Delta \mid \Gamma \triangleleft B, \Delta}{G \mid \Gamma \triangleleft A \vee B, \Delta}$$

Observe that the rules for $\to$, $\wedge$ and $\vee$ have the *subformula property*, i.e., all formulas occurring in the premises of a rule occur as subformulas of formulas in the conclusion. The rules for $\odot$ do not have this property, since $\bot$ appears in the premises and possibly not the conclusion. Nevertheless, the right premise in $(\odot, \triangleleft, l)$, and $\Gamma \triangleleft \bot, \Delta$ in the premise of $(\odot, \triangleleft, r)$ may be removed with no loss of soundness for $\mathbf{G}$ and $\mathbf{\Pi}$. Moreover, since $\mathbf{Ł}$ can be based on a language without $\odot$, non-uniform rules with the subformula property can be given for all three logics.

Definition 6. *A rule* $\frac{G_1 \ \ldots \ G_n}{G}$ *is* sound *for a logic* L *if whenever* $\models_L G_i$ *for* $i = 1, \ldots, n$*, then* $\models_L G$*, and* invertible *if whenever* $\models_L G$*, then* $\models_L G_i$ *for* $i = 1, \ldots, n$*.*

Lemma 1. *If* $\frac{G_1 \ \ldots \ G_n}{G}$ *is sound (invertible) for* L*, then so is* $\frac{H|G_1 \ \ldots \ H|G_n}{H|G}$*.*

Proof. Follows directly from Definition 4. $\square$

Theorem 1. *The uniform logical rules are sound and invertible for* $\mathbf{Ł}$*,* $\mathbf{G}$*, and* $\mathbf{\Pi}$*.*

Proof. We consider only the rules for $\to$ (the cases for $\odot$ being similar), using Lemma 1 to disregard side r-hypersequents. Let v be a valuation for $\mathbf{Ł}$, $\mathbf{G}$, or $\mathbf{\Pi}$. If $v(A) \leq v(B)$, then $v(A \to B) = 1$, and clearly for both $(\to, \triangleleft, l)$ and $(\to, \triangleleft, r)$, the premises hold iff the conclusion holds. Now suppose that $v(A) > v(B)$. We consider each rule in turn:

- $(\to, \triangleleft, l)$. The right premise clearly holds. For $\mathbf{Ł}$ and $\mathbf{\Pi}$, by simple arithmetic, the conclusion holds iff the left premise holds. For $\mathbf{G}$, $v(A \to B) = v(B)$ and $min(\#^v_{\mathbf{G}}(\Gamma), v(B)) \triangleleft min(v(A), \#^v_{\mathbf{G}}(\Delta))$ iff $min(\#^v_{\mathbf{G}}(\Gamma), v(B)) \triangleleft v(A)$ and $min(\#^v_{\mathbf{G}}(\Gamma), v(B)) \triangleleft \#^v_{\mathbf{G}}(\Delta)$. However, $min(\#^v_{\mathbf{G}}(\Gamma), v(B)) \triangleleft v(A)$ since $v(A) > v(B)$, so we have that the left premise holds iff the conclusion holds.

– $(\to, \triangleleft, r)$. If the conclusion holds, then the left premise, and (by simple arithmetic) the right premise hold. For **Ł** and **Π**, by simple arithmetic, the conclusion holds iff the right premise holds. For $\mathbf{G}$, if $min(\#^v_{\mathbf{G}}(\Gamma), v(A)) \triangleleft min(v(B), \#^v_{\mathbf{G}}(\Delta))$ then $min(\#^v_{\mathbf{G}}(\Gamma), v(A)) \triangleleft v(B)$ holds, and, since $v(A) > v(B)$, $min(\#^v_{\mathbf{G}}(\Gamma), v(A)) = \#^v_{\mathbf{G}}(\Gamma)$. Hence the right premise holds iff the conclusion holds. □

Example 1. The uniform logical rules may be applied upwards exhaustively to reduce r-hypersequents to atomic r-hypersequents, e.g.,

$$\dfrac{\dfrac{p \leq q \mid p, q \leq p, q \qquad p \leq q \mid q < p}{p, p \to q \leq q}\,{\scriptstyle(\to,\leq,l)} \qquad \bot \leq q}{p \odot (p \to q) \leq q}\,{\scriptstyle(\odot,\leq,l)}$$

Proposition 1. *Applying the uniform logical rules upwards to r-hypersequents terminates with atomic r-hypersequents.*

Proof. We define the following measures and well-orderings:

$c(q) = 1$ for q atomic, $c(A \odot B) = c(A \to B) = c(A) + c(B) + 1$ for formulas A, B.
$mc(\Gamma \triangleleft \Delta) = \{c(A) \mid A \in \Gamma \cup \Delta\}$ for multisets Γ, Δ, and $\triangleleft \in \{<, \leq\}$.
$mmc(G) = \{mc(\Gamma \triangleleft \Delta) \mid \Gamma \triangleleft \Delta \in G\}$ for an r-hypersequent G.

For multisets α, β of integers: $\alpha <_m \beta$ iff (1) $\alpha \subset \beta$, or (2) $\alpha <_m \gamma$ where $\gamma = (\beta - \{j\}) \cup \{i, \ldots, i\}$, and $i < j$.

For multisets ϕ, ψ of multisets of integers, $\phi <_{mm} \psi$ iff (1) $\phi \subset \psi$, or (2) $\phi <_{mm} \chi$ where $\chi = (\psi - \{\alpha\}) \cup \{\beta, \ldots, \beta\}$ and $\beta <_m \alpha$.

For each uniform logical rule $\frac{G_1 \ \ldots \ G_n}{G}$ it is easy to check that $mmc(G_i) <_{mm} mmc(G)$ for $i = 1, \ldots, n$. Hence, since there is always a rule for any non-atomic formula, the rules applied upwards terminate with atomic r-hypersequents. □

4 Evaluating Atomic Relational Hypersequents

Let us take stock of what we have achieved so far. By providing uniform rules for **Ł**, $\mathbf{G}$, and **Π**, that are sound and invertible, we are able to reduce the validity problem (i.e., checking the validity of a formula) in these logics to checking the validity of atomic r-hypersequents. We might also view the atomic r-hypersequents thus obtained as a sort of "uniform normal form" for these logics. This is a pleasant enough achievement in itself but it is only really useful *computationally* if we can show that checking the validity of atomic r-hypersequents is less complex than deciding the validity problem for each logic. In fact, while it is well-known that the validity problem for all these logics is Co-NP complete (see e.g., [10] for proofs and references), we show here that checking validity for atomic r-hypersequents is in each case *polynomial*.

We begin with a useful translation of atomic r-hypersequents into a set of inequations, where an atomic r-hypersequent is valid in a logic iff the associated set is inconsistent over $[0, 1]$.

Definition 7. *For atomic $G = \Gamma_1 \triangleleft_1 \Delta_1 \mid \ldots \mid \Gamma_n \triangleleft_n \Delta_n$ and $L \in \{\mathbf{Ł}, \mathbf{G}, \mathbf{\Pi}\}$:*

$$S_G = \{\circ_L \Gamma_1 \not\triangleleft_1 \circ_L \Delta_1, \ldots, \circ_L \Gamma_n \not\triangleleft_n \circ_L \Delta_n\}$$

where $\not\leq$ is $>$ and $\not<$ is $\geq$, $\circ_L \emptyset = 1$, and

$$\circ_{\mathbf{Ł}}(\Gamma) = 1 + \sum_{q \in \Gamma} \{x_q - 1\} \quad \circ_{\mathbf{G}}(\Gamma) = min_{q \in \Gamma}\{x_q\} \quad \circ_{\mathbf{\Pi}}(\Gamma) = \prod_{q \in \Gamma} \{x_q\}$$

where x_q is a real-valued variable for all propositional variables q, and $x_\perp = 0$.

Lemma 2. *For atomic G and $L \in \{\mathbf{Ł}, \mathbf{G}, \mathbf{\Pi}\}$, $\models_L G$ iff S_G is inconsistent over $[0, 1]$.*

Proof. Immediate from Definition 4. □

For **Ł** we obtain the desired result using linear programming methods.

Theorem 2. *Checking $\models_{\mathbf{Ł}} G$ for an atomic r-hypersequent G is polynomial.*

Proof. By Lemma 2, since linear programming is polynomial, see e.g., [17]. □

To show that checking the validity of atomic r-hypersequents for $\mathbf{G}$ is polynomial, we use a result of Jeavons et al. [11] concerning relations over a finite domain.

Definition 8. *Let R be an n-ary relation over a domain D and $\otimes : D^2 \to D$ be an* ACI operation*, i.e., a binary idempotent, associative, and commutative operation. We say that R is* closed under $\otimes$ *if $(t_1, \ldots t_n), (t'_1, \ldots t'_n) \in R$ implies $(t_1 \otimes t'_1, \ldots, t_n \otimes t'_n) \in R$. A set of relations S is closed under $\otimes$ iff R is closed under $\otimes$ for all $R \in S$.*

Theorem 3 ([11]). *If a set of relations Γ over a finite domain D is closed under some ACI operation, then its constraint satisfaction problem is solvable in polynomial time.*

Theorem 4. *Checking $\models_{\mathbf{G}} G$ for an atomic r-hypersequent G is polynomial.*

Proof. Let $x_1, \ldots, x_n$ be the distinct variables occurring in S_G. It can be shown that S_G is inconsistent over $[0, 1]$ iff S_G is inconsistent over the set $D_n = \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$. Associate with each $\circ_{\mathbf{G}} \Gamma \not\triangleleft \circ_{\mathbf{G}} \Delta \in S_G$ a relation $R(x_1, \ldots, x_n)$ such that $R(a_1, \ldots, a_n)$ for $a_i \in D_n$, $i = 1, \ldots, n$, holds iff $\circ_{\mathbf{G}} \Gamma \not\triangleleft \circ_{\mathbf{G}} \Delta$ holds when x_i is replaced by a_i. Moreover, if $R(a_1, \ldots, a_n)$ and $R(b_1, \ldots, b_n)$ hold for $a_i, b_i \in D_n$, $i = 1, \ldots, n$, then also $R(min(a_1, b_1), \ldots, min(a_n, b_n))$ holds. Hence the set of relations associated with S_G is closed under the ACI operation $min : D_n^2 \to D_n$, and, by Theorem 3, its constraint satisfaction problem is solvable in polynomial time. However, this problem is equivalent to checking the inconsistency of S_G which, by Lemma 2, is equivalent to checking the validity of G. □

For $\mathbf{\Pi}$ we again use linear programming methods, dealing separately with the cases where propositional variables are assigned the value 0.

Definition 9. *Let G be an atomic r-hypersequent. An atomic formula q is:*

– 0-zero-ok *for G if $\Gamma, q \leq \Delta \in G$.*

– n-zero-ok *for G if $\Gamma, q < \Delta \in G$, and for all $p \in \Delta$, p is m-zero-ok for G for some $m \in \mathbb{N}$, $m < n$, where $n = 1 + \sum_{p \in \Delta} min\{k \mid p$ is k-zero-ok for $G\}$.*
– zero-ok *for G if q is n-zero-ok for G for some $n \in \mathbb{N}$.*

Lemma 3. *Let $H = G \mid \Gamma < \Delta$ be an atomic r-hypersequent, and $p \in \Gamma \cup \Delta$ where p is not zero-ok for H. If $\models_{\mathbf{\Pi}} H$, then $\models_{\mathbf{\Pi}} G$.*

Proof. Note first that if $p \in \Gamma$ is not zero-ok, then there must be $q \in \Delta$ such that q is not zero-ok for H. Hence we can assume that $p \in \Delta$. Suppose $\not\models_{\mathbf{\Pi}} G$, i.e., there is a valuation v for $\mathbf{\Pi}$ such that for all $\Gamma' \lhd \Delta' \in G$, $\#^v_{\mathbf{\Pi}}\Gamma' \not\lhd \#^v_{\mathbf{\Pi}}\Delta'$. We define a valuation v' such that $v'(q) = 0$ if q is not zero-ok, $v'(q) = v(q)$ otherwise. Clearly, $\#^{v'}_{\mathbf{\Pi}}\Gamma \not< \#^{v'}_{\mathbf{\Pi}}\Delta = 0$. Consider $\Gamma' \lhd \Delta' \in G$. If all $q \in \Gamma'$ are zero-ok, then $\#^{v'}_{\mathbf{\Pi}}\Gamma' = \#^v_{\mathbf{\Pi}}\Gamma'$. If $q \in \Gamma'$ is not zero-ok, then $\lhd$ is $<$, and for some not zero-ok $q' \in \Delta'$, $v'(q') = 0$. In both cases $\#^{v'}_{\mathbf{\Pi}}\Gamma' \not\lhd \#^v_{\mathbf{\Pi}}\Delta' \geq \#^v_{\mathbf{\Pi}}\Delta'$. Hence $\not\models_{\mathbf{\Pi}} H$ as required. □

Lemma 4. *Let G be an atomic r-hypersequent where p is zero-ok for G. For all valuations v for $\mathbf{\Pi}$, if $v(p) = 0$, then $\#^v_{\mathbf{\Pi}}(\Gamma) \lhd \#^v_{\mathbf{\Pi}}(\Delta)$ for some $\Gamma \lhd \Delta \in G$.*

Proof. A simple induction on n where p is n-zero-ok. □

Theorem 5. *Checking $\models_{\mathbf{\Pi}} G$ for an atomic r-hypersequent G is polynomial.*

Proof. It is straightforward to show that finding the zero-ok atomic formulas of G is polynomial in the size of G. Moreover, by repeated applications of Lemma 3, $\models_{\mathbf{\Pi}} G$ iff $\models_{\mathbf{\Pi}} G'$ for some $G' \subseteq G$ containing only zero-ok atomic formulas. If $\bot$ occurs in G' (which can be checked in polynomial time) then by Lemma 4, G' is valid. If $\bot$ does not occur in G', by Lemma 2, $\models_{\mathbf{\Pi}} G'$ iff $S_{G'}$ is inconsistent over $[0,1]$ iff, by Lemma 4, $S_{G'}$ is inconsistent over $(0,1]$. However this latter problem is isomorphic to a linear programming problem over the positive reals, known to be polynomial. □

5 Co-NP Calculi

Despite having invertible rules and polynomially decidable atomic r-hypersequents, we do not yet have Co-NP calculi for **Ł**, **Π**, and **G**, since the rules applied upwards may increase the size of r-hypersequents exponentially. This problem is overcome by giving rules that make use of new propositional variables.

Definition 10. *We define the following* revised logical rules *for $\lhd \in \{<, \leq\}$, where p and q are propositional variables not occurring in the conclusions of the rules:*

$$(\to, \lhd, l)' \quad \frac{G \mid \Gamma, q \lhd \Delta \mid B < q, A}{G \mid \Gamma, A \to B \lhd \Delta}$$

$$(\to, \lhd, r)' \quad \frac{G \mid \Gamma \lhd \Delta \quad G \mid \Gamma, p \lhd q, \Delta \mid p \leq q \mid A < p \mid q < B}{G \mid \Gamma \lhd A \to B, \Delta}$$

$$(\odot, \lhd, l) \quad \frac{G \mid \Gamma, A, B \lhd \Delta \quad G \mid \Gamma, \bot \lhd \Delta}{G \mid \Gamma, A \odot B \lhd \Delta} \qquad (\odot, \lhd, r)' \quad \frac{G \mid \Gamma \lhd q, \Delta \mid q < A, B \mid q < \bot}{G \mid \Gamma \lhd A \odot B, \Delta}$$

Theorem 6. *The revised logical rules are sound and invertible for* **Ł**, **G**, *and* **Π**.

Proof. We consider just the rules for $\to$ (the cases for $\odot$ being similar), using Lemma 1 to disregard side r-hypersequents. Let $L \in \{\mathbf{Ł}, \mathbf{G}, \mathbf{\Pi}\}$.

- $(\to, \triangleleft, l)'$. For soundness, given a valuation v, we can assume (since q does not occur in the conclusion) that $v(q) = v(A \to B)$. From $v(B) \geq \#^v_L(q, A)$ we get $\#^v_L(\Gamma, A \to B) \triangleleft \#^v_L(\Delta)$ as required. For invertibility, given a valuation v, if $v(B) < \#^v_L(q, A)$ then we are done, otherwise we must have $v(q) \leq v(A \to B)$ and hence, $\#^v_L(\Gamma, q) \triangleleft \#^v_L(\Delta)$ as required.
- $(\to, \triangleleft, r)'$. For soundness, consider a valuation v. If $v(A) \leq v(B)$, then $v(A \to B) = 1$ and we are done by the first premise. If $v(A) > v(B)$, then we can assume (since p and q do not occur in the conclusion) that $v(p) = v(A)$ and $v(q) = v(B)$. Hence, $\#^v_L(\Gamma, p) \triangleleft \#^v_L(q, \Delta)$ and, similarly to the case of $(\to, \triangleleft, r)$ in Theorem 1, $\#^v_L(\Gamma) \triangleleft \#^v_L(A \to B, \Delta)$ as required. For invertibility, the left premise is obvious, for the right premise consider a valuation v. If $v(A) < v(p)$, $v(q) < v(B)$, or $v(p) \leq v(q)$, then we are done. Otherwise, $\#^v_L(\Gamma, A) \triangleleft \#^v_L(B, \Delta)$ and, similarly to the case of $(\to, \triangleleft, r)$ in Theorem 1, $\#^v_L(\Gamma, p) \triangleleft \#^v_L(q, \Delta)$ as required. □

Proposition 2. *Applying the revised logical rules upwards to an r-hypersequent G terminates with atomic r-hypersequents of size polynomial in the size of G.*

Proof. Similar to the proof of Proposition 1, except that also each upward application of a rule gives only a constant increase in the size of the r-hypersequent. □

Theorem 7. *The revised logical rules provide Co-NP decision procedures for the validity problems for* **Ł**, **G**, *and* **Π**.

Proof. To show that a formula is not valid we apply the revised logical rules upwards exhaustively, making a non-deterministic choice of two branches where necessary. The result follows from Proposition 2, and Theorems 2, 4, and 5. □

6 Structural Rules

The aim of this section is to use the uniform logical rules to give purely syntactic calculi for **Ł**, **G**, and **Π** with very simple axioms and structural rules.

Definition 11. *We define the following* uniform axioms and structural rules*:*

$$(ID)\ A \leq A \qquad (\bot)\ \bot \leq A \qquad (\Lambda)\ \leq \qquad (<)\ \bot <$$

$$(EW)\ \frac{G}{G \mid \Gamma \triangleleft \Delta} \qquad (EC)\ \frac{G \mid \Gamma \triangleleft \Delta \mid \Gamma \triangleleft \Delta}{G \mid \Gamma \triangleleft \Delta} \qquad (WL)\ \frac{G \mid \Gamma \triangleleft \Delta}{G \mid \Gamma, A \triangleleft \Delta}$$

$$(S_{\leq})\ \frac{G \mid \Gamma_1, \Gamma_2 \leq \Delta_1, \Delta_2}{G \mid \Gamma_1 \leq \Delta_1 \mid \Gamma_2 \leq \Delta_2} \qquad (M)\ \frac{G \mid \Gamma_1 \triangleleft \Delta_1 \quad G \mid \Gamma_2 \triangleleft \Delta_2}{G \mid \Gamma_1, \Gamma_2 \triangleleft \Delta_1, \Delta_2}$$

Lemma 5. *The uniform axioms and rules are sound for* **Ł**, **G**, *and* **Π**.

Proof. Straightforward using Definition 4. $\square$

We now define calculi for **Ł**, **G**, and **Π** by extending the core uniform axioms and rules with further structural rules reflecting the characteristic properties of each logic.

Definition 12. **rHŁ** *consists of the uniform axioms and rules together with:*

$$(S_{\mathbf{Ł}})\ \frac{G \mid \Gamma_1, \Gamma_2 \leq \Delta_1, \Delta_2}{G \mid \Gamma_1 \leq \Delta_1 \mid \Gamma_2 < \Delta_2} \qquad (W\bot)\ \frac{G \mid \Gamma \leq \Delta}{G \mid \Gamma, \bot < \Delta}$$

Theorem 8. *An r-hypersequent G is derivable in* **rHŁ** *iff* $\models_{\mathbf{Ł}} G$.

Proof. For soundness it is enough and easy to show that $(S_{\mathbf{Ł}})$ and $(W\bot)$ are sound. For completeness we apply the invertible logical rules to G upwards to obtain valid atomic r-hypersequents. For each atomic r-hypersequent $H = \Gamma_1 \lhd_1 \Delta_1 \mid \ldots \mid \Gamma_n \lhd_n \Delta_n$, $\models_{\mathbf{Ł}} H$ iff S_H is inconsistent over $[0,1]$. By linear programming methods [17], this holds iff there exist $\lambda, \lambda_1, \ldots, \lambda_n \in \mathbb{N}$ where either $\lambda > 0$, or $\lambda_i > 0$ and $\lhd_i$ is $\leq$ for some i, $1 \leq i \leq n$, and:

$$\lambda\bot \cup \bigcup_{i=1}^{n} \lambda_i \Delta_i \subseteq^* \bigcup_{i=1}^{n} \lambda_i \Gamma_i$$

where (1) $\Delta \subseteq^* \Gamma$ if $\Delta \subseteq \Gamma$, and (2) $\Delta \cup \{A\} \subseteq^* \Gamma \cup \{\bot\}$ if $\Delta \subseteq^* \Gamma$. If $\lambda > 0$, then we choose any i such that $\bot \in \Gamma_i$ and apply $(W\bot)$ upwards to get an r-hypersequent H' where $S_{H'}$ meets the conditions of the second case. If $\lambda_i > 0$ and $\lhd_i$ is $\leq$ for some i, $1 \leq i \leq n$, then we apply (EW) and (EC) upwards to get λ_i copies of $\Gamma_i \lhd_i \Delta_i$. Applying $(S_{\mathbf{Ł}})$ and $(S_{\leq})$ upwards we have that H is derivable if $H' = \lambda_1\Gamma_1, \ldots, \lambda_n\Gamma_n \leq \lambda_1\Delta_1, \ldots, \lambda_n\Delta_n$ is derivable. However, H' is derivable by repeated applications of (M), (WL), (ID), $(\bot)$, and (Λ). $\square$

Definition 13. **rHG** *consists of the uniform rules and axioms together with:*

$$(S_{\mathbf{G}}, \lhd)\ \frac{G \mid \Gamma_1, \Gamma_2 \lhd \Delta_1 \quad G \mid \Gamma_1 \leq \Delta_2}{G \mid \Gamma_1 \lhd \Delta_1 \mid \Gamma_2 < \Delta_2} \qquad (CL)\ \frac{G \mid \Gamma, A, A \lhd \Delta}{G \mid \Gamma, A \lhd \Delta}$$

Lemma 6. *The following rules are invertible for* **G***, and derivable in* **rHG***:*

$$(M, \lhd, l)\ \frac{G \mid \Gamma_1 \lhd \Delta \mid \Gamma_2 \lhd \Delta}{G \mid \Gamma_1, \Gamma_2 \lhd \Delta} \qquad (M, \lhd, r)\ \frac{G \mid \Gamma \lhd \Delta_1 \quad G \mid \Gamma \lhd \Delta_2}{G \mid \Gamma \lhd \Delta_1, \Delta_2}$$

Proof. It is straightforward to show that $(M, \lhd, l)$ and $(M, \lhd, r)$ are invertible for **G**. They are derivable in **rHG** as follows, where we write $(WL)^*$ and $(CL)^*$ for multiple applications of (WL) and (CL) respectively:

$$\dfrac{\dfrac{G \mid \Gamma_1 \lhd \Delta \mid \Gamma_2 \lhd \Delta}{G \mid \Gamma_1, \Gamma_2 \lhd \Delta \mid \Gamma_1, \Gamma_2 \lhd \Delta}\,{\scriptstyle (WL)^*}}{G \mid \Gamma_1, \Gamma_2 \lhd \Delta}\,{\scriptstyle (EC)} \qquad \dfrac{\dfrac{G \mid \Gamma \lhd \Delta_1 \quad G \mid \Gamma \lhd \Delta_2}{G \mid \Gamma, \Gamma \lhd \Delta_1, \Delta_2}\,{\scriptstyle (M)}}{G \mid \Gamma \lhd \Delta_1, \Delta_2}\,{\scriptstyle (CL)^*}$$

$\square$

Theorem 9. *An r-hypersequent G is derivable in* **rHG** *iff* $\models_{\mathbf{G}} G$.

Proof. For soundness, it suffices and is easy to show that (CL) and $(S_{\mathbf{G}}, \lhd)$ are sound for $\mathbf{G}$. For completeness, we first apply the invertible logical rules to G upwards to obtain valid atomic r-hypersequents. By Lemma 6, applying $(M, \lhd, l)$ and $(M, \lhd, r)$ upwards, atomic r-hypersequents are derivable if valid r-hypersequents in which all multisets contain at most one atomic formula are derivable. Such an r-hypersequent H is valid iff the sequent-of-relations obtained by replacing the empty set by $\top$ is valid, and hence, using a result of [4] for sequents-of-relations, we get that H must have one of the following forms, where $\lhd_i \in \{<, \leq\}$ for $i = 1, \ldots, n$, and we allow $C, C_1, \ldots, C_n$ to stand for multisets containing at most one formula.

1. *(cycles)* $G' \mid C \leq C$ or $G' \mid C_1 \lhd_1 C_2 \mid \ldots \mid C_{n-1} \lhd_{n-1} C_n \mid C_n \leq C_1$.
2. (1-*chains*) $G' \mid C \leq$ or $G' \mid C_1 \leq C_2 \mid C_2 < C_3 \mid \ldots \mid C_{n-1} < C_n \mid C_n <$
3. (0-*chains*) $G' \mid \bot \leq C$ or $G' \mid \bot < C_1 \mid C_1 < C_2 \mid \ldots \mid C_{n-1} < C_n \mid C_n \leq C$.
4. (0-1-*chains*) $G' \mid \bot <$ or $G' \mid \bot < C_1 \mid C_1 < C_2 \mid \ldots \mid C_n <$.

It is straightforward to show that the above r-hypersequents are derivable in $\mathbf{rHG}$. $\square$

Definition 14. $\mathbf{rH\Pi}$ *consists of the uniform rules together with:*

$$(S_{\mathbf{\Pi}})\ \frac{G \mid \Gamma_1, \Gamma_2 \leq \Delta_1, \Delta_2 \qquad G \mid \Gamma_3 \leq \Delta_2}{G \mid \Gamma_1 \leq \Delta_1 \mid \Gamma_2 < \Delta_2 \mid \Gamma_3 \leq \Delta_3} \qquad (RCL)\ \frac{G \mid \Gamma, \bot, \bot \lhd \Delta}{G \mid \Gamma, \bot \lhd \Delta}$$

Lemma 7. *If* $G \mid \Gamma_1, \Gamma_2 \leq \Delta_1, \Delta_2$ *is atomic and derivable in* $\mathbf{rH\Pi}$ *and* p *is zero-ok for all* $p \in \Delta_2$*, then* $G \mid \Gamma_1 \leq \Delta_1 \mid \Gamma_2 < \Delta_2$ *is derivable in* $\mathbf{rH\Pi}$.

Proof. We proceed by induction on $n = 1 + \sum_{p \in \Delta_2} min\{m \mid p$ is m-zero-ok for $G\}$ For each $p \in \Delta_2$, we have two cases. If p is 0-zero-ok, then $\Gamma', p \leq \Delta' \in G$. If p is m-zero-ok for some $m > 0$, then $\Gamma', p < \Delta' \in G$ where all $q \in \Delta'$ are zero-ok. Repeatedly applying $(S_{\leq})$ upwards in the former case, and the induction hypothesis in the latter, plus repeated applications of (EC) and (EW) upwards, we get that $G \mid \Gamma_1 \leq \Delta_1 \mid \Gamma_2 < \Delta_2$ is derivable if $H = G \mid \Gamma_1 \leq \Delta_1 \mid \Gamma_2 < \Delta_2 \mid \Gamma_3 \leq \Delta_3$ is derivable where $\Delta_2 \subseteq \Gamma_3$. Now applying $(S_{\mathbf{\Pi}})$ upwards, since $G \mid \Gamma_3 \leq \Delta_2$ is derivable, we get that H is derivable if $G \mid \Gamma_1, \Gamma_2 \leq \Delta_1, \Delta_2$ is derivable. $\square$

Theorem 10. *An r-hypersequent* G *is derivable in* $\mathbf{rH\Pi}$ *iff* $\models_{\mathbf{\Pi}} G$.

Proof. It is easy to show that (RCL) is sound for $\mathbf{\Pi}$. For $(S_{\mathbf{\Pi}})$, if v is a valuation for $\mathbf{\Pi}$ in which the conclusion does not hold, and $\#^v_{\mathbf{\Pi}}(\Gamma_1) \cdot \#^v_{\mathbf{\Pi}}(\Gamma_2) \leq \#^v_{\mathbf{\Pi}}(\Delta_1) \cdot \#^v_{\mathbf{\Pi}}(\Delta_2)$, then, since $\#^v_{\mathbf{\Pi}}(\Gamma_1) > \#^v_{\mathbf{\Pi}}(\Delta_1)$ and $\#^v_{\mathbf{\Pi}}(\Gamma_2) \geq \#^v_{\mathbf{\Pi}}(\Delta_2)$, we must have $\#^v_{\mathbf{\Pi}}(\Delta_2) = 0$. Hence, since $\#^v_{\mathbf{\Pi}}(\Gamma_3) > \#^v_{\mathbf{\Pi}}(\Delta_3) \geq 0$, the right premise cannot hold. For completeness, we apply the invertible logical rules to G upwards to obtain valid atomic r-hypersequents. By Lemma 3, for each valid atomic r-hypersequent H, $\models_{\mathbf{\Pi}} H$ implies $\models_{\mathbf{\Pi}} H'$ for some $H' \subseteq H$ such that H' contains only zero-ok atomic formulae. If H' contains $\bot$ then it is easy to prove that H' is derivable as required. Otherwise $S_{H'}$ is inconsistent over $(0, 1]$ and by linear programming methods there exist $\lambda_1, \ldots, \lambda_n \in \mathbb{N}$ with $\lambda_i > 0$, where $\lhd_i$ is $\leq$ for some i, $1 \leq i \leq n$, and

$$\bigcup_{i=1}^{n} \lambda_i \Delta_i \subseteq \bigcup_{i=1}^{n} \lambda_i \Gamma_i$$

By (EC) applied upwards to obtain λ_i copies of $\Gamma_i \triangleleft_i \Delta_i$, then multiple applications of Lemma 7 and $(S_{\leq})$, and (EW) applied upwards, H is derivable if $\lambda_1 \Gamma_1, \ldots, \lambda_n \Gamma_n \leq \lambda_1 \Delta_1, \ldots, \lambda_n \Delta_n$ is derivable. But this r-hypersequent is derivable using (M), (WL), (ID) and (Λ). $\square$

It is important to note that for each logic there may be considerable redundancy in the rules presented. For example, for **Ł** we can drop the right premise of $(\rightarrow, l)$ and maintain soundness; we are then able to drop all rules and axioms referring to $<$. What we obtain is essentially the hypersequent calculus presented in [16]. For **Π** our pruning leads to a calculus that, unlike the sequent or hypersequent calculi of [14], has the subformula property, albeit with more complicated structures. For **G** simplifications lead to a calculus very similar to the sequent-of-relations calculus presented in [5].

7 Game Interpretation

In the 1970s [7, 8] Robin Giles presented a characterization of **Ł** in terms of a dialogue game combined with bets. In this section we review (very briefly) Giles's game and generalize it with the aim of revealing a deep connection between our uniform r-hypersequent rules and the search for winning strategies in versions of the game for **Ł**, **Π**, and **G**.[2] Giles's game consists of two largely independent building blocks:

1. Betting for positive results of experiments. There are two players — say, me and you — who agree to pay 1$ to the opponent player for every false statement that they assert.[3] By $[p_1, \ldots, p_m \| q_1, \ldots, q_n]$ we denote an *elementary state* in the game, where I assert each of the q_i in the multiset $\{q_1, \ldots, q_n\}$ of statements (atomic formulas), and you assert each $p_i \in \{p_1, \ldots, p_m\}$.

Each statement q refers to an experiment E_q with a binary (yes/no) result: q can be read as 'E_q yields a positive result'. The same experiment may yield different results when repeated. However, for every run of the game, a certain *risk value* $\langle q \rangle^* \in [0, 1]$ is associated with q, denoting the probability that E_q yields a negative result. For the special atomic formula $\perp$ (*falsum*) we define $\langle \perp \rangle^* = 1$. The risk associated with a multiset $\{p_1, \ldots, p_m\}$ of atomic formulas is defined as $\langle p_1, \ldots, p_m \rangle^* = \sum_{i=1}^{m} \langle p_i \rangle^*$. The risk $\langle \rangle^*$ associated with $\emptyset$ is defined as 0. The risk associated with an elementary state $[p_1, \ldots, p_m \| q_1, \ldots, q_n]$ is calculated from my point of view. Therefore the condition $\langle p_1, \ldots, p_m \rangle^* \geq \langle q_1, \ldots, q_n \rangle^*$ expresses that I do not expect any loss (but possibly some gain) when betting as explained above.

2. A Lorenzen-style dialogue game for compound formulas. Giles follows Paul Lorenzen (see e.g., [12]) in implicitly defining the meaning of logical connectives by reference to rules of a dialogue game that proceeds by systematically reducing arguments about compound formulas to arguments about their subformulas.

[2] We also generalize the results of [6] that relate a dialogue game for **G** to the sequents-of-relations calculus of [5].

[3] For a detailed motivation and explanation of the game we refer to [8].

To assist a concise presentation, we will only consider implication ($\to$), noting that in **Ł** all other connectives can be defined from $\to$ and $\bot$. The central dialogue rule can be stated as follows:

(R) If I assert $A \to B$, then whenever you choose to attack this assertion by asserting A, I have to assert also B. (And *vice versa*, i.e., for the roles of me and you switched.)

No special regulations on the succession of moves in the dialogue game are required. However, each assertion is attacked at most once: this is reflected by the removal of $A \to B$ from the multiset of all formulas asserted by a player during a run of the game, as soon as the other player has either attacked by asserting A, or indicated that she will not attack $A \to B$ at all. Observe that these stipulations ensure that every run of the dialogue game ends in an elementary state $[p_1, \ldots, p_m \| q_1, \ldots, q_n]$. Given an assignment $\langle \cdot \rangle^*$ of risk values to the p_is and q_is we say that I *win* the game if I do not expect any loss, i.e., if $\langle p_1, \ldots, p_m \rangle^* \geq \langle q_1, \ldots, q_n \rangle^*$.

As an almost trivial example consider the game with intial state $[\| p \to q]$; i.e., I initially assert $p \to q$, for some atomic formulas p and q. In response, you can either assert p in order to force me to assert q, or explicitly refuse to attack $p \to q$. In the first case the game ends in the elementary state $[p \| q]$; in the second case it ends in $[\|]$. If an assignment $\langle \cdot \rangle^*$ of risk values gives $\langle p \rangle^* \geq \langle q \rangle^*$, then I win the game, whatever move you choose to make. In other words: I have a winning strategy associated with $p \to q$ for assignments of risk values such that $\langle p \rangle^* \geq \langle q \rangle^*$.

Theorem 11 (R. Giles [7, 8]). *A formula A is valid in* **Ł** *iff for all assignments of risk values to atomic formulas occurring in A, I have a winning strategy.*

Giles proved the theorem without formalizing the concept of strategies. However, to reveal the connection to analytic proof systems we need to define structures that register possible choices for both players. These structures, called *disjunctive strategies* or, for short, *d-strategies*, appear at a different level of abstraction to strategies. The latter are only defined with respect to given assignments of risk values (and may be different for different assignments), whereas d-strategies abstract away from particular assignments.

Definition 15. *A* d-strategy *(for me) is a tree whose nodes are disjunctions of states:*

$$[A_1^1, \ldots, A_{m_1}^1 \| B_1^1, \ldots, B_{n_1}^1] \bigvee \ \ldots \ \bigvee [A_1^k, \ldots, A_{m_k}^k \| B_1^k, \ldots, B_{n_k}^k]$$

which fulfill the following conditions:

1. *All leaf nodes of a d-strategy denote disjunctions of elementary states.*
2. *Internal nodes are partitioned into I-nodes and you-nodes.*
3. *Any I-node is of the form $\mathcal{G} \bigvee [A \to B, \Gamma \| \Delta]$ and has exactly one successor node of the form $\mathcal{G} \bigvee [\Gamma, B \| A, \Delta] \bigvee [\Gamma \| \Delta]$, where $\mathcal{G}$ denotes a (possibly empty) disjunction of states, and Γ, Δ denote (possibly empty) multisets of formulas.*
4. *For every state $[\Gamma \| \Delta]$ of a you-node and every occurrence of $A \to B$ in Δ, the you-node has a successor node of the form $\mathcal{G} \bigvee [\Gamma, A \| B, \Delta]$ as well as a successor node of the form $\mathcal{G} \bigvee [\Gamma \| \Delta]$. Moreover, there is at least one occurrence of an implication on the right hand side of some disjunct (i.e., state) of a you-node.*[4]

[4] If there is a total of n occurrences of compound formulas on the right hand sides of states in a you-node, then it has $2n$ successor nodes, i.e., corresponding to $2n$ possible moves for you.

We call a d-strategy winning *(for me) if, for all leaf nodes ν and for all possible assignments $\langle\cdot\rangle^*$ of risk values to atomic formulas, there is a disjunct $[p_1, \ldots, p_m \| q_1, \ldots, q_n]$ in ν, such that $\langle p_1, \ldots, p_m\rangle^* \geq \langle q_1, \ldots, q_n\rangle^*$.*

In game theory a winning strategy (for me) is usually defined as a function from all possible states where I have a choice, into the set of my possible moves. Note that winning strategies in the latter sense exist for all assignments of risk values if and only if a winning d-strategy exists.

Strictly speaking we have only defined d-strategies (and therefore, implicitly, also strategies) with respect to some given regulation that, for each possible state, determines who is to move next. Each consistent partition of internal nodes into I-nodes and you-nodes corresponds to such a regulation. However, it has been (implicitly) proved by Giles that the order of moves is irrelevant. Therefore no loss of generality is involved.

The defining conditions for I-nodes and you-nodes not only correspond to possible moves in the dialogue game, but also to the introduction rules for implication in the hypersequent calculus for **Ł** defined in [16]. In fact, every winning d-strategy corresponds to a family of proofs in that hypersequent calculus. In order to establish a similar relation between our uniform r-hypersequent rules and game based characterizations of **Ł**, $\mathbf{\Pi}$, and $\mathbf{G}$, we start by observing that the phrase 'betting for a positive result of (a multiset of) experiments' is ambiguous. As we have seen, Giles identified the combined risk associated with such a bet with the *sum* of risks associated with the single experiments. However, other ways of interpreting the combined risk are worth exploring. In particular, we are interested in a second version of the game, where an elementary state $[p_1, \ldots, p_m \| q_1, \ldots, q_n]$ corresponds to my single bet that *all* experiments associated with the q_is ($1 \leq i \leq n$) show a positive result, against your single bet that *all* experiments associated with the p_is ($1 \leq i \leq m$) show a positive result. A third form of the game arises if one decides to perform only *one* experiment for each of the two players, where the relevant experiment is chosen by the opponent.

To achieve a direct correspondence between the three versions of the game and the standard t-norm based semantics for **Ł**, $\mathbf{\Pi}$, and $\mathbf{G}$, respectively, we invert risk values into probabilites of *positive* results of associated experiments. More formally, the value of an atomic formula q is defined as $\langle q\rangle = 1 - \langle q\rangle^*$; in particular, $\langle\bot\rangle = 0$.

My expected gain in the elementary state $[p_1, \ldots, p_m \| q_1, \ldots, q_n]$ in Giles's game for **Ł** is the sum of money that I expect you to have pay me minus the sum that I expect to have to pay you. This amounts to $\sum_{i=1}^{m}(1 - \langle p_i\rangle) - \sum_{i=1}^{n}(1 - \langle q_i\rangle)$ \$. Therefore my expected gain is greater or equal to zero if and only if the condition $1+\sum_{i=1}^{m}(\langle p_i\rangle - 1) \leq 1 + \sum_{i=1}^{n}(\langle q_i\rangle - 1)$ holds.

In the second version of the game, you have to pay me 1\$ unless all experiments associated with the p_is test positively, and I have to pay you 1\$ unless all experiments associated with the q_is test positively. My expected gain is therefore $1 - \prod_{i=1}^{m}\langle p_i\rangle - (1 - \prod_{i=1}^{n}\langle q_i\rangle)$ \$. The corresponding winning condition is $\prod_{i=1}^{m}\langle p_i\rangle \leq \prod_{i=1}^{n}\langle q_i\rangle$.

To maximize the expected gain in the third version of the game I will choose a $p_i \in \{p_1, \ldots, p_m\}$ where the probability of a positive result of the associated experiment is least; and you will do the same for the q_is that I have asserted. Therefore my expected gain is $(1 - \min_{1\leq i\leq m}\langle p_i\rangle) - (1 - \min_{1\leq i\leq n}\langle q_i\rangle)$ \$. Hence the corresponding winning condition is $\min_{1\leq i\leq m}\langle p_i\rangle \leq \min_{1\leq i\leq n}\langle q_i\rangle$.

We thus arrive at the following definitions for the value of a multiset $\{p_1,\ldots,p_n\}$ of atomic formulas, according to the three versions of the game:

$$\langle p_1,\ldots,p_n\rangle_{\mathbf{Ł}} = 1+\sum_{i=1}^{n}(\langle p_i\rangle - 1) \quad \langle p_1,\ldots,p_n\rangle_{\mathbf{\Pi}} = \prod_{i=1}^{n}\langle p_i\rangle \quad \langle p_1,\ldots,p_n\rangle_{\mathbf{G}} = \min_{1\leq i\leq n}\langle p_i\rangle$$

For the empty multiset we define $\langle\rangle_{\mathbf{Ł}} = \langle\rangle_{\mathbf{\Pi}} = \langle\rangle_{\mathbf{G}} = 1$.

A disjunction of elementary states ν is now called *winning according to logic* $L \in \{$**Ł**, $\mathbf{\Pi}$, $\mathbf{G}\}$ if for every assignment $\langle\cdot\rangle$ of values there is a state $[p_1,\ldots,p_m \| q_1,\ldots,q_n]$ in ν where $\langle p_1,\ldots,p_m\rangle_L \leq \langle q_1,\ldots,q_n\rangle_L$.

It turns out that, in order to characterize $\mathbf{\Pi}$ and $\mathbf{G}$, the dialogue game rule (R) has to be augmented[5] by the following additional rule:

(Q) If I have a strategy for winning the game starting in the state $[A\|B]$, then I am not allowed to attack your assertion of $A \to B$. (And *vice versa*.)[6]

The trees of disjunctive states as presented in Definition 15 do not yet contain all the information that is needed to formulate winning d-strategies for the new versions of the game. To see what kind of information is missing, observe that rule (Q), at the meta-level, corresponds to

– if $v(A) \leq v(B)$, then I have to quit on your assertions of $A \to B$, and you have to quit on my assertions of $A \to B$,

where v is the valuation extending the relevant assignment $\langle\cdot\rangle$ from atomic formulas to arbitrary formulas. Incorporating this fact into the definition of d-strategies seems, at first glance, to require additional notation for *conditions* of the form 'if $A \leq B$'. However, we can use the fact that 'if X then Y' (at the classical meta-level) is equivalent to 'not X or Y'. Thus we remain within the notation for *disjunctive* states, as long as we are willing to use also the strict inequality $<$, in order to be able to express 'not $A \leq B$' as '$B < A$'. Consequently, states $[\Gamma\|\Delta]$ now come in two different forms: $[\Gamma < \Delta]$ and $[\Gamma \leq \Delta]$.

Taking into account these modifications, condition 3 of Definition 15 is replaced by

3'. Any I-node is of the form $\mathcal{G} \bigvee [A \to B, \Gamma \lhd \Delta]$, where $\lhd$ is either $\leq$ or $\leq$. It has exactly two successor nodes: one of the form $\mathcal{G} \bigvee [\Gamma, B \lhd A, \Delta] \bigvee [\Gamma \lhd \Delta]$ and one of the form $\mathcal{G} \bigvee [B < A] \bigvee [\Gamma \lhd \Delta]$.

Note that this new condition corresponds directly to the uniform logical rules $(\to, \lhd, l)$ for r-hypersequents.

In Definition 15 conditions 3 and 4 are dual. In fact, the availabilty of both inequality relations allows us to express the dual to conjunctions of disjunctive states as conjunctions of disjunctive states, by pushing negations inside and finally expressing 'not $\Gamma \leq \Delta$' as '$\Delta < \Gamma$'. After removing some redundancies, the result of this purely

[5] We could have used rule (Q) already in Giles's original game. However, in contrast to the game for $\mathbf{\Pi}$ and $\mathbf{G}$, (Q) does not affect the existence of winning strategies for formulas valid in **Ł**.

[6] Recall that the strategies mentioned in (Q) refer to a given assignment $\langle\cdot\rangle$ of values.

mechanical dualization of condition 3’ results in a version 4’ that corresponds to rule $(\rightarrow, \triangleleft, r)$.

Concluding Remark. We have presented invertible uniform logical rules for the fundamental t-norm based fuzzy logics **Ł**, $\mathbf{G}$, and $\mathbf{\Pi}$, that both provide the basis for Co-NP decision procedures, and may be interpreted within a framework of dialogue games with bets. However, these rules are also sound and invertible for a number of related logics. This raises the interesting question as to which other logics can be characterized in an analogous way. In particular we hope to find a first natural calculus for Hájek’s Basic logic $\mathbf{BL}$ [10], the logic characterizing all logics based on continuous t-norms.

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