Exercise 31

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Problem 13.1: Prove the maximal principle for the case where I is enumerable.

Proof: Enumerate the elements $i_0, i_1, i_2, \ldots$ of I, and build Γ^* as the union of sets Γ_n in P, where $\Gamma_0 = \Gamma$, and $\Gamma_{n+1} = \Gamma_n \cup \{i_n\}$ if $\Gamma_n \cup \{i_n\}$ is in P, and $= \Gamma_n$ otherwise. It is clear that Γ^* contains Γ, and it is left to prove that Γ^* is maximal with respect to P. Suppose that Γ^* were *not* maximal, i.e., there exists a subset Δ of I that is in P and properly includes Γ^*. Since each element of I is considered for inclusion in the construction of Γ^*, there then must be an element i_k with smallest index k that can be found in Δ, but was not included in Γ^* on the ground that $\Gamma_k \cup \{i_k\}$ was not in P. Since i_k is in Δ, and Δ was assumed to (properly) include Γ^*, we obtain $\Gamma_k \cup \{i_k\} \subseteq \Delta$. Since $\Delta \in P$, all its finite subsets are also in P, and in particular, all finite subsets of $\Gamma_k \cup \{i_k\}$ are also in P. Therefore $\Gamma_k \cup \{i_k\}$ is itself in P, and this contradicts the assumption that i_k was rejected in the construction of Γ^*.