Exercise 26

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Exercise 12.12: Show that if two models of Eq are isomorphic, then the equivalence relations of the models have the same signature.

Proof: Let $\mathcal{A}$ and $\mathcal{B}$ be two isomorphic models of Eq, that is, there exists a correspondence j between the individuals of $|\mathcal{A}|$ and those of $|\mathcal{B}|$ subject to properties (I1), (I2) and (I3). It suffices to show that there exists a correspondence c between the equivalence classes in $|\mathcal{A}|$ and $|\mathcal{B}|$ such that for every equivalence class $C \subseteq |\mathcal{A}|$, $\text{card}(C) = \text{card}(c(C))$.

Let $C \subseteq |\mathcal{A}|$ be an equivalence class in $|\mathcal{A}|$. For any two $e_1, e_2 \in C$, $e_1 \equiv^{\mathcal{A}} e_2$, and by property (I1), $j(e_1) \equiv^{\mathcal{B}} j(e_2)$. Define a mapping c as $c(C) = \{j(e)|e \in C\}$. $c(C)$ is an equivalence class in $|\mathcal{B}|$, for suppose there were an element $b \notin c(C)$ such that for some $d \in c(C)$ it would hold that $d \equiv^{\mathcal{B}} b$, it would follow that $j^{-1}(b)$ were not in C (where j^{-1} denotes the inverse of j, which is known to exist since j is onto and one-to-one), and that would contradict property (I1). It is clear that $\text{card}(C) = \text{card}(c(C))$, due to the way c was defined. Furthermore, the mapping c is one-to-one, since equivalence classes in $|\mathcal{A}|$ are disjoint, and j is one-to-one. It is also onto, since the set of all equivalence classes in $|\mathcal{A}|$ is exhaustive, and j is onto. Hence, c is a correspondence between the equivalence classes in $|\mathcal{A}|$ and $|\mathcal{B}|$, and the equivalence relations induced by the respective sets of equivalence classes must have the same signature.