Exercise 15

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Exercise: Let A be the following sentence:
(1) $\forall x_1 \forall x_2 (f(x_1) = f(x_2) \rightarrow x_1 = x_2)$ &
(2) $\forall y \exists x (f(x) = y)$ &
(3) $\forall x \forall y (f(x) = y \rightarrow (Px \leftrightarrow \sim Py))$.
Show that the spectrum of A is the set of all even positive integers.

Proof: Given an even positive integer n, we can construct a model $\mathcal{M}$ for A with $\text{card}(|\mathcal{M}|) = n$ as follows: Let $|\mathcal{M}| = \{m_1, m_2, \ldots, m_n\}$. Let P denote a one-place predicate such that $P^{\mathcal{M}}(m_i)$ for all even $i \in \{1, 2, \ldots n\}$ and *not* $P^{\mathcal{M}}(m_j)$ for all odd $j \in \{1, 2, \ldots n\}$. Let $f^{\mathcal{M}}(m_k) = m_{k-1}$ for all $k \in \{2, \ldots n\}$ and $f^{\mathcal{M}}(m_1) = m_n$. It is easy to see that $\mathcal{M} \models A$.

It is now left to prove that $\mathcal{M}$ can not be a model of A if $\text{card}(|\mathcal{M}|)$ is odd. It is clear that $\text{card}(|\mathcal{M}|) > 1$: Let $\mathcal{M}$ be a finite model of A and choose an individual $m \in |\mathcal{M}|$. From (2) it follows that there must be an individual $n \in |\mathcal{M}|$ such that $f^{\mathcal{M}}(n) = m$, and from (3) it follows that m and n are distinct, for suppose $m = n$, i.e., $f^{\mathcal{M}}(m) = m$, then (3) would imply that $P^{\mathcal{M}}(m) \leftrightarrow \sim P^{\mathcal{M}}(m)$, which can not be the case. Therefore, $\text{card}(|\mathcal{M}|) \geq 2$ for all models $\mathcal{M}$ of A, and it also follows that there must exist an individual—call it p_1—in $|\mathcal{M}|$ such that $P^{\mathcal{M}}(p_1)$.

We will now enumerate the individuals in $|\mathcal{M}|$ in a certain way: Start with p_1; next, write down the individual p_2, where $f^{\mathcal{M}}(p_2) = p_1$. The existence of p_2 was established above, and its uniqueness follows from (1). From (3) we can deduce that *not* $P^{\mathcal{M}}(p_2)$. Continuing in this manner, we obtain a sequence of individuals p_i, yielding alternating truth values of $P^{\mathcal{M}}(p_i)$. This sequence can come to a halt if and only if for some p_j, $f^{\mathcal{M}}(p_1) = p_j$ (the sequence must stop eventually, since the domain is finite). If this happens, it must consist of an even number of individuals, because $P^{\mathcal{M}}(p_1)$ is true, the truth values alternate, and $P^{\mathcal{M}}(p_j)$ must be false by (3). If a sequence comes to a halt, we remove the participating individuals from the domain (its individuals can be seen as "connected", and by uniqueness demanded in (1) none of the remaining elements in the domain can be "connected" with any element of the sequence) and apply the same procedure to the remaining elements. Since all emerging sequences consist of an even number of elements and eventually all individuals of the domain are used up, the number of elements in the domain must as well be even.