CONTINUOUS FRAÏSSÉ CONJECTURE

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PART I

THE STARTING POINT

GÖDEL LOGICS

Syntax and Semantics

Truth value set	$\{0,1\} \subseteq V \subseteq [0,1]$, V closed
Language	countable quantified propositional
$\wedge$	min
$\vee$	max
$\to$	projection on smaller or 1
$\forall x A$	infimum
$\exists x A$	supremum

SEMANTICS CONT.

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$$\begin{aligned} V_\infty &= [0,1] \\ V_\downarrow &= \{0\} \cup \{1/n : n \geq 1\} \\ V_\uparrow &= \{1\} \cup \{1 - 1/n : n \geq 1\} \\ V_k &= \{1\} \cup \{1 - 1/n : n = 1, \ldots, k-1\} \end{aligned}$$

NUMBER OF GÖDEL LOGICS?

- Question not useless – it's about the expressive power of Gödel logics
- the results known show only that there are at least $\aleph_0$ with infinite truth value set (see e.g. Baaz et al. (1996); P. 2002)
- Evidence for $\aleph_1$: Extension with principles (intermediate logics, modal logics)

EXTENSIONAL DEFINITION – ORDER STRUCTURE

absolute values — relative position

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$$V_{\uparrow} = \{1 - \frac{1}{k} : k \geq 1\} \cup \{1\}$$

$$V = V_{\uparrow} \setminus \{\frac{1}{2}\}$$

COUNTING GÖDEL LOGICS

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$V_1 \neq V_2$

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$V_1 \neq V_2$

Gödel sets + smc-embeddability = quasi ordering, ...

PART II

BACK TO SCHOOL

ORDER THEORY

QUASI ORDERINGS

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A $(Q, \leq)$ with reflexive and transitive $\leq$ is a *quasi-ordering*.

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Example

all linear orderings containing infinite descending chains, e.g., the order types of dense suborderings of $\mathbb{R}$.

WELL-QUASI-ORDERINGS

Conjecture (Fraïssé (1948); Laver (1971))

The set of scattered linear orderings is more well-behaved, i.e., it is a well-quasi-ordering.

WELL-QUASI-ORDERINGS CONT.

Definition

A set Q is a *well-quasi-ordering* (wqo) if one of the following conditions hold

- There are no infinite antichains and no infinite descending chains in Q.
- All sequences of elements in Q are *good* (i.e., if there are n and k such that $n < k$ and $q_n \leq q_k$).
- All sequences of elements in Q contain either a strictly increasing subsequence of a constant subsequence.

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Question: What about Gödel sets with smc-embeddability?

ORDERING FINITE SEQUENCES

Definition

If Q is a quasi-ordering, define the quasi-ordering of $Q^{<\omega}$ of finite sequences of in Q

$$\langle p_0, p_1, \ldots, p_{n-1} \rangle \leq \langle q_0, q_1, \ldots, q_{m-1} \rangle$$

if there is a strictly increasing h with

$$h : n \to m, \quad p_i \leq_Q q_{h(i)} \text{ for all } i < n.$$

ORDERING ω SEQUENCES

Definition

The quasi-ordering Q^ω of ω-sequences of elements of Q:

$$\langle p_n : n < \omega\rangle \leq \langle q_n : n < \omega\rangle$$

if there is a strictly increasing h with

$$h : \omega \to \omega, \quad p_n \leq_Q q_{h(n)} \text{ for all } n < \omega.$$

WHY THIS STRANGE SEQUENCE ORDERING

Let $\langle L_0, L_1, \ldots, L_{n-1} \rangle$ and $\langle M_0, M_1, \ldots, M_{m-1} \rangle$ be two finite sequences of linear orderings.

One way for $L_0 + L_1 + \cdots + L_{n-1}$ to be embeddable in $M_0 + M_1 + \cdots + M_{m_1}$ is if

$$\langle L_0, L_1, \ldots, L_{n-1} \rangle \leq \langle M_0, M_1, \ldots, M_{m-1} \rangle$$

decomposition into infinite sums with smaller ‘rank’

RESULTS FOR WQO

Theorem

If Q is a wqo, then $Q^{<\omega}$ also is a wqo.

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Unfortunately this is not true for Q^ω:

Let $Q = \{(i, j) : i < j < \omega\}$ be quasi-ordered by

$$(i, j) \leq (k, l) \text{ iff either } j < k \text{ or } i = k \text{ and } j \leq l.$$

Define

$$\vec{q}_i = \langle (i, i+1), (i, i+2), (i, i+3), \ldots \rangle,$$

then

$$\langle \vec{q}_1, \vec{q}_2, \ldots \rangle$$

is a bad sequence of elements in Q^ω.

BETTER QUASI ORDERINGS

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bqo: $h : B \to L$ such that if $b_1 \triangleleft b_2$, then $h(b_1) \leq h(b_2)$.

BETTER QUASI ORDERINGS

wqo: $h : \omega \to L$ such that if $i < j$, then $h(i) \leq h(j)$.

bqo: $h : B \to L$ such that if $b_1 \lhd b_2$, then $h(b_1) \leq h(b_2)$.

Definition

An infinite set B of finite subsets of $\mathbb{N}$ is a *block* if every infinite subset X of $\bigcup B := \bigcup \{b : b \in B\}$ has an initial segment in B.

A block B is called a *barrier* if no two elements of B are comparable w.r.t. inclusion.

EXAMPLES FOR BLOCKS AND BARRIERS

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$\{0\}, \{1\}, \{2, 3\}, \{2, 4\}, \ldots, \{3, 4\}, \{3, 5\}, \ldots, \ldots$ is a barrier for $\mathbb{N}$.

BQO CONT.

Definition B barrier, $h : B \to Q$

$$\begin{array}{lll} h \text{ bad} & \text{iff} & b_1, b_2 \in B \\ & & b_1 \lhd b_2 \\ & & h(b_1) \not\leq_Q h(b_2) \end{array}$$

$$\begin{array}{lll} b_1 \lhd b_2 & \text{iff} & i_1 < i_2 < \cdots < i_m \\ & & b_1 = \{i_1, i_2, \ldots, i_k\} \\ & & b_2 = \{i_2, \ldots, i_m\} \end{array}$$

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Example $\langle 2, 4, 6, 8\rangle \lhd \langle 4, 6, 8, 11, 42\rangle$, $\langle i\rangle \lhd \langle j\rangle$

BQO CONT.

Definition

We say that Q is a *better-quasi-ordering* (bqo) if every $h : B \to Q$ is good, for every barrier B of finite subsets of $\mathbb{N}$.

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Theorem (Nash-Williams (1968))

If Q is a bqo, then $Q^{<\omega}$ and Q^{ω} are bqos.

COUNTABLE CLOSED LINEAR ORDERINGS

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labeled cclo: fix a qo Q, a Q-cclo is a function A whose domain $\operatorname{dom} A$ is a cclo and whose range is contained in Q.

$A \preceq B$ (Q-smc-embeddable) iff there is a smc-embedding h from $\operatorname{dom} A$ to $\operatorname{dom} B$ and $A(a) \leq_Q B(h(a))$ for all $a \in \operatorname{dom} A$.

EXAMPLES FOR Q-CCLOS

If Q is a singleton, then $A \preceq B$ is just smc-embedding from $\operatorname{dom} A$ to $\operatorname{dom} B$.

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If Q is a singleton, then $A \preceq B$ is just smc-embedding from $\operatorname{dom} A$ to $\operatorname{dom} B$.

If $Q = \{p, q\}$ is an antichain, and $A(0) = A(1) = p$ and $A(b) = q$ for $b \neq 0, 1$, then $A \preceq B$ is a smc-embedding from $\operatorname{dom} A$ into $\operatorname{dom} B$ which additionally preserves 0 and 1.

NOTATIONS OF Q-CCLO TERMS

We will use the following notation:

$$L_0 + L_1 + L_2 + \ldots + p + \ldots + {}_2L + {}_1L + {}_0L$$

or

$$\sum L_i + p + \sum{}^{*}{}_iL$$

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Following conditions are imposed:

- p is an element of Q.
- All the L_i and ${}_iL$ are Q-cclo's.
- Either all L_i are empty, or none of them are empty.
- $\operatorname{dom} L_i < \operatorname{dom} L_{i+1} < \operatorname{dom} {}_{i+1}L < \operatorname{dom} {}_iL$ for all i
- $\lim_{n\to\infty} a_n = \lim_{n\to\infty} {}_na$

TWO OPERATIONS TO BUILD Q-CCLOS

Sums from $\mathscr{O}$: Let $\mathscr{O}$ be a class of Q-cclo's. The set $S(\mathscr{O})$ be the set of all Q-cclo's which are finite sums of Q-cclo's from $\mathscr{O}$, plus the set of all Q-cclo's of the form

$$L_0 + L_1 + L_2 + \ldots + p + \ldots + {}_2L + {}_1L + {}_0L$$

where $p \in Q$ and all L_n and all ${}_nL$ are in $\mathscr{O}$.

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where $p \in Q$ and all L_n and all ${}_nL$ are in $\mathscr{O}$.

Unbounded sums from $\mathscr{O}$: Additional requirement

$$\forall n\, \exists k > n\ L_n \preceq L_k \ \text{ and } \ \forall n\, \exists k > n\ {}_nL \preceq {}_kL.$$

CLASSES OF Q-CCLOS

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Let $\mathscr{C}$ be the set of all Q-cclo's.

Let $\mathscr{C}_0 = \mathscr{C}_0'$ be the set of all Q-cclo's with singleton or empty domain. For any $\alpha \leq \omega_1$ let

$$\mathscr{C}_{\alpha+1} = S(\mathscr{C}_\alpha) \cup \mathscr{C}_\alpha \qquad \mathscr{C}_{\alpha+1}' = S'(\mathscr{C}_\alpha') \cup \mathscr{C}_\alpha'$$

and for limit ordinals $\delta > 0$ let $\mathscr{C}_\delta = \bigcup_{\alpha<\delta} \mathscr{C}_\alpha$, $\mathscr{C}_\delta' = \bigcup_{\alpha<\delta} \mathscr{C}_\alpha'$.

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Rank are defined as usual for such a inductive definition.

RESULTS ON Q-CCLOS

Theorem (all Beckmann et al. (20??))

$\mathscr{C} = \mathscr{C}_{\omega_1}$, and every order in $\mathscr{C}$ can be written as finite sum of orders from $\mathscr{C}'_{\omega_1}$.

Theorem

If $(\mathscr{C}', \preceq)$ is a bqo, then $(\mathscr{C}, \preceq)$ is a wqo.

Theorem

$(\mathscr{C}', \preceq)$ is a bqo.

RESULTS ON Q-CCLOS CONT.

Corollary

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Corollary

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Lemma

Let $(Q, \leq)$ be a wqo with uncountable many $\equiv$-equivalence classes. Then there exists a 1-1 monotone map $f : \omega_1 \to Q$.

PROOF IDEA FOR $(\mathscr{C}', \preceq)$ IS A BQO

Let $\mathcal{F}$ denote the set of all functions $g : B \to Q$ where B is an arbitrary barrier for $\mathbb{N}$.

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Let $\mathcal{F}$ denote the set of all functions $g : B \to Q$ where B is an arbitrary barrier for $\mathbb{N}$.

Let $g : B \to Q$ and $h : C \to Q$ be two functions from $\mathcal{F}$. We can define a notion of 'shorter' on these functions (C is proper extended sub-barrier, g and h coincide on the intersection, and if c extends b, then $h(c) \leq g(b)$, and $h(c)$ has lower rank than $g(b)$).

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If $\mathcal{F}$ contains some bad function, than it contains some minimal bad function (minimal w.r.t. shorter).

PROOF IDEA CONT.

Now assume that there is a bad function, then there is a minimal one f. We can find a sub-barrier B' such that the place of 'evil' (not embedabbility) occurs in one of the three parts of the sum-terms.

PROOF IDEA CONT.

Now assume that there is a bad function, then there is a minimal one f. We can find a sub-barrier B' such that the place of ‘evil’ (not embedabbility) occurs in one of the three parts of the sum-terms.

Then we consider the barrier $B'(2) = \{b_1 \cup b_2 : b_1, b_2 \in B' \text{ and } b_1 \lhd b_2\}$ and define $g : B'(2) \to C'$ by letting $g(b_1 \cup b_2)$ be the first representative of evil. Then g is shorter than f, but also bad, which contradicts the minimality of f.

PART III

BACK TO GÖDEL LOGICS

ORDERING GÖDEL LOGICS

V_1 and V_2 Gödel sets

$Q = \{0, 1\}$ with $0 <_Q 1$

A_1 and A_2 be Q-labeled cclos
$\mathrm{dom}(A_i) = V_i$, $A_i(0) = A_i(1) = 1$ and $A_i(x) = 0$ o.w.

If $A_1 \preceq A_2$ then $G(V_1) \supseteq G(V_2)$.

WQORDERING GÖDEL LOGICS

Lemma

The class of countable Gödel logics, ordered by $\supseteq$, is a wqo.

Proof.

Take an infinite sequence of countable Gödel logics $\langle \mathbf{G}_n : n \in \omega \rangle$. Define the respective Q-labeled cclos V_n as above. Since the Q-cclos are a wqo, this sequence must be good, hence also the sequence of Gödel logics. □

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Lemma

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Proof.

Each countable Gödel logic is a subset of the fixed countable set of all formulas, thus it cannot contain a copy of ω_1, thus it is countable. □

THE NUMBER OF GÖDEL LOGICS

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Proof.

The Gödel logics determined by $V = P \cup C$ and $V' = V \cup [\inf P, 1]$ are the same. Thus, only the countable part $V \cap [0, \inf P]$ is relevant for discerning logics. Thus we have $2 \cdot \aleph_0 + 1$, i.e. $\aleph_0$ many Gödel logics. □

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